NLO corrections to W boson production in polarised proton-proton collisions

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Contents

Abstract

In this thesis, I calculate the spin-dependent next-to-leading order QCD corrections to the differential scattering cross section for W boson production by proton-proton collisions. Apart from the relevant Feynman diagrams, the crossing functions are derived as well, allowing a complete description of the hadronic scattering cross section to NLO as a function of hadron polarisation. This quantity can be used to calculate the polarisation asymmetry, which is very well suited for direct experimental measurement. Exploiting this possibility to directly compare theoretical results with experimental findings, one will be able to gain deeper insight into the spin-dependent parton distribution functions, which in turn will help in making further progress towards solving the proton spin puzzle.

Preface

This is the account of how
all was in suspense,
all calm,
in silence;
all motionless,
all pulsating,
and empty was the expanse of the sky.

Popol Vuh, creation story

The task of my PhD thesis was to calculate the different contributions at next-to-leading order (short "NLO") to the differential scattering cross section for the process $pp \to W^\pm \to \ell^\pm \nu$:

- the partonic scattering cross sections following from the NLO Feynman diagrams:
  - the vertex diagram (virtual correction), $q\bar{q} \to W^\pm \to \ell^\pm \nu$
  - the gluon bremsstrahlung diagram (real correction), $q\bar{q} \to W^\pm g \to \ell^\pm \nu g$
  - the quark bremsstrahlung diagram (real correction), $qg \to W^\pm q \to \ell^\pm \nu q$ or with $q \leftrightarrow \bar{q}$;
- the phase spaces for these diagrams, especially for the $2 \to 3$ processes, with appropriate slicing (separation of soft and collinear parts) where necessary;
- the crossing functions, which include contributions from unresolved particles.

Combining these parts makes it possible to express the hadronic differential scattering cross section at next-to-leading order.

This thesis is organised as follows:
The first chapter is dedicated to a short introduction into the subject, a motivation for the task undertaken and its connection to experiments as well as an outlook on what can be learned from the comparison of these theoretical predictions to experimental findings. In chapter 2, the different diagrams are drawn and the hadronic differential scattering cross section is derived to NLO. All the required quantities are listed, including the crossing functions, which are introduced and calculated. The next three chapters (3, 4, 5) show in detail the calculation of the matrix elements squared (or multiplied with the LO matrix element) and the relevant phase spaces, including slicing. From these quantities, the contributions to the differential scattering cross section are then derived. Chapter 6 collects the results from the calculations of the Feynman diagrams and shows that the singularities cancel, as expected. Picking up the findings of chapter 2, the hadronic scattering cross section is finally written to NLO and the components shown. The conclusions from the calculation and a short outlook can be found in chapter 7.

"FRANKLY, I HAVE A LOT OF TROUBLE RELATING TO ALL THIS SYMBOLISM."

Acknowledgements

So long, and thanks for all the fish.

DOUGLAS ADAMS: Hitchhiker's Guide to the Galaxy, Book 4

The calculations shown below, or more precisely the task to carry them out, have given me the possibility to focus on a single subject for a longer period of time and delve into the very interesting but also challenging world of regularisation, renormalisation and the treatment of singularities in the context of QCD corrections. It has been a great opportunity to do some work on my own, use and enlarge my knowledge, and work at the level of research. Furthermore, I could profit extensively from this new collaboration we established with Prof. Dr. Thomas Gehrmann of the University of Zurich. Without it, this work could never have been achieved. During my studies, I received steady support and encouragement from PD Dr. Andreas Aste and Thomas Gehrmann. To them I would like to express my sincere gratitude! Many thanks also go to Prof. Dr. Dirk Trautmann for giving me the possibility to do this thesis in the first place and providing a very stimulating working environment. Financial support has been granted by the Swiss National Science Foundation.

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- Sid Harris of http://ScienceCartoonsPlus.com for allowing me to use some of his cartoons in my thesis.

"Quarks. Neutrinos. Mesons. All those damn particles
you can't see. That's what drove me to drink.
But now I can see them!"

1 Introduction

Beginning my studies the first step pleas'd me so much,
I have hardly gone and hardly wish'd to go any farther,
But stop and loiter all the time to sing it in ecstatic songs.

Walt Whitman: Beginning my studies

Within the standard model, the hadrons (protons and neutrons) are made up of three valence quarks, characterising their parent hadron, and sea quarks, which spring in and out of existence as quark-antiquark pairs and are the product of vacuum fluctuations. The quarks, denoted generically by q, and their antiparticles, denoted by $\bar{q}$, come in six flavours: up (u) and down (d) in the first generation, strange (s) and charm (c) in the second generation, and bottom (b) and top (t) in the third generation. Quark masses range from $m_u \approx 1.5$ MeV through $m_c \approx 1.3$ GeV up to $m_t \approx 180$ GeV. They are held together by massless gluons (g) which are their own antiparticles. Collectively, quarks and gluons are called partons, a name conceived in the first days of the model when an experimental verification still had to be found.
A natural assumption then is that the different properties of a hadron are made up from those of its constituent partons. Of interest here is the spin: being a fermion, a proton has a spin of 1/2. The quarks themselves, since they are fermions as well, have a spin of 1/2, too, and by combining their spins make up the hadron's spin. Unfortunately, as experiments have shown, this simple model does not correspond to reality, i.e. measurements.
With the notation (a proper definition of the entity q follows further down)

$$\Delta q := q_+^+ + q_-^- - q_-^+ - q_+^- \quad \text{and} \quad q := q_+^+ + q_-^- + q_-^+ + q_+^- \ ,$$

where the upper index describes the helicity (whether the spin is parallel or anti-parallel to the momentum) of the parton and the lower one the helicity of the parent hadron, the flavour-singlet axial charge can be defined:

$$\Delta\Sigma := \Delta u + \Delta d + \Delta s$$

which is a measure of the spin distributed over the valence quarks.
While one would expect

$$\Delta\Sigma = 1$$

if the hadron's entire spin were distributed solely among these quarks, experimental measurements, starting with the European Muon Collaboration [30,31] in 1988, have found quite a different figure for the proton, with a modern value of

$$\Delta\Sigma = 0.33 \pm 0.03 \text{ (stat.)} \pm 0.05 \text{ (syst.)} \ .$$

Why the valence quarks contribute so little to their hadron's spin and where the missing spin contribution is to be found came to be known as the *proton spin puzzle* (for a recent review, see e.g. [3] and references therein). Since the occurrence of this enigma, a lot of effort has been made towards understanding the spin content of the hadrons and how it is distributed among their constituent partons. Possible solutions include strong gluon polarisation, sea quark polarisation,

topological effects and contributions from angular momentum as well as relativistic motion of the partons. Apart from theoretical work, dedicated experiments have been carried out at CERN, DESY, JLab, BNL's RHIC and SLAC.
From a theorist's point of view, the constituents' spin is contained inside the parton distribution functions (PDFs), usually denoted $f_h^H(x)$ or $h(x)$, which describe the probability of finding a parton of flavour h inside a hadron H carrying the momentum fraction x of its parent hadron. The PDFs are the non-perturbative part of the expression connecting the hadrons' differential scattering cross section with the partons' – perturbatively calculable – cross section:

$$d\sigma_{AB} = \sum_{ab} \int_0^1 dx_1 \int_0^1 dx_2 \, f_a^A(x_1) f_b^B(x_2) \, d\sigma_{ab}(x_1, x_2)$$

Finding suitable parametrisations of the PDFs by fitting the predictions to the measurements and thereby refining the functions derived from theory has been a major topic in research [35, 21]. In a first step, these parametrisations have been constructed for spin-averaged partons and only relatively recently have the groups begun to incorporate spin-dependency into these functions [27]. This in turn makes the comparison of polarised measurements with predictions from theory possible and at the same time very important. After all, if one is able to quantitatively describe the PDFs, the distribution of spin inside a hadron can be very accurately described and a large step towards the solution of the spin puzzle has been made. Of course, the other way round works as well: By finding a solution to the spin problem, one will gain greater insight into the spin-dependency of the partons which in turn can be used to increase the precision of the PDFs. The path taken by research has been something intermediate and while one still has large uncertainties in the PDFs, they are no longer large enough to be able to swallow the missing spin. These findings have led to further progress and very recently a solution to the spin problem has been proposed [22]. This in turn will renew interest in the PDFs and make it even more important to find possible processes which can be described theoretically as well as measured with great precision.

One such process is the production of W bosons by proton-proton (pp) collisions. Currently, such experiments are carried out at Brookhaven's RHIC, in an energy range allowing this process. Furthermore, W boson production is a tool very well suited to explore the PDFs because these particles are produced by the weak interaction, which is a pure V-A interaction (i.e. "vector − axial vector" coupling within the standard model, conserving helicities). Thus, the helicities of the participating quarks and antiquarks are fixed in the reaction. In addition, the W couples to a weak charge that correlates directly to flavours, if one concentrates on one generation. Indeed, the production of Ws in pp collisions is dominated by u, d, $\bar{u}$, and $\bar{d}$, with some contamination from s, c, $\bar{s}$, and $\bar{c}$, mostly through quark mixing. With a W boson mass of $M_W \approx 80$ GeV, the masses of these quarks can be neglected, resulting in greatly simplified expressions (the error introduced lies well below the order of the series expanson in the coupling constant). Therefore, W production is considered an ideal tool to study the spin-flavour structure of the nucleon.
The asymmetry of W boson production with respect to the leptons' rapidity distribution,

$$A(y_\ell) = \frac{d\sigma(\ell^+)/dy_\ell - d\sigma(\ell^-)/dy_\ell}{d\sigma(\ell^+)/dy_\ell + d\sigma(\ell^-)/dy_\ell}$$

with $d\sigma(\ell^\pm)/dy_\ell$ the differential scattering cross section for the process $p\bar{p} \to W^\pm \to \ell^\pm\nu$ as a function of lepton rapidity, has been studied at leading order [19] in conjunction with measurements by the CDF collaboration at Fermilab's Tevatron collider.
In the case of only one proton polarised, the leading-order production of W^+s via $u\bar{d} \to W^+$ (and

W^-s completely analogously via $\bar{u}d \to W^-$) can be built up from four different cases where either the u or the $\bar{d}$ stem from the polarised proton and have either positive or negative helicity. Colliding with the other parton coming from the unpolarised proton, they form the W^+ boson, which decays subsequently:

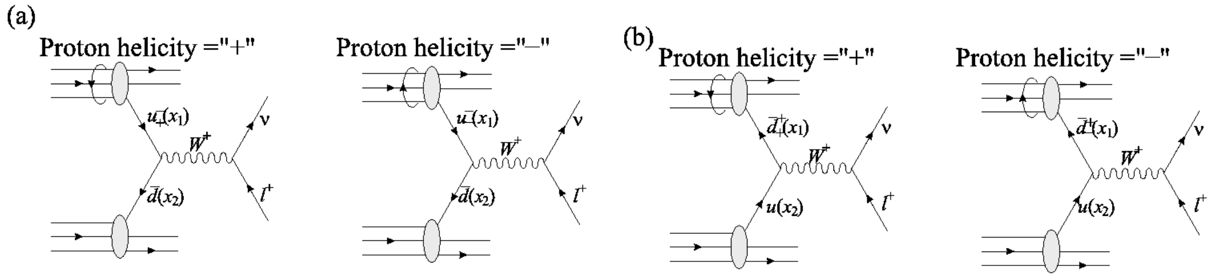


Figure 1.1: *The four possible situations for the case of one proton polarised and one unpolarised (+ is right-handed, − is left-handed helicity).*

The parity-violating asymmetry is defined as the difference of left-handed and right-handed production of Ws, divided by the sum and normalised by the beam polarisation:

$$A_L^{W^+} = \frac{1}{P} \times \frac{N_-(W^+) - N_+(W^+)}{N_-(W^+) + N_+(W^+)} \; .$$

One can construct this asymmetry from either one of the polarised beams and by summing over the helicity states of the other beam. The production of left-handed weak bosons violates parity maximally (because the antilepton in the final state would then also be left-handed). Therefore, if for example the production of the W^+ proceeded only through the process where the u quark is polarised (as depicted in Figure 1a), the parity-violating asymmetry would directly equal the longitudinal polarisation asymmetry of the u quark in the proton:

$$A_L^{W^+} = \frac{u_-^-(x_1)\bar{d}(x_2) - u_+^-(x_1)\bar{d}(x_2)}{u_-^-(x_1)\bar{d}(x_2) + u_+^-(x_1)\bar{d}(x_2)} = \frac{\Delta u(x_1)}{u(x_1)} \; .$$

Similarly for the case where only the $\bar{d}$ is polarised (Figure 1b):

$$A_L^{W^+} = \frac{\bar{d}_-^+(x_1)u(x_2) - \bar{d}_+^+(x_1)u(x_2)}{\bar{d}_-^+(x_1)u(x_2) - \bar{d}_+^+(x_1)u(x_2)} = -\frac{\Delta\bar{d}(x_1)}{\bar{d}(x_1)} \; .$$

In general, the asymmetry is a superposition of the two cases [6], expressed with the corresponding scale M_W and as a function of the vector boson rapidity y ($x_{1,2} \propto e^{\pm y}$):

$$A_L^{W^+}(y) = \frac{\Delta u(x_1, M_{W^2})\bar{d}(x_2, M_{W^2}) - \Delta\bar{d}(x_1, M_{W^2})u(x_2, M_{W^2})}{u(x_1, M_{W^2})\bar{d}(x_2, M_{W^2}) + \bar{d}(x_1, M_{W^2})u(x_2, M_{W^2})} \; .$$

For W^-, the asymmetry is obtained by interchanging u and d.

By identifying the rapidity of the W boson relative to the polarised proton, it is possible to obtain a direct measurement of the quark and antiquark polarisations while distinguishing the different quark flavours. $A_L^{W^+}$ approaches $\Delta u/u$ in the limit $y \gg 0$, whereas for $y \ll 0$ the asymmetry becomes $-\Delta\bar{d}/\bar{d}$. In practice one can probe, e.g., the polarised antiquark distributions at RHIC for $x \leq 0.12$ from $A_L(y \leq 0)$ [13].

Because a direct detection of the W boson is impossible, one has to infer the boson rapidity y from the measurable lepton rapidity y_ℓ. The W's rapidity is related to the lepton rapidity in the W rest frame (y_ℓ^*) and in the laboratory frame (y_ℓ^{lab}) by $y_\ell^{lab} = y_\ell^* + y$, where $y_\ell^* = 1/2 \cdot \ln[(1+\cos\theta^*)/(1-\cos\theta^*)]$, with θ^* the decay angle of the lepton in the W rest frame, and $\cos\theta^*$ can be determined from the transverse momentum of the lepton with an irreducible uncertainty of the sign, if one neglects the transverse momentum of the W.

Since RHIC is a pp collider, the antiquarks stem from a proton as well and a measurement of the asymmetry will therefore reveal further information on the distribution of sea quarks. In the $p\bar{p}$ experiments, the contribution of sea quarks has been strongly suppressed with respect to valence quarks and the measurements have permitted only little insight into the (anti)quark sea. RHIC is now aiming at filling this important gap, which is all the more relevant because recent experiments have shown a large SU(2) symmetry breaking in the antiquark sea [32].

The aim of my PhD thesis is to increase the precision of the predictions by calculating the polarisation-dependent differential scattering cross section to next-to-leading order. An increase in precision for this quantity directly results in a greater precision in the asymmetry and thus a stronger constraint on the PDF fits. This, however, comes at a price: One has to deal with infrared and ultraviolet divergences in the mathematical expressions and find suitable methods to obtain sensible – i.e. finite or at least mathematically well-defined – results. All the relevant diagrams have been considered already in my M.S. thesis [36], but the calculations for the ones contributing at NLO have been carried out only up to the scattering matrix element. The mathematically sensitive process of squaring the expressions, carrying out the integrations, and dealing with the phase space has been left for this work. In addition to the Feynman diagrams, the crossing functions have to be calculated as well, completing the list of ingredients necessary for a description of the hadronic differential scattering cross section to NLO. Only the calculations for the production of W^+ bosons will be shown explicitly; by choosing the appropriate flavour of the PDF, the results can be trivially adapted to describe W^- bosons instead.

2 NLO corrections and perturbative QCD

Latin me that, my trinity scholard, out of eure sanscreed into oure eryan!

JAMES JOYCE: Finnegans Wake

2.1 Next-to-leading-order approximation of the scattering cross section

The NLO approximation of the differential scattering cross section for hadron collisions can be written (within the QCD-improved parton model) as the convolution of the partonic cross section with the parton structure functions (which are effective parton distribution functions) $\mathcal{F}$:

$$d\sigma_{H_1H_2} = \sum_{ab} \int dx_1 dx_2 \, \mathcal{F}_a^{H_1}(x_1)\mathcal{F}_b^{H_2}(x_2)\, d\sigma_{ab}(x_1, x_2) \; ,$$

where the x_i are the partons' momentum fractions of their parent hadrons H_i. Expanding the differential cross section and the effective structure functions in a series in the coupling constant yields:

$$\begin{aligned} d\sigma_{ab} &= d\sigma_{ab}^{LO} + \alpha_s d\sigma_{ab}^{NLO} + \mathcal{O}(\alpha_s^2) \\ \mathcal{F}_a^{H_1}(x_1) &= f_a^{H_1}(x_1, \mu_F) + \alpha_s C_a^{H_1}(x_1, \mu_F) + \mathcal{O}(\alpha_s^2) \; . \end{aligned}$$

The $C_a^H(x, \mu_F)$ are called *crossing functions* and account for unresolved partons, as shown below; the $f_a^H(x, \mu_F)$ are the parton distribution functions (PDFs) appearing in the DGLAP equations. Both are dependent on the factorisation scale μ_F.
Combining the three expressions, we get for the cross section:

$$\begin{aligned} d\sigma_{H_1H_2} = \sum_{ab} \int dx_1 dx_2 \Big\{ & f_a^{H_1}(x_1, \mu_F) f_b^{H_2}(x_2, \mu_F) \left[d\sigma_{ab}^{LO}(x_1, x_2) + \alpha_s d\sigma_{ab}^{NLO}(x_1, x_2) \right] \\ & + \alpha_s \Big[C_a^{H_1}(x_1, \mu_F) f_b^{H_2}(x_2, \mu_F) + f_a^{H_1}(x_1, \mu_F) C_b^{H_2}(x_2, \mu_F) \Big] d\sigma_{ab}^{LO}(x_1, x_2) + \mathcal{O}(\alpha_s^2) \Big\} \; . \end{aligned} \tag{2.1}$$

Thus, for the complete differential cross section at next-to-leading order one needs the NLO contribution to the cross section as well as the crossing functions. The expression $d\sigma_{ab}^{NLO}$ contains the NLO diagrams, shown in the next section; the crossing functions will be calculated in the remainder of this chapter. To calculate with well-defined expressions and pole structures, one applies a trick, detailed in [16], when calculating the phase space for the bremsstrahlung contributions: Simply put, one calculates the contribution to $d\sigma_{ab}^{NLO}$ with the wrong phase space and cancels this mistake with the appropriate contribution to the crossing function. This procedure ensures that the crossing function is a finite quantity and that the pole structure of the gluon bremsstrahlung diagram exactly cancels the one of the vertex diagram. Without it, one would have to balance these three quantities against each other to obtain a finite contribution to the differential scattering cross section, making the calculations cumbersome. This procedure only works because the NLO contributions to the differential cross section can be written as a prefactor times the LO cross section! Thus, we may safely move the prefactors from one bracket to another in eq. 2.1.

Going to higher orders in the coupling constant not only increases the precision of the prediction, it also lessens the dependence on the factorisation scale μ_F since this scale-dependence tends to cancel among the contributions of different order. Therefore, higher-order corrections are important not only from a point of view of precision and predictive power but also in light of an unphysical scheme dependence.

2.2 The relevant diagrams

At NLO, the relevant Feynman diagrams making up $d\Delta\sigma_{ab}^{NLO}$ are the vertex correction and the contributions from bremsstrahlung:

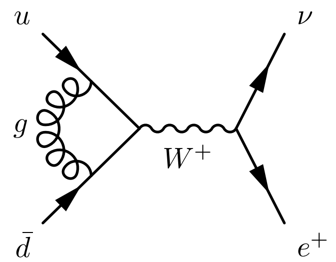


Figure 2.1: *The vertex correction*

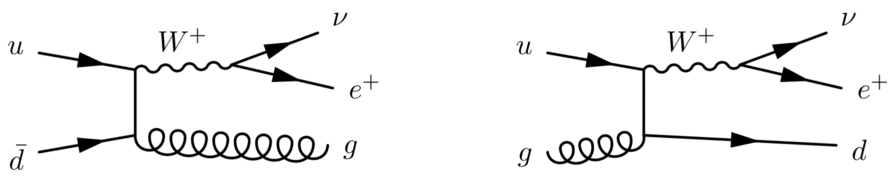


Figure 2.2: *The bremsstrahlung corrections (plus the $u \leftrightarrow \bar{d}$ diagrams)*

Only these diagrams contribute because the scattering matrix to NLO squared can be written as

$$\begin{aligned} \left|S^{NLO}\right|^2 &= \left|S^{LO} + S^V + \mathcal{O}(\alpha_s^2)\right|^2 + \left|S^B\right|^2 \\ &= \left|S^{LO}\right|^2 + 2\left(S^{LO}\right)^\dagger S^V + \left|S^B\right|^2 + \mathcal{O}(\alpha_s^2)\,, \end{aligned}$$

where S^V and S^B are the contributions of the vertex and the (two) bremsstrahlung diagrams, respectively. Since all combinations not shown are of higher order, only three terms contribute at $\mathcal{O}(\alpha_s)$.
The corresponding matrix elements have already been calculated in my M.S. thesis [36] and will be taken from there.

2.3 Organisation of the calculation

The expressions resulting from these diagrams are mostly divergent, as has to be expected in such calculations. However, through careful treatment and with a proper technique, these divergences can be made manifest and therefore be dealt with. For that purpose, the integrals appearing in the vertex correction (chapter 3) will be calculated using dimensional regularisation ($d = 4 - 2\varepsilon$) to express the divergences in poles of ε (for other regularisation procedures see appendix A). In the case of gluon bremsstrahlung (chapter 4), the poles will be a result of the phase space which has to be treated with care. Upon summation, the pole structure of the different contributions to the differential scattering cross section should cancel and leave a finite expression in which the limit $\varepsilon \to 0$ can be taken safely, according to the theorems by Block & Nordsieck [5] and Kinoshita, Lee & Nauenberg [17, 20]. However, the phase space is not as straight-forward as one would expect for all diagrams and there's also a calculational trick necessary in the case of the bremsstrahlung processes, developed by [16]. Furthermore, as shown above, the total NLO cross section requires not only the calculation of the contribution from these diagrams but also the so-called crossing functions. This task will be carried out in the remaining sections of this chapter. In the following chapters (3, 4, and 5), the contributions from the different diagrams will be derived, including the phase spaces. Finally, in chapter 6, the different parts will be summed up appropriately, as shown in this chapter, to give the complete NLO scattering cross section.

2.4 Crossing functions and treatment of phase space

The crossing function receives two contributions owing to the fact that we cannot distinguish two partons whose invariant mass is smaller than some resolution parameter s_{min}. Thus, we get a contribution from the case where one initial-state parton emits collinear radiation with an invariant mass smaller than s_{min}. Already mentioned above, the trick with the phase space generates the second contribution to the crossing functions: By crossing a pair of collinear final-state partons to the initial state, one gets the proper pole structure such that the gluon bremsstrahlung expression cancels the poles of the vertex correction and at the same time ensures that the crossing function is a finite quantity. This, however, requires that we make good for the wrong phase space (we do not have two collinear initial-state partons but one initial-state parton which emits collinear radiation into the final state) by subtracting this contribution from the crossing function. Schematically, the crossing function looks like

$$C_a^H(x) \sim \sum_c \left[\int_x^1 \frac{dz}{z} f_c^H\!\left(\frac{x}{z}\right) \hat{P}^d_{c\to a}(z) - f_a^H(x) \int_0^1 dz\, \hat{P}^d_{a\to c}(z) \right] \frac{s_{min}^{-\varepsilon}}{\varepsilon} \,,$$

where both contributions are divergent but sum up to a finite expression; c is the unobserved parton in the initial state. To lowest order, the (d-dimensional) splitting function $\hat{P}^d_{c\to a}(z)$ can be interpreted as the probability of finding a parton of type c inside a parton of type a with the fraction z of its parent parton's momentum. Its index has to be read as 'final state → initial state'.

As already mentioned, if two of the participating particles are collinear, the matrix element factorises into one factor containing the collinear behaviour and one the process without radiation: In the case in question where parton a emits a collinear parton 1 and parton c, the latter contributing to the reaction with parton b, one finds

$$|M(a+b \to 1+2+3)|^2 = \hat{c}^{a\to 1c}\, |M(b+c \to 2+3)|^2$$

where

$$\hat{c}^{a\to 1c} := 8\pi\alpha_s \frac{\hat{P}^d_{c1\to a}(z)}{z} \frac{1}{|s_{a1}|} \,.$$

If two initial-state particles a and 1 are collinear and combine to c, a similar expression

$$|M(\ldots, a, 1, \ldots)|^2 = \hat{c}^{a1\to c}\, |M(\ldots, c, \ldots)|^2$$

with

$$\hat{c}^{a1\to c} := 8\pi\alpha_s \hat{P}^d_{a1\to c}(z)\frac{1}{s_{a1}}$$

is found.

The phase space for the decay $p \to u + h$ with u collinear to p is

$$dP^{4-2\varepsilon}_{coll.}(p \to h+u) = \frac{(4\pi)^\varepsilon}{16\pi^2\Gamma(1-\varepsilon)}\, z\, dz\, d|s_{pu}|\ [(1-z)|s_{pu}|]^{-\varepsilon}\ \Theta(s_{min}-|s_{pu}|)\ .$$

2.4.1 Initial-state collinear radiation

The differential cross section for initial-state collinear radiation can be written as

$$d\sigma_{\text{initial}} = \int dx_1 dx_2 \sum_a f_b^{H_2}(x_2)\left\{f_a^{H_1}(y)\hat{c}^{a\to 1c} dP^d_{coll.}(a \to c+1)\,\delta(x_1 - yz)\,dy\right\} d\sigma^{LO}_{bc}(x_1,x_2)\ ,$$

from which follows the crossing function

$$\begin{aligned}\alpha_s C^{H_1}_{c,\text{initial}}(x_1) &= \sum_a f_a^{H_1}(y)\hat{c}^{a\to 1c} dP^d_{coll.}(a\to c+1)\,\delta(x_1-yz)\,dy \\ &= -\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^\varepsilon \frac{1}{\varepsilon}\sum_a \int_{x_1}^{1-z_2} dz\, \hat{P}^{4-2\varepsilon}_{c1\to a}(z)\frac{(1-z)^{-\varepsilon}}{z} f_a^{H_1}\Big(\frac{x_1}{z}\Big)\ .\end{aligned}$$

The upper integration boundary follows from the constraint that the unobserved parton 1 must not be soft with regard to its neighbouring parton:

$$s_{12} > s_{min} \quad \Leftrightarrow \quad (1-z)s_{c2} > s_{min} \quad \Leftrightarrow \quad z < 1 - \frac{s_{min}}{s_{c2}} =: 1 - z_2\ ,$$

where one assumes there is no possibility to distinguish whether a or c are in the initial state. Since one has the two processes $q\bar{q} \to W^\pm g \to \ell^\pm \nu g$ and $gq \to W^\pm q \to \ell^\pm \nu q$ (as well as the ones with $q \leftrightarrow \bar{q}$), the index a can be either a quark or a gluon and the sum contains the corresponding two splitting functions

$$\begin{aligned}\hat{P}^{4-2\varepsilon}_{qg\to q}(z) &= C_F\left[\frac{1+z^2}{1-z} - \varepsilon(1-z)\right] \\ \hat{P}^{4-2\varepsilon}_{q\bar{q}\to g}(z) &= T_F\frac{z^2+(1-z)^2-\varepsilon}{1-\varepsilon}\ ;\end{aligned}$$

the index c, on the other hand, is always a quark (or antiquark, but that only changes the PDF involved).
With the definition

$$[F(z)]_+ := \lim_{\beta\to 0}\left[\Theta(1-z-\beta)F(z) - \delta(1-z-\beta)\int_0^{1-\beta} dy\, F(y)\right]$$

and using the identity

$$\int_{x_1}^{1-z_2} dz\, \frac{g(z)}{(1-z)^{1+\varepsilon}} = \int_{x_1}^{1} dz\, \frac{g(z)}{[(1-z)^{1+\varepsilon}]_+} + g(1)\frac{z_2^{-1}-1}{\varepsilon}\ ,$$

one is able to rewrite the z integral over $\hat{P}^{4-2\varepsilon}_{qg\to q}$ in the crossing function above as

$$\int_{x_1}^{1-z_2} \frac{dz}{z}\left[\frac{1+z^2}{(1-z)^{1+\varepsilon}} - \varepsilon(1-z)^{1-\varepsilon}\right] = \int_{x_1}^{1} dz \left\{2\frac{z_2^{-\varepsilon}-1}{\varepsilon}\,\delta(1-z) + \frac{1+z^2}{z\left[(1-z)^{1+\varepsilon}\right]_+} - \varepsilon\frac{(1-z)^{1-\varepsilon}}{z}\right\}\,.$$

The integral over the other contributing splitting function, $\hat{P}^{4-2\varepsilon}_{q\bar{q}\to g}$, need not be evaluated because it contains no divergence.
Thus, the crossing function for initial-state collinear radiation is:

$$C^{H_1}_{q,\text{initial}}(x_1) = -\frac{1}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon}\sum_a\int_{x_1}^{1}\frac{dz}{z}f_a^{H_1}\!\left(\frac{x_1}{z}\right)\,J_{a\to q1}(z,z_2) \tag{2.2}$$

with

$$\begin{aligned}J_{q\to qg}(z,z_2) &:= C_F\left[2\frac{z_2^{-\varepsilon}-1}{\varepsilon}\,\delta(1-z) + \frac{1+z^2}{\left[(1-z)^{1+\varepsilon}\right]_+} - \varepsilon(1-z)^{1-\varepsilon}\right]\\ J_{g\to q\bar{q}}(z) &:= \hat{P}^{4-2\varepsilon}_{q\bar{q}\to g}(z)\,(1-z)^{-\varepsilon}\,.\end{aligned}$$

2.4.2 Correction for two collinear final-state partons crossed into initial state

The differential cross section for the case where one crosses two collinear final-state particles into the initial state is

$$d\sigma_{\text{final}} = \int dx_1 dx_2 \sum_a f_b^{H_2}(x_2)\left\{f_c^{H_1}(x_1)\,\hat{c}^{a1\to c}\,dP^d_{coll.,\text{final}}(c\to a+1)\right\}d\sigma^{LO}_{bc}(x_1,x_2)\,,$$

from which follows the crossing function (again, c is a quark)

$$\begin{aligned}\alpha_s C^{H_1}_{q,\text{final}}(x_1) &= f_q^{H_1}(x_1)\sum_a \hat{c}^{a1\to q}\,dP^d_{coll.,\text{final}}(c\to a+1)\\ &= -\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon}\,f_q^{H_1}(x_1)\sum_a\int_0^{1-z_2}dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\hat{P}^{4-2\varepsilon}_{a1\to q}(z)\,,\end{aligned}$$

where the integration boundaries result from the requirement that the hard partons be resolved

$$s_{12} > s_{min} \quad\Leftrightarrow\quad (1-z)s_{c2} > s_{min} \quad\Leftrightarrow\quad z < 1-\frac{s_{min}}{s_{c2}} =: 1-z_2$$

and the phase space is

$$dP^{4-2\varepsilon}_{coll.,\text{final}} = z^{-\varepsilon}\,dP^{4-2\varepsilon}_{coll.}\,.$$

Since the parton c is fixed to be a quark (or antiquark) and the splitting function is associated with it, there's no contribution from the process $gq\to W^\pm q\to \ell^\pm\nu q$ to this part of the crossing function. As a consequence, the summation over a vanishes as well.
The z integration (over the remaining splitting function) can be carried out with the help of the incomplete beta function

$$\int_0^{1-z_2} dz\,z^{-\varepsilon}\left[\frac{1+z^2}{(1-z)^{1+\varepsilon}} - \varepsilon(1-z)^{1-\varepsilon}\right] = 2\left[\frac{z_2^{-\varepsilon}-1}{\varepsilon} - \frac{3}{4} + \left(\frac{\pi^2}{6}-\frac{7}{4}\right)\varepsilon + \mathcal{O}(\varepsilon^2) + \mathcal{O}(s_{min})\right]\,.$$

Thus, the crossing function in this case is:

$$C^{H_1}_{q,\text{final}}(x_1) = -\frac{1}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon}\, f^{H_1}_q(x_1)\, I_{a1\to q}(z_2) \tag{2.3}$$

with

$$I_{qg\to q}(z_2) := C_F\cdot 2\left[\frac{z_2^{-\varepsilon}-1}{\varepsilon} - \frac{3}{4} + \left(\frac{\pi^2}{6}-\frac{7}{4}\right)\varepsilon + \mathcal{O}(\varepsilon^2) + \mathcal{O}(s_{min})\right] .$$

2.4.3 The crossing function

Combining the expressions above, eqs. 2.2 and 2.3, one finds for the total crossing function

$$\begin{aligned} C^{H_1}_q(x_1) &= C^{H_1}_{q,\text{initial}}(x_1) - C^{H_1}_{q,\text{final}}(x_1) \\ &= \sum_a \int_{x_1}^{1} \frac{dz}{z} f^{H_1}_a\Big(\frac{x_1}{z}\Big)\; X_{a\to q}(z) , \end{aligned}$$

with

$$X_{a\to q}(z) := -\frac{1}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon}\left[J_{a\to q1}(z,z_2) - I_{a1\to q}(z_2)\,\delta(1-z)\right] \tag{2.4}$$

and

$$\begin{aligned} X_{q\to q}(z) &= -\frac{1}{2\pi}\frac{C_F}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon} \\ &\quad\cdot\left\{\frac{1+z^2}{[(1-z)^{1+\varepsilon}]_+} - \varepsilon(1-z)^{1-\varepsilon} + \left[\frac{3}{2} - \left(\frac{\pi^2}{3}-\frac{7}{2}\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right]\delta(1-z)\right\} \\ X_{g\to q}(z) &= -\frac{1}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon}\,\hat{P}^{4-2\varepsilon}_{q\bar{q}\to g}(z)\,(1-z)^{-\varepsilon} . \end{aligned}$$

As a result of this procedure, the double pole cancels in the sum. The remaining one will be absorbed in the mass factorisation.

2.5 Mass factorisation and the crossing function

The parton distribution function is conventionally renormalised at the factorisation scale μ_F and thus made finite

$$f^H_h(x) = f^H_h(x,\mu_F) + \alpha_s \sum_p \int_x^1 \frac{dz}{z} f^H_p\Big(\frac{x}{z},\mu_F\Big)\, R_{p\to h}(z) + \mathcal{O}(\alpha_s^2)$$

by using the appropriate counterfunction R. For the processes in question, these are

$$\begin{aligned} R^{\text{scheme}}_{q\to q}(z,\mu_F) &= \frac{1}{2\pi}\frac{C_F}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{\mu_F}\right)^{\varepsilon}\frac{1}{\varepsilon}\left\{\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) + \varepsilon f^{\text{scheme}}_{q\to q}(z)\right\} \\ R^{\text{scheme}}_{g\to q}(z,\mu_F) &= \frac{1}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{\mu_F}\right)^{\varepsilon}\frac{1}{\varepsilon}\left\{\hat{P}^4_{q\bar{q}\to g}(z) + \varepsilon f^{\text{scheme}}_{g\to q}(z)\right\} , \end{aligned}$$

where $f^{\text{scheme}}_{a\to q}$ is the renormalisation scheme-dependent mass-factorisation term chosen such that $f^{\overline{MS}}_{a\to q}(z) \equiv 0 \quad \forall a$. This can be inserted into the expression for the effective structure function – which is the only physical quantity and therefore has to be finite – and leads to

$$\begin{aligned}\mathcal{F}^H_h(x) &= f^H_h(x) + \alpha_s C^H_h(x)\\ &= f^H_h(x,\mu_F) + \alpha_s C^H_h(x,\mu_F) + \mathcal{O}(\alpha_s^2)\ ,\end{aligned}$$

where the $\mathcal{O}(\alpha_s)$ contribution has been absorbed into the crossing function, making it factorisation-scale dependent:

$$C^H_h(x,\mu_F) = \sum_p \int_x^1 \frac{dz}{z} f^H_p\Big(\frac{x}{z},\mu_F\Big)\,[X_{p\to h}(z) + R_{p\to h}(z,\mu_F)]\ .$$

In our processes $h \equiv c = q$, but there are two possibilities for $p \equiv a$. For the case where $a = q$, the expression with the + description in eq. 2.4 has to be expanded according to

$$\int_x^1 dz \frac{1+z^2}{[(1-z)^{1+\varepsilon}]_+} = \int_x^1 dz \frac{1+z^2}{(1-z)_+} - \varepsilon \int_x^1 dz\,(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ + \mathcal{O}(\varepsilon^2)$$

and one finds after a series expansion in ε

$$\begin{aligned}X_{q\to q} + R_{q\to q} = \frac{C_F}{2\pi\,\Gamma(1-\varepsilon)} &\left\{\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]\ln\frac{s_{min}}{\mu_F^2}\right.\\ &\left. +(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ + (1-z) + \left(\frac{\pi^2}{3} - \frac{7}{2}\right)\delta(1-z) + \mathcal{O}(\varepsilon)\right\}\ .\end{aligned}$$

In the case of $a = g$, the corresponding expression becomes

$$X_{g\to q} + R_{g\to q} = \frac{1}{2\pi\,\Gamma(1-\varepsilon)}\left\{\hat{P}^4_{q\bar{q}\to g}(z)\ln\frac{s_{min}}{\mu_F^2} + \hat{P}^4_{q\bar{q}\to g}(z)\ln(1-z) - \hat{P}^\varepsilon_{q\bar{q}\to g}(z) + \mathcal{O}(\varepsilon)\right\}\ .$$

Summing up the two contributions, one finds the total crossing function (n.b. after mass factorisation and where the limit $\varepsilon \to 0$ has been taken):

$$\begin{aligned}C^{H_1}_q(x_1,\mu_F) = &\int_{x_1}^1 \frac{dz}{z} f^{H_1}_q\Big(\frac{x_1}{z},\mu_F\Big)\cdot\frac{C_F}{2\pi}\left\{\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]\ln\frac{s_{min}}{\mu_F^2}\right.\\ &\left. +(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ + (1-z) + \left(\frac{\pi^2}{3} - \frac{7}{2}\right)\delta(1-z)\right\}\\ &+\int_{x_1}^1 \frac{dz}{z} f^{H_1}_g\Big(\frac{x_1}{z},\mu_F\Big)\cdot\frac{1}{2\pi}\left\{\hat{P}^4_{q\bar{q}\to g}(z)\ln\frac{s_{min}}{\mu_F^2} + \hat{P}^4_{q\bar{q}\to g}(z)\ln(1-z)\right\}\ .\end{aligned} \tag{2.5}$$

As can be seen, all the poles either cancelled or have been absorbed, leaving a mathematically well-defined expression in which the limit $\varepsilon \to 0$ could be taken. The crossing function for an antiquark is obtained by simply replacing the quark PDF by the one for an antiquark in this expression,

$$C^{H_1}_{\bar{q}}(x_1,\mu_F) = C^{H_1}_q\left(f^{H_1}_q \to f^{H_1}_{\bar{q}}\right)\ .$$

2.6 Spin-dependency

2.6.1 NLO approximation of the scattering cross section

If the hadrons are polarised, this spin-dependency is passed down to the partons and one finds for the differential scattering cross section

$$d\Delta\sigma_{H_1H_2} = \sum_{ab} \int dx_1 dx_2 \; \Delta\mathcal{F}_a^{H_1}(x_1)\Delta\mathcal{F}_b^{H_2}(x_2)\; d\Delta\sigma_{ab}(x_1, x_2) \;,$$

with

$$\begin{aligned} d\Delta\sigma_{ab} &= d\Delta\sigma_{ab}^{LO} + \alpha_s d\Delta\sigma_{ab}^{NLO} + \mathcal{O}(\alpha_s^2) \\ \Delta\mathcal{F}_a^{H_1}(x_1) &= \Delta f_a^{H_1}(x_1, \mu_F) + \alpha_s \Delta C_a^{H_1}(x_1, \mu_F) + \mathcal{O}(\alpha_s^2) \end{aligned}$$

and the definitions already introduced above

$$f_q^H \equiv q := q_+^+ + q_-^- + q_-^+ + q_+^- \quad \text{and} \quad \Delta f_q^H \equiv \Delta q := q_+^+ + q_-^- - q_-^+ - q_+^- \;.$$

Combining these expressions, the spin-dependent cross section can be written as

$$\begin{aligned} d\Delta\sigma_{H_1H_2} = \sum_{ab} \int dx_1 dx_2 \Big\{ &\Delta f_a^{H_1}(x_1, \mu_F)\Delta f_b^{H_2}(x_2, \mu_F) \left[d\Delta\sigma_{ab}^{LO}(x_1, x_2) + \alpha_s d\Delta\sigma_{ab}^{NLO}(x_1, x_2) \right] \\ &+ \alpha_s \left[\Delta C_a^{H_1}(x_1, \mu_F)\Delta f_b^{H_2}(x_2, \mu_F) + \Delta f_a^{H_1}(x_1, \mu_F)\Delta C_b^{H_2}(x_2, \mu_F) \right] d\Delta\sigma_{ab}^{LO}(x_1, x_2) \\ &+ \mathcal{O}(\alpha_s^2) \Big\} \;. \end{aligned} \tag{2.6}$$

2.6.2 Crossing function

In the case of spin-dependent partons, the contributions to the crossing functions include the spin-dependent splitting functions (see e.g. [10, 34])

$$\begin{aligned} \Delta\hat{P}_{qg\to q}^{4-2\varepsilon}(z) &= C_F \left[\frac{1+z^2}{1-z} + 3\varepsilon(1-z) \right] \\ \Delta\hat{P}_{q\bar{q}\to g}^{4-2\varepsilon}(z) &= T_F \left[2z - 1 - 2\varepsilon(1-z) \right] \end{aligned}$$

and lead to the two expressions:

$$\begin{aligned} \Delta C_{q,\text{initial}}^{H_1}(x_1) &= -\frac{1}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{s_{min}} \right)^{\varepsilon} \frac{1}{\varepsilon} \sum_a \int_{x_1}^{1} \frac{dz}{z} \Delta f_a^{H_1}\Big(\frac{x_1}{z}\Big) \; \Delta J_{a\to q1}(z, z_2) \\ \Delta C_{q,\text{final}}^{H_1}(x_1) &= -\frac{1}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{s_{min}} \right)^{\varepsilon} \frac{1}{\varepsilon} \; \Delta f_q^{H_1}(x_1) \; \Delta I_{a1\to q}(z_2) \end{aligned}$$

with

$$\begin{aligned} \Delta J_{q\to qg}(z, z_2) &:= C_F \left[2\frac{z_2^{-\varepsilon} - 1}{\varepsilon} \, \delta(1-z) + \frac{1+z^2}{\left[(1-z)^{1+\varepsilon}\right]_+} + 3\varepsilon(1-z)^{1-\varepsilon} \right] \\ \Delta J_{g\to q\bar{q}}(z) &:= \Delta\hat{P}_{q\bar{q}\to g}^{4-2\varepsilon}(z) \; (1-z)^{-\varepsilon} \\ \Delta I_{qg\to q}(z_2) &:= C_F \cdot 2 \left[\frac{z_2^{-\varepsilon} - 1}{\varepsilon} - \frac{3}{4} + \left(\frac{\pi^2}{6} - \frac{3}{4} \right) \varepsilon + \mathcal{O}(\varepsilon^2) + \mathcal{O}(s_{min}) \right] \;. \end{aligned}$$

In complete analogy to the spin-independent case, the renormalisation terms contain the spin-dependent splitting functions instead of the spin-independent ones and also a spin-dependent mass-factorisation term. Thus, one finds for the total spin-dependent crossing function:

$$\Delta C_q^{H_1}(x_1,\mu_F) = \int_{x_1}^{1} \frac{dz}{z} \Delta f_q^{H_1}\Big(\frac{x_1}{z},\mu_F\Big) \cdot \frac{C_F}{2\pi} \left\{ \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] \ln \frac{s_{min}}{\mu_F^2} \right.$$
$$\left. +(1+z^2) \left[\frac{\ln(1-z)}{1-z} \right]_+ - 3(1-z) + \left(\frac{\pi^2}{3} - \frac{3}{2} \right) \delta(1-z) \right\}$$
$$+ \int_{x_1}^{1} \frac{dz}{z} \Delta f_g^{H_1}\Big(\frac{x_1}{z},\mu_F\Big) \cdot \frac{1}{2\pi} \left\{ \Delta\hat{P}^4_{q\bar{q}\to g}(z) \ln \frac{s_{min}}{\mu_F^2} + \Delta\hat{P}^4_{q\bar{q}\to g}(z) \ln(1-z) \right\} . \quad (2.7)$$

With these results (eqs. 2.5 and 2.7) we are now able to calculate the cross section to NLO, as shown in eqs. 2.1 and 2.6. The second – still missing – ingredient for the complete differential scattering cross section at NLO is $d\sigma_{ab}^{NLO}$; its components will be calculated in the next three chapters.

3 Vertex correction

[...] On a huge hill,
Cragged and steep, Truth stands, and he that will
Reach her, about must and about must go,
And what the hill's suddenness resists, win so.

Dr. John Donne: Seek True Religion!

3.1 Dissecting the scattering matrix element

From my M.S. [36] thesis follows the scattering matrix element of the vertex graph (in Feynman gauge):

$$
\begin{aligned}
S_{fi} = \frac{g_s^2 g_w^2}{8} V_{qq'} C_F \, \delta^{(4)}(k_1^i + k_2^i - k_1^f - k_2^f) \, \frac{1}{(k_1^f + k_2^f)^2 - M_W^2 + i0} \\
\cdot \int d^4q \, \frac{1}{q^2 + i0} \, \bar{v}(k_2^i) \gamma_\mu \frac{\not{q} - \not{k}_2^i + m_d}{(q - k_2^i)^2 - m_d^2 + i0} \gamma^\nu \left(1 - \gamma^5\right) \frac{\not{q} + \not{k}_1^i + m_u}{(q + k_1^i)^2 - m_u^2 + i0} \gamma^\mu u(k_1^i) \\
\cdot \bar{u}(k_2^f) \gamma_\nu \left(1 - \gamma^5\right) v(k_1^f) \; .
\end{aligned}
$$

The integral to be solved is therefore:

$$
I^\nu := \int d^4q \, \frac{\bar{v}(k_2^i)\gamma_\mu(\not{q} - \not{k}_2^i + m_d)\gamma^\nu(1-\gamma^5)(\not{q} + \not{k}_1^i + m_u)\gamma^\mu u(k_1^i)}{\left[(q - k_2^i)^2 - m_d^2 + i0\right]\left[(q + k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} \; . \tag{3.1}
$$

Defining the integrals (n.b.: with a negative sign before k_1^i to recover symmetry)

$$
K^{(0)} := \int d^4q \, \frac{1}{\left[(q - k_2^i)^2 - m_d^2 + i0\right]\left[(q - k_1^i)^2 - m_u^2 + i0\right]} \tag{3.2}
$$

$$
J_\mu^{(1)} := \int d^4q \, \frac{q_\mu}{\left[(q - k_2^i)^2 - m_d^2 + i0\right]\left[(q - k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} \tag{3.3}
$$

$$
J_{\mu\nu}^{(2)} := \int d^4q \, \frac{q_\mu q_\nu}{\left[(q - k_2^i)^2 - m_d^2 + i0\right]\left[(q - k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} \tag{3.4}
$$

$$
J^{(0)} := \int d^4q \, \frac{1}{\left[(q - k_2^i)^2 - m_d^2 + i0\right]\left[(q - k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} \tag{3.5}
$$

and using dimensional continuation ($d = 4 - 2\varepsilon$), one can write the original integral (eq. 3.1) as

$$
\begin{aligned}
I^\nu = \bar{v}(k_2^i) \Big\{ 2(1-\varepsilon) K^{(0)} \gamma^\nu (1-\gamma^5) - 4(1-\varepsilon) J^{(2)\,\nu\alpha} \gamma_\alpha (1-\gamma^5) - 4(k_1^i k_2^i) J^{(0)} \gamma^\nu (1-\gamma^5) \\
- 2 J^{(1)\,\alpha} \gamma_\alpha \left[2(k_1^{i\,\nu} - k_2^{i\,\nu}) + m_d \gamma^\nu\right](1-\gamma^5) + 4 J^{(1)\,\alpha} (k_{1\,\alpha}^i - k_{2\,\alpha}^i) \gamma^\nu (1-\gamma^5) \\
+ 2 m_u J^{(1)\,\alpha} \gamma_\alpha \gamma^\nu (1+\gamma^5) + 4 m_d J^{(1)\,\nu} (1-\gamma^5) \Big\} u(k_1^i) \; ,
\end{aligned} \tag{3.6}
$$

where the following relations have been used:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \{\gamma^\mu, \gamma^5\} = 0$$
$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2-d)\gamma_\alpha$$
$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4g_{\alpha\beta} + (d-4)\gamma_\alpha \gamma_\beta$$
$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma^\mu = -2\gamma_\lambda \gamma_\beta \gamma_\alpha - (d-4)\gamma_\alpha \gamma_\beta \gamma_\lambda$$
$$(\not{k}_j - m_j)u(k_j) = 0 \qquad \bar{u}(k_j)(\not{k}_j - m_j) = 0$$
$$(\not{k}_j + m_j)v(k_j) = 0 \qquad \bar{v}(k_j)(\not{k}_j + m_j) = 0 \ .$$

3.2 On-shell or off-shell? Massive or massless?

At this point, one has to choose whether to calculate with quarks on- or off-shell because the on-shell limit does not commute with dimensional continuation ($d = 4 - 2\varepsilon$) of the integration.
In principle, theory cannot neglect quark masses – on the other hand, the expressions including a nonvanishing quark mass become very cumbersome and the mass-dependent terms contribute only little if not negligibly to the differential cross section. Especially in comparison to the W mass, one may safely neglect the masses of the contributing quarks (the heavier quarks' contribution is strongly suppressed by the CKM matrix elements).
To get a hand on the relevant technique, the integrals are first calculated off-shell and massive, i.e. the most general case, and afterwards on-shell and massless. The calculation proceeds along the lines of [29, 18].

3.3 Off-shell and massive integrals

3.3.1 Calculating $K^{(0)}$

The first integral to be calculated is

$$K^{(0)} := \int d^4q \, \frac{1}{\left[(q-k_2^i)^2 - m_d^2 + i0\right]\left[(q-k_1^i)^2 - m_u^2 + i0\right]} \ . \tag{3.2}$$

Using Feynman parametrisation first and shifting the integration variable by
$q \mapsto q + \alpha k_1^i + (1-\alpha)k_2^i$ gives

$$\begin{aligned} K^{(0)} &= \int d^4q \int_0^1 d\alpha \, \left\{\alpha\left[(q-k_1^i)^2 - m_u^2 + i0\right] + (1-\alpha)\left[(q-k_2^i)^2 - m_d^2 + i0\right]\right\}^{-2} \\ &= \int d^4q \int_0^1 d\alpha \, \left\{q^2 - \alpha^2(k_1^i - k_2^i)^2 + \alpha\left[(k_1^i - k_2^i)^2 - m_u^2 + m_d^2\right] - m_d^2 + i0\right\}^{-2} \\ &=: \int d^4q \int_0^1 d\alpha \, \left\{q^2 - a^2(\alpha) + i0\right\}^{-2} \ . \end{aligned}$$

The integration over q can now be carried out, using dimensional regularisation, with the dimension $4 - 2\varepsilon$:

$$\int d^{4-2\varepsilon}q \, (q^2 - a^2 + i0)^{-2} = i\pi^{2-\varepsilon}\,\Gamma(\varepsilon)\,(a^2)^{-\varepsilon} = i\pi^2\left(\frac{1}{\varepsilon} - \gamma_E - \ln\pi - \ln a^2 + \mathcal{O}(\varepsilon)\right) \ ,$$

where γ_E is the Euler-Mascheroni constant appearing in the Laurent series expansion of $(a^2)^{-\varepsilon}$. For convenience, the divergent term $1/\varepsilon$ is combined with γ_E and $\ln\pi$ to give $N_\varepsilon := 1/\varepsilon - \gamma_E - \ln\pi$. To take into account the unit-dependency of a^2, one writes $a^2 = \mu^2 \cdot \frac{a^2}{\mu^2}$ with μ some regularisation mass, leading to

$$K^{(0)} = i\pi^2\,\mu^{-2\varepsilon} \int_0^1 d\alpha\,\left[N_\epsilon - \ln\frac{a^2(\alpha)}{\mu^2}\right] ,$$

which can be integrated out straight-forward to give, with $q := k_1^i - k_2^i$:

$$\begin{aligned} K^{(0)} = i\pi^2\,\mu^{-2\varepsilon} \Bigg\{ & N_\varepsilon + 2 - \ln\frac{m_u m_d}{\mu^2} - \frac{m_u^2 - m_d^2}{(k_1^i - k_2^i)^2} \ln\frac{m_u}{m_d} \\ & - \frac{\sqrt{q^4 - 2q^2(m_u^2 + m_d^2) + (m_u^2 - m_d^2)^2}}{2q^2} \\ & \cdot \left[\ln\frac{\sqrt{q^4 - 2q^2(m_u^2 + m_d^2) + (m_u^2 - m_d^2)^2} - q^2 - m_u^2 + m_d^2}{\sqrt{q^4 - 2q^2(m_u^2 + m_d^2) + (m_u^2 - m_d^2)^2} + q^2 + m_u^2 - m_d^2}\right. \\ & \left. - \ln\frac{\sqrt{q^4 - 2q^2(m_u^2 + m_d^2) + (m_u^2 - m_d^2)^2} + q^2 - m_u^2 + m_d^2}{\sqrt{q^4 - 2q^2(m_u^2 + m_d^2) + (m_u^2 - m_d^2)^2} - q^2 + m_u^2 - m_d^2}\right] + \mathcal{O}(\varepsilon) \Bigg\} . \end{aligned} \tag{3.7}$$

3.3.2 Calculating $J_\mu^{(1)}$

Because of its Lorentz structure, the integral

$$J_\mu^{(1)} := \int d^4q\, \frac{q_\mu}{\left[(q-k_2^i)^2 - m_d^2 + i0\right]\left[(q-k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} \tag{3.3}$$

can be decomposed as

$$J_\mu^{(1)} = k^i_{1\,\mu}\, J_A + k^i_{2\,\mu}\, J_B ,$$

from which J_A and J_B follow, with $N := (k_1^i k_2^i)^2 - k_1^{i\,2} k_2^{i\,2}$:

$$\begin{aligned} J_A &= \frac{1}{N}\left[(k_1^i k_2^i)\, k_2^{i\,\mu}\, J_\mu^{(1)} - k_2^{i\,2}\, k_1^{i\,\mu}\, J_\mu^{(1)}\right] \\ J_B &= \frac{1}{N}\left[(k_1^i k_2^i)\, k_1^{i\,\mu}\, J_\mu^{(1)} - k_1^{i\,2}\, k_2^{i\,\mu}\, J_\mu^{(1)}\right] . \end{aligned}$$

Since obviously $J_B = J_A(k_1^i \leftrightarrow k_2^i, m_u \leftrightarrow m_d)$, it is sufficient to calculate J_A, which can be written using the integrals defined above (eqs. 3.2, 3.5):

$$\begin{aligned} J_A = \frac{1}{2N} \Bigg\{ & \left[(k_1^i k_2^i) - k_2^{i\,2}\right] K^{(0)} + \left[(k_1^i k_2^i)(k_2^{i\,2} - m_d^2) - k_2^{i\,2}(k_1^{i\,2} - m_u^2)\right] J^{(0)} \\ & -(k_1^i k_2^i) \int d^4q\, \frac{1}{\left[(q-k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} + k_2^{i\,2} \int d^4q\, \frac{1}{\left[(q-k_2^i)^2 - m_d^2 + i0\right](q^2 + i0)} \Bigg\} . \end{aligned}$$

The two intergals to be carried out are identical, up to $k_1^i \leftrightarrow k_2^i$ and $m_u \leftrightarrow m_d$:

$$\int d^4q \frac{1}{\left[(q-k_1^i)^2 - m_u^2 + i0\right](q^2 + i0)} = i\pi^2\mu^{-2\varepsilon}\left[N_\varepsilon + 2 - \ln\frac{m_u^2}{\mu^2} - \frac{k_1^{i\,2} - m_u^2}{k_1^{i\,2}} \ln\frac{m_u^2 - k_1^{i\,2}}{m_u^2} + \mathcal{O}(\varepsilon)\right] .$$

From this follows, by combining the results:

$$\begin{aligned}
J^{(1)}_\mu = &\frac{(k^i_1 k^i_2)\,(k^i_1 + k^i_2)_\mu - k^{i\,2}_2 k^i_{1\,\mu} - k^{i\,2}_1 k^i_{2\,\mu}}{2N} K^{(0)} \\
&+ \; k^i_{1\,\mu} \frac{(k^i_1 k^i_2)(k^{i\,2}_2 - m^2_d) - k^{i\,2}_2 (k^{i\,2}_1 - m^2_u)}{2N} J^{(0)} \\
&+ \; k^i_{2\,\mu} \frac{(k^i_1 k^i_2)(k^{i\,2}_1 - m^2_u) - k^{i\,2}_1 (k^{i\,2}_2 - m^2_d)}{2N} J^{(0)} \\
&- \; i\pi^2\,\mu^{-2\varepsilon} \frac{k^i_{1\,\mu}(k^i_1 k^i_2) - k^i_{2\,\mu} k^{i\,2}_1}{2N} \left[N_\varepsilon + 2 - \ln\frac{m^2_u}{\mu^2} - \frac{k^{i\,2}_1 - m^2_u}{k^{i\,2}_1} \ln\frac{m^2_u - k^{i\,2}_1}{m^2_u} + \mathcal{O}(\varepsilon) \right] \\
&- \; i\pi^2\,\mu^{-2\varepsilon} \frac{k^i_{2\,\mu}(k^i_1 k^i_2) - k^i_{1\,\mu} k^{i\,2}_2}{2N} \left[N_\varepsilon + 2 - \ln\frac{m^2_d}{\mu^2} - \frac{k^{i\,2}_2 - m^2_d}{k^{i\,2}_2} \ln\frac{m^2_d - k^{i\,2}_2}{m^2_d} + \mathcal{O}(\varepsilon) \right] \; . \qquad (3.8)
\end{aligned}$$

3.3.3 Calculating $J^{(2)}_{\mu\nu}$

In analogous fashion to $J^{(1)}_\mu$, this integral (eq. 3.4) can be decomposed as well:

$$\begin{aligned}
J^{(2)}_{\mu\nu} = &\frac{g_{\mu\nu}}{4} K^{(0)} + \left(k^i_{1\,\mu} k^i_{1\,\nu} - g_{\mu\nu} \frac{k^{i\,2}_1}{4} \right) J_C \\
&+ \left(k^i_{2\,\mu} k^i_{2\,\nu} - g_{\mu\nu} \frac{k^{i\,2}_2}{4} \right) J_D + \left(k^i_{1\,\mu} k^i_{2\,\nu} + k^i_{2\,\mu} k^i_{1\,\nu} - g_{\mu\nu} \frac{(k^i_1 k^i_2)}{2} \right) J_E \; . \qquad (3.9)
\end{aligned}$$

From this follows

$$\begin{aligned}
J_E = &- \frac{(k^i_1 k^i_2)}{2N} K^{(0)} \\
&+ \frac{2N + 3k^{i\,2}_1 k^{i\,2}_2}{2N^2} \Bigg\{ k^i_{2\,\nu} \Bigg[\int d^4q \frac{q^\nu}{\left[(q - k^i_1)^2 - m^2_u + i0\right] \left[(q - k^i_2)^2 - m^2_d + i0\right]} \\
&\quad + (k^{i\,2}_1 - m^2_u) J^{(1)\,\nu} - \int d^4q \frac{q^\nu}{(q^2 + i0) \left[(q - k^i_2)^2 - m^2_d + i0\right]} \Bigg] \\
&\quad + k^i_{1\,\nu} \Bigg[\int d^4q \frac{q^\nu}{\left[(q - k^i_1)^2 - m^2_u + i0\right] \left[(q - k^i_2)^2 - m^2_d + i0\right]} \\
&\quad + (k^{i\,2}_2 - m^2_d) J^{(1)\,\nu} - \int d^4q \frac{q^\nu}{(q^2 + i0) \left[(q - k^i_1)^2 - m^2_u + i0\right]} \Bigg] \Bigg\} \\
&- \frac{3(k^i_1 k^i_2)}{4N^2} \Bigg\{ \left[k^{i\,2}_1 k^i_{2\,\nu} + k^{i\,2}_2 k^i_{1\,\nu} \right] \int d^4q \frac{q^\nu}{\left[(q - k^i_1)^2 - m^2_u + i0\right] \left[(q - k^i_2)^2 - m^2_d + i0\right]} \\
&\quad + k^{i\,2}_2 (k^{i\,2}_1 - m^2_u) \left[k^{i\,2}_1 J_A + (k^i_1 k^i_2) J_B \right] + k^{i\,2}_1 (k^{i\,2}_2 - m^2_d) \left[(k^i_1 k^i_2) J_A + k^{i\,2}_2 J_B \right] \\
&\quad - k^{i\,2}_2 k^i_{1\,\nu} \int d^4q \frac{q^\nu}{(q^2 + i0) \left[(q - k^i_2)^2 - m^2_d + i0\right]} - k^{i\,2}_1 k^i_{2\,\nu} \int d^4q \frac{q^\nu}{(q^2 + i0) \left[(q - k^i_1)^2 - m^2_u + i0\right]} \Bigg\} \; ,
\end{aligned}$$

with the integrals

$$\int d^4q \frac{q^\nu}{\left[(q-k_1^i)^2 - m_u^2 + i0\right]\left[(q-k_2^i)^2 - m_d^2 + i0\right]}$$
$$= i\pi^2\mu^{-2\varepsilon}\left\{\frac{k_1^{i\,\nu}+k_2^{i\,\nu}}{2}\left[N_\varepsilon - \ln\frac{q^2(1-u_1)(1-u_2)}{\mu^2}\right] + k_2^{i\,\nu}\left[u_1\ln\frac{u_1-1}{u_1} + u_2\ln\frac{u_2-1}{u_2} + 2\right]\right.$$
$$\left. + \frac{k_1^{i\,\nu}-k_2^{i\,\nu}}{2}\left[u_1^2\ln\frac{u_1-1}{u_1} + u_2^2\ln\frac{u_2-1}{u_2} + u_1 + u_2 + 1\right] + \mathcal{O}(\varepsilon)\right\}$$

$$\int d^4q \frac{q^\nu}{(q^2+i0)\left[(q-k_2^i)^2 - m_d^2 + i0\right]}$$
$$= i\pi^2\mu^{-2\varepsilon}\frac{k_2^{i\,\nu}}{2}\left\{N_\varepsilon + \frac{m_d^2}{k_2^{i\,2}}\left(2 - \frac{m_d^2}{k_2^{i\,2}}\right)\ln\frac{m_d^2 - k_2^{i\,2}}{m_d^2} - \ln\frac{m_d^2 - k_2^{i\,2}}{\mu^2} + \frac{2k_2^{i\,2} - m_d^2}{k_2^{i\,2}} + \mathcal{O}(\varepsilon)\right\} ,$$

where u_1 and u_2 have been defined as

$$u_{1,2} := \frac{q^2 - m_u^2 + m_d^2 \pm \sqrt{(q^2 - m_u^2 + m_d^2)^2 - 4q^2 m_d^2}}{2q^2} .$$

Knowing J_E, one can calculate J_C

$$J_C = \frac{k_2^{i\,2}}{N}\left(\frac{k^i_{2\,\mu}k^i_{2\,\nu}}{k_2^{i\,2}} - \frac{k^i_{1\,\mu}k^i_{2\,\nu}}{(k_1^i k_2^i)}\right)J^{(2)\,\mu\nu} - \frac{k_2^{i\,2}}{(k_1^i k_2^i)}J_E$$

and $J_D = J_C(k_1^i \leftrightarrow k_2^i, m_u \leftrightarrow m_d)$. With these ($J_E$, J_C, and J_D), the integral $J^{(2)}_{\mu\nu}$, eq. 3.9, is solved completely and analytically.

3.3.4 Calculating $J^{(0)}$

The most tedious integral in this context is

$$J^{(0)} := \int d^4q \, \frac{1}{\left[(q-k_1^i)^2 - m_u^2 + i0\right]\left[(q-k_2^i)^2 - m_d^2 + i0\right](q^2+i0)} , \tag{3.5}$$

which can be written, again using Feynman parametrisation, as

$$J^{(0)} = 2\int_0^1 d\alpha \int_0^\alpha d\beta \int d^4q \left\{(1-\alpha)\left[(q-k_1^i)^2 - m_u^2 + i0\right]\right.$$
$$\left. +(\alpha-\beta)\left[(q-k_2^i)^2 - m_d^2 + i0\right] + \beta\left(q^2 + i0\right)\right\}^{-3} .$$

Shifting the variable $q \mapsto q + (1-\alpha)k_1^i + (\alpha-\beta)k_2^i$ enables one to carry out the q integration, while the second shift $\beta \mapsto \beta + \alpha x$ transforms the integrand into

$$\beta^2 k_2^{i\,2} + \beta\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] - \alpha\left\{k_1^{i\,2} + m_u^2 + (1-x)\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right]\right\}$$
$$- 2\alpha\beta\left[(1-x)k_2^{i\,2} - (k_1^i k_2^i)\right] + m_u^2 ,$$

with the definition

$$x := 1 + \frac{-(k_1^i k_2^i) + \sqrt{N}}{k_2^{i\,2}} \; .$$

Abbreviating $\widetilde{N}(\beta) := k_1^{i\,2} + m_u^2 + (1-x)\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] + 2\beta\left[(1-x)k_2^{i\,2} - (k_1^i k_2^i)\right]$ and inverting the order of integration

$$\int_0^1 d\alpha \int_{-\alpha}^{(1-x)\alpha} d\beta = \int_0^{1-x} d\beta \int_{\beta/(1-x)}^{1} d\alpha - \int_0^{-x} d\beta \int_{-\beta/x}^{1} d\alpha \; ,$$

the α integral can be carried out, leading to

$$\begin{aligned} J^{(0)} = i\pi^2 &\int_{-x}^{1-x} d\beta \, \frac{\ln\left\{-\beta^2 k_2^{i\,2} - \beta\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] + \widetilde{N}(\beta) - m_u^2\right\}}{\widetilde{N}(\beta)} \\ - i\pi^2 &\int_0^{1-x} d\beta \, \frac{\ln\left\{-\beta^2 k_2^{i\,2} - \beta\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] + \frac{\beta}{1-x}\widetilde{N}(\beta) - m_u^2\right\}}{\widetilde{N}(\beta)} \\ + i\pi^2 &\int_0^{-x} d\beta \, \frac{\ln\left\{-\beta^2 k_2^{i\,2} - \beta\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] - \frac{\beta}{x}\widetilde{N}(\beta) - m_u^2\right\}}{\widetilde{N}(\beta)} \; . \end{aligned}$$

Subtracting zero in the form of

$$\begin{aligned} 0 = i\pi^2 &\int_{-x}^{1-x} d\beta \, \frac{\ln\left\{-\beta_0^2 k_2^{i\,2} - \beta_0\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] - m_u^2\right\}}{\widetilde{N}(\beta)} \\ - i\pi^2 &\int_0^{1-x} d\beta \, \frac{\ln\left\{-\beta_0^2 k_2^{i\,2} - \beta_0\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] - m_u^2\right\}}{\widetilde{N}(\beta)} \\ + i\pi^2 &\int_0^{-x} d\beta \, \frac{\ln\left\{-\beta_0^2 k_2^{i\,2} - \beta_0\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right] - m_u^2\right\}}{\widetilde{N}(\beta)} \end{aligned}$$

with β_0 chosen such, that $\widetilde{N}(\beta_0) \equiv 0$, i.e.

$$\beta_0 := \frac{k_1^{i\,2} + m_u^2 + (1-x)\left[k_2^{i\,2} - m_d^2 - 2(k_1^i k_2^i)\right]}{2\sqrt{N}} \; , \tag{3.10}$$

and performing the transformations (and using the definitions) in the different integrals, respectively,

$$\begin{aligned} &\beta \mapsto \beta + x && \beta_1 := \beta_0 + x && \widetilde{N}(\beta - x) - \widetilde{N}(\beta_1 - x) = 2\sqrt{N}(\beta - \beta_1) \\ &\beta \mapsto \frac{\beta}{1-x} && \beta_2 := \frac{\beta_0}{1-x} && \widetilde{N}((1-x)\beta) - \widetilde{N}((1-x)\beta_2) = -2\sqrt{N}(1-x)(\beta - \beta_2) \\ &\beta \mapsto -\frac{\beta}{x} && \beta_3 := -\frac{\beta_0}{x} && \widetilde{N}(-x\beta) - \widetilde{N}(-x\beta_3 = 2\sqrt{N}x(\beta - \beta_3) \; , \end{aligned}$$

leads to

$$\begin{aligned} J^{(0)} &= -\frac{i\pi^2}{2\sqrt{N}}\int_0^1 d\beta\,\frac{1}{\beta-\beta_1}\ln\frac{(1-\beta)^2 k_2^{i\,2}-(1-\beta)(k_2^{i\,2}-m_d^2)}{(1-\beta_1)^2 k_2^{i\,2}-(1-\beta_1)(k_2^{i\,2}-m_d^2)} \\ &\quad+\frac{i\pi^2}{2\sqrt{N}}\int_0^1 d\beta\,\frac{1}{\beta-\beta_2}\ln\frac{(1-\beta)^2 k_1^{i\,2}-(1-\beta)(k_1^{i\,2}-m_u^2)}{(1-\beta_2)^2 k_1^{i\,2}-(1-\beta_2)(k_1^{i\,2}-m_u^2)} \\ &\quad-\frac{i\pi^2}{2\sqrt{N}}\int_0^1 d\beta\,\frac{1}{\beta-\beta_3}\ln\frac{(\beta-u_1)(\beta-u_2)}{(\beta_3-u_1)(\beta_3-u_2)}\,, \end{aligned}$$

with

$$u_{1,2} := \frac{q^2+m_u^2-m_d^2\pm\sqrt{(q^2+m_u^2-m_d^2)^2-4q^2m_u^2}}{2q^2} \qquad \text{and} \qquad q := k_1^i - k_2^i\,.$$

These three integrals all lead to Spence functions, defined as

$$\mathrm{Sp}\,(x) := -\int_0^x dt\,\frac{\ln(1-t)}{t}\,,$$

which appear directly in the final result

$$\begin{aligned} J^{(0)} = -\frac{i\pi^2}{2\sqrt{N}}\Bigg\{&\mathrm{Sp}\left(\frac{\beta_1}{\beta_1-1}\right)+\mathrm{Sp}\left(\frac{\beta_1}{\beta_1-\widetilde{\beta}_1}\right)-\mathrm{Sp}\left(\frac{\beta_1-1}{\beta_1-\widetilde{\beta}_1}\right) \\ &-\mathrm{Sp}\left(\frac{\beta_2}{\beta_2-1}\right)-\mathrm{Sp}\left(\frac{\beta_2}{\beta_2-\widetilde{\beta}_2}\right)+\mathrm{Sp}\left(\frac{\beta_2-1}{\beta_2-\widetilde{\beta}_2}\right) \\ &+\mathrm{Sp}\left(\frac{\beta_3}{\beta_3-u_1}\right)-\mathrm{Sp}\left(\frac{\beta_3-1}{\beta_3-u_1}\right)+\mathrm{Sp}\left(\frac{\beta_3}{\beta_3-u_2}\right)-\mathrm{Sp}\left(\frac{\beta_3-1}{\beta_3-u_2}\right)\Bigg\}\,, \end{aligned} \tag{3.11}$$

where $\widetilde{\beta}_1 := m_d^2/k_2^{i\,2}$ and $\widetilde{\beta}_2 := m_u^2/k_1^{i\,2}$.

3.3.5 Combining the integrals

Putting the results of the integrals $K^{(0)}$ (eq. 3.7), $J_\mu^{(1)}$ (eq. 3.8), $J_{\mu\nu}^{(2)}$ (eq. 3.9), and $J^{(0)}$ (eq. 3.11) into the expression for the original integral (eq. 3.6) enables one to further simplify it. Writing it down explicitely would be page-filling, though, and has been left out at this place.
The results have been compared to [2, 18], taking into account the well-known typos in the first and finding two discrepancies with respect to the latter: Comparing J_D to their eq. 50, I don't get the summand -1 in the first bracket, and in their expression for y_0 in eq. A17, which corresponds to my β_0 in eq. 3.10, there seems to be a summand $-p^2$ missing in their last bracket.

Finally, the scattering matrix element is thus

$$S_{fi} = \frac{g_s^2 g_w^2}{8} C_F V_{qq'}\,\delta^{(4)}(k_1^i+k_2^i-k_1^f-k_2^f)\,\frac{I^\nu\cdot\bar{u}(k_2^f)\gamma_\nu\left(1-\gamma^5\right)v(k_1^f)}{(k_1^f+k_2^f)^2-M_W^2+i0}\,. \tag{3.12}$$

3.4 On-shell and massless integrals

In these two limits, taken in that order, one takes $k_n^{i\,2} = m_n^2 = 0 \quad \forall n$. Through this, the analytical expressions for the integrals in question become much simpler than in the most general case (see above). Since these two limits don't commute with dimensional continuation, one cannot just take them in the expressions already calculated. But by now the treatment of such integrals is well known and the calculations can be further shortened by using the following formulæ (for the sake of generality including the mass terms [24]):

$$\int d^dp \left[p^2 + 2(pq) - m^2\right]^{-\alpha} = (-1)^{d/2}\, i\pi^{d/2}\, \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \left[-q^2 - m^2\right]^{-\alpha + d/2}$$

$$\int d^dp\, p^\mu \left[p^2 + 2(pq) - m^2\right]^{-\alpha} = (-1)^{1+d/2}\, i\pi^{d/2}\, \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} q^\mu \left[-q^2 - m^2\right]^{-\alpha + d/2}$$

$$\int d^dp\, p^\mu p^\nu \left[p^2 + 2(pq) - m^2\right]^{-\alpha} = (-1)^{d/2}\, i\pi^{d/2}\, \frac{1}{\Gamma(\alpha)} \left[-q^2 - m^2\right]^{-\alpha + d/2} \cdot \left\{ q^\mu q^\nu \Gamma\left(\alpha - \frac{d}{2}\right) + \frac{1}{2} g^{\mu\nu}(-q^2 - m^2) \Gamma\left(\alpha - 1 - \frac{d}{2}\right) \right\} .$$

3.4.1 Calculating $K^{(0)}$

Taking these two limits in the integral

$$K^{(0)} := \int d^4q\, \frac{1}{\left[(q - k_2^i)^2 + i0\right]\left[(q - k_1^i)^2 + i0\right]}$$

and using Feynman parametrisation we find

$$K^{(0)} = i\pi^{2-\varepsilon}\, \Gamma(\varepsilon) \left[2(k_1^i k_2^i)\right]^{-\varepsilon} \int_0^1 d\alpha\, \alpha^{-\varepsilon}(1 - \alpha)^{-\varepsilon} .$$

With the help of the Euler β-function integral

$$\int_0^1 dx\, x^m (1 - x)^n = \frac{\Gamma(1 + m)\Gamma(1 + n)}{\Gamma(2 + m + n)} \quad \text{for} \quad \mathrm{Re}\{m, n\} > -1$$

one can rewrite the expression above into

$$K^{(0)} = i\pi^{2-\varepsilon} \left[2(k_1^i k_2^i)\right]^{-\varepsilon} \frac{\Gamma(\varepsilon)\left[\Gamma(1 - \varepsilon)\right]^2}{\Gamma(2 - 2\varepsilon)} ,$$

whose factors depending on ε are expanded in a Laurent series, leading to

$$K^{(0)} = i\pi^2 \mu^{-2\varepsilon} \left\{ N_\varepsilon - \ln \frac{2(k_1^i k_2^i)}{\mu^2} + 2 + \mathcal{O}(\varepsilon) \right\}$$

with $N_\varepsilon = 1/\varepsilon - \gamma_E - \ln \pi$ as defined above.
Expanding only the gamma functions yields:

$$K^{(0)} = i\pi^2 \mu^{-2\varepsilon} \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \cdot \left[\frac{1}{\varepsilon} + 2 + \mathcal{O}(\varepsilon)\right] \pi^{-\varepsilon} e^{-\gamma_E \varepsilon} , \tag{3.13}$$

where $Q^2 := -q^2 = 2(k_1^i k_2^i)$ has been taken into account.

3.4.2 Calculating $J^{(1)}_\mu$

The integral

$$J^{(1)}_\mu := \int d^4q \, \frac{q_\mu}{\left[(q-k_2^i)^2+i0\right]\left[(q-k_1^i)^2+i0\right](q^2+i0)}$$

can be calculated as shown above:

$$J^{(1)}_\mu = i\pi^2\mu^{-2\varepsilon}\,\frac{k^i_{1\,\mu}+k^i_{2\,\mu}}{2(k_1^ik_2^i)}\left\{N_\varepsilon - \ln\frac{2(k_1^ik_2^i)}{\mu^2}+2+\mathcal{O}(\varepsilon)\right\} .$$

Again, this expression can be written as

$$J^{(1)}_\mu = i\pi^2\mu^{-2\varepsilon}\,\frac{k^i_{1\,\mu}+k^i_{2\,\mu}}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon\cdot\left[\frac{1}{\varepsilon}+2+\mathcal{O}(\varepsilon)\right]\pi^{-\varepsilon}e^{-\gamma_E\varepsilon} . \tag{3.14}$$

3.4.3 Calculating $J^{(2)}_{\mu\nu}$

The same goes for this integral, which turns out to be

$$J^{(2)}_{\mu\nu} = i\pi^2\frac{\mu^{-2\varepsilon}}{4(k_1^ik_2^i)}\Big\{\left[k^i_{1\,\mu}k^i_{1\,\nu}+k^i_{2\,\mu}k^i_{2\,\nu}+g_{\mu\nu}(k_1^ik_2^i)\right]\left[N_\varepsilon-\ln\frac{2(k_1^ik_2^i)}{\mu^2}+2+\mathcal{O}(\varepsilon)\right] \\ -\left[k^i_{1\,\mu}k^i_{2\,\nu}+k^i_{2\,\mu}k^i_{1\,\nu}-g_{\mu\nu}(k_1^ik_2^i)\right]\left[1+\mathcal{O}(\varepsilon)\right]\Big\}$$

or, once again leaving certain factors exact,

$$J^{(2)}_{\mu\nu} = \frac{i\pi^2\mu^{-2\varepsilon}}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon\Big\{\left[k^i_{1\,\mu}k^i_{1\,\nu}+k^i_{2\,\mu}k^i_{2\,\nu}+g_{\mu\nu}(k_1^ik_2^i)\right]\left[\frac{1}{\varepsilon}+2+\mathcal{O}(\varepsilon)\right] \\ -\left[k^i_{1\,\mu}k^i_{2\,\nu}+k^i_{2\,\mu}k^i_{1\,\nu}-g_{\mu\nu}(k_1^ik_2^i)\right]\left[1+\mathcal{O}(\varepsilon)\right]\Big\}\pi^{-\varepsilon}e^{-\gamma_E\varepsilon} . \tag{3.15}$$

3.4.4 Calculating $J^{(0)}$

This integral, which is by far the most tedious one in the general case, is solved quite easily in the on-shell, massless case:

$$J^{(0)} = -i\pi^2\,\frac{\mu^{-2\varepsilon}}{4(k_1^ik_2^i)}\left\{\frac{1}{\varepsilon^2}+N_\varepsilon^2+\ln^2\frac{2(k_1^ik_2^i)}{\mu^2}\;-2N_\varepsilon\ln\frac{2(k_1^ik_2^i)}{\mu^2}-\frac{\pi^2}{6}+\mathcal{O}(\varepsilon)\right\} ,$$

which can also be formulated as

$$J^{(0)} = -\frac{i\pi^2\mu^{-2\varepsilon}}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon\cdot\left[\frac{1}{\varepsilon^2}-\frac{\pi^2}{12}+\mathcal{O}(\varepsilon)\right]\pi^{-\varepsilon}e^{-\gamma_E\varepsilon} . \tag{3.16}$$

3.4.5 Combining the integrals

Putting the integrals (eqs. 3.13 - 3.16) into the original expression for the matrix element (eq. 3.6), one has to take into account the negative sign of k_1^i. Use of the relations $\bar{v}(k)\not{k}=0$ and $\not{k}\,u(k)=0$ leads to

$$I^\nu = -i\pi^2\mu^{-2\varepsilon}\left(\frac{\mu^2}{-Q^2}\right)^\varepsilon\cdot\bar{v}(k_2^i)\gamma^\nu(1-\gamma^5)u(k_1^i)\cdot\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8-\frac{\pi^2}{6}+\mathcal{O}(\varepsilon)\right]\pi^{-\varepsilon}e^{-\gamma_E\varepsilon} . \tag{3.17}$$

The last factor is often written as $e^{-\gamma_E\varepsilon}=1/\Gamma(1-\varepsilon)\left[1+\varepsilon^2\pi^2/12+\mathcal{O}(\varepsilon^3)\right]$.
In principle, one has additional real terms in eq. 3.17, which stem from the negative sign inside the logarithm, if one makes a series expansion of the factor $(-\mu^2/Q^2)^\varepsilon$. These do not contribute, however, since the scattering matrix element is $|S_1+S_3|^2=|S_1|^2+2\,\mathrm{Re}\{S_1^*S_3\}+|S_3|^2$, and can therefore be neglected.

3.5 Contribution to the differential cross section

To calculate the vertex's contribution to the differential cross section, one has to multiply it with the corresponding expression of the leading-order contribution. As in the calculation of the LO, we write $S_{fi} =: i(2\pi)^d\, \delta^{(d)}(k_1^i + k_2^i - k_1^f - k_2^f)\, M$, thereby defining M. With the definitions $g_w^2 = 4\sqrt{2}G_F M_W^2$, $g_s^2 = 4\pi\alpha_s$, and coupling constant renormalisation $g_s \mapsto g_s\mu^{d-4}$, from above (eq. 3.12) follows

$$M^V = -i\frac{\sqrt{2}\alpha_s}{(2\pi)^{3+2\varepsilon}} C_F V_{qq'} \frac{\mu^{2\varepsilon} G_F M_W^2}{(k_1^f + k_2^f)^2 - M_W^2 + i0}\, I^\nu \cdot \bar{u}(k_2^f)\gamma_\nu \left(1-\gamma^5\right) v(k_1^f)\,.$$

The corresponding expression of the LO graph is:

$$M^{LO} = \frac{1}{\sqrt{2}} V_{qq'} \frac{G_F M_W^2}{(k_1^f + k_2^f)^2 - M_W^2 + i0}\; \bar{v}(k_2^i)\gamma^\mu \left(1-\gamma^5\right) u(k_1^i) \cdot \bar{u}(k_2^f)\gamma_\mu \left(1-\gamma^5\right) v(k_1^f)\,,$$

and therefore one can write

$$M^V = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{-Q^2}\right)^\varepsilon \left[-\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right] (4\pi)^\varepsilon e^{-\gamma_E \varepsilon} \cdot M^{LO}\,.$$

The different polarisation states in the quark current can be projected out by means of

$$u_R = \frac{1+\gamma^5}{2}(u_L + u_R) \qquad \bar{u}_R = (\bar{u}_L + \bar{u}_R)\frac{1-\gamma^5}{2}$$
$$u_L = \frac{1-\gamma^5}{2}(u_L + u_R) \qquad \bar{u}_L = (\bar{u}_L + \bar{u}_R)\frac{1+\gamma^5}{2}$$

and analogous relations for v, which restore the summation over spin states needed for Casimir's theorem. Thus, it is evident that in the the LO matrix element the only contributing polarisations are $v \equiv v_L$ and $u \equiv u_L$, since $(1-\gamma^5)(1+\gamma^5) = 0$; it is non-vanishing only for left-handed initial-state particles. The same holds for the vertex matrix element (see, for example, eq. 3.17), because it has an identical spinor term.

For the NLO correction, one needs the *polarisation difference* of the matrix elements squared, which is defined by

$$\Delta\left|\mathcal{M}\right|^2 := \left|\mathcal{M}\right|^2_{LL} - \left|\mathcal{M}\right|^2_{LR} - \left|\mathcal{M}\right|^2_{RL} + \left|\mathcal{M}\right|^2_{RR}\,.$$

In this case, due to the vanishing of all matrix elements with right-handed initial-state particles, it is identical to the matrix element squared calculated above.

Finally, the polarisation difference of the differential cross section can be written as

$$d\Delta\sigma^V = -\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^\varepsilon \left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + 4 - \frac{7\pi^2}{12} + \mathcal{O}(\varepsilon)\right] e^{-\gamma_E\varepsilon} \cdot d\Delta\sigma^{LO}\,. \tag{3.18}$$

3.6 Physical Gauge

Calculating in physical (axial) gauge, the polarisation sum has additional terms according to

$$\sum_\lambda \epsilon_\mu^\dagger(\vec{q},\lambda)\, \epsilon_\nu(\vec{q},\lambda) = -g_{\mu\nu} + \frac{q_\mu n_\nu + n_\mu q_\nu}{(qn)}\,,$$

where n is an arbitrary, light-like gauge-fixing vector, constrained by physical considerations. This leads to an additional term in the numerator of S_{fi} which can be treated in complete analogy to above: We define

$$\tilde{I}^{\nu} := \int d^4q \frac{\bar{v}(k_2^i)\gamma^{\alpha}(\not{q} - \not{k}_2^i)\gamma^{\nu}(1-\gamma^5)(\not{q} + \not{k}_1^i)\gamma^{\beta}u(k_1^i)}{\left[(q-k_2^i)^2 + i0\right]\left[(q+k_1^i)^2 + i0\right](q^2+i0)} \cdot \frac{q_{\alpha}n_{\beta} + n_{\alpha}q_{\beta}}{(qn)} \tag{3.19}$$

and are thus able to write the entire scattering matrix element as $S_{fi} \propto (I^{\nu} + \tilde{I}^{\nu}) \cdot \bar{u}(k_2^f)\gamma_{\nu}\left(1-\gamma^5\right)v(k_1^f)$, where I^{ν} stems from Feynman gauge and $\tilde{I}^{\nu}$ from the additional terms in physical gauge.
Defining the following integrals

$$\begin{aligned}
I^{(0)}(k_1^i, k_2^i) &:= \int d^4q \frac{1}{\left[(q-k_2^i)^2 + i0\right]\left[(q+k_1^i)^2 + i0\right](qn)} \\
I_{\mu}^{(1)}(k_1^i, k_2^i) &:= \int d^4q \frac{q_{\mu}}{\left[(q-k_2^i)^2 + i0\right]\left[(q+k_1^i)^2 + i0\right](qn)} \\
H_{\mu}^{(1)}(k_1^i, k_2^i) &:= \int d^4q \frac{q_{\mu}}{\left[(q-k_2^i)^2 + i0\right]\left[(q+k_1^i)^2 + i0\right][q^2+i0](qn)} \\
H_{\mu\nu}^{(2)}(k_1^i, k_2^i) &:= \int d^4q \frac{q_{\mu}q_{\nu}}{\left[(q-k_2^i)^2 + i0\right]\left[(q+k_1^i)^2 + i0\right][q^2+i0](qn)} ,
\end{aligned}$$

the original intergal (3.19) can be expressed by these quantities:

$$\begin{aligned}
\tilde{I}^{\nu} = \ \bar{v}(k_2^i)\Big\{ & 2K^{(0)}\gamma^{\nu} + 4J^{(1)\,\alpha}k^i_{1\,\alpha}\gamma^{\nu} + 2I^{(1)\,\nu}\not{n} \\
& + 2I^{(0)}\left[(k_1^i n) - (k_2^i n)\right]\gamma^{\nu} - 4H^{(1)\,\alpha}\left[k^i_{2\,\alpha}(k_1^i n) + k^i_{1\,\alpha}(k_2^i n)\right]\gamma^{\nu} \\
& - 4H^{(2)\,\alpha\nu}k^i_{2\,\alpha}\not{n} + 2H^{(2)\,\alpha\beta}k^i_{2\,\alpha}\gamma_{\beta}\gamma^{\nu}\not{n} - 2H^{(2)\,\alpha\beta}k^i_{1\,\alpha}\gamma_{\beta}\not{n}\gamma^{\nu}\Big\}(1-\gamma^5)u(k_1^i) .
\end{aligned} \tag{3.20}$$

3.6.1 Calculating $I^{(0)}$

The calculation of this integral proceeds along the general lines used above, leading to

$$I^{(0)} = 2i\pi^{2-\varepsilon}\frac{\mu^{-2\varepsilon}}{Q^2}\left(\frac{\mu^2}{-Q^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\int_0^1 \frac{d\alpha}{(1-\alpha)^2} ,$$

where one demands that $(k_1^i n) \stackrel{!}{=} 0 \stackrel{!}{=} (k_2^i n)$ to ensure physical gluons. To deal with the divergence in the α integral, we integrate from 0 to $1-\delta$, $\delta > 0$, which leads to:

$$I^{(0)} = 2i\pi^{2-\varepsilon}\frac{\mu^{-2\varepsilon}}{Q^2}\left(\frac{\mu^2}{-Q^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{1}{\delta} - 1\right) . \tag{3.21}$$

3.6.2 Calculating $I_{\mu}^{(1)}$

In analogous fashion follows

$$I_{\mu}^{(1)} = -i\pi^{2-\varepsilon}\frac{\mu^{-2\varepsilon}}{Q^2}\left(\frac{\mu^2}{-Q^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)} \cdot \left(\frac{1}{\delta} - 1\right)\left\{k^i_{1\,\mu} - k^i_{2\,\mu} + \frac{n_{\mu}}{2}\left(\frac{1}{\delta} - 1\right)\right\} . \tag{3.22}$$

3.6.3 Calculating $H_\mu^{(1)}$

In this case, on needs the Feynman parametrisation for four propagators, which is

$$\frac{1}{a_1 a_2 a_3 a_4} = \Gamma(3) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_0^{1-\alpha-\beta} d\gamma \left\{ \alpha a_1 + \beta a_2 + \gamma a_3 + (1-\alpha-\beta-\gamma) a_4 \right\}^{-4} .$$

This leads to

$$H_\mu^{(1)} = -2i\pi^{2-\varepsilon} \frac{\mu^{-2\varepsilon}}{Q^4} \left(\frac{\mu^2}{-Q^2} \right)^\varepsilon \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)} \cdot \left(\frac{1}{\delta} - 1 \right) \left\{ k^i_{1\,\mu} - k^i_{2\,\mu} + \frac{n_\mu}{4} \left(\frac{1}{\delta} - 1 \right) \frac{1+2\varepsilon}{1+\varepsilon} \right\} . \quad (3.23)$$

3.6.4 Calculating $H_{\mu\nu}^{(2)}$

Here, we find

$$H_{\mu\nu}^{(2)} = -i\pi^{2-\varepsilon} \frac{\mu^{-2\varepsilon}}{Q^4} \left(\frac{\mu^2}{-Q^2} \right)^\varepsilon \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{1}{\delta} - 1 \right) \left\{ \frac{n_\mu n_\nu}{6} \left(\frac{1}{\delta} - 1 \right)^2 \frac{1+2\varepsilon}{1+\varepsilon} + k^i_{1\,\mu} k^i_{1\,\nu} + k^i_{2\,\mu} k^i_{2\,\nu} + \frac{1}{2} \left(n_\mu k^i_{1\,\nu} + k^i_{1\,\mu} n_\nu - n_\mu k^i_{2\,\nu} - k^i_{2\,\mu} n_\nu \right) \left(\frac{1}{\delta} - 1 \right) - \left[k^i_{1\,\mu} k^i_{2\,\nu} + k^i_{2\,\mu} k^i_{1\,\nu} - g_{\mu\nu} \frac{(k^i_1 k^i_2)}{1+\varepsilon} \right] \frac{1+\varepsilon}{\varepsilon} \right\} . \quad (3.24)$$

3.6.5 The total contribution from physical gauge

Inserting eqs. (3.21) - (3.24) into $\tilde{I}^\nu$ (eq. 3.20), the entire expression vanishes. Hence, there is no contribution from the additional terms of the physical gauge (relative to the Feynman gauge).

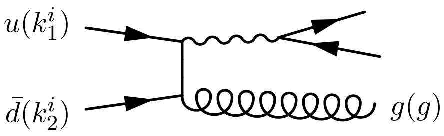
4 Gluon bremsstrahlung correction

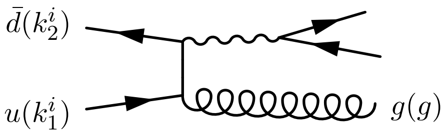
Sing out the song; sing to the end and sing
The strange reward of all that discipline.

William Butler Yeats: The Phases of the Moon

4.1 The matrix element

The scattering matrix element for the two possible gluon bremsstrahlung diagrams (the $\bar{d}$ or the u quark emitting the gluon)





has already been calculated in my M.S. thesis [36] as well:

$$
\begin{aligned}
S_{fi} = i(2\pi)^4\, \delta^{(4)}(k_1^i + k_2^i - k_1^f - k_2^f - g) \cdot \frac{g_s g_w^2}{8} V_{qq'}\, T_a \frac{1}{q_1^2 - M_W^2 + i0} \\
\cdot\, \bar{v}(k_2^i) \left[\not{\epsilon}(k_2^i, \lambda) \frac{\not{q}_2 - m_d}{q_2^2 - m_d^2} \gamma^\mu (1-\gamma^5) - \gamma^\mu (1-\gamma^5) \frac{\not{q}_3 + m_u}{q_3^2 - m_u^2} \not{\epsilon}(k_2^i, \lambda) \right] u(k_1^i) \\
\cdot\, \bar{u}(k_2^f) \gamma_\mu \left(1 - \gamma^5\right) v(k_1^f)\ ,
\end{aligned}
$$

with $q_1 := k_1^f + k_2^f$, $q_2 := k_2^i - g$, and $q_3 := k_1^i - g$.
For convenience, we define again $S_{fi} =: (2\pi)^4\, i\, \delta^{(4)}(k^i - k^f) \cdot M$, as well as the lepton and quark currents

$$
\begin{aligned}
L_\mu &:= \bar{u}(k_2^f) \gamma_\mu \left(1 - \gamma^5\right) v(k_1^f) \\
J^\mu &:= \bar{v}(k_2^i) \left[\not{\epsilon}(k_2^i, \lambda) \frac{\not{q}_2 - m_d}{q_2^2 - m_d^2} \gamma^\mu (1-\gamma^5) - \gamma^\mu (1-\gamma^5) \frac{\not{q}_3 + m_u}{q_3^2 - m_u^2} \not{\epsilon}(k_2^i, \lambda) \right] u(k_1^i)\ .
\end{aligned}
$$

To calculate the contribution to the differential cross section, one needs

$$
\left|\mathcal{M}^B\right|^2 := \sum_{s,\lambda,a} |M|^2 = \sum_{s,\lambda,a} \left|V_{qq'}\right|^2 \operatorname{tr}\left[T_a^* T_a\right] \left[\frac{g_s g_w^2}{8} \frac{1}{q_1^2 - M_W^2 + i0} \right]^2 \cdot J^{\mu\,\dagger} J^\nu \cdot L_\mu^\dagger L_\nu\ ,
$$

with the sum running over final state spins, polarisation of the gluon, and colour (the colour factor is $\operatorname{tr}\left[T_a^* T_a\right] = C_F$). The sum over final state spins yields

$$
\sum_s L_\mu^\dagger L_\nu = 8 \left[k^f_{1\,\mu} k^f_{2\,\nu} + k^f_{1\,\nu} k^f_{2\,\mu} - (k_1^f k_2^f)\, g_{\mu\nu} + i \epsilon_{\alpha\beta\mu\nu} k_1^{f\,\alpha} k_2^{f\,\beta} \right]\ . \tag{4.1}
$$

The quarks' polarisation will be incorporated using the projection operator, as shown above (chapter 3).

Calculating in Feynman gauge (physical gauge will be considered in Section 4.4), the polarisation sum over the product of the gluon's polarisation vectors is

$$\sum_\lambda \epsilon^\dagger_\mu(\vec{k},\lambda)\,\epsilon_\nu(\vec{k},\lambda) = -g_{\mu\nu}\ . \tag{4.2}$$

For $(\mathbf{R},\mathbf{R})$, where the first index describes the polarisation of k_1^i and the second one the spin orientation of k_2^i, we find

$$J^{\mu\,\dagger}J^\nu = 0 \quad \text{and thus} \quad \left|\mathcal{M}^B\right|^2_{RR} = 0\ .$$

For $(\mathbf{L},\mathbf{R})$, the result is

$$J^{\mu\,\dagger}J^\nu = 16\left(\frac{m_d}{q_2^2-m_d^2}\right)^2\left[k_1^{i\,\mu}k_2^{i\,\nu} + k_2^{i\,\mu}k_1^{i\,\nu} - (k_1^ik_2^i)g^{\mu\nu} + i\epsilon^{\alpha\beta\mu\nu}k^i_{1\,\alpha}k^i_{2\,\beta}\right]\ ,$$

leading to

$$\left|\mathcal{M}^B\right|^2_{LR} = 8C_F\left|V_{qq'}\right|^2\left[\frac{g_sg_w^2}{q_1^2-M_W^2+i0}\frac{m_d}{q_2^2-m_d^2+i0}\right]^2(k_1^ik_1^f)(k_2^ik_2^f)\ .$$

For $(\mathbf{R},\mathbf{L})$, the symmetry in quark masses is restored, as

$$\left|\mathcal{M}^B\right|^2_{RL} = 8C_F\left|V_{qq'}\right|^2\left[\frac{g_sg_w^2}{q_1^2-M_W^2+i0}\frac{m_u}{q_3^2-m_u^2+i0}\right]^2(k_1^ik_1^f)(k_2^ik_2^f)\ .$$

For $(\mathbf{L},\mathbf{L})$, the calculation becomes a bit more involved. A first result is

$$\begin{aligned}\sum_{s,\epsilon} J^{\mu\,\dagger}J^\nu = {}& 4\frac{\mathrm{tr}\left[\not{q}_2\not{k}_2^i\not{q}_2\gamma^\nu\not{k}_1^i\gamma^\mu(1-\gamma^5)\right]}{(q_2^2-m_d^2)^2} + 4\frac{\mathrm{tr}\left[\not{k}_2^i\gamma^\nu\not{q}_3\not{k}_1^i\not{q}_3\gamma^\mu(1-\gamma^5)\right]}{(q_3^2-m_u^2)^2}\\ &- 4\frac{\mathrm{tr}\left[\not{k}_1^i\not{k}_2^i\gamma^\mu\not{q}_3\not{q}_2\gamma^\nu(1-\gamma^5)\right] + \mathrm{tr}\left[\not{k}_2^i\not{k}_1^i\gamma^\mu\not{q}_2\not{q}_3\gamma^\nu(1+\gamma^5)\right]}{(q_2^2-m_d^2)(q_3^2-m_u^2)}\ .\end{aligned}$$

Using trace theorems and the well-known relations among Dirac matrices, one then finds

$$\begin{aligned}\sum_{s,\epsilon} J^{\mu\,\dagger}J^\nu = {}& -16\left[\frac{q_2^2}{(q_2^2-m_d^2)^2}+\frac{q_3^2}{(q_3^2-m_u^2)^2}\right]\left[k_1^{i\,\mu}k_2^{i\,\nu} + k_2^{i\,\mu}k_1^{i\,\nu} - (k_1^ik_2^i)g^{\mu\nu} - i\epsilon^{\alpha\beta\nu\mu}k^i_{1\,\alpha}k^i_{2\,\beta}\right]\\ &+ 32\frac{(k_2^iq_2)}{(q_2^2-m_d^2)^2}\left[k_1^{i\,\mu}q_2^\nu + q_2^\mu k_1^{i\,\nu} - (k_1^iq_2)g^{\mu\nu} - i\epsilon^{\alpha\beta\nu\mu}k^i_{1\,\alpha}q_{2\,\beta}\right]\\ &+ 32\frac{(k_1^iq_3)}{(q_3^2-m_u^2)^2}\left[q_3^\mu k_2^{i\,\nu} + k_2^{i\,\mu}q_3^\nu - (q_3k_2^i)g^{\mu\nu} - i\epsilon^{\alpha\beta\nu\mu}q_{3\,\alpha}k^i_{2\,\beta}\right]\\ &- \frac{32}{(q_2^2-m_d^2)(q_3^2-m_u^2)}\Big\{-2(k_1^ik_2^i)\left[k_1^{i\,\mu}k_2^{i\,\nu} + k_2^{i\,\mu}k_1^{i\,\nu} - (k_1^ik_2^i)g^{\mu\nu} - i\epsilon^{\alpha\beta\nu\mu}k^i_{1\,\alpha}k^i_{2\,\beta}\right]\\ &\quad + (k_1^ik_2^i)\Big[(k_1^{i\,\mu}+k_2^{i\,\mu})g^\nu + g^\mu(k_1^{i\,\nu}+k_2^{i\,\nu}) - (k_1^ig)g^{\mu\nu} - (k_2^ig)g^{\mu\nu}\\ &\qquad\qquad -i\epsilon^{\alpha\beta\nu\mu}(k^i_{1\,\alpha}g_\beta + g_\alpha k^i_{2\,\beta})\Big]\\ &\quad + \left[(k_1^ig)+(k_2^ig)\right]\left[k_1^{i\,\mu}k_2^{i\,\nu} + k_2^{i\,\mu}k_1^{i\,\nu} - (k_1^ik_2^i)g^{\mu\nu} - i\epsilon^{\alpha\beta\nu\mu}k^i_{1\,\alpha}k^i_{2\,\beta}\right]\\ &\quad - (k_1^ig)\left[2k_2^{i\,\mu}k_2^{i\,\nu} - k_2^{i\,2}g^{\mu\nu}\right] - (k_2^ig)\left[2k_1^{i\,\mu}k_1^{i\,\nu} - k_1^{i\,2}g^{\mu\nu}\right]\\ &\quad -k_1^{i\,2}\left[k_2^{i\,\mu}g^\nu + g^\mu k_2^{i\,\nu} - 2k_2^{i\,\mu}k_2^{i\,\nu}\right] - k_2^{i\,2}\left[k_1^{i\,\mu}g^\nu + g^\mu k_1^{i\nu} - 2k_1^{i\,\mu}k_1^{i\,\nu}\right] - k_1^{i\,2}k_2^{i\,2}g^{\mu\nu}\Big\}\ .\end{aligned}$$

Assuming $k_{1,2}^{i\,2} = m_{u,d}^2$ and multiplying the above result with eq. 4.1, the squared and summed-over matrix element is

$$\begin{aligned}
\left|\mathcal{M}^B\right|^2_{LL} = 2C_F\left|V_{qq'}\right|^2\left[\frac{g_s g_w^2}{q_1^2 - M_W^2}\right]^2 \\
&\left\{\frac{2\left[(k_1^i k_1^f)(k_1^i k_2^f) - (k_1^i k_2^f)(k_2^i k_1^f) + (k_2^i k_1^f)(g k_2^f)\right] - m_u^2(k_1^f k_2^f)}{(k_1^i g)}\right.\\
&+\frac{2\left[(k_2^i k_1^f)(k_2^i k_2^f) - (k_1^i k_2^f)(k_2^i k_1^f) + (k_1^i k_2^f)(g k_1^f)\right] - m_d^2(k_1^f k_2^f)}{(k_2^i g)}\\
&+2(k_1^i k_2^i)\frac{2(k_1^i k_2^f)(k_2^i k_1^f) - (k_1^i k_2^f)(g k_1^f) - (k_2^i k_1^f)(g k_2^f)}{(k_1^i g)(k_2^i g)}\\
&+\frac{(k_1^i k_2^f) - 2(g k_2^f)}{(k_1^i g)^2}(k_2^i k_1^f)\, m_u^2 + \frac{(k_2^i k_1^f) - 2(g k_1^f)}{(k_2^i g)}(k_1^i k_2^f)\, m_d^2\\
&+m_u^2\frac{(k_2^i k_1^f)(g k_2^f) + (k_2^i k_2^f)(g k_1^f) - 2(k_2^i k_1^f)(k_2^i k_2^f)}{(k_1^i g)(k_2^i g)}\\
&+m_d^2\frac{(k_1^i k_1^f)(g k_2^f) + (k_1^i k_2^f)(g k_1^f) - 2(k_1^i k_1^f)(k_1^i k_2^f)}{(k_1^i g)(k_2^i g)}\\
&\left.+m_u^2 m_d^2\frac{(k_1^f k_2^f)}{(k_1^i g)(k_2^i g)}\right\} .
\end{aligned}$$

Using the definitions $g_s^2 = 4\pi\alpha_s \cdot \mu^{2\varepsilon}$ (for $d = 4 - 2\varepsilon$) and $g_w^2 = 4\sqrt{2}\, G_F M_W^2$ brings the prefactor into the usual form:

$$\left[\frac{g_s g_w^2}{q_1^2 - M_W^2 + i0}\right]^2 \to 128\,\pi\,\alpha_s\mu^{2\varepsilon}\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 .$$

At this point it is sensible to neglect quark masses, which simplifies the result somewhat:

$$\begin{aligned}
\frac{1}{256}\sum J^{\mu\,\dagger} J^\nu L_\mu^\dagger L_\nu = -(k_1^i k_1^f)(k_2^i k_2^f)\left[\frac{1}{(k_1^i g)} + \frac{1}{(k_2^i g)}\right] + \frac{(k_1^i k_1^f)(k_2^f g)}{(k_2^i g)} + \frac{(k_2^i k_2^f)(k_1^f g)}{(k_1^i g)}\\
+\frac{(k_2^i k_1^f)(k_2^i k_2^f)}{(k_2^i g)} + \frac{(k_1^i k_1^f)(k_1^i k_2^f)}{(k_1^i g)} - \frac{(k_1^i k_2^i)}{(k_1^i g)(k_2^i g)}\left[-2(k_1^i k_1^f)(k_2^i k_2^f) + (k_1^i k_1^f)(k_2^f g) + (k_2^i k_2^f)(k_1^f g)\right]
\end{aligned} \tag{4.3}$$

Expressing the scalar products with the help of scalar invariants of the process [7]

$$\begin{aligned}
2(k_1^i k_2^i) &= s = s_{ab} & 2(k_1^f k_2^f) &= s_2 = s_{23}\\
2(k_1^i k_1^f) &= -t_2 = s_{b3} & 2(k_2^i k_1^f) &= s - s_1 + t_2 = s_{a3}\\
2(k_1^i k_2^f) &= s_2 + t_2 - t_1 = s_{b2} & 2(k_2^i k_2^f) &= s_1 + t_1 - t_2 = s_{a2}\\
2(k_1^i g) &= s - s_2 + t_1 = s_{b1} & 2(k_2^i g) &= -t_1 = s_{a1}\\
2(k_1^f g) &= s - s_1 - s_2 = s_{13} & 2(k_2^f g) &= s_1 = s_{12}
\end{aligned}$$

we find

$$\sum J^{\mu\,\dagger} J^\nu L_\mu^\dagger L_\nu = -128\,\frac{(s_1 + t_1 - t_2)^2 + t_2^2}{(s - s_2 + t_1)\, t_1}\, s_2 .$$

From this follows the only non-vanishing matrix element squared for the bremsstrahlung process (neglecting quark masses)

$$\left|\mathcal{M}^B\right|^2_{LL} = 256\pi\alpha_s\mu^{2\varepsilon}C_F\left|V_{qq'}\right|^2\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 \cdot \frac{s_{a2}^2 + s_{b3}^2}{s_{a1}s_{b1}} s_{23} \,,$$

where the index on the Mandelstam variable s represents the particles of the process $a+b \to 1+2+3$ with 1 being the gluon.

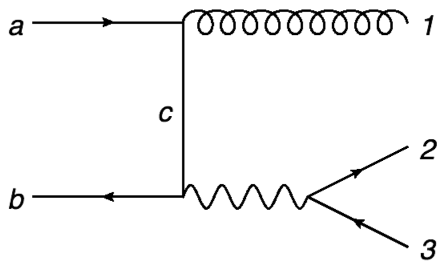


Figure 4.1: *Feynman diagram of the process in question with particle labels.*

In analogous fashion, the leading-order matrix element squared can be written as

$$\left|\mathcal{M}^{LO}\right|^2_{LL} = 32\left|V_{qq'}\right|^2\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 \cdot s_{a2}s_{b3} \equiv \Delta\left|\mathcal{M}^{LO}\right|^2 \,,$$

(the second relation has been shown above) and thus, one may write the bremsstrahlung matrix element squared as

$$\left|\mathcal{M}^B\right|^2_{LL} = 8\alpha_s\mu^{2\varepsilon}C_F \cdot \frac{s_{a2}^2 + s_{b3}^2}{s_{a1}s_{b1}}\frac{s_{23}}{s_{a2}s_{b3}} \cdot \left|\mathcal{M}^{LO}\right|^2_{LL} \,. \tag{4.4}$$

The Mandelstam variables s_{a1}, s_{b1}, s_{a2}, and s_{b3} are all negative since they combine an initial-state momentum with one from the final state.

Again, the quantity required for the NLO corrections is the polarisation difference:

$$\Delta\left|\mathcal{M}\right|^2 := \left|\mathcal{M}\right|^2_{LL} - \left|\mathcal{M}\right|^2_{LR} - \left|\mathcal{M}\right|^2_{RL} + \left|\mathcal{M}\right|^2_{RR} \,,$$

which in this case is just

$$\Delta\left|\mathcal{M}^B\right|^2 \equiv \left|\mathcal{M}^B\right|^2_{LL} \,.$$

4.2 Contributions to the scattering matrix element

For this process, the phase space has to be sliced into two separate parts: one where the emitted gluon is collinear to one of the quarks and one where the gluon is soft. Very much care has to be taken to match these two disjoint regions onto each other without having them overlap.

4.2.1 The collinear region – behaviour of the phase space

In the case of a collinear final-state particle, one defines the fraction of the incoming particle's four-momentum transferred to the W boson as z by $s_{bc} =: z s_{ab}$, where 1 is collinear to a. The region itself is then defined by $|s_{a1}| < s_{min}$ with s_{min} some cutoff. Therefore, the $(d = 4-2\varepsilon)$-dimensional phase space factorises according to [15, 16]:

$$\frac{1}{2s_{ab}} dP^d(a+b \to 1+2+3) = dP^d_{coll.}(a \to 1+c) \cdot \frac{1}{2s_{bc}} dP^d(b+c \to 2+3) \; .$$

The two factors on the right-hand side are

$$dP^{4-2\varepsilon}_{coll.}(a \to 1+c) = \frac{(4\pi)^{-2+\varepsilon}}{\Gamma(1-\varepsilon)} \cdot [(1-z)|s_{a1}|]^{-\varepsilon} \cdot z\, dz \cdot d|s_{a1}|\; \Theta(s_{min} - |s_{a1}|)$$

$$\begin{aligned} dP^{4-2\varepsilon}(b+c \to 2+3) &= dP^{4-2\varepsilon}(b+c \to Q) \cdot \frac{dQ^2}{2\pi} \cdot dP^{4-2\varepsilon}(Q \to 2+3) \\ &= \left(\frac{s_{bc}}{s_{ab}}\right)^{-\varepsilon} dP^{4-2\varepsilon}(a+b \to 2+3) \; . \end{aligned}$$

Combining the two formulas above, the phase space (including the flux factor) needed for this diagram follows as:

$$\begin{aligned} \frac{1}{2s_{ab}} dP^{4-2\varepsilon}(a+b \to 1+2+3) = \frac{[z(1-z)|s_{a1}|]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}\Gamma(1-\varepsilon)}\, dz\; d|s_{a1}|\; \Theta(s_{min} - |s_{a1}|) \\ \cdot \frac{1}{2s_{ab}} dP^{4-2\varepsilon}(a+b \to 2+3) \; . \end{aligned} \tag{4.5}$$

4.2.2 The collinear region – behaviour of the matrix element

From eq. 4.4 follows the polarisation difference of the matrix element squared for the bremsstrahlung process

$$\Delta \left|\mathcal{M}^B\right|^2_{coll.} = 8\pi\alpha_s\mu^{2\varepsilon} \hat{P}^4_{qg\to q}(z) \frac{1}{|s_{a1}|} \cdot \Delta \left|\mathcal{M}^{LO}\right|^2 \; , \tag{4.6}$$

taking into account $s_{b3} = zs_{a2}$, $|s_{b1}| = (1-z)s_{ab}$ and with the (unregularised), four-dimensional Altarelli splitting function

$$\hat{P}^4_{qg\to q}(z) = C_F(1+z^2)/(1-z) \; .$$

If we had calculated the traces in $d = 4-2\varepsilon$ dimensions, we would have found

$$\hat{P}^{4-2\varepsilon}_{qg\to q}(z) = C_F\left[(1+z^2)/(1-z) - \varepsilon(1-z)\right]$$

instead. To take this dimensional dependence into account we simply replace $\hat{P}^4$ by $\hat{P}^{4-2\varepsilon}$.

4.2.3 The collinear region – contribution to the differential scattering cross section

The combination of the expressions above (eqs. 4.6 and 4.5) requires integrating out the unobserved gluon. This, however, has to be done with care, as one must distinguish the collinear region from the soft one. The soft region for this process is defined by, additionally, $s_{b1} < s_{min}$. Thus, to stay in the collinear part of phase space we have to enforce

$$|s_{b1}| > s_{min} \quad \Leftrightarrow \quad 1 - \frac{s_{min}}{s_{ab}} > z \; .$$

The collinear part of the differential scattering cross section is therefore

$$d\Delta\sigma^B_{coll.} = d\Delta\sigma^{LO} \cdot \frac{\alpha_s\mu^{2\varepsilon}}{2\pi} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \cdot \int\limits_0^{1-\frac{s_{min}}{s_{ab}}} dz\, \hat{P}^{4-2\varepsilon}_{qg\to q}(z)\, [z(1-z)]^{-\varepsilon} \cdot \int\limits_0^{s_{min}} d|s_{a1}|\, |s_{a1}|^{-1-\varepsilon} \; .$$

Integration over z can be done by means of the identity for the incomplete beta function

$$\int_0^{1-\delta} dz\, z^{a-1}(1-z)^{b-1} = \int_0^1 dz\, z^{a-1}(1-z)^{b-1} - \frac{\delta^b}{b}\left[1+\mathcal{O}(\delta)\right] ,$$

hence we find

$$\int_0^{1-\delta} dz\, \hat{P}^{4-2\varepsilon}_{qg\to q}(z)\,[z(1-z)]^{-\varepsilon} = 2\frac{\delta^{-\varepsilon}}{\varepsilon} - \frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\frac{(1-\varepsilon)(4-\varepsilon)}{2\varepsilon} + \frac{1}{\varepsilon}\mathcal{O}(\delta) .$$

The integral over $|s_{a1}|$ yields

$$\int_0^{s_{min}} d|s_{a1}|\, |s_{a1}|^{-1-\varepsilon} = -\frac{1}{\varepsilon}s_{min}^{-\varepsilon} .$$

Together, this leads to the following contribution to the differential scattering cross section:

$$d\Delta\sigma^B_{coll.} = -d\Delta\sigma^{LO}\cdot\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{s_{min}}\right)^\varepsilon\left[\frac{4}{\varepsilon^2}\left(\frac{s_{ab}}{s_{min}}\right)^\varepsilon\frac{1}{\Gamma(1-\varepsilon)}\right.$$
$$\left. - \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\frac{(1-\varepsilon)(4-\varepsilon)}{\varepsilon^2(1-2\varepsilon)} + \mathcal{O}\left(\frac{s_{min}}{s_{ab}}\right)\right] , \quad (4.7)$$

where we have taken into account the two possible diagram layouts (i.e. either the quark or the antiquark emitting the gluon) which result in a factor of 2.

4.2.4 The soft region – behaviour of the phase space

The phase space factorises again, as in the collinear case [15]:

$$dP^{4-2\varepsilon}(a+b\to 1+2+3) = dP^{4-2\varepsilon}_{soft}(a\to 1+c,b)\cdot dP^{4-2\varepsilon}(c+b\to 2+3) , \quad (4.8)$$

with the soft factor taking the form

$$dP^{4-2\varepsilon}_{soft}(a\to 1+c,b) = \frac{(4\pi)^{-2+\varepsilon}}{\Gamma(1-\varepsilon)}\frac{d|s_{a1}|d|s_{b1}|}{s_{ab}}\left[\frac{|s_{a1}||s_{b1}|}{s_{ab}}\right]^{-\varepsilon}\Theta(s_{min}-|s_{a1}|)\Theta(s_{min}-|s_{b1}|) .$$

4.2.5 The soft region – behaviour of the matrix element

The soft behaviour is dominated by the most divergent term in eq. 4.3

$$\sum J^{\mu\,\dagger}J^\nu L^\dagger_\mu L_\nu \sim 512\frac{(k_1^i k_2^i)(k_1^i k_1^f)(k_2^i k_2^f)}{(k_1^i g)(k_2^i g)} = 256\frac{s_{ab}}{s_{a1}s_{b1}}s_{a2}s_{b3}$$

and therefore, the soft contribution to the matrix element squared is

$$\Delta\left|\mathcal{M}^B\right|^2_{soft} = 16\pi\alpha_s\mu^{2\varepsilon}C_F\cdot\frac{s_{ab}}{|s_{a1}||s_{b1}|}\cdot\Delta\left|\mathcal{M}^{LO}\right|^2 . \quad (4.9)$$

4.2.6 The soft region – contribution to the differential scattering cross section

Combining eqs. 4.8 and 4.9, we find for the soft contribution to the scattering cross section

$$d\Delta\sigma^B_{soft} = d\Delta\sigma^{LO} \cdot \frac{\alpha_s \mu^{2\varepsilon}}{2\pi} C_F \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \, 2 s_{ab}^\varepsilon \cdot \int\limits_0^{s_{min}} d|s_{a1}| \, |s_{a1}|^{-1-\varepsilon} \cdot \int\limits_0^{s_{min}} d|s_{b1}| \, |s_{b1}|^{-1-\varepsilon} \, ,$$

which leads to

$$d\Delta\sigma^B_{soft} = d\Delta\sigma^{LO} \cdot \frac{\alpha_s}{2\pi} \frac{C_F}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{s_{min}}\right)^\varepsilon \left(\frac{s_{ab}}{s_{min}}\right)^\varepsilon \cdot \frac{2}{\varepsilon^2} \, . \tag{4.10}$$

4.3 Combination of collinear and soft contributions

The polarisation difference of the differential cross section for this bremsstrahlung process is the combination of eqs. 4.7 and 4.10:

$$d\Delta\sigma^B = d\Delta\sigma^{LO} \cdot \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s_{min}}\right)^\varepsilon \left[-\frac{2}{\varepsilon^2} \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{s_{ab}}{s_{min}}\right)^\varepsilon + \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{(1-\varepsilon)(4-\varepsilon)}{\varepsilon^2(1-2\varepsilon)} + \mathcal{O}\left(\frac{s_{min}}{s_{ab}}\right) \right] \, ,$$

which can be brought to its final form by expanding in a series in ε and neglecting terms of $\mathcal{O}(s_{min}/s_{ab})$:

$$d\Delta\sigma^B = \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s_{ab}}\right)^\varepsilon \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 7 - \frac{5\pi^2}{6} + 3 \ln \frac{s_{ab}}{s_{min}} - 2 \ln^2 \frac{s_{ab}}{s_{min}} + \mathcal{O}(\varepsilon) \right] e^{-\gamma_E \varepsilon} \cdot d\Delta\sigma^{LO} \, . \tag{4.11}$$

4.4 Physical gauge

Calculating in physical (axial) gauge changes the polarisation sum (cf. eq. 4.2) into

$$\sum_\lambda \epsilon^\dagger_\mu(\vec{k}, \lambda) \, \epsilon_\nu(\vec{k}, \lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{(kn)} \, ,$$

where one has introduced the light-like gauge vector n. The additional terms, however, only contribute to the imaginary part of the matrix element squared and hence do not change the contribution to the scattering cross section.

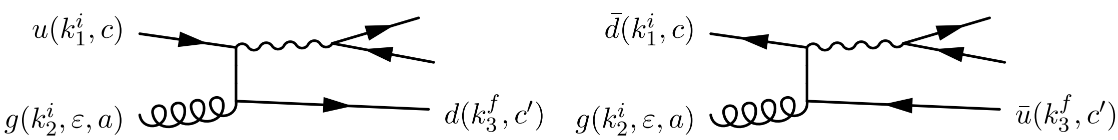
5 Quark bremsstrahlung correction

La volpe lascia il pelo,
non abbandona il vizio.
Marchese mio, giudizio,
o vi farò pentir!

Giuseppe Verdi: La Traviata

5.1 Dissecting the matrix element

For the parton subprocess of $p+p \to W+X$ with a gluon in the initial state, one has to take into account two possible diagrams:



This leads to the following matrix element:

$$M_{fi} = -\frac{g_s g_w^2}{8} V_{qq'}^* \, T_{cc'}^a \frac{1}{q_1^2 - M_W^2 + i0} \cdot \bar{u}(k_2^f)\gamma_\mu(1-\gamma^5)v(k_1^f) \\ \cdot \left[\bar{u}(k_3^f)\not{\varepsilon}\frac{\not{q}_2 - m_d}{q_2^2 - m_d^2 + i0}\gamma^\mu(1-\gamma^5)u(k_1^i) - \bar{v}(k_1^i)\gamma^\mu(1-\gamma^5)\frac{\not{q}_2 + m_u}{q_2^2 - m_u^2 + i0}\not{\varepsilon}v(k_3^f)\right]$$

with $q_1 := k_1^f + k_2^f$ the four-momentum of the W boson and $q_2 := k_2^i - k_3^f$ the one of the (anti)quark drawn vertically. Here, the calculation will be made in $d = 4$ dimensions and quark/lepton masses shall be neglected after the first fews steps.
The contribution to the differential cross section is therefore

$$|\mathcal{M}_{fi}|^2 = \left|V_{qq'}\right|^2 T_{cc'}^{a\,\dagger} T_{cc'}^a \left[\frac{g_s g_w^2}{8}\frac{1}{q_1^2 - M_W^2 + i0}\right]^2 \cdot J^{\mu\,\dagger} J^\nu \cdot L_\mu^\dagger L_\nu \tag{5.1}$$

where the lepton and quark currents have been abbreviated:

$$\begin{aligned} L_\mu &:= \bar{u}(k_2^f)\gamma_\mu(1-\gamma^5)v(k_1^f) \\ J^\mu &:= \bar{u}(k_3^f)\not{\varepsilon}\frac{\not{q}_2 - m_d}{q_2^2 - m_d^2 + i0}\gamma^\mu(1-\gamma^5)u(k_1^i) - \bar{v}(k_1^i)\gamma^\mu(1-\gamma^5)\frac{\not{q}_2 + m_u}{q_2^2 - m_u^2 + i0}\not{\varepsilon}v(k_3^f) \\ &=: J_1^\mu + J_2^\mu \,. \end{aligned}$$

Because of the different Dirac spinors in $J_{1,2}^\mu$, one gets

$$J^{\mu\,\dagger} J^\nu = J_1^{\mu\,\dagger} J_1^\nu + J_2^{\mu\,\dagger} J_2^\nu \,,$$

which is sensible since one doesn't have the same initial-state particles in the two diagrams.
And because final-state polarisation of the leptons is not observed, we may sum over it, leading to

$$\sum_s L^\dagger_\mu L_\nu = 8\left[k^f_{1\,\mu}k^f_{2\,\nu} + k^f_{1\,\nu}k^f_{2\,\mu} - (k^f_1 k^f_2)\, g_{\mu\nu} + i\epsilon_{\alpha\beta\mu\nu}k^{f\,\alpha}_1 k^{f\,\beta}_2\right] \,. \tag{5.2}$$

The polarisation of the gluon, however, is an observed quantity and can be treated in the following way (see, e.g., [14, 38, 4, 9]):

$$\varepsilon^\pm_\rho(k^i_2, n)\left[\varepsilon^\pm_\lambda(k^i_2, n)\right]^\dagger = \mathrm{tr}\left[\frac{1\mp\gamma^5}{2}\gamma_\rho \not{k}^i_2 \gamma_\lambda \not{n}\right]\frac{1}{4(k^i_2 n)}\,,$$

where n is some arbitrary, light-like vector to ensure physical polarisation (i.e. one has to calculate in physical gauge). Finally, the colour factor can also be computed since this degree of freedom cannot be measured:

$$\sum_{c,c'} T^{a\,\dagger}_{cc'} T^b_{cc'} = \delta^{ab}\, T_F\,.$$

5.2 The different spin combinations

The polarisation state of the quarks can be accounted for by using the – by now – well-known polarisation projection operator.

For $(\mathbf{R},\mathbf{R})$, where the first letter describes the polarisation state of the initial-state (anti)quark's Dirac spinor and the second one the polarisation of the gluon, the expression for J^μ vanishes because of $(1+\gamma^5)(1-\gamma^5) = 0$ and therefore

$$|\mathcal{M}_{fi}|^2_{RR} = |\mathcal{M}_1|^2_{RR} + |\mathcal{M}_2|^2_{RR} = 0\,.$$

For $(\mathbf{R},\mathbf{L})$ the same mechanism produces

$$|\mathcal{M}_{fi}|^2_{RL} = |\mathcal{M}_1|^2_{RL} + |\mathcal{M}_2|^2_{RL} = 0\,.$$

For $(\mathbf{L},\mathbf{R})$ we get

$$\begin{aligned} J^{\mu\,\dagger}_1 J^\nu_1 &= \frac{1}{(q_2^2 - m_d^2 + i0)^2}\cdot\frac{1}{4(k^i_2 n)}\,\mathrm{tr}\left[\frac{1+\gamma^5}{2}\gamma_\rho \not{k}^i_2\gamma_\lambda\not{n}\right] \\ &\quad\cdot \mathrm{tr}\left[(\not{k}^i_1 + m_u)\gamma^\mu(1-\gamma^5)(\not{q}_2 - m_d)\gamma^\lambda(\not{k}^f_3 + m_d)\gamma^\rho(\not{q}_2 - m_d)\gamma^\nu(1-\gamma^5)\right] \\ &= \frac{1}{(q_2^2 - m_d^2 + i0)^2}\cdot\frac{1}{(k^i_2 n)}\cdot\left[k^i_{2\,\rho}n_\lambda + n_\rho k^i_{2\,\lambda} - (k^i_2 n)g_{\rho\lambda} + i\epsilon_{\alpha\beta\lambda\rho}k^{i\,\alpha}_2 n^\beta\right] \\ &\quad\cdot\left\{\mathrm{tr}\left[\not{k}^i_1\gamma^\mu\not{q}_2\gamma^\lambda\not{k}^f_3\gamma^\rho\not{q}_2\gamma^\nu(1-\gamma^5)\right] + m_d^2\,\mathrm{tr}\left[\not{k}^i_1\gamma^\mu\gamma^\lambda\not{k}^f_3\gamma^\rho\gamma^\nu(1-\gamma^5)\right]\right\}\,. \end{aligned} \tag{5.3}$$

To deal with the trace of six or eight gamma matrices, one has to use some ingenious tricks:

$$\begin{gathered} \not{q}_2\gamma^\lambda\not{k}^f_3\gamma^\rho\not{q}_2 = 2q_2^\lambda\cdot\not{k}^f_3\gamma^\rho\not{q}_2 + 2q_2^\rho\cdot\gamma^\lambda\not{k}^f_3\not{q}_2 - 2(q_2 k^f_3)\cdot\gamma^\lambda\gamma^\rho\not{q}_2 - q_2^2\cdot\gamma^\lambda\not{k}^f_3\gamma^\rho \\ \gamma^\lambda\not{k}^f_3\gamma^\rho + \gamma^\rho\not{k}^f_3\gamma^\lambda = 2\left(k^{f\,\rho}_3\gamma^\lambda + \gamma^\rho k^{f\,\lambda}_3 - g^{\lambda\rho}\not{k}^f_3\right)\,. \end{gathered}$$

Applying them yields, with $\gamma_\alpha \gamma^\nu \gamma^\alpha = -2\gamma^\nu$,

$$\left[k^i_{2\,\rho} n_\lambda + n_\rho k^i_{2\,\lambda} - (k^i_2 n) g_{\rho\lambda}\right] \mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\lambda \not{k}^f_3 \gamma^\rho \not{q}_2 \gamma^\nu (1-\gamma^5)\right]$$
$$= 2(k^i_2 k^f_3)\left\{2(q_2 n)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5)\right] - q_2^2\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{n} \gamma^\nu (1-\gamma^5)\right]\right\}$$
$$+\ 2(k^f_3 n)\left\{2(k^i_2 q_2)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5)\right] - q_2^2\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{k}^i_2 \gamma^\nu (1-\gamma^5)\right]\right\}$$

and analogously

$$\left[k^i_{2\,\rho} n_\lambda + n_\rho k^i_{2\,\lambda} - (k^i_2 n) g_{\rho\lambda}\right] \mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \gamma^\lambda \not{k}^f_3 \gamma^\rho \gamma^\nu (1-\gamma^5)\right]$$
$$= 2(k^f_3 n)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{k}^i_2 \gamma^\nu (1-\gamma^5)\right] + 2(k^i_2 k^f_3)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{n} \gamma^\nu (1-\gamma^5)\right]\ ,$$

which can be put into eq. 5.3. From here onwards, it is sensible to neglect quark masses, as done in the other diagrams as well. This gives

$$J_1^{\mu\,\dagger} J_1^\nu = \frac{2}{(q_2^2+i0)^2}\cdot\frac{1}{(k^i_2 n)}\Big\{2(k^i_2 k^f_3)(q_2 n)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5)\right] - q_2^2 (k^i_2 k^f_3)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{n} \gamma^\nu (1-\gamma^5)\right]$$
$$+ 2(k^i_2 q_2)(k^f_3 n)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5)\right] - q_2^2 (k^f_3 n)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{k}^i_2 \gamma^\nu (1-\gamma^5)\right]$$
$$+\frac{i}{2}\epsilon_{\alpha\beta\lambda\rho} k_2^{i\,\alpha} n^\beta\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\lambda \not{k}^f_3 \gamma^\rho \not{q}_2 \gamma^\nu (1-\gamma^5)\right]\Big\}\ . \tag{5.4}$$

The trace in the last line of eq. 5.4 can be expanded using the formulas above, the definition $q_2^2 = (k^f_3)^2 - 2(g k^f_3)$, and the on-shell and massless limits $(k^f_3)^2 = 0$, resulting in

$$\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\lambda \not{k}^f_3 \gamma^\rho \not{q}_2 \gamma^\nu (1-\gamma^5)\right]$$
$$= -4 q_2^\lambda\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{k}^f_3 \not{k}^i_2 \gamma^\rho \gamma^\nu (1-\gamma^5)\right] + 4 k_2^{i\,\rho} k_3^{f\,\lambda}\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5)\right]$$
$$- 4(k^i_2 k^f_3) q_2^\rho\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \gamma^\lambda \gamma^\nu (1-\gamma^5)\right] + 2(k^i_2 k^f_3)\,\mathrm{tr}\left[\not{k}^i_1 \gamma^\mu \gamma^\lambda \not{k}^i_2 \gamma^\rho \gamma^\nu (1-\gamma^5)\right]\ , \tag{5.5}$$

where the second term vanishes when put into eq. 5.4 due to the antisymmetry of the Levi-Civita symbol. Traces of six gamma matrices can be expressed as a sum over traces of four gamma matrices:

$$\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda\gamma^\mu\gamma^\nu\right] = g^{\mu\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda\right] - g^{\lambda\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\mu\right]$$
$$+ g^{\delta\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\mu\right] - g^{\beta\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\delta\gamma^\lambda\gamma^\mu\right] + g^{\alpha\nu}\,\mathrm{tr}\left[\gamma^\beta\gamma^\delta\gamma^\lambda\gamma^\mu\right]$$

$$\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^5\right] = g^{\alpha\beta}\,\mathrm{tr}\left[\gamma^\delta\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^5\right] - g^{\alpha\delta}\,\mathrm{tr}\left[\gamma^\beta\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^5\right] + g^{\beta\delta}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^5\right]$$
$$+ g^{\lambda\mu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\nu\gamma^5\right] - g^{\lambda\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\mu\gamma^5\right] + g^{\mu\nu}\,\mathrm{tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\lambda\gamma^5\right]\ .$$

Expressing the two remaining traces of six gamma matrices in eq. 5.5 with these expansions, putting it back into eq. 5.4, and using antisymmetry again, we find

$$\begin{aligned}
J_1^{\mu\,\dagger} J_1^\nu =& \frac{1}{2(k_2^i k_3^f)^2 (k_2^i n)} \Big\{ 2(k_2^i k_3^f) \left[(k_2^i n) - 2(k_3^f n) \right] \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \not{q}_2 \gamma^\nu (1-\gamma^5) \right] \\
& + 2(k_2^i k_3^f)^2 \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \not{n} \gamma^\nu (1-\gamma^5) \right] + 2(k_2^i k_3^f)(k_3^f n) \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \not{k}_2^i \gamma^\nu (1-\gamma^5) \right] \\
& + 2i\epsilon_{\alpha\beta\lambda\rho} k_2^{i\,\alpha} n^\beta k_3^{f\,\lambda} \Big(g^{\mu\rho} \operatorname{tr}\left[\not{k}_1^i \not{k}_3^f \not{k}_2^i \gamma^\nu \right] + g^{\nu\rho} \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \not{k}_3^f \not{k}_2^i \right] - g^{\mu\nu} \operatorname{tr}\left[\not{k}_1^i \not{k}_3^f \not{k}_2^i \gamma^\rho \gamma^5 \right] \\
& \quad - k_1^{i\,\rho} \operatorname{tr}\left[\gamma^\mu \not{k}_3^f \not{k}_2^i \gamma^\nu \right] + k_1^{i\,\nu} \operatorname{tr}\left[\gamma^\mu \not{k}_3^f \not{k}_2^i \gamma^\rho \gamma^5 \right] - k_1^{i\,\mu} \operatorname{tr}\left[\not{k}_3^f \not{k}_2^i \gamma^\rho \gamma^\nu \gamma^5 \right] \Big) \\
& + 2i\epsilon_{\alpha\beta\lambda\rho} k_2^{i\,\alpha} n^\beta k_3^{f\,\rho} (k_2^i k_3^f) \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \gamma^\lambda \gamma^\nu \right] \\
& + i\epsilon_{\alpha\beta\lambda\rho} k_2^{i\,\alpha} n^\beta (k_2^i k_3^f) \Big(g^{\mu\rho} \operatorname{tr}\left[\not{k}_1^i \gamma^\lambda \not{k}_2^i \gamma^\nu \right] + g^{\nu\rho} \operatorname{tr}\left[\not{k}_1^i \gamma^\mu \gamma^\lambda \not{k}_2^i \right] - g^{\mu\nu} \operatorname{tr}\left[\not{k}_1^i \gamma^\lambda \not{k}_2^i \gamma^\rho \gamma^5 \right] \\
& \quad - k_1^{i\,\rho} \operatorname{tr}\left[\gamma^\mu \gamma^\lambda \not{k}_2^i \gamma^\nu \right] + k_1^{i\,\nu} \operatorname{tr}\left[\gamma^\mu \gamma^\lambda \not{k}_2^i \gamma^\rho \gamma^5 \right] - k_1^{i\,\mu} \operatorname{tr}\left[\gamma^\lambda \not{k}_2^i \gamma^\rho \gamma^\nu \gamma^5 \right] \Big) \Big\} \; .
\end{aligned}$$

Now is the time to write the traces and the products of Levi-Civita tensors explicitly:

$$\begin{aligned}
\operatorname{tr}\left[\gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\lambda \right] &= 4 \left(g^{\alpha\beta} g^{\delta\lambda} - g^{\alpha\delta} g^{\beta\lambda} + g^{\alpha\lambda} g^{\beta\delta} \right) \\
\operatorname{tr}\left[\gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\lambda \gamma^5 \right] &= 4i\epsilon^{\alpha\beta\delta\lambda} \\
\epsilon^{\mu\nu\delta\lambda} \epsilon_{\mu\nu\sigma\tau} &= -2 \left(\delta_\sigma^\delta \delta_\tau^\lambda - \delta_\tau^\delta \delta_\sigma^\lambda \right) \\
\epsilon^{\mu\nu\delta\lambda} \epsilon_{\mu\kappa\sigma\tau} &= - \delta_\tau^\lambda \delta_\sigma^\delta \delta_\kappa^\nu + \delta_\sigma^\lambda \delta_\tau^\delta \delta_\kappa^\nu + \delta_\tau^\lambda \delta_\kappa^\delta \delta_\sigma^\nu \\
& \quad - \delta_\kappa^\lambda \delta_\tau^\delta \delta_\sigma^\nu - \delta_\sigma^\lambda \delta_\kappa^\delta \delta_\tau^\nu + \delta_\kappa^\lambda \delta_\sigma^\delta \delta_\tau^\nu \; .
\end{aligned}$$

From this follows

$$\begin{aligned}
J_1^{\mu\,\dagger} J_1^\nu =& \frac{4}{(k_2^i k_3^f)^2 (k_2^i n)} \Big\{ (k_2^i k_3^f)(k_2^i n)\; i\epsilon^{\alpha\beta\mu\nu} k^i_{1\;\alpha} (k^i_{2\;\beta} - k^f_{3\;\beta}) - (k_2^i k_3^f)(k_3^f n)\; i\epsilon^{\alpha\beta\mu\nu} k^i_{1\;\alpha} k^i_{2\;\beta} \\
& + \; 2(k_2^i k_3^f)(k_3^f n) \left[k_1^{i\,\mu} k_3^{f\,\nu} + k_1^{i\,\nu} k_3^{f\,\mu} - (k_1^i k_3^f) g^{\mu\nu} + i\epsilon^{\alpha\beta\mu\nu} k^i_{1\;\alpha} k^f_{3\;\beta} \right] + (k_2^i k_3^f)^2 \; i\epsilon^{\alpha\beta\mu\nu} k^i_{1\;\alpha} n_\beta \\
& + \; i\epsilon_{\alpha\beta\lambda\rho} k_2^{i\alpha} n^\beta k_3^{f\,\lambda} \Big[(k_1^i k_3^f)(k_2^{i\,\nu} g^{\mu\rho} - k_2^{i\,\mu} g^{\nu\rho}) - (k_1^i k_2^i)(k_3^{f\,\nu} g^{\mu\rho} - k_3^{f\,\mu} g^{\nu\rho}) - k_1^{i\,\rho} (k_3^{f\,\mu} k_2^{i\,\nu} - k_3^{f\,\nu} k_2^{i\,\mu}) \Big] \\
& - \; (k_2^i k_3^f)\; i\epsilon_{\alpha\beta\lambda\rho} k_2^{i\,\alpha} n^\beta \left[(k_1^i k_2^i) g^{\mu\rho} g^{\nu\lambda} - k_1^{i\,\lambda} (k_2^{i\,\nu} g^{\mu\rho} - k_2^{i\,\mu} g^{\nu\rho}) \right] \Big\} \; .
\end{aligned}$$

Combining this result with the lepton current $L_\mu^\dagger L_\nu$, eq. 5.2, leads to the comparably simple result

$$J_1^{\mu\,\dagger} J_1^\nu \; L_\mu^\dagger L_\nu = 256 \frac{(k_1^i k_1^f)(k_2^f k_3^f)(k_3^f n)}{(k_2^i k_3^f)(k_2^i n)} \; ,$$

which has been checked with the help of FORM [33]. Putting it into eq. 5.1 yields:

$$|\mathcal{M}_1|^2_{LR} = \left| V_{qq'} \right|^2 T_F \left[\frac{g_s g_w^2}{q_1^2 - M_W^2 + i0} \right]^2 \cdot 4 \frac{(k_1^i k_1^f)(k_2^f k_3^f)(k_3^f n)}{(k_2^i k_3^f)(k_2^i n)} \; .$$

To find the final expression, we use the definitions $g_s^2 = 4\pi\alpha_s$ and $g_w^2 = 4\sqrt{2}\, G_F M_W^2$, which gives

$$|\mathcal{M}_1|^2_{LR} = 512\; \alpha_s \pi \left| V_{qq'} \right|^2 T_F \left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0} \right]^2 \frac{(k_1^i k_1^f)(k_2^f k_3^f)(k_3^f n)}{(k_2^i k_3^f)(k_2^i n)} \; .$$

All the other contributions to the matrix element can be computed in analogous fashion. The results are:

$$|\mathcal{M}_2|^2_{LR} = 512\ \alpha_s\pi\left|V_{qq'}\right|^2 T_F\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 (k_1^i k_2^f)\left\{\frac{(k_2^i k_1^f) - (k_1^f k_3^f)}{(k_2^i k_3^f)}\left[1 - \frac{(k_3^f n)}{(k_2^i n)}\right] + \frac{(k_1^f n)}{(k_2^i n)}\right\} ,$$

whereas for $(\mathbf{L},\mathbf{L})$ we get

$$|\mathcal{M}_1|^2_{LL} = 512\ \alpha_s\pi\left|V_{qq'}\right|^2 T_F\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 (k_1^i k_1^f)\left\{\frac{(k_2^i k_2^f) - (k_2^f k_3^f)}{(k_2^i k_3^f)}\left[1 - \frac{(k_3^f n)}{(k_2^i n)}\right] + \frac{(k_2^f n)}{(k_2^i n)}\right\}$$

and

$$|\mathcal{M}_2|^2_{LL} = 512\ \alpha_s\pi\left|V_{qq'}\right|^2 T_F\left[\frac{G_F M_W^2}{q_1^2 - M_W^2 + i0}\right]^2 \frac{(k_1^i k_2^f)(k_1^f k_3^f)(k_3^f n)}{(k_2^i k_3^f)(k_2^i n)} .$$

They, too, have been checked with the help of FORM.

According to literature [38, 4, 9], the choice of the gauge vector n is entirely arbitrary as long as it suffices the two conditions imposed: $n^2 = 0$ and $n \neq k_2^i$. What is more, a judicious choice may simplify the results very much. Thus, one may choose $n := k_2^f$, which leads to the following expressions:

$$\begin{aligned}
|\mathcal{M}_1|^2_{LR} &\propto s_{b3}\frac{s_{12}^2}{s_{a1}s_{a2}} \\
|\mathcal{M}_1|^2_{LL} &\propto s_{b3}\frac{s_{c2}^2}{s_{a1}s_{a2}} \\
|\mathcal{M}_2|^2_{LR} &\propto s_{b2}\frac{s_{c3}^2}{s_{a1}s_{a3}} \\
|\mathcal{M}_2|^2_{LL} &\propto s_{b2}\frac{s_{13}^2}{s_{a1}s_{a3}} ,
\end{aligned}$$

where the definitions and labels of Figure 4.1 have been used. Since in the process $g+\bar{q} \to \bar{q}+\ell^\pm+\nu$ the label of the initial-state $\bar{q}$ is the same as the one of the initial-state q in the LO matrix element, the final-state labels 2 and 3 will be interchanged relative to the LO expression.

5.3 Contributions to the scattering cross section

Like in the gluon bremsstrahlung diagram, one has to distinguish between soft and collinear divergences. As the procedures have been shown in some detail above, the elaborations will be kept rather short here.

5.3.1 The collinear region

Again we denote the fraction of the radiated (anti)quark's four-momentum to be $(1-z)$ times the gluon's:

$$\begin{aligned}
|s_{b1}| &= (1-z)s_{ab} \\
|s_{c2}| &= z|s_{ab}| \\
s_{12} &= (1-z)|s_{a2}| \\
s_{bc} &= z s_{ab} .
\end{aligned}$$

Thus, one finds

$$|\mathcal{M}_1|^2_{LR} \propto -(1-z)^2 \frac{|s_{a2}||s_{b3}|}{|s_{a1}|}$$
$$|\mathcal{M}_1|^2_{LL} \propto -z^2 \frac{|s_{a2}||s_{b3}|}{|s_{a1}|}$$
$$|\mathcal{M}_2|^2_{LR} \propto -z^2 \frac{|s_{a3}||s_{b2}|}{|s_{a1}|}$$
$$|\mathcal{M}_2|^2_{LL} \propto -(1-z)^2 \frac{|s_{a3}||s_{b2}|}{|s_{a1}|} \ ,$$

from which follow, with the definition

$$\Delta |\mathcal{M}_i|^2 := |\mathcal{M}_i|^2_{LL} - |\mathcal{M}_i|^2_{LR} - |\mathcal{M}_i|^2_{RL} + |\mathcal{M}_i|^2_{RR} \qquad \text{for} \qquad i = 1, 2 \ ,$$

the polarisation differences

$$\Delta |\mathcal{M}_1|^2 = -8\pi\alpha_s \mu^{2\varepsilon} T_F \frac{1}{|s_{a1}|} \ \Delta\hat{P}^{4-2\varepsilon}_{q\to g}(z) \cdot \Delta \left|\mathcal{M}^{LO}\right|^2$$
$$\Delta |\mathcal{M}_2|^2 = 8\pi\alpha_s \mu^{2\varepsilon} T_F \frac{1}{|s_{a1}|} \ \Delta\hat{P}^{4-2\varepsilon}_{q\to g}(z) \cdot \Delta \left|\mathcal{M}^{LO}\right|^2 \ ,$$

where we have denoted the spin-dependent splitting function [34] as

$$\Delta\hat{P}^{4-2\varepsilon}_{q\to g}(z) = 2z - 1 - 2\varepsilon(1-z) \ .$$

The phase space remains the same as in the gluon bremsstrahlung diagram, eq. 4.5, and we find for the polarisation-difference of the differential cross sections

$$d\Delta\sigma_{1,coll.} = -\frac{\alpha_s}{2\pi} T_F \frac{(4\pi\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)} \int\limits_0^{1-z_2} dz\, [z(1-z)]^{-\varepsilon} \Delta\hat{P}^{4-2\varepsilon}_{q\to g}(z) \cdot \int\limits_0^{s_{min}} d|s_{a1}|\, |s_{a1}|^{-1-\varepsilon} \cdot d\Delta\sigma^{LO}$$
$$d\Delta\sigma_{2,coll.} = -d\Delta\sigma_{1,coll.} \ .$$

Evaluating the intergals in the manner detailed in chapter 4, they lead to

$$\int\limits_0^{1-z_2} dz\, [z(1-z)]^{-\varepsilon} \Delta\hat{P}^{4-2\varepsilon}_{q\to g}(z) = -\frac{1}{1-\varepsilon} z_2^{1-\varepsilon} + \frac{2\varepsilon}{2-\varepsilon} z_2^{2-\varepsilon} - \frac{\varepsilon}{1-2\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \mathcal{O}(z_2)$$
$$\int\limits_0^{s_{min}} d|s_{a1}|\, |s_{a1}|^{-1-\varepsilon} = -\frac{1}{\varepsilon} s_{min}^{-\varepsilon} \ ,$$

which, inserted in the expression above, result in

$$d\Delta\sigma_{1,coll.} = \frac{\alpha_s}{2\pi} \frac{T_F}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{s_{min}}\right)^\varepsilon \left\{ \frac{2}{2-\varepsilon} z_2^{2-\varepsilon} - \frac{1}{\varepsilon(1-\varepsilon)} z_2^{1-\varepsilon} - \frac{1}{1-2\varepsilon} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \mathcal{O}(z_2) \right\} \cdot d\Delta\sigma^{LO} \tag{5.6a}$$
$$d\Delta\sigma_{2,coll.} = -d\Delta\sigma_{1,coll.} \ , \tag{5.6b}$$

with $z_2 := s_{min}/s_{ab}$.

5.3.2 The soft region

Taking the soft-quark limit, $k_3^f \to 0$, leads to the expressions

$$\begin{aligned}
|\mathcal{M}_1|^2_{LR} &\to 0\\
|\mathcal{M}_1|^2_{LL} &\to -\frac{|s_{a2}||s_{b3}|}{|s_{a1}|}\\
|\mathcal{M}_2|^2_{LR} &\to -\frac{|s_{a3}||s_{b2}|}{|s_{a1}|}\\
|\mathcal{M}_2|^2_{LL} &\to 0\ .
\end{aligned}$$

Using the phase space expression derived above (eq. 4.8), and evaluating the integrals yields

$$d\Delta\sigma_{1,soft} = \frac{\alpha_s}{2\pi}\frac{T_F}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\frac{1}{\varepsilon(1-\varepsilon)}\left(\frac{s_{min}}{s_{ab}}\right)^{1-\varepsilon}\cdot d\Delta\sigma^{LO} \tag{5.7a}$$

$$d\Delta\sigma_{2,soft} = -d\Delta\sigma_{1,soft}\ . \tag{5.7b}$$

5.3.3 Combination of collinear and soft contributions

Combining the collinear and soft expressions, eqs. 5.6 and 5.7 respectively, the poles cancel, leaving well-defined quantities:

$$\begin{aligned}
d\Delta\sigma_1 &= \frac{\alpha_s}{2\pi}\frac{T_F}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{min}}\right)^{\varepsilon}\left\{\frac{2}{2-\varepsilon}\left(\frac{s_{min}}{s_{ab}}\right)^{2-\varepsilon} - \frac{1}{1-2\varepsilon}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \mathcal{O}\left(\frac{s_{min}}{s_{ab}}\right)\right\}\cdot d\Delta\sigma^{LO}\\
d\Delta\sigma_2 &= -d\Delta\sigma_1\ .
\end{aligned}$$

Expanding these terms in a series in ε, we find

$$d\Delta\sigma_1 = -\frac{\alpha_s}{2\pi}\frac{T_F}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\varepsilon}\left\{1 + \mathcal{O}\left[\left(\frac{s_{min}}{s_{ab}}\right)^2\right] + \mathcal{O}(\varepsilon)\right\}\cdot d\Delta\sigma^{LO}\ ,$$

where one may safely ignore terms of $\mathcal{O}\left[(s_{min}/s_{ab})^2\right]$ as well as of $\mathcal{O}(\varepsilon)$ and expand the gamma function to obtain the final results:

$$d\Delta\sigma_{qg} \equiv d\Delta\sigma_1 = -\frac{\alpha_s}{2\pi}T_F\left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\varepsilon} e^{-\gamma_E\varepsilon}\cdot d\Delta\sigma^{LO} \tag{5.8a}$$

$$d\Delta\sigma_{\bar{q}g} \equiv d\Delta\sigma_2 = -d\Delta\sigma_1\ . \tag{5.8b}$$

6 The total NLO correction

[...] catholicus interpres [...] solidam etiam explicationem reperire entiatur, quæ [...] certis quoque profanarum disciplinarum conclusionibus debito modo satisfaciat.

Pope Pius XII: Divino afflante Spiritu

The different contributions to the NLO correction $d\Delta\sigma_{ab}^{NLO}$, see eq. 2.6, calculated in the previous chapters now have to be added up to give the total contribution at $\mathcal{O}(\alpha_s)$. The indices a and b can in this case be either a quark, an antiquark, or a gluon.

If we have only quarks in the initial state, there are contributions from the vertex as well as from the gluon bremsstrahlung:
From the vertex follows, eq. 3.18,

$$d\Delta\sigma^V = -\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + 4 - \frac{7\pi^2}{12} + \mathcal{O}(\varepsilon)\right] e^{-\gamma_E \varepsilon} \cdot d\Delta\sigma^{LO} \ ,$$

where $Q^2 = s_{ab}$ is the center-of-mass energy, and from the gluon bremsstrahlung, eq. 4.11,

$$d\Delta\sigma^B = \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\varepsilon} \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 7 - \frac{5\pi^2}{6} + 3\ln\frac{s_{ab}}{s_{min}} - 2\ln^2\frac{s_{ab}}{s_{min}} + \mathcal{O}(\varepsilon)\right] e^{-\gamma_E \varepsilon} \cdot d\Delta\sigma^{LO} \ .$$

The total contribution to the scattering matrix at NLO squared can be written as

$$\left|S^{NLO}\right|^2 = \left|S^{LO}\right|^2 + 2\left(S^{LO}\right)^\dagger S^V + \left|S^B\right|^2 + \mathcal{O}(\alpha_s^2) \ ,$$

from which follows directly the scattering cross section. Summing up the contributions from vertex and gluon bremsstrahlung, we find:

$$\begin{aligned} \alpha_s \, d\Delta\sigma_{q\bar{q}}^{NLO} &\equiv d\Delta\sigma^B + 2\, d\Delta\sigma^V \\ &= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\varepsilon} \left[-1 + \frac{\pi^2}{3} + 3\ln\frac{s_{ab}}{s_{min}} - 2\ln^2\frac{s_{ab}}{s_{min}} + \mathcal{O}(\varepsilon)\right] e^{-\gamma_E \varepsilon} \cdot d\Delta\sigma^{LO} \ . \end{aligned}$$

As can be seen, the poles in ε cancel against each other and the resulting expression is free of any singularities in the limit $\varepsilon \to 0$. It is very important to notice that *all* the poles cancel, i.e. the UV as well as the IR ones, as expected by Bloch & Nordsieck [5] and Kinoshita, Lee & Nauenberg [17,20].

The contributions from the diagrams with a gluon in the initial state are (eqs. 5.8):

$$\begin{aligned} \alpha_s \, d\Delta\sigma_{qg}^{NLO} &\equiv d\Delta\sigma_{qg} = -\frac{\alpha_s}{2\pi} T_F \left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\varepsilon} e^{-\gamma_E \varepsilon} \cdot d\Delta\sigma^{LO} \\ \alpha_s \, d\Delta\sigma_{\bar{q}g}^{NLO} &\equiv d\Delta\sigma_{\bar{q}g} = -d\Delta\sigma_{gq} \ . \end{aligned}$$

From my M.S. thesis [36] follows the polarisation difference of the leading-order differential cross section

$$d\Delta\sigma^{LO} = \frac{1}{12\pi}\left(\frac{G_F M_W^2}{\hat{s} - M_W^2}\right)^2 \left|V_{qq'}\right|^2 s_{ab} \left(\frac{1 - \tanh \hat{y}_\ell}{\cosh \hat{y}_\ell}\right)^2 dy_\ell$$

as a function of lepton rapidity y_ℓ with

$$\hat{y}_\ell := y_\ell - \frac{1}{2}\ln\frac{x_a}{x_b}$$
$$s_{ab} = s \cdot x_a x_b \ ,$$

where s is the proton-proton center-of-mass energy and x_i the i^{th} parton's momentum fraction of its parent hadron. With that result, all the NLO contributions are now completely known.

Taking all these corrections into account, one is finally able to write the differential scattering cross section for W boson production by proton-proton collision at next-to-leading order (eq. 2.6):

$$\begin{aligned} d\Delta\sigma_{p_1p_2} = \int dx_1 dx_2 \, \Big\{ & \Delta q(x_1,\mu_F)\Delta\bar{q}(x_2,\mu_F)\left[d\Delta\sigma^{LO}_{q\bar{q}}(x_1,x_2) + \alpha_s d\Delta\sigma^{NLO}_{q\bar{q}}(x_1,x_2)\right] \\ & + \ \alpha_s \left[\Delta C^{p_1}_q(x_1,\mu_F)\Delta\bar{q}(x_2,\mu_F) + \Delta q(x_1,\mu_F)\Delta C^{p_2}_{\bar{q}}(x_2,\mu_F)\right] d\Delta\sigma^{LO}_{q\bar{q}}(x_1,x_2) \\ & + \Delta q(x_1,\mu_F)\Delta g(x_2,\mu_F)\alpha_s d\Delta\sigma^{NLO}_{qg}(x_1,x_2) \\ & + \Delta\bar{q}(x_1,\mu_F)\Delta g(x_2,\mu_F)\alpha_s d\Delta\sigma^{NLO}_{\bar{q}g}(x_1,x_2) + (1 \leftrightarrow 2) + \mathcal{O}(\alpha_s^2) \Big\} \ , \end{aligned}$$

where $q(x) \equiv f^H_q(x)$ and similar for the other partons. All the perturbative quantities appearing in this expression have been calculated; the non-perturbative ones (the PDFs) need to be parametrised (e.g. [35, 21, 27]). The quark flavours are u and d with some small contamination from c and s (suppressed by the corresponding CKM matrix element). Thus, the scattering matrix element at next-to-leading order is completely described, can in principle be calculated – or used to calculate other quantities – and compared to experimental measurements.

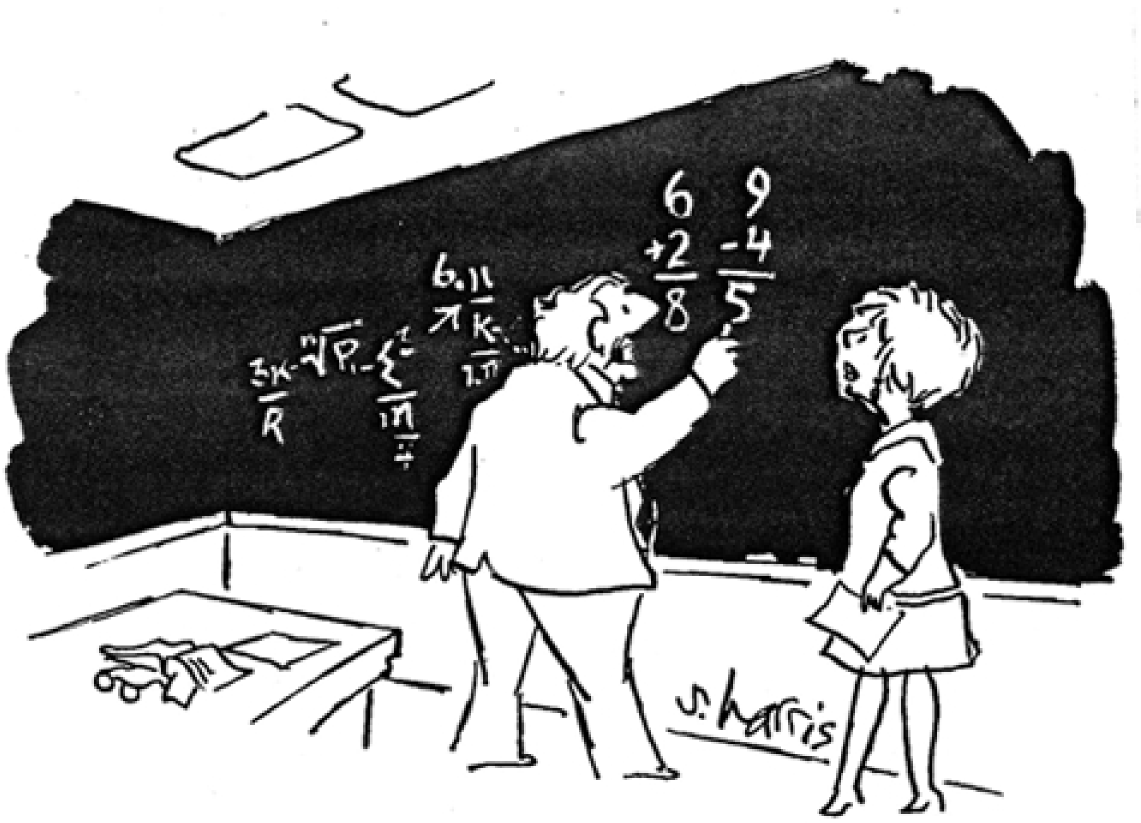
7 Conclusion and Outlook

If we shadows have offended,
Think but this, and all is mended,
That you have but slumber'd here
While these visions did appear.
And this weak and idle theme,
No more yielding but a dream.

William Shakespeare: A Midsummer Night's Dream

In the previous chapters the different contributions to the spin-dependent differential scattering cross section at next-to-leading order have been calculated: the NLO Feynman diagrams as well as the crossing functions. Combined together, they allow a complete description of the hadronic scattering cross section at NLO, which in turn can be used to calculate the polarisation asymmetry. As expected from the theorems of Bloch & Nordsieck and Kinoshita, Lee & Nauenberg, the UV and IR divergences have cancelled and left a mathematically well-defined expression in the limit of (exactly) 4 dimensions, i.e. the physical limit. Obtaining the pole structures has not been straight-forward in all the diagrams: Some sophisticated tricks for a correct phase space slicing and matching, introduced in the literature, have had to be applied to remain on firm mathematical – and ultimately physical – ground. Notwithstanding these difficulties, it has been a very interesting and instructive endeavour, in the course of which I have been amazed by the robustness and the well-tunedness of the model as well as the elaborate methods and procedures specifically developed for such calculations.
A further point of note is the large difference between the solution of the integrals in the massive and in the massless case. This clearly highlights the strong dependence on the particle masses, most of which can only be neglected in comparison to the mass of the produced intermediate W boson. Again, this is an argument strongly in favour of W physics and its possibilities.

Having calculated the spin-dependent differential scattering cross section for W boson production by proton-proton collisions to NLO, the next step would be to use these expressions to implement the polarisation asymmetry numerically in order to take care of detector properties and obtain results comparable to experimental measurements. This, however, lies beyond the scope of this work.



"EVERY ONCE IN A WHILE I JUST LIKE TO UNWIND WITH A LITTLE ADDITION AND SUBTRACTION."

A Regularisation methods – theory and example

Viel erforscht' ich,
erkannte viel;
wichtiges konnt' ich
machem künden,
machem wehren,
was ihn mühte,
nagende Herzens-Not.

RICHARD WAGNER: Siegfried

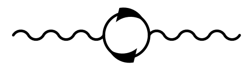
To get familiar with and see how the most important regularisation procedures work, I applied them to scalar three-dimensional QED where I calculated the vacuum polarisation and regularised the occurring divergences.

In scalar QED_3, the Lagrangian and (perturbative) interaction Hamiltonian (to leading order in e) are given by:

$$\mathcal{L} := \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi + ie\, \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi A^\mu + e^2 \Phi^\dagger \Phi A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{H}_{int} = -ie\ :\Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi:\ A^\mu \quad \text{with} \quad \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi := \Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi \ .$$

Second-order terms in the interaction Hamiltonian appear as a normalisation term of the electron-electron scattering tree diagram.

The Feynman diagram for the vacuum polarisation p  p' leads to:

$$S_{fi} = \frac{e^2}{2}\, \delta^{(3)}\left(p - p'\right) \left[\varepsilon_\mu\left(\vec{p}, \kappa\right) \varepsilon_\nu^*\left(\vec{p}', \kappa'\right) + \varepsilon_\nu\left(\vec{p}, \kappa\right) \varepsilon_\mu^*\left(\vec{p}', \kappa'\right)\right] \cdot \int d^3k \frac{4k^\mu k^\nu - 2k^\mu p^\nu - 2k^\nu p^\mu + p^\mu p^\nu}{\left(k^2 - m^2 + i0\right)\left[(k-p)^2 - m^2 + i0\right]} \ .$$

We are mainly interested in the integral part of S_{fi}, which is

$$\mathcal{I}^{\mu\nu}(p) := \int d^3k \frac{p^\mu p^\nu - 2k^\mu p^\nu - 2k^\nu p^\mu + 4k^\mu k^\nu}{\left(k^2 - m^2 + i0\right)\left[(k-p)^2 - m^2 + i0\right]} \ ,$$

and shows IR as well as UV divergent behaviour. It can be split into three separate components:

$$\mathcal{I}^{\mu\nu}(p) := p^\mu p^\nu \mathcal{I}_1(p) - 2p^\nu \mathcal{I}_2^\mu(p) - 2p^\mu \mathcal{I}_2^\nu(p) + 4\mathcal{I}_3^{\mu\nu}(p) \ ,$$

with the definitions

$$\mathcal{I}_1(p) := \int d^3k \frac{1}{(k^2 - m^2 + i0)\left[(k-p)^2 - m^2 + i0\right]}$$

$$\mathcal{I}_2^\mu(p) := \int d^3k \frac{k^\mu}{(k^2 - m^2 + i0)\left[(k-p)^2 - m^2 + i0\right]}$$

$$\mathcal{I}_3^{\mu\nu}(p) := \int d^3k \frac{k^\mu k^\nu}{(k^2 - m^2 + i0)\left[(k-p)^2 - m^2 + i0\right]} .$$

According to naive dimensional analysis, the most UV divergent expression behaves like:

$$\mathcal{I}_3^{\mu\nu} \propto \int_0^\infty k^2 \, dk \, \frac{k^2}{k^4} = \int_0^\infty dk \, ,$$

clearly showing the occurence of an infinity at the upper integration boundary.

A.1 Pauli-Villars regularisation

Introduced in 1949 by W. Pauli and F. Villars [23] building on work by E.C.G. Stückelberg, R.P. Feynman and D. Rivier, the propagator is replaced by

$$\frac{1}{q^2 - m^2 + i0} \mapsto \frac{1}{q^2 - m^2 + i0} - \frac{1}{q^2 - \Lambda^2 + i0}$$

with $\Lambda \gg 1$ (in the end $\Lambda \to \infty$ to recover the original expression). The integrand doesn't change for small q, but is cut off smoothly for large $|q| \gtrsim \Lambda$.
Generalising this procedure, one finds the substitution prescription

$$\mathcal{I}(m) \mapsto \sum_{i=0}^{n} c_i \, \mathcal{I}\left(\Lambda_i\right)$$

with $c_0 = 1$ and $\Lambda_0 = m$, and the following conditions to eliminate linear and logarithmic divergences:

$$\sum_{i=0}^{n} c_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} c_i \Lambda_i = 0 \, .$$

A.1.1 $\mathcal{I}_1$

Applying naive power counting to the first integral, we see that it is convergent and can therefore be calculated using Feynman parameters:

$$\mathcal{I}_1(p) = \frac{i\pi^2}{p} \ln \left| \frac{p + 2m}{p - 2m} \right| \, ,$$

with the short-hand notation $p := \sqrt{p^2}$. This result (as well as the following ones) holds for the kinematic range of $0 \leq p^2 \leq 4m^2$.

A.1.2 $\mathcal{I}_2^\mu$

In this case, the procedure outlined above has to be applied: Transformation of the propagators

$$\frac{1}{q^2-m^2+i0} \mapsto \frac{1}{q^2-m^2+i0} - \frac{1}{q^2-\Lambda^2+i0}$$

leads to

$$\mathcal{I}_2^\mu = \int d^3k \frac{k^\mu \left(m^2-\Lambda^2\right)^2}{(k^2-m^2+i0)\,(k^2-\Lambda^2+i0)\,[(k-p)^2-m^2+i0]\,[(k-p)^2-\Lambda^2+i0]} \; .$$

Subtraction at $p=0$ defines

$$\mathcal{I}_2^\mu(p) =: \mathcal{I}_2^\mu(0) + \widetilde{\mathcal{I}}_2^\mu(p)$$

with the components

$$\begin{aligned}
\mathcal{I}_2^\mu(0) &= \int d^3k \frac{k^\mu \left(m^2-\Lambda^2\right)^2}{(k^2-m^2+i0)^2\,(k^2-\Lambda^2+i0)^2} \\
\widetilde{\mathcal{I}}_2^\mu(p) &= \int d^3k \frac{k^\mu \left(m^2-\Lambda^2\right)^2}{(k^2-m^2+i0)\,(k^2-\Lambda^2+i0)} \\
&\quad \cdot \left\{ \frac{-1}{(k^2-m^2+i0)\,(k^2-\Lambda^2+i0)} + \frac{1}{[(k-p)^2-m^2+i0]\,[(k-p)^2-\Lambda^2+i0]} \right\} \; .
\end{aligned}$$

Taking the limit $\Lambda \to \infty$ in both of these integrals yields

$$\begin{aligned}
\mathcal{I}_2^\mu(0) &= 0 \\
\widetilde{\mathcal{I}}_2^\mu(p) &= \int d^3k \frac{-k^\mu \left[p^2-2(kp)\right]}{(k^2-m^2+i0)^2\,[(k-p)^2-m^2+i0]} \; ,
\end{aligned}$$

the latter of which is now convergent. Using Feynman parametrisation

$$\frac{1}{ABC} = 2\int_0^1 d\alpha \int_0^\alpha d\beta\,[(\alpha-\beta)A + \beta B + (1-\alpha)C]^{-3}$$

and momentum translation leads to

$$\begin{aligned}
\widetilde{\mathcal{I}}_2^\mu = \int d^3k\; 2\int_0^1 d\alpha \int_0^\alpha d\beta \left[2(pk)\,k^\mu - (1-2\beta)p^2k^\mu + 2\beta p^\mu(pk) - (1-2\beta)\beta p^2 p^\mu\right] \\
\cdot \left[k^2 - \beta(\beta-1)p^2 - m^2 + i0\right]^{-3} \; .
\end{aligned}$$

Because the integration is symmetric, the integrals proportional to an odd power of k disappear and we find, with the replacement $k^\mu k^\nu = g^{\mu\nu}k^2/3$,

$$\widetilde{\mathcal{I}}_2^\mu = 2p^\mu \int_0^1 d\alpha \int_0^\alpha d\beta \int d^3k \left[\frac{2}{3}k^2 - (1-2\beta)\beta p^2\right] \cdot \left[k^2 - \beta(\beta-1)p^2 - m^2 + i0\right]^{-3} \; ,$$

which can now be Wick rotated ($k^0 \mapsto ik^4$) and integrated over k to

$$\begin{aligned}
\widetilde{\mathcal{I}}_2^\mu &= \frac{i\pi^2}{2} p^\mu \int_0^1 d\alpha \int_0^\alpha d\beta \left\{ \frac{2}{\sqrt{\beta(\beta-1)p^2+m^2}} + \frac{(1-2\beta)\beta p^2}{[\beta(\beta-1)p^2+m^2]^{3/2}} \right\} \\
&= i\pi^2 p^\mu \int_0^1 d\alpha \frac{\alpha}{\sqrt{\alpha(\alpha-1)p^2+m^2}} \\
&= \frac{i\pi^2 p^\mu}{2p} \ln\left|\frac{p+2m}{p-2m}\right| \quad = \quad \mathcal{I}_2^\mu(p) \,.
\end{aligned}$$

A.1.3 $\mathcal{I}_3^{\mu\nu}$

The same procedure may be applied in this case, too, leading to

$$\mathcal{I}_3^{\mu\nu} = \frac{i\pi^2}{4} p \left\{ -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left[\frac{2m}{p} + \frac{p^2-4m^2}{2p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right] + \frac{p^\mu p^\nu}{p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right\} \,.$$

A.1.4 Combining the integrals

Putting the three integrals together, one obtains

$$\mathcal{I}^{\mu\nu} = -i\pi^2 p \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2-4m^2}{2p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right] \,,$$

again for the kinematic range of $0 \leq p^2 \leq 4m^2$.

A.2 Dimensional regularisation

This method, also called *dimensional continuation*, has been used for quite some time in statistical mechanics (e.g. [12, 37]). It was first applied to quantum field theory by G. 't Hooft and others in 1972 [28].

Changing the dimension of an integral, e.g.

$$\int \frac{d^3q}{q^3} \mapsto \int \frac{d^dq}{q^3} \,,$$

makes it (UV) convergent for certain values of d (in the example above for $d < 3$).

This already leads to the general procedure of this method: calculate in $d = n - 2\varepsilon$ instead of n dimensions which makes the divergences manifest in poles of ε and – if possible – take the limit $\varepsilon \to 0$.

If one has dimension-dependent quantities (like Dirac gamma matrices), care has to be taken to properly continue them into arbitrary dimensions!

A.2.1 $\mathcal{I}_1$

Using Feynman parametrisation

$$\frac{1}{AB} = \int_0^1 d\alpha \, [\alpha A + (1-\alpha)B]^{-2}$$

and performing the momentum translation $k^\mu \mapsto k^\mu + \alpha p^\mu$, the integral can be written as

$$\mathcal{I}_1(p) = \int d^3k \int_0^1 d\alpha \left[k^2 - m^2 + \alpha(1-\alpha)p^2 + i0\right]^{-2} .$$

Changing the integration dimension to $d = 3 - 2\varepsilon$ and using the general relation (for a derivation see e.g. [24, 8])

$$\int d^dk \frac{\left(k^2\right)^r}{\left(k^2 - a^2\right)^m} = i(-1)^{r-m} \, \pi^{\frac{d}{2}} \, \frac{\Gamma\left(r + \frac{d}{2}\right)\Gamma\left(m - r - \frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(m)\left(a^2\right)^{m-r-\frac{d}{2}}} ,$$

the momentum integral can be carried out (with the limit $\varepsilon \to 0$), after which the α-integral becomes simple:

$$\mathcal{I}_1(p) = i\pi^2 \int_0^1 \frac{d\alpha}{\sqrt{\alpha(\alpha-1)p^2 + m^2}} = \frac{i\pi^2}{p} \ln\left|\frac{p+2m}{p-2m}\right| .$$

The fact that the limit $\varepsilon \to 0$ could be taken without generating divergences highlights the fact that this integral in only apparently divergent, in fact can be calculated (with proper technique) and leads to a finite result.

A.2.2 $\mathcal{I}_2^\mu$

The same procedure as before is applied to this integral, leading to

$$\mathcal{I}_2^\mu(p) = \int d^3k \int_0^1 d\alpha \, (k^\mu + \alpha p^\mu) \left[k^2 - m^2 + \alpha(1-\alpha)p^2 + i0\right]^{-2} .$$

Integrating symmetrically makes the intergal proportional to k^μ disappear, leaving

$$\mathcal{I}_2^\mu(p) = i\pi^2 \int_0^1 d\alpha \frac{\alpha p^\mu}{\sqrt{\alpha(\alpha-1)p^2 + m^2}} = \frac{i\pi^2 p^\mu}{2p} \ln\left|\frac{p+2m}{p-2m}\right| .$$

A.2.3 $\mathcal{I}_3^{\mu\nu}$

Once again we use Feynman parametrisation and momentum translation to obtain

$$\mathcal{I}_3^{\mu\nu}(p) = \int d^3k \int_0^1 d\alpha \left[k^\mu k^\nu + \alpha\left(k^\mu p^\nu + p^\mu k^\nu\right) + \alpha^2 p^\nu p^\nu\right] \cdot \left[k^2 - m^2 + \alpha(1-\alpha)p^2 + i0\right]^{-2} ,$$

where the intergals proportional to an odd power of k vanish.
Rewriting $k^\mu k^\nu = g^{\mu\nu}k^2/3$ and performing $(d = 3 - 2\varepsilon)$-dimensional integration, we get

$$\mathcal{I}_3^{\mu\nu}(p) = i\pi^2 \int_0^1 d\alpha \left[g^{\mu\nu}\sqrt{\alpha(\alpha-1)p^2+m^2} + p^\mu p^\nu \frac{\alpha^2}{\sqrt{\alpha(\alpha-1)p^2+m^2}} \right] ,$$

which can be integrated-out straightforward to

$$\mathcal{I}_3^{\mu\nu}(p) = \frac{i\pi^2}{4} p \left\{ -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2-4m^2}{2p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right] + g^{\mu\nu}\frac{4m}{p} + \frac{p^\mu p^\nu}{p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right\} .$$

A.2.4 Combining the integrals

Recalling the splitting of the original integral and combining the different results, we find

$$\begin{aligned} \mathcal{I}^{\mu\nu} &= p^\mu p^\nu \mathcal{I}_1 - 2p^\mu \mathcal{I}_2^\nu - 2p^\nu \mathcal{I}_2^\mu + 4\mathcal{I}_3^{\mu\nu} \\ &= -i\pi^2 p \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2-4m^2}{2p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right] + 4i\pi^2 g^{\mu\nu} m , \end{aligned}$$

which differs from the solution with Pauli-Villars regularisation (see above) by a local term $4i\pi^2 g^{\mu\nu} m$. Such local terms are permitted by the theory because of the integral's degree of divergence (see below) and have to be fixed using physical constraints.
Following from this, the scattering matrix element is

$$\begin{aligned} S_{fi} = -e^2\pi\, \delta^{(3)}\left(p-p'\right) \frac{g_{\kappa\kappa}g_{\kappa'\kappa'}}{2\omega_p} & \left[\varepsilon_\mu(\vec{p},\kappa)\, \varepsilon_\nu^*\left(\vec{p}\,',\kappa'\right) + \varepsilon_\nu(\vec{p},\kappa)\, \varepsilon_\mu^*\left(\vec{p}\,',\kappa'\right) \right] \\ & \cdot \left\{ i\pi^2 p \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2-4m^2}{2p^2} \ln\left|\frac{p+2m}{p-2m}\right| \right] - 4i\pi^2 g^{\mu\nu} m \right\} . \end{aligned}$$

A.2.5 The seagull graph

Calculating the contribution of the fermion-line contraction in the seagull graph (one of the above-mentioned higher-order terms: $\mathcal{H}_{int} = -e^2 \; \mathbf{:}\, \Phi^\dagger \Phi \; A_\mu A^\mu \,\mathbf{:}$) gives

$$\begin{aligned} S_{fi}^{seagull} &= -2\pi e^2 \frac{g_{\kappa\kappa}g_{\kappa'\kappa'}}{2\omega_p} 2\varepsilon_\mu(\vec{p},\kappa)\, \varepsilon^{\mu\,*}\left(\vec{p}\,',\kappa'\right) \delta^{(3)}(p-p') \int \frac{d^3k}{k^2-m^2+i\varepsilon} \\ &= -e^2\pi\delta^{(3)}(p-p') \frac{g_{\kappa\kappa}g_{\kappa'\kappa'}}{2\omega_p} 2\varepsilon_\mu(\vec{p},\kappa)\, \varepsilon^{\mu\,*}\left(\vec{p}\,',\kappa'\right) \cdot 4i\pi^2 m, \end{aligned}$$

which exactly cancels the local term in the scattering matrix element of the vacuum polarisation, that only appears if one uses dimensional regularisation.

A.3 The causal approach

A.3.1 An introduction

This method is the only mathematically well-defined procedure to cope with the divergences. As we will see, it actually avoids creating them in the first place! But its machinery is quive involved, making it very complicated to calculate higher-order diagrams. For a detailed introduction and further applications see e.g. [25, 1].

The usual formal way of writing the S-matrix is

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \; \mathrm{T}\left[\mathcal{H}_{int}(x_1) \cdots \mathcal{H}_{int}(x_n)\right] \; .$$

Since we end up with divergences, there must be something wrong with the expression above. The problem lies with the definition of the time-ordered product of interaction Hamiltonians

$$\begin{aligned} &\mathrm{T}\left[\mathcal{H}_{int}(x_1) \cdots \mathcal{H}_{int}(x_n)\right] \\ &\qquad \overset{def.}{=} \sum_{\Pi} \Theta\left(x^0_{\Pi 1} - x^0_{\Pi 2}\right) \cdots \Theta\left(x^0_{\Pi(n-1)} - x^0_{\Pi n}\right) \cdot \mathcal{H}_{int}\left(x_{\Pi 1}\right) \cdots \mathcal{H}_{int}\left(x_{\Pi n}\right) \; , \end{aligned}$$

where one multiplies the operator-valued distributions $T_1 := -i\,\mathcal{H}_{int}$ with Heaviside distributions, which isn't mathematically sound (i.e. well-defined)! E.C.G. Stückelberg, N.N. Bogoljubov, H. Epstein and V.J. Glaser (see. [11] and references therein) noted that it's possible to avoid the UV divergences if one used the causal structure of the theory. The IR divergences do not seem to appear either at first glance, but they are hidden in the adiabatic limit $g \to 1$ and have to be treated separately.
Starting from a mathematically well-formed expression

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, T_n(x_1, \ldots, x_n) \cdot g(x_1) \cdots g(x_n) \; ,$$

with T_n the n-point distribution and $g \in \mathcal{S}\left(\mathbb{R}^4\right)$, the inverse of S is given by

$$S^{-1}(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \widetilde{T}_n(x_1, \ldots, x_n) \cdot g(x_1) \cdots g(x_n) \; ,$$

where, using the formal series expansion $1/(1+T) = 1 + \sum_n (-T)^n$,

$$\widetilde{T}_n(X) := \sum_{r=1}^{n} (-1)^r \sum_{P_r} T_{n_1}(X_1) \cdots T_{n_r}(X_r)$$

and the sum runs over all partitions of X into r disjoint, non-empty subsets. Avoiding a direct definition of T_n, one defines instead

$$\begin{aligned} A'_n(x_1, \ldots, x_n) &:= \sum_{P_2} \widetilde{T}_{n_1}(X)\, T_{n-n_1}(Y, x_n) \\ R'_n(x_1, \ldots, x_n) &:= \sum_{P_2} T_{n-n_1}(Y, x_n)\, \widetilde{T}_{n_1}(X), \end{aligned}$$

where the sum runs over all partitions of $\{x_1, \ldots, x_{n-1}\} = X \cup Y$, $X \neq \emptyset$, into two disjoint subsets with $|X| = n_1 \geq 1$ and $|Y| \leq n-2$.

Allowing for $X = \emptyset$ yields

$$\begin{aligned} A_n(x_1, \ldots, x_n) &:= \sum_{P_2^0} \widetilde{T}_{n_1}(X)\, T_{n-n_1}(Y, x_n) = A'_n + T_n(x_1, \ldots, x_n) \\ R_n(x_1, \ldots, x_n) &:= \sum_{P_2^0} T_{n-n_1}(Y, x_n)\, \widetilde{T}_{n_1}(X) = R'_n + T_n(x_1, \ldots, x_n). \end{aligned}$$

Exploiting causality reveals the support properties:

$$\text{supp } R_n(Y, x_n) \subseteq \Gamma^+_{n-1}(x_n)$$
$$\text{supp } A_n(Y, x_n) \subseteq \Gamma^-_{n-1}(x_n),$$

with the n-dimensional generalisation $\Gamma^\pm_n$ of the closed forward (+) and backward (−) light cone.

While A_n and R_n are not known (because they contain the T_n we are looking for), A'_n and R'_n can be constructed by induction. Their difference, however, is identical:

$$D_n = R'_n - A'_n = R_n - A_n \; ,$$

with causal support in $\Gamma^+_{n-1} \cup \Gamma^-_{n-1}$.

The exact way in which D_n is split into A_n and R_n using the support properties contains the crux of the whole process!
Expand the distribution in external fields:

$$D_n(x_1, \ldots, x_n) = \sum_k : \prod_i \bar{\Psi}(x_i) \, d^k_n(x_1, \ldots, x_n) \prod_j \Psi(x_j) : \prod_m A(x_m) \; ,$$

with $d^k_n(x) = r(x) - a(x)$ having the same causal support as $D_n(x)$.

Analysing the behaviour of $\hat{d}^k_n(p)$ in the limit $p \to \infty$ determines its so-called *degree of divergence* ω:

- if $\omega < 0$, the trivial splitting with Heaviside distributions poses no problems and one finds

$$\hat{r}(p) = \frac{i}{2\pi} \int\limits_{-\infty}^{\infty} dt \, \frac{\hat{d}(tp)}{1 - t + i0}$$

- if $\omega \geq 0$, one has to amend this according to

$$\hat{r}(p) = \frac{i}{2\pi} \int\limits_{-\infty}^{\infty} dt \, \frac{\hat{d}(tp)}{(t - i0)^{\omega+1} \, (1 - t + i0)} \tag{A.1}$$

Combining all the above, one gets the following “recipe”:

- from $T_1 = -i \, \mathcal{H}_{int}$ construct R'_n and A'_n inductively, from which r' and a' follow
- write down D_n and consecutively d^k_n, determine the degree of divergence ω, and split into retarded and advanced parts, r and a, respectively
- $t = r - r' = a - a'$ from which T_n follows.

Another important finding is (r and $\tilde{r}$ are obtained using different ways of splitting):

$$r - \tilde{r} = \sum_{|a| \leq \omega} C_a \, D^a \delta(x) \; ,$$

with $C_a \in \mathbb{R}$ and D^a the multi-index differentiation operator. This means that in the case of $\omega \geq 0$, r is determined only up to local terms, which have to be fixed using physical constraints!

A.3.2 Example: the causal approach to vacuum polarisation

Starting all over again (see also [26]), we remember the interaction Hamiltonian $T_1(x) := -i\mathcal{H}_{int}(x) = -e\ :\Phi^\dagger(x)\overleftrightarrow{\partial}_\mu\Phi(x):\ A^\mu(x)$ and write thus

$$A'_2(x,y) = -e^2 :\Phi^\dagger(x)\overleftrightarrow{\partial}_\mu\Phi(x)::\Phi^\dagger(y)\overleftrightarrow{\partial}_\nu\Phi(y)::A^\mu(x)A^\nu(y):,$$
$$R'_2(x,y) = -e^2 :\Phi^\dagger(y)\overleftrightarrow{\partial}_\mu\Phi(y)::\Phi^\dagger(x)\overleftrightarrow{\partial}_\nu\Phi(x)::A^\mu(x)A^\nu(y):,$$

and

$$D_2(x,y) = R'_2(x,y) - A'_2(x,y)\,.$$

Using Wick's theorem on normal ordering, this leads to, e.g., the first term in A'_2

$$\begin{aligned}&:\Phi^\dagger(x)\,\partial^\mu\Phi(x)::\Phi^\dagger(y)\,\partial^\nu\Phi(y): = :\Phi^\dagger(x)\,\partial^\mu\Phi(x)\,\Phi^\dagger(y)\,\partial^\nu\Phi(y):\\ &\qquad + :\Phi^\dagger(x)\,\partial^\nu\Phi(y):\, i\partial_x^\mu D^{(+)}(x-y) - :\partial^\mu\Phi(x)\,\Phi^\dagger(y):\, i\partial_y^\nu D^{(-)}(y-x)\\ &\qquad\qquad + \partial_x^\mu D^{(+)}(x-y)\,\partial_y^\nu D^{(-)}(y-x)\,,\end{aligned}$$

with $D^{(\pm)}$ the positive/negative frequency part of the Pauli-Jordan distribution.
Since normal ordering of the terms in D_n doesn't obscure causal support, one can do distribution splitting for every diagram separately.

We're only interested in the vacuum polarisation, whose D_2 is

$$\begin{aligned}D_2(x,y) = -e^2 &:A_\mu(x)\,A_\nu(y):\\ &\cdot\Big[\partial_y^\nu D^{(+)}(y-x)\,\partial_x^\mu D^{(-)}(x-y) - \partial_y^\nu D^{(-)}(y-x)\,\partial_x^\mu D^{(+)}(x-y)\\ &+ \partial_y^\mu D^{(+)}(y-x)\,\partial_y^\nu D^{(-)}(x-y) - \partial_x^\mu D^{(-)}(y-x)\,\partial_y^\nu D^{(+)}(x-y)\\ &- \partial_x^\mu\partial_y^\nu D^{(+)}(y-x)\,D^{(-)}(x-y) + \partial_x^\mu\partial_y^\nu D^{(-)}(y-x)\,D^{(+)}(x-y)\\ &\quad -D^{(+)}(y-x)\,\partial_x^\mu\partial_y^\nu D^{(-)}(x-y) + D^{(-)}(y-x)\,\partial_x^\mu\partial_y^\nu D^{(+)}(x-y)\Big]\,.\end{aligned}$$

For further calculation, we change to momentum space and write:

$$D_2(x,y) =: d^{\mu\nu}(x,y)\ :A_\mu(x)A_\nu(y): \quad \text{with} \quad \hat{d}^{\mu\nu}(k) =: \frac{e^2}{(4\pi)^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\hat{d}(k)\,.$$

After some calculation, one finds

$$\hat{d}(k) = \frac{\sqrt{k^2}}{2}\Theta(k^2-4m^2)\,\mathrm{sgn}(k_0)\,\left(1-\frac{4m^2}{k^2}\right)\,,$$

which behaves like $|k|$ for $k\to\infty$. Therefore, $\omega = 1$.

Remembering eq. A.1, we now easily write

$$\begin{aligned}\hat{r}(k_0) &= \frac{ik_0^2}{2\pi}\int_{-\infty}^{\infty} dp_0\,\frac{|p_0|\,\Theta(p_0^2-4m^2)\,\mathrm{sgn}(p_0)}{(p_0-i0)^2\,(k_0-p_0+i0)}\cdot\frac{1}{2}\left(1-\frac{4m^2}{p_0^2}\right)\\ &= \frac{ik_0^2}{4\pi}\int_{4m^2}^{\infty}\frac{ds}{s^{3/2}}\,\frac{s-4m^2}{k_0^2-s+i0k_0}\,.\end{aligned}$$

From R_2' follows (with similar definitions as for D_2)

$$\hat{r}'(k_0) = -\frac{k_0}{2}\Theta(k_0^2 - 4m^2)\Theta(-k_0)\left(1 - \frac{4m^2}{k_0^2}\right)$$

and therefore

$$\hat{t}(k_0) = \hat{r}(k_0) - \hat{r}'(k_0) = \frac{ik_0^2}{4\pi}\int\limits_{4m^2}^{\infty}\frac{ds}{s^{3/2}}\frac{s-4m^2}{k_0^2 - s + i0}\,.$$

Integrating out and using the definition of $\hat{t}^{\mu\nu}$ (similar to $\hat{d}^{\mu\nu}$) gives:

$$\hat{t}^{\mu\nu}(k) = -2\frac{ie^2}{(4\pi)^3}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\sqrt{k^2}\left[\frac{2m}{\sqrt{k^2}} + \frac{k^2 - 4m^2}{2k^2}\ln\left|\frac{\sqrt{k^2}+2m}{\sqrt{k^2}-2m}\right|\right]\,.$$

Comparison

Comparing the results of all three procedures, one sees that they all lead to the same result (up to local terms and normalisation). It is also clear that, whilst providing the mathematically most sound path, the causal approach becomes cumbersome for higher-order calculations. Thus, the most widely used way to deal with divergences is dimensional regularisation, which is also used in this work. Nevertheless, it has been very instructive to see how these different approaches work in detail and also to see that they really lead to the same result, making the choice of procedure a matter of taste.

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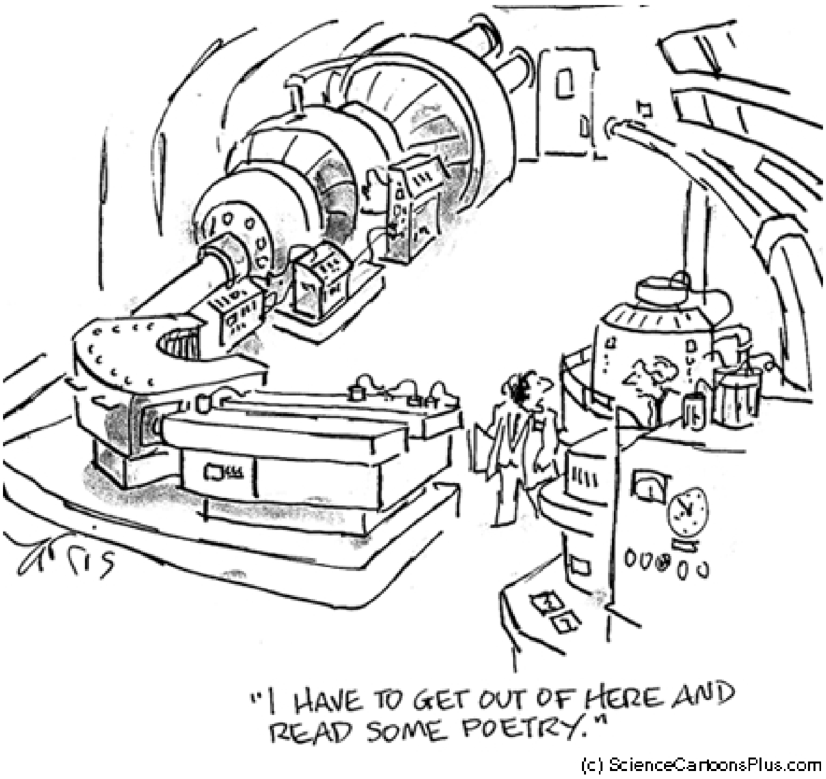
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