Mathematical Research Letters **8**, 641–649 (2001)

RATIONAL COVARIANTS OF REDUCTIVE GROUPS AND HOMALOIDAL POLYNOMIALS

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Abstract. Let G be a complex reductive group, V a G-module and $f \in \mathcal{O}(V)^G$ a nonconstant homogeneous invariant. We investigate relations between the following properties:

- $df: V \to V^*$ is dominant,
- f is *homaloidal*, i.e., df induces a birational map $\mathbb{P}(V) \to \mathbb{P}(V^*)$,
- V is stable, i.e., the generic G-orbit is closed.

If f generates $\mathcal{O}(V)^G$, we show that the properties are equivalent, generalizing results of Sato-Kimura on prehomogeneous vector spaces.

Introduction

Our ground field is $\mathbb{C}$, the field of complex numbers. Throughout this note, G will be a reductive (complex) algebraic group, and V a (finite dimensional algebraic) G-module. We denote by $\mathcal{O}(V)$ the algebra of polynomial functions on V and by $\mathcal{O}(V)^G$ the subalgebra of G-invariant functions. The *algebraic quotient* (of V by G) is the morphism $\pi_V: V \to V/\!\!/G$ corresponding to the inclusion $\mathcal{O}(V)^G \subset \mathcal{O}(V)$. Similar definitions and notation apply to an affine G-variety, i.e., an affine variety X equipped with an algebraic action of G.

If $f \in \mathcal{O}(V)^G$, then the differential

$$df: V \to V^*, \quad v \mapsto df(v): (T_v V \simeq V) \to \mathbb{C}$$

is G-equivariant, where V^* denotes the dual G-module. The (schematic) fiber $F := f^{-1}(f(v))$ is non-singular at v if and only if $df(v) \neq 0$, in which case $\operatorname{Ker} df(v) = T_v F$, the tangent space to F at v. Note that if f is homogeneous and df is dominant, then df defines a rational map $\Phi_f: \mathbb{P}(V) \to \mathbb{P}(V^*)$ of finite degree. If Φ_f is birational, then f is said to be *homaloidal*.

We say that V is *stable* if the set of closed G-orbits contains a non-empty open subset of V. The main results of this paper are the following.

Theorem A. *Let $f \in \mathcal{O}(V)^G$ be a homogeneous invariant. If the differential $df: V \to V^*$ is dominant, then V is a stable G-module.*

Received April 5, 2001

Both authors thank the Mathematical Sciences Research Institute and the Erwin Schrödinger Institute for hospitality, and they thank the SNF and NSF, respectively, for partial support.

Theorem B. *Assume that the invariant ring $\mathcal{O}(V)^G$ is generated by a homogeneous function f. Then the following are equivalent:*

(i) *The differential $df\colon V \to V^*$ is dominant;*
(ii) *The differential df induces a birational map $\Phi_f\colon \mathbb{P}(V) \to \mathbb{P}(V^*)$, i.e., f is homaloidal;*
(iii) *The G-module V is stable.*

A crucial step in the proof of Theorem B is Proposition C, below, which is interesting in its own right. In a way it gives conditions for Luna's Slice Theorem ([Lu73], see also [Sl89]) to hold "generically." Recall that a G-equivariant morphism $\varphi\colon X \to Y$ is *excellent* if the induced morphism $\bar{\varphi}\colon X/\!\!/G \to Y/\!\!/G$ is étale and the following diagram is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/\!\!/G & \xrightarrow{\bar{\varphi}} & Y/\!\!/G \end{array}$$

In particular, excellent morphisms are étale. A subset S of an affine G-variety X is *saturated* if $S = \pi_X^{-1}(\pi_X(S))$.

Proposition C. *Let X and Y be two smooth affine G-varieties of the same dimension and let $\varphi\colon X \to Y$ be a G-equivariant dominant morphism. Assume that the generic fibers (see §1) of π_X and π_Y are G-isomorphic. Then there is a non-empty saturated open set $U \subset Y$ such that the induced morphism $\varphi|_{\varphi^{-1}(U)}\colon \varphi^{-1}(U) \to U$ is excellent.*

In the special case of a G-module and its dual the assumption on the generic fibers is automatically fulfilled.

Proposition D. *Let $\varphi\colon V \to V^*$ be a G-equivariant dominant morphism. Then the generic fibers of π_V and π_{V^*} are isomorphic.*

The following relations between V and V^* come in handy:

Proposition E. *Let V be a G-module. Then*

(1) *V is stable if and only if V^* is stable.*
(2) *The principal isotropy groups (see §1) of V and V^* are the same.*

Outline of the paper. In §1 we state and prove a generalization (Proposition 1) of Proposition C, and we prove Propositions D and E and Theorem A. In §2 we consider the special case of G-modules with one-dimensional quotient, and we give the proof of Theorem B, which generalizes results obtained by Sato-Kimura [SaK77] for prehomogeneous vector spaces. In §3 we show how to obtain the Sato-Kimura results from Theorem B.

§1. Dominant Covariants

Let X be an affine G-variety. Note that any fiber F of $\pi_X: X \to X/\!\!/G$ is *without invariants* in the sense that $\mathcal{O}(F)^G = \mathbb{C}$. We say that F is a *generic fiber of* π_X *(or of* X*)* if there is a nonempty open subset $Z \subset X/\!\!/G$ such that $\pi_X^{-1}(Z) \to Z$ is a G-fiber bundle with fiber F (in the étale topology). LUNA's slice theorem [Lu73] shows that any smooth affine G-variety has a generic fiber F. Moreover, F has the form $G *^H W$ where $H \subset G$ is reductive and W is an H-module without invariants. In particular, F is smooth and G acts transitively on the irreducible components of F. The subgroup H (or any G-conjugate of H) is called a *principal isotropy group* of X. A nonsmooth affine G-variety may not have a generic fiber.

For completeness, we add the following well-known result:

Lemma 1. *Let $\varphi: F \to F'$ be a dominant G-equivariant morphism where F and F' are G-isomorphic affine varieties without invariants. Then φ is an isomorphism.*

Proof. Since φ is dominant, we have a G-linear inclusion $\varphi^*: \mathcal{O}(F') \hookrightarrow \mathcal{O}(F)$. By assumption, the isotypic components of $\mathcal{O}(F)$ and $\mathcal{O}(F')$ are finite dimensional of the same dimension. Thus φ^* is an isomorphism. □

The next result is a generalization of Proposition C of the introduction.

Proposition 1. *Let X and Y be irreducible affine G-varieties of the same dimension which have generic fibers F_X and F_Y, respectively. Assume that F_X and F_Y are G-isomorphic and that G acts transitively on the irreducible components of F_X. If $\varphi: X \to Y$ is a dominant G-equivariant morphism, then there is a non-empty saturated open set $U \subset Y$ such that φ is excellent on $\varphi^{-1}(U)$.*

Proof. It suffices to find a (non-empty) saturated open set $X' \subset X$ such that $\varphi|_{X'}: X' \to Y$ is excellent, since every such X' contains a dense subset of the form $\varphi^{-1}(U)$ where $U \subset Y$ is open and saturated. Therefore, we can assume that $\pi_X: X \to X/\!\!/G$ and $\pi_Y: Y \to Y/\!\!/G$ are both fiber bundles over smooth bases with fibers F_X and F_Y, respectively, and that $\overline{\varphi}: X/\!\!/G \to Y/\!\!/G$ is dominant. Shrinking further we can assume that $\overline{\varphi}$ is étale.

Let $X_0 \subset X$ be the (non-empty) G-stable open set where φ is smooth. If a fiber F of π_X meets X_0, then $\overline{\varphi(F)}$ has the same dimension as F, hence contains an irreducible component of a fiber F' of π_Y. Since G acts transitively on the irreducible components of the fibers, we get that $\overline{\varphi(F)} = F'$, and Lemma 1 shows that φ induces an isomorphism $F \xrightarrow{\sim} F'$.

The argument above shows that we can assume that every fiber of π_X is mapped isomorphically onto a fiber of π_Y. Thus we obtain a map $\tilde{\varphi}: X \to X/\!\!/G \times_{Y/\!\!/G} Y$ of fiber bundles over $X/\!\!/G$ inducing isomorphisms on the fibers. The following lemma shows that $\tilde{\varphi}$ is an isomorphism, i.e., that φ is excellent. □

The lemma below is certainly well-known. Since we could not find a suitable reference, we include a short proof.

Lemma 2. *Let $p\colon X \to S$ and $q\colon Y \to S$ be two fiber bundles with fiber F and let $\varphi\colon X \to Y$ be a morphism of fiber bundles (i.e., $q \circ \varphi = p$) inducing isomorphisms $\varphi_s\colon X_s \xrightarrow{\sim} Y_s$ on the fibers for all $s \in S$. Then φ is an isomorphism.*

Proof. (a) Let us first assume that φ is a finite morphism and let $x \in X$, $y := \varphi(x) \in Y$ and $s := p(x) = q(y)$. Then $\varphi^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite and injective and induces, by assumption, an isomorphism $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$. It follows that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y} + \mathfrak{m}_y\mathcal{O}_{X,x}$ and so $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$ by NAKAYAMA's Lemma. Hence φ is an isomorphism.

(b) Next we remark that the lemma also holds if Y is normal, since every bijective morphism onto a normal variety is an isomorphism, by ZARISKI's Main Theorem (see [Mu88, Ch. 3, §9]).

(c) In general, let $\tilde{X} \to X$ be the normalization of X and $\tilde{S} \to S$ the normalization of S. We obtain a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\eta_X} & X \\ \Big\downarrow{\scriptstyle \tilde{p}} & & {\scriptstyle p}\Big\downarrow \\ \tilde{S} & \longrightarrow & S \end{array}$$

where all horizontal maps are finite and where $\tilde{p}$ is a fiber bundle with fiber $\tilde{F}$, the normalization of F. A similar diagram is obtained for Y. Moreover, φ induces a fiber bundle morphism $\tilde{\varphi}\colon \tilde{X} \to \tilde{Y}$ which is again an isomorphism on the fibers. It now follows from (b) that $\tilde{\varphi}$ is an isomorphism:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\eta_X} & X \\ \simeq\Big\downarrow{\scriptstyle \tilde{\varphi}} & & {\scriptstyle \varphi}\Big\downarrow \\ \tilde{Y} & \xrightarrow{\eta_Y} & Y \end{array}$$

Therefore, φ is finite and the lemma follows from (a). □

In order to prove Propositions D and E of the introduction, we use the following well-known lemma:

Lemma 3. *Let G be a connected reductive group. Then there exists an automorphism $\tau\colon G \xrightarrow{\sim} G$ with the following property: If $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then $\rho \circ \tau\colon G \to \mathrm{GL}(V)$ is equivalent to the dual representation $\rho^*\colon G \to \mathrm{GL}(V^*)$.*

As a consequence, we see that for a connected reductive group G any covariant $\varphi\colon V \to W$ can be regarded as a covariant $\varphi^*\colon V^* \to W^*$ by simply changing the action of G by the automorphism τ. Moreover, the quotient map $\pi_V\colon V \to V/\!\!/G$ can be identified with the quotient map $\pi_{V^*}\colon V^* \to V^*/\!\!/G$.

Remark. It is essential in the lemma above that G be connected. In fact, the smallest Mathieu group M_{11} has a non self-dual representation W of dimension 10, but M_{11} has no outer automorphism. (This example was shown to us by Guralnick.) We do not know if the invariant rings of W and W^* are isomorphic.

Proof of Proposition D. Let $\varphi: V \to V^*$ be a dominant covariant. We want to show that the generic fibers of π_V and π_{V^*} are isomorphic. We first assume that G is connected. We have seen above that φ can be regarded as a covariant $\varphi^*: V^* \to V$, and the composition $\varphi^* \circ \varphi: V \to V$ is a dominant covariant. It follows from Proposition 1 that for a generic fiber F of V the image $F' := \overline{\varphi^*(\varphi(F))}$ is also a generic fiber of V, and the composition $F \to \overline{\varphi(F)} \to F'$ is an isomorphism. Hence $F \xrightarrow{\sim} \overline{\varphi(F)}$. Since the generic fibers of V and V^* are irreducible and of the same dimension, it follows that $\overline{\varphi(F)}$ is a generic fiber F^* of V^* and that $F \xrightarrow{\sim} F^*$.

In case of a general reductive group G we can apply Propositions C and D to G^0 and find a G^0-saturated open set $U \subset Y$ such that φ is excellent on $\varphi^{-1}(U)$ with respect to the action of G^0. Replacing U be GU we may assume that U is G-saturated. In the following diagram

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\varphi} & U \\ \downarrow{\scriptstyle \pi_V^0} & & \downarrow{\scriptstyle \pi_{V^*}^0} \\ \varphi^{-1}(U)/\!\!/G^0 & \xrightarrow{\bar\varphi^0} & U/\!\!/G^0 \\ \downarrow{\scriptstyle \bar\pi_V} & & \downarrow{\scriptstyle \bar\pi_{V^*}} \\ \varphi^{-1}(U)/\!\!/G & \xrightarrow{\bar\varphi} & U/\!\!/G \end{array}$$

the upper square is cartesian, the maps π_V^0 and $\pi_{V^*}^0$ are the quotients of the G^0-action, and the maps $\bar\pi_V$ and $\bar\pi_{V^*}$ are quotients by the finite group $\bar G := G/G^0$. It suffices to show that the generic fibers of $\bar\pi_V$ and $\bar\pi_{V^*}$ have the same number of elements, or, equivalently, that the generic fibers of π_V and π_{V^*} have the same number of components.

Let G' denote the kernel of the action of G on $V/\!\!/G^0$. Then the number n_V of components of the generic fiber of π_V equals $[G : G']$. The following lemma shows that $n_V = n_{V^*}$, proving the proposition. □

Lemma 4. *Let $N \subset G$ be a normal subgroup, and let W be a G-module. Then $(W^N)^*$ and $(W^*)^N$ are isomorphic G/N-modules. In particular, the kernels of the actions of G on W^N and $(W^*)^N$ are the same.*

Proof. One easily reduces to the case that W is an irreducible G-module. Since W^N is G-stable, either N acts trivially on W, or N acts nontrivially on W and $W^N = (0)$. But W^* is also an irreducible G-module, and N acts trivially on W^*

if and only if it acts trivially on W. Thus, $(W^N)^*$ and $(W^*)^N$ are either both zero or W^*. □

Proof of Proposition E. Let V be a G-module, and let V_0 denote V considered as a G^0-module. Then V is stable if and only if V_0 is stable. Since V_0 is stable if and only if the generic fiber of π_{V_0} is an orbit, the comments following Lemma 3 show that V_0 is stable if and only if V_0^* is stable. Thus stability of V and V^* are equivalent.

Let H be a principal isotropy group of V, and let N denote $N_G(H)/H$. By the Luna-Richardson Theorem [LuR79], the restriction map $\mathcal{O}(V)^G \to \mathcal{O}(V^H)^N$ is an isomorphism where the action of N on V^H has trivial principal isotropy groups (hence is stable). The action of N on $(V^*)^H \simeq (V^H)^*$ is then also stable with finite principal isotropy groups. By [Lu75], $\mathcal{O}(V^*)^G \to \mathcal{O}((V^*)^H)^N$ is finite, and every closed N-orbit in $(V^*)^H$ lies in a closed G-orbit in V^*. Since $\mathcal{O}((V^*)^H)^N$ and $\mathcal{O}(V^*)^G$ have the same dimension ($= \dim V/\!\!/G$), some closed N-orbit in $(V^*)^H$ intersects the closed orbit of a generic fiber of V^*. It follows that $H \subset H^*$, where H^* is a principal isotropy group of V^*. Repeating the argument for V^* shows that H^* is contained in a conjugate of H, hence $H = H^*$. □

We now give the proof of Theorem A of the introduction.

Proof of Theorem A. Let $\varphi\colon V \to V^*$ be a covariant. For any point $v \in V$, the derivative $d\varphi(v)\colon T_vV \to T_{\varphi(v)}V^* \simeq V^*$ gives a bilinear form β_v on $T_vV \simeq V$, namely, $\beta_v(w_1, w_2) := d\varphi(v)(w_1)(w_2)$. Since φ is a covariant, β_v is G_v-invariant, and β_v is non-degenerate precisely at the points v where φ is étale. Now assume that $f \in \mathcal{O}(V)^G$ and $\varphi := df$. Then the form β_v is the Hessian of f and is symmetric. If φ is dominant, Propositions C and D give us an open G-saturated subset $U \subset V^*$ such that $\varphi|_{\varphi^{-1}(U)}\colon \varphi^{-1}(U) \to U$ is excellent, and we may assume that U and $\varphi^{-1}(U)$ are affine. The following result of Luna (see [Lu72], [Lu73]) shows that the action of G on $\varphi^{-1}(U)$ is stable, proving Theorem A. □

Theorem (Luna)**.** *Let X be a smooth affine G-variety. Assume that for each $x \in X$ there is a G_x-invariant non-degenerate symmetric bilinear form on the tangent space T_xX. Then X is stable.*

§2. One-dimensional quotients

Let V be a G-module such that the invariant ring $\mathcal{O}(V)^G$ is generated by the homogeneous function f of degree d. Then $\mathcal{O}(V^*)^G$ is also generated by a homogeneous function f^* of degree d.

Proposition 2. *Let V, G, f and f^* be as above and let $Gv = \overline{Gv}$ be a closed orbit in V. Then*

(1) *$f^* \circ df$ is a non-zero multiple of f^{d-1}.*
(2) *$df^*(df(v)) = c_v v$ for some non-zero $c_v \in \mathbb{C}$.*
(3) *If V is stable, then df is dominant.*

Proof. We may choose an isomorphism $V \simeq \mathbb{C}^n$ such that $G \subset \mathrm{GL}_n(\mathbb{C})$ is the complexification of a compact subgroup $K \subset \mathrm{U}_n(\mathbb{C})$. Let $(\ ,\ )$ denote the standard hermitian inner product on $\mathbb{C}^n$. Then Kempf-Ness theory ([KeN79], cf. [Sch89]) tells us the following: Let $v \in V$. Then

$$(*) \quad T_v(Gv) \perp v \textit{ if and only if } Gv \textit{ is a closed orbit and } (v,v) = \inf_{g \in G}(gv, gv).$$

Assume that V is stable, and choose $0 \neq v \in V$ such that Gv is closed. We may assume that (gv, gv) achieves its minimum at v. If $df(v) = 0$, then df vanishes on $G(\mathbb{C}v)$ which is dense in V. Thus $df \equiv 0$, which is impossible. Let λ denote the linear form $df(v) \neq 0$. Then $T_v(Gv) = \operatorname{Ker}\lambda$ has dimension $n-1$, and so $\lambda(w) = c(w,v)$ for all $w \in V$ and a suitable $c \in \mathbb{C}$. Considered as an element of V^{**}, v lies in the annihilator of $T_\lambda(G\lambda)$: if $A \in \mathfrak{g}$, then $(A\lambda)(v) = \lambda(-Av) = c(-Av, v) \in (\mathfrak{g}v, v) = \{0\}$. But this shows that $\lambda \perp T_\lambda(G\lambda)$ (using the induced hermitian inner product on V^*). Thus $G\lambda$ is a closed orbit in V^* and $\lambda \in G\lambda$ is a point of minimal norm. Hence $f^*(\lambda) \neq 0$, giving (1). Since V^* is also stable, we may apply the reasoning above to $v' := df^*(\lambda) \in V^{**}$. We obtain that v' generates the annihilator of $T_\lambda(G\lambda)$, so that v' is a non-zero multiple of $v \in V^{**}$. We have (2), and (3) follows since $G(\mathbb{C}\lambda)$ is dense in V^*.

In general, let H be a principal isotropy group of V. Then the action of $N := N_G(H)/H$ on V^H is stable, and by the Luna-Richardson Theorem [LuR79], the restriction $\bar{f}$ of f to V^H generates the invariant ring $\mathcal{O}(V^H)^N$. For $v \in V^H$, we have $df(v) \in (V^*)^H$, and using the canonical isomorphism $(V^*)^H \simeq (V^H)^*$ this linear function identifies with $df(v)|_{V^H} = d\bar{f}(v)$. Proposition E shows that H is also the principal isotropy group of V^*. Hence $\bar{f}^* = f^*|_{V^{*H}}$ generates the invariant ring $\mathcal{O}(V^{*H})^N$. By the first part of our proof, $f^*(df(v)) = \bar{f}^*(d\bar{f}(v)) \neq 0$ and $df^*(df(v)) = d\bar{f}^*(d\bar{f}(v))$ is a multiple of v, so we have (1) and (2). □

Corollary 1. *Let V, f and f^* be as above. If $df\colon V \to V^*$ is dominant, then $df^* \circ df = cf^{d-2}\,\mathrm{Id}_V$ for a non-zero constant $c \in \mathbb{C}$.*

Proof. Since df is dominant, V is stable by Proposition 1 and $df^*(df(v))$ is a nonzero multiple of v for every v not in the nullcone $\mathcal{N}_V = f^{-1}(0)$ of V. Since $G \times \mathbb{C}^*$ has a dense orbit in V and since $df^* \circ df$ is G-equivariant and homogeneous, $df^* \circ df(w) = h(w)w$ for all $w \in V$, where $h(w)$ is a nonzero homogeneous polynomial of degree $d(d-2)$. Clearly, h is G-invariant, hence it has the given form. □

We now give the proof of Theorem B of the introduction.

Proof of Theorem B. If df induces a birational map $\Phi_f\colon \mathbb{P}(V) \to \mathbb{P}(V^*)$, then df is dominant, and conversely by Corollary 1. Dominance of df and stability of V are equivalent by Proposition 2 and Theorem A. □

Examples. It is easy to find cases where df is not dominant. Start with any G-module V with a one-dimensional quotient $f\colon V \to \mathbb{C}$, let H be a reductive group and W a nonzero H-module without invariants. Then $\tilde{V} := V \oplus W$ is a $\tilde{G} := G \times H$-module with a one-dimensional quotient given by $\tilde{f}\colon (v, w) \mapsto f(v)$. Clearly $d\tilde{f}(\tilde{V}) \subset V$, so df is not dominant.

§3. Prehomogeneous vector spaces

Let V be a prehomogeneous G-module and χ a character of G. A function $f\colon V \to \mathbb{C}$ is called a *semi-invariant (with character χ)* if $f(gv) = \chi(g)f(v)$ for all $g \in G$. It is easy to see that semi-invariants have to be homogeneous functions. Note that the subalgebra $\mathbb{C}[f] \subset \mathcal{O}(V)$ is the invariant ring under $G_0 := \operatorname{Ker}\chi$. In fact, f is G_0-invariant and the fibers of the restriction of f to the dense orbit Gv are exactly the G_0-orbits.

We say that V is *regular* if there is a semi-invariant f whose Hessian does not vanish identically; equivalently (by Theorem B), f is homaloidal. The following result is essentially contained in the work [SaK77] of SATO-KIMURA.

Theorem 3. *Let V be prehomogeneous, and let $v \in V$ such that Gv is dense in V. Then the following are equivalent:*

(1) *V is regular.*
(2) *There is a semi-invariant f with character χ such that V is a stable G_0-module, where $G_0 := \operatorname{Ker}\chi$ (a reductive group).*
(3) *The isotropy group G_v is reductive.*
(4) *The complement of Gv is a hypersurface.*

If these conditions hold then a semi-invariant f is homaloidal if and only if its zero set is the complement of the dense orbit Gv.

Proof. Given a semi-invariant f, let χ_f denote the corresponding character. If V is regular, then we have a semi-invariant f such that $df\colon V \to V^*$ is dominant, and an application of Theorem A (or B) shows that V is a stable $(G_0 := \operatorname{Ker}\chi_f)$-module. Thus (1) implies (2). If G_0 and f are as in (2), then the isotropy group $(G_0)_v$ is reductive. Since $f(v) \neq 0$, we have $G_v = (G_0)_v$, giving (3). If (3), then $Gv \simeq G/G_v$ is affine, which forces the complement $V \setminus Gv$ to be a hypersurface, giving (4). If $S := V \setminus Gv$ is a hypersurface, let f be a function defining S. Then f is a homogeneous semi-invariant, and the fibers of $f\colon V \to \mathbb{C}$ over $\mathbb{C} \setminus \{0\}$ are the $(G_0 := \operatorname{Ker}\chi_f)$-orbits in Gv. Since all of these orbits have dimension $\dim V - 1$, they are all closed. Hence the action of G_0 on V is stable, and V is regular by Theorem B. Thus (4) implies (1).

Let f be a semi-invariant. If $f^{-1}(0) = S := V \setminus Gv$, we have shown that f is homaloidal. Suppose that $f^{-1}(0) \neq S$. Then there is an irreducible component S' of S such that $f|_{S'}\colon S' \to \mathbb{C}$ is dominant. It follows that the generic fiber of f consists of a $(G_0 := \operatorname{Ker}\chi_f)$-orbit in Gv, together with at least one other orbit. Hence V is not a stable G_0-module, and f is not homaloidal. □

Remark. The last statement of the theorem above proves a very special case of a conjecture by DOLGACHEV claiming that a homogeneous polynomial f with prime decomposition $\prod_i f_i^{n_i}$ is homaloidal if and only if $\prod_i f_i$ is homaloidal (see [Do00], end of section 3).

Example. Let $G = \mathrm{SL}_2 \times (\mathbb{C}^*)^3$ and $V = (\mathbb{C}^2)^3$. The group SL_2 acts on each copy of $\mathbb{C}^2$ in the standard way, and $(t_1, t_2, t_3) \in (\mathbb{C}^*)^3$ sends $(v_1, v_2, v_3) \in V$ to $(t_1 v_1, t_2 v_2, t_3 v_3)$. Clearly V is a prehomogeneous G-module. Let $f_{ij}(v_1, v_2, v_3)$ denote the determinant $\langle v_i, v_j \rangle$, $1 \le i < j \le 3$. Then the semi-invariants are the products of the f_{ij}, and a semi-invariant is homaloidal if and only if it has each of f_{12}, f_{13} and f_{23} as a factor.

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