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Existence of Linear Equilibria in the Kyle Model with Multiple Informed Traders

by

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Abstract:
We consider Kyle's market order model of insider trading with multiple informed traders and show: if a linear equilibrium exists for two different numbers of informed traders, asset payoff and noise trading are independent and have finite second moments, then these random variables are normally distributed.
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1   Introduction

In the Kyle [8] model of informed trading in a financial market a risk neutral insider, who receives private information about the payoff of a risky asset, and noise traders submit order quantities to risk neutral market makers, who set prices competitively. An equilibrium in the Kyle model is characterized by the conditions that the price of the asset is equal to its expected value conditional on the information contained in the order flow, and that the insider chooses his quantity optimally as a function of his information, given the dependence of price on order flow.

Kyle's paper and most of the subsequent literature building on his model (see O'Hara [10] for a survey) impose normality of the distribution of the exogenous random variables and investigate equilibria in which price is a linear function of order flow. Such linear equilibria exist under the normality assumption. They are attractive because of their tractability and have been used as a versatile tool to analyze how changes in the environment affect financial market equilibrium under asymmetric information and, in particular, to analyze an informed trader's ability to profit from his private information.

In this paper we identify simple additional conditions under which normality is not only sufficient but also necessary for the existence of linear equilibria in the Kyle model. We do so in a version of the Kyle model in which there is a single round of trading and N informed traders. We assume that noise trading and the asset payoff (the information of an insider) are stochastically independent and have finite second moments, but impose no additional restrictions on the distributions of these random variables. For given distributions of asset payoff and noise trade satisfying these assumptions, we prove that the existence of a linear equilibrium for two different numbers of informed traders implies that the underlying random variables are normally distributed. A similar, but weaker characterization of the normal distribution was obtained independently by Bagnoli, Viswanathan and Holden [2, Theorem 5]. Besides our distributional assumptions Bagnoli, Viswanathan and Holden impose and make substantial use of a technical condition on the characteristic functions of the relevant random variables and then show that the existence of a linear equilibrium for all N implies normality.

The assumptions which yield our characterization of the normal distribution are tight in the sense that (i) there are non-normal distributions with finite second moments, but violating the independence assumption, such that linear equilibria exist for all N (Foster and Viswanathan [6]), (ii) there are distributions with infinite second moments, satisfying the independence assumption, such that linear equilibria exist for all N (Bagnoli, Viswanathan and Holden [2]), and (iii) for any given N there exist independent, non-normal distributions for asset payoff and noise trading with finite second moments such that linear equilibria exist (Bagnoli, Viswanathan and Holden [2]; for the case $N = 1$ see Pagano and Röell [11, Proposition 3]).

2 The Model

The model follows the single auction setting considered in Kyle [8, Section 2], but allows for multiple informed traders as in Holden and Subrahmanyan [7] and Foster and Viswanathan [6]. There are three types of traders: noise traders, risk neutral market makers, and $N \geq 1$ risk neutral informed traders (the insiders). The aggregate quantity traded by noise traders and the payoff of the risky asset are given by exogeneous random variables.[1] Noise trading is denoted by $\tilde{u}$. The payoff of the risky asset is denoted by $\tilde{v}$. In contrast to Kyle we do not impose normality on the joint distribution of $\tilde{u}$ and $\tilde{v}$. We do however maintain his independence assumption and assume that noise trade and payoff of the risky asset have strictly positive, finite second moments. No further distributional assumptions (e.g., existence of a density) are made.

The realization of $\tilde{v}$, but not of $\tilde{u}$, is observed by all insiders, who then simultaneously decide on the market order they submit. A strategy for insider n is given by a Lebesgue measurable function $X_n : \mathbb{R} \to \mathbb{R}$, determining his market order as a function of the observed payoff. For a given strategy X_n, let $\tilde{x}_n = X_n(\tilde{v})$. A strategy combination $(X_1, \ldots, X_N)$ determines the order flow as $\tilde{y} = \sum_n \tilde{x}_n + \tilde{u}$.

Market makers observe the realization of the order flow, but not any of its components, and engage in a competitive auction to serve the order flow. The outcome of this competition is described by a Lebesgue measurable function $P : \mathbb{R} \to \mathbb{R}$, called the pricing rule. Given $(P, X_1, \ldots, X_N)$ define $\tilde{p} = P(\tilde{y})$ and let $\tilde{\pi}_n = (\tilde{v} - \tilde{p})\tilde{x}_n$ denote the resulting trading profit of insider n. To ensure that the expected profit of an insider is well-defined for all feasible $(P, X_1, \ldots, X_N)$, we restrict the strategy set of an insider to $\mathcal{X} = \{X_n \mid E[\tilde{x}_n^2] < \infty\}$, and the set of pricing rules to $\mathcal{P} = \{P \mid \forall (X_1, \ldots, X_N) \in \mathcal{X}^N : E[\tilde{p}^2] < \infty\}$.

The equilibrium conditions are that the competition between market makers drives their expected profits to zero conditional on the order flow and that each insider chooses his trading strategy to maximize his expected profits. Following the convention in the existing literature, an equilibrium is said to be linear if the pricing rule is an affine function of the order flow.

Definition 1 $(P, X_1, \ldots, X_N) \in \mathcal{X}^N \times \mathcal{P}$ *is an equilibrium for the model* $(\tilde{u}, \tilde{v}, N)$ *if*

$$E[\tilde{v} - \tilde{p} \mid \tilde{y}] = 0 \tag{1}$$

and, for all n and $x \in \mathcal{X}$,

$$E[\tilde{\pi}_n] \geq E[(\tilde{v} - P(\sum_{m \neq n} \tilde{x}_m + X(\tilde{v}) + \tilde{u}))X(\tilde{v})]. \tag{2}$$

An equilibrium is linear if there exist constants μ, λ such that

$$\forall y : P(y) = \mu + \lambda y.$$

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[1]As it is customary we omit the explicit reference to the underlying probability space and identify random variables that coincide with probability one.

We refer to $(\tilde{p}, \tilde{x}_1, \ldots, \tilde{x}_N)$ as an equilibrium outcome of the model $(\tilde{u}, \tilde{v}, N)$ if there exists an equilibrium $(P, X_1, \ldots, X_N)$ for this model such that $\tilde{p} = P(\tilde{y})$ and $\tilde{x}_n = X_n(\tilde{v})$ for all n.

3   Results

Before we proceed to the study of linear equilibria it is convenient to note the following result, which generalizes some familiar comparative statics properties for the linear equilibria in the model with normally distributed $(\tilde{u}, \tilde{v})$ to arbitrary equilibria and distributions.

Lemma 1 *Let* $(\tilde{p}, \tilde{x}_1, \cdots, \tilde{x}_N)$ *be an equilibrium outcome for the model* $(\tilde{u}, \tilde{v}, N)$ *and* (a, b, c, d) *be constants with* $b, d > 0$. *Then* $(\hat{p}, \hat{x}_1, \cdot \hat{x}_N)$ *is an equilibrium outcome for the model* $(\hat{u}, \hat{v}, N)$, *where* $\hat{u} = a + b\tilde{u}$, $\hat{v} = c + d\tilde{v}$, $\hat{p} = c + d\tilde{p}$ *and* $\hat{x}_n = b\tilde{x}_n$.

Proof: Let $(P, X_1, \cdots, X_N)$ be an equilibrium in the model $(\tilde{u}, \tilde{v}, N)$ resulting in the outcome $(\tilde{p}, \tilde{x}_1, \cdots, \tilde{x}_N)$. Define $(\hat{P}, \hat{X}_1, \cdots, \hat{X}_N)$ by setting $\hat{P}(v) = c + dP((y - a)/b)$ and $\hat{X}_n(v) = bX_n((v - c)/d)$. Then $\hat{X}_n(\hat{v}) = \hat{X}_n(c + d\tilde{v}) = b\tilde{x}_n = \hat{x}_n$ and $\hat{P}(\hat{y}) = \hat{P}(a + b\tilde{y}) = c + d\tilde{p} = \hat{p}$, where $\hat{y} = \sum_n \hat{x}_n + \hat{u}$, showing that $(\hat{P}, \hat{X}_1, \cdots, \hat{X}_N)$ results in the outcome $(\hat{p}, \hat{x}_1, \cdots, \hat{x}_N)$ in the model $(\hat{u}, \hat{v}, N)$. Because $E[\hat{v} - \hat{p} \mid \hat{y}] = dE[\tilde{v} - \tilde{p} \mid \tilde{y}] = 0$, $(\hat{P}, \hat{X}_1, \cdots, \hat{X}_N)$ satisfies equilibrium condition (1) in the model $(\hat{u}, \hat{v}, N)$. Let $\hat{X}$ be any strategy in the model $(\hat{u}, \hat{v}, N)$. Because $\hat{\pi}_n = (\hat{v} - \hat{p})\hat{x}_n = bd\tilde{\pi}_n$ and $X(v) = \hat{X}(c + dv)/b$ is a strategy in the model $(\tilde{u}, \tilde{v}, N)$, we have

$$E[\hat{\pi}_n] = bdE[\tilde{\pi}_n] \geq$$
$$bdE[(\tilde{v} - P(\sum_{m \neq n} \tilde{x}_m + X(\tilde{v}) + \tilde{u}))X(\tilde{v})] =$$
$$E[\hat{v} - \hat{P}(\sum_{m \neq n} \hat{x}_m + \hat{X}(\hat{v}) + \hat{u}))\hat{X}(\hat{v})]$$

showing that condition (2) is satisfied.

In particular, as in Kyle [8], changes in the standard deviation of noise trading or the asset payoff (modeled as a linear rescaling of the underlying distributions and corresponding to b and d in the statement of the lemma) affect equilibrium behavior as follows. An increase in the standard deviation of noise trading results in a proportional increase in the standard deviations of the insiders' orders without affecting the distribution of the equilibrium price. An increase in the standard deviation of the asset payoff does not affect the distribution of the insiders' orders, but results in a proportional increase in the standard deviation of the equilibrium price. Taken together these two properties imply that the insiders' profits are proportional to the standard deviation of both noise trade and the asset payoff: $\hat{\pi}_n = (\hat{v} - \hat{p})\hat{x}_n = bd\tilde{x}_n$.

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From Lemma 1 it is without further loss of generality to restrict attention to distributions with expectation zero and unit variance in the following result, providing a simple necessary and sufficient condition for the existence of linear equilibria.

Lemma 2 *Suppose $\tilde{u}$ and $\tilde{v}$ satisfy $E[\tilde{u}] = E[\tilde{v}] = 0$ and $Var[\tilde{u}] = Var[\tilde{v}] = 1$. Then a linear equilibrium in the model $(\tilde{u}, \tilde{v}, N)$ exists if and only if*

$$E[\sqrt{N}\tilde{u} - \tilde{v} \mid \tilde{u} + \sqrt{N}\tilde{v}] = 0. \tag{3}$$

The proof of Lemma 2 is given in the Appendix. The idea is the following. Given any linear pricing rule with positive slope (nonpositive slope implies that an insider's maximization problem has no solution) and independence of noise trading and payoff information, condition (2) uniquely determines the order flow as a linear combination of $\tilde{u}$ and $\tilde{v}$. The coefficients in this linear combination depend only on the number of informed traders and the parameters of the price function, but not on the underlying distribution. The conditions that the "forecast error" $\tilde{v} - \tilde{p}$ has expected value zero and is uncorrelated with the order flow. both implied by the market efficiency condition (1), yield two equations, which only depend on the number of informed traders and the parameters of the price function. Solving these equations for μ and λ shows that in any linear equilibrium the order flow is given by $\tilde{y} = \sqrt{N}\tilde{v} + \tilde{u}$ whereas the equilibrium price is given by $\tilde{p} = \sqrt{N}\tilde{y}/(N+1)$. Substituting these values into (1) yields (3), proving necessity of this condition for the existence of a linear equilibrium. Sufficiency is then easily verified.

To obtain our main result, we use Theorem 6.1.1. in Lukacs and Laha [9, p. 103] which asserts that for any random variables $(\tilde{u}, \tilde{v})$ condition (3) is satisfied if and only if $i = \sqrt{-1}$ denotes the imaginary unit)[2]

$$\forall t \in \mathcal{R} : E[(\sqrt{N}\tilde{u} - \tilde{v})e^{it(\tilde{u} + \sqrt{N}\tilde{v})}] = 0 \tag{4}$$

The proof of the following proposition shows that the only case in which condition (4) holds for two distinct values of N is the one in which $\tilde{u}$ and $\tilde{v}$ are both normally distributed. Related characterizations of the normal distribution are given in Bryc [4, Chapter 7].

Proposition 1 *Let $N_1 < N_2$ and suppose there exist linear equilibria in the models $(\tilde{u}, \tilde{v}, N_1)$ and $(\tilde{u}, \tilde{v}, N_2)$. Then $\tilde{u}$ and $\tilde{v}$ are normally distributed.*

Proof: Let $E[\tilde{u}] = E[\tilde{v}] = 0$ and $Var[\tilde{u}] = Var[\tilde{v}] = 1$. Let $f(t) = E[e^{it\tilde{u}}]$ and $g(t) = E[e^{it\tilde{v}}]$ be the characteristic functions of $\tilde{u}$ and $\tilde{v}$, respectively. From the existence of the second moments these characteristic functions are twice

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[2]Related results are used in Bagnoli, Viswanathan and Holden [2] to obtain an explicit characterization of those distributions for which a linear equilibrium exists for a given number of informed traders. Their characterization [2, Corrollary 1] requires an additional assumption on the distribution of $\tilde{u}$ and $\tilde{v}$, namely that there is no point at which the characteristic functions and all of its derivatives are equal to zero.

continuously differentiable (see Billingsley [3, Section 26]). Using independence of $\tilde{u}$ and $\tilde{v}$ (and multiplying by i), (4) implies

$$\forall t, N = N_1, N_2 : \sqrt{N} E[i\tilde{u}e^{it\tilde{u}}]E[e^{i\sqrt{N}t\tilde{v}}] = E[i\tilde{v}e^{i\sqrt{N}t\tilde{v}}]E[e^{it\tilde{u}}],$$

or

$$\forall t, N = N_1, N_2 : \sqrt{N} f'(t)g(\sqrt{N}t) = g'(\sqrt{N}t)f(t), \tag{5}$$

where dashes indicate derivatives.

Let T be a neighborhood of zero on which $f(t)$ and $g(\sqrt{N_2}t)$ are strictly positive. Such a neighborhood exists because the characteristic functions satisfy $f(0) = g(0) = 1$. Define $F(t) = \log f(t)$ for $t \in T$, and $G(t) = \log g(t)$ for $t/\sqrt{N_2} \in T$. From (5) these functions satisfy

$$\forall t \in T, N = N_1, N_2 : G'(\sqrt{N}t) = \sqrt{N}F'(t), \tag{6}$$

implying

$$\forall t \in T : G''(\sqrt{N_1}t) = F''(t) = G''(\sqrt{N_2}t). \tag{7}$$

As $G''(t)$ is continuous at zero, (7) implies

$$\forall t \in T : G''(t) = G''(\alpha t) = G''(\alpha^2 t) = G''(\alpha^3 t) = \ldots = G''(0),$$

where $\alpha = \sqrt{N_1}/\sqrt{N_2} < 1$. Together with $G(0) = 0$, $G'(0) = 0$ and $G''(0) = -1$ (where the latter two conditions follow from $g'(0) = iE[\tilde{v}] = 0$ and $g''(0) = -E[\tilde{v}^2] = -1$), it follows that

$$\forall t \in T : G(t) = -\frac{t^2}{2}. \tag{8}$$

Using (6) and $F(0) = 0$, (8) also implies that $F(t) = -t^2/2$. Hence,

$$\forall t \in T : g(t) = f(t) = e^{-t^2/2}.$$

Consequently, the moments of $\tilde{u}$ and $\tilde{v}$ coincide with the moments of the standard normal distribution, implying that $\tilde{u}$ and $\tilde{v}$ are normally distributed (see Billingsley [3, Section 30]).

4   Conclusion

This note has considered the simplest version of the Kyle model in which all traders are risk neutral, noise trading is exogenous, and there is only a single auction. In this setting we have provided simple additional conditions under which normality is not only sufficient but also necessary for the existence of a linear equilibrium. In our view this result suggests that further fruitful investigations of the Kyle model should focus on the characterization of nonlinear equilibria as in Cho and El Karoui [5], Back [1], and Rochet and Vila [12].

Appendix: Proof of Lemma 2

To show necessity, suppose $(P, X_1, \cdots, X_N)$ is an equilibrium with $P(y) = \mu + \lambda y$. We have $\lambda > 0$: otherwise the strategy $X(v) = kv$ would be a profitable deviation for every insider if k is sufficiently large.

Because $X_n(v)$ is an equilibrium strategy,

$$X_n(v) \in \arg\max_x E[v - \mu - \lambda(\sum_{m \neq n} X_m(v) + \tilde{u} + x))x \mid \tilde{v} = v], \ \tilde{v} - a.e. \quad (9)$$

The first order condition and independence of $\tilde{u}$ and $\tilde{v}$ yield

$$\forall n : \tilde{v} - \mu - \lambda(\sum_m \tilde{x}_m + \tilde{x}_n) = 0.$$

Hence $\tilde{x}_n$ is independent of n and thus

$$\forall n : \tilde{x}_n = \frac{1}{\lambda(N+1)}(\tilde{v} - \mu). \quad (10)$$

Consequently, order flow and price are given by

$$\tilde{y} = \frac{N}{\lambda(N+1)}(\tilde{v} - \mu) + \tilde{u}, \ \tilde{p} = \mu + \frac{N}{N+1}(\tilde{v} - \mu) + \lambda\tilde{u}. \quad (11)$$

The market efficiency condition (1) implies

$$E[\tilde{v} - \tilde{p}] = 0, \ E[(\tilde{v} - \tilde{p})\tilde{y}] = 0. \quad (12)$$

Substituting (11) into (12) yields two equations for λ and μ. Using that $\tilde{u}$ and $\tilde{v}$ are independent and both standardized, one obtains the solution

$$\mu = 0, \ \lambda = \frac{\sqrt{N}}{N+1}.$$

Substituting these equilibrium values of μ and λ back into (11) yields

$$\tilde{y} = \sqrt{N}\tilde{v} + \tilde{u}, \ \tilde{p} = \frac{N\tilde{v} + \sqrt{N}\tilde{u}}{N+1}. \quad (13)$$

Hence,

$$E[\tilde{v} - \tilde{p} \mid \tilde{y}] = \frac{1}{N+1}E[\tilde{v} - \sqrt{N}\tilde{u} \mid \sqrt{N}\tilde{v} + \tilde{u}]. \quad (14)$$

In particular, (1) implies (3).

To show sufficiency, define $P(y) = \sqrt{N}y/(N+1)$ and $X_n(v) = v/\sqrt{N}$ for all n. The induced order flow $\tilde{y}$ and price $\tilde{p}$ satisfy (13). By (14) it follows that (3) implies (1). It remains to verify (2). But this is immediate from the fact that

$$\forall v : X_n(v) \in \arg\max_x(v - \frac{\sqrt{N}}{N+1}(\sum_{m \neq n} X_m(v) + x))x.$$

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