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Properties and Examples of FCR-Algebras

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Abstract. An algebra A over a field k is FCR if every finite dimensional representation of A is completely reducible and the intersection of the kernels of these representations is zero. We give a useful characterization of FCR-algebras and apply this to C^*-algebras and to localizations. Moreover, we show that "small" products and sums of FCR-algebras are again FCR.

1. Introduction

It is well-known that the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ in characteristic zero has the following remarkable properties:

- Every finite dimensional representation of $\mathfrak{U}(\mathfrak{g})$ is completely reducible.
- $\mathfrak{U}(\mathfrak{g})$ is *residually finite dimensional*, i.e., the intersection of the kernels of all finite dimensional representations is zero.

The first property is due to Weyl, the second to Harish-Chandra (see [Dix77], 2.5.7, p. 84).

Question. *Are there other algebras satisfying these properties and how do they look?*

We introduced the following notion in [KrS94] in order to study this question.

Definition. An algebra A over an arbitrary field k is an FCR-*algebra* if every finite dimensional representation of A is completely reducible and A is residually finite-dimensional.
(FCR = "**F**inite dimensional representations are **C**ompletely **R**educible")

At that time the only known examples of FCR-algebras beside the finite dimensional semisimple algebras were the enveloping algebras of semisimple

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Lie algebras (in characteristic zero) and the quantum enveloping algebras $\mathfrak{U}_q(\mathfrak{g})$ for q not a root of unity. Since then a number of new examples of FCR-algebras have been discovered: Invariant algebras of enveloping algebras and direct summands ([KrS94], [KSW99]), algebras of invariant differential operators ([MvB98], [Sch00]), generalized Weyl algebras and orthosymplectic Lie superalgebras ([KiS00]).

The aim of this paper is to give a simple, but useful characterization of FCR-algebras (section 3) and to apply this to C^*-algebras (section 3) and to localizations (section 4). Moreover, we show that "small" direct products and direct sums (with a unit adjoined) of FCR-algebras are again FCR (section 2).

Remark 1. The classification of (affine Noetherian) FCR-algebras is an open problem, even in small dimensions. It is not hard to see that there are no such algebras in Gelfand-Kirillow-dimension (GK-dimension) equal to 1. (In fact, it follows from [SSW85] that any such algebra is PI and from [Fa87] that a (finitely generated) FCR-PI-algebra is finite dimensional.)

The smallest examples known so far are $\mathfrak{U}(\mathfrak{sl}_2)$ and its invariant subalgebras under a finite group of automorphism, all in GK-dimension 3 (see [KrS94]). *We conjecture that there are no* FCR-*algebras in GK-dimension 2.* At this time, we can only show that any *residually finite homomorphic image of an enveloping algebra* has GK-dimension ≥ 3.

2. Products and sums

Theorem 1. *Let $(A_i)_{i\in I}$ be a family of* FCR-*algebras over a field k. Assume that the cardinality of k is greater or equal than the cardinality of the index set I. Then the product $\prod_{i\in I} A_i$ is an* FCR-*algebra.*

Theorem 2. *Let $(A_i)_{i\in I}$ be a family of* FCR-*algebras over a field k. Then the sum $k \oplus \bigoplus_{i\in I} A_i$ is an* FCR-*algebra.*

(We add a copy of k and define multiplication by $(\lambda, a)\cdot(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab)$ so that we have again a k-algebra with a unit. Clearly, $k\oplus\bigoplus_{i\in I} A_i \subset \prod_{i\in I} A_i$ is a (unital) subalgebra in a natural way.)

It is clear that both algebras are residually finite dimensional, since the intersection of the kernels of the projection homomorphisms

$$\mathrm{pr}_j \colon A := \prod_{i\in I} A_i \to A_j$$

and of the homomorphisms

$$p_j \colon A := k \oplus \bigoplus_{i\in I} A_i \to A_j,\ (\lambda, a) \mapsto \lambda + a_j \quad \text{and} \quad \mathrm{pr}\colon A := k \oplus \bigoplus_{i\in I} A_i \to k$$

is zero in both cases. So the main point is to prove that every finite dimensional representation is completely reducible. We will do this by showing that for every surjective k-algebra homomorphism $A \to B$ where B is a finite dimensional algebra, there is a finite subproduct $\prod_{\text{finite}} A_{i_\nu} \subset A$ which maps surjectively onto B. This reduces the proof to the following result.

Proposition 1. *A finite product of* FCR*-algebras is* FCR.

Proof. This is clear: Every homomorphism $\prod_{i=1}^n A_i \to B$ onto a finite dimensional algebra B factors through a product $\prod_{i=1}^n A_i/\mathfrak{a}_i$ where the algebras $A_i/\mathfrak{a}_i$ are finite dimensional and hence semisimple. □

Proof of Theorem 2. Let $\varphi\colon A := k \oplus \bigoplus_{i\in I} A_i \to B$ be a homomorphism onto a finite dimensional algebra B. Lifting a k-basis of B to A we see that there is a finite sum $k \oplus \bigoplus_{\text{finite}} A_{i_\nu}$ which contains all these representatives. Therefore, $\varphi(k \oplus \bigoplus_{\text{finite}} A_{i_\nu}) = B$. As a k-algebra, the sum $k \oplus \bigoplus_{\text{finite}} A_{i_\nu}$ is isomorphic to $k \times \prod_{\text{finite}} A_{i_\nu}$, hence is FCR by Proposition 1. □

For the proof of the first theorem we need some preparation.

Lemma 1. *Let k be a field and consider the product $C := \prod_I k$ where the index set I has a cardinality less or equal to the cardinality of k. Then every k-homomorphism $C \to B$ into a finite dimensional k-algebra B factors through a projection onto a finite product $\prod_I k \to \prod_{\text{finite}} k$. In particular, the image of C in B is a finite product $k \times k \times \cdots \times k$.*

Proof. For any subset $J \subset I$ we denote by e_J the idempotent defined by

$$(e_J)_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

We have to show that the kernel of any k-homomorphism $\varphi\colon \prod_I k \to B$, B finite dimensional, contains an idempotent e_J where $J \subset I$ is cofinite. In fact, this means that the kernel contains $Ce_J = \prod_{j\in J} k$, hence the homomorphism factors through $C/Ce_J = \prod_{i\in I\setminus J} k$.

Because of the assumption we can choose an element $a \in C$ such that all components a_i are distinct. The image $b := \varphi(a) \in B$ satisfies an equation $\sum_{s=0}^n \lambda_s b^s = 0$ where $\lambda_s \in k$ and $\lambda_n = 1$. The polynomial $p(x) := \sum_{s=0}^n \lambda_s x^s$ has only finitely many zeros in k. Therefore, $p(a_i) = 0$ for only finitely many $i \in I$. This implies that the support J of the element $p(a) \in \ker\varphi$ is cofinite. (The *support* of an element $c \in C$ is defined to be the subset $\{i \in I \mid c_i \neq 0\}$.) Therefore, the kernel contains $Cp(a) = Ce_J$ and we are done. □

Proof of Theorem 1. Let $\varphi\colon \prod_{i\in I} A_i \to B$ be a surjective homomorphism onto a finite dimensional k-algebra B. We have $\prod_I k \subset \prod_{i\in I} A_i$ in a natural way. By our Lemma 1 above there is an idempotent $e_J \in \ker\varphi \cap \prod_I k$ with cofinite $J \subset I$. Hence, the homomorphism φ factors through a finite product

$$\varphi\colon \prod_{i\in I} A_i \to \prod_{\text{finite}} A_{i_\nu} \to B$$

and we are done by Proposition 1. □

Remark 2. We do not know if Theorem 1 holds without the assumption on the cardinality of the field k. However, our argument breaks down as shown by the following example. Consider the product $A := \prod_{\mathbb{N}} \mathbb{F}_2$. The maximal ideals in this algebra correspond bijectively to the ultrafilters of $\mathbb{N}$. It is well-known that there are many ultrafilters which are different from the *principal* filters (i.e. those filters consisting of all subsets containing a given element $i \in \mathbb{N}$, corresponding to the maximal ideal $\prod_{\mathbb{N}\setminus\{i\}} \mathbb{F}_2$.). However, we have $A/M = \mathbb{F}_2$ for every maximal ideal M of A.

On the other hand we will see later that *every product of matrix rings over division rings is* FCR (section 3, Corollary 1).

3. A characterization of FCR-algebras

Theorem 3. *Let A be a k-algebra where k is an arbitrary field. The following assertions are equivalent:*

(i) *Every finite dimensional representation of A is completely reducible.*
(ii) *For every two-sided ideal $I \subset A$ of finite codimension we have $I^2 = I$.*
(iii) *If M_1, M_2 are two-sided maximal ideals of finite codimension then we have $M_1 \cap M_2 = M_1 M_2 = M_2 M_1$.*

(The last statement includes the condition $M^2 = M$ for every two-sided maximal ideal of finite codimension.)

Proof. (i) $\Rightarrow$ (ii): Let $I \subset A$ be a two-sided ideal of finite codimension. Then A/I is a finite dimensional semisimple algebra and I/I^2 is a semisimple A/I-module. If I where different from I^2 then we can find an A-submodule $K/I^2 \subset I/I^2$ of finite codimension such that the quotient $I/(K+I^2)$ is non-trivial. If follows that $M := A/(I^2+K)$ is finite dimensional. Therefore, we have an exact sequence

$$0 \to I/K \to M \to A/I \to 0$$

of finite dimensional A-modules. By assumption, these modules are semisimple and thus the sequence splits. Hence M is annihilated by I, and so $I/(I^2+K) = IM = 0$, in contradiction to the choice of K. Thus we have $I = I^2$.

(ii) $\Rightarrow$ (iii): Put $I := M_1 \cap M_2$. Then I has finite codimension and we obtain $I^2 \subset M_1M_2 \cap M_2M_1 \subset I$. Since $I^2 = I$ by assumption, the claim follows.

(iii) $\Rightarrow$ (i): Let $J \subset A$ be a two-sided ideal of finite codimension. We have to show that the algebra A/J is semisimple. Since A/J is finite dimensional there are only finitely many two-sided maximal ideals $M_1, M_2, \ldots, M_s$ containing J, and

$$(M_1 \cap M_2 \cap \ldots \cap M_s)/J \subset A/J$$

is the nilradical of A/J. From the assumption we obtain, by an easy induction, that

$$D := M_1 \cap M_2 \cap \ldots \cap M_s = M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(s)}$$

for any permutation σ of $\{1, 2, \ldots, s\}$. Therefore, $D^2 = D$ and so the nilradical D/I is trivial. □

We will now give an interpretation of statement (iii) in the Theorem above. Let us first look at an example.

Example 1. Consider the k-algebra $B := \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ and the two two-sided maximal ideals $M_1 := \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$ and $M_2 := \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$. We find

$$M_1 M_2 = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} = M_1 \cap M_2 \quad \text{and} \quad M_2 M_1 = (0)$$

On the other hand, there is an exact non-split sequence of (left) A-modules

$$0 \to A/M_2 \to \begin{bmatrix} k \\ k \end{bmatrix} \to A/M_1 \to 0$$

whereas $\mathrm{Ext}^1_A(A/M_2, A/M_1) = 0$. This is a general fact as seen from the following lemma.

Lemma 2. *Let A be a k-algebra and P_1, P_2 two simple A-modules. Put $M_i := \mathrm{Ann}_A P_i$. Then we have*

$$\mathrm{Ext}^1_A(P_2, P_1) = 0 \qquad \Longleftrightarrow \qquad M_1 M_2 = M_1 \cap M_2.$$

Outline of Proof. Given any exact sequence $0 \to P_1 \to P \to P_2 \to 0$ of A-modules we see that P is annihilated by the product $M_1 M_2$. So if $M_1 M_2 = M_1 \cap M_2$ we obtain a surjective homomorphism $A/M_1 \cap M_2 = A/M_1 \oplus A/M_2 \to P$ which implies that P is semisimple.

On the other hand, every module P annihilated by $M_1 M_2$ admits an exact sequence $0 \to M_2 P \to P \to P/M_2 P \to 0$. Hence P is an extension between an A/M_2-module $P/M_2 P$ and an A/M_1-module $M_2 P$. So if every such sequence splits, it follows that P is annihilated by $M_1 \cap M_2$ and so $M_1 M_2 = M_1 \cap M_2$. □

As an application of Theorem 3 we will show that all finite dimensional representations of a C^*-algebra are completely reducible.

Proposition 2. *Let A be a C^*-algebra with unit. Then every two-sided ideal I of finite codimension is closed, $*$-invariant and satisfies $I^2 = I$. In particular, every finite dimensional representation of A is completely reducible.*

Proof. (a) We start with a general remark about ideals in C^*-algebras. If $I \subset A$ is a closed two-sided ideal then I is automatically $*$-invariant by Proposition 1.8.2 of [Dix64]. It follows that I is generated by *hermitian* elements, i.e. those $x \in I$ which satisfy $x^* = x$. Moreover, every hermitian $x \in I$ can be written in the form $x = x^+ - x^-$ where both x^+ and x^- have a positive spectrum ([Dix64, **?**]) and belong to I. (We regard I itself as a C^*-algebra.) Now we can use Proposition 1.6.1 of [Dix64] and conclude that $x = a^2 - b^2$ where $a, b \in I$. (In particular, we have $I^2 = I$.)

(b) Let M be a two-sided maximal ideal of A. Then M is closed: It's closure is again a two-sided ideal and cannot contain the unit element 1 of A, because every element of the form $1 + x$ is invertible in A for $||x|| < 1$.

Now let M_1, M_2 be two two-sided maximal ideals in A. Then $M_1 \cap M_2$ is closed and therefore generated by elements of the form $x = a^2 - b^2$ where $a, b \in M_1 \cap M_2$ by (a). But this implies that $M_1 \cap M_2 \subset M_1 M_2$. Thus statement (iii) of Theorem 3 is satisfied and the claim follows. □

Let us recall here that a von Neumann *regular algebra* A is defined to be a k-algebra which satisfies the following condition: *For every element* $a \in A$ *there is an* $x \in A$ *such that* $a = axa$ (see [La91] Theorem 4.23). This implies that $I = I^2$ for every (left or right) ideal. Thus we obtain the following results.

Proposition 3. *If A is a* von Neumann *regular algebra then every finite dimensional representation of A is completely reducible.*

Corollary 1. *An arbitrary product $A = \prod_{i \in I} A_i$ of* von Neumann *regular algebras A_i is again* von Neumann *regular and so every finite dimensional representation of A is completely reducible. In particular, every product $\prod_{i \in I} M_{n_i}(K_i)$ of matrix rings over division rings K_i is* FCR.

(Cf. section 2, Remark 2)

4. Localization

We add some results about the localization of an FCR-algebra R with respect to a (right) Ore set $S \subset R$. (For definitions and basic properties of non-commutative localization we refer to [La98, Chap. 4]. Since we do not exclude 0-divisors in S we always assume that S satisfies the following additional condition: If $sr = 0$ for some $s \in S$ and $r \in R$ then there is an $s' \in S$ such that $rs' = 0$.)

Proposition 4. *Let R be an* FCR*-algebra, $S \subset R$ be a right Ore set and R_S the corresponding localization.*

(1) *Every finite-dimensional representation of R_S is completely reducible.*
(2) *If $J \subset R_S$ is a two-sided ideal of finite codimension and $I := J \cap R$ then $J = IR_S$ and $R/I \xrightarrow{\simeq} R_S/J$.*

(3) *If S is generated by finitely many normalizing elements (i.e. elements s satisfying $sR = Rs$), then R_S is* FCR*. In this case, the two-sided maximal ideals of finite codimension of R_S are of the form $M' = MR_S = R_S M$ where M is a two-sided maximal ideal of R of finite codimension which does not contain s.*

Proof. Let $J \subset R_S$ be a two-sided ideal of finite codimension. Then $I := J \cap R$ has finite codimension in R, hence $I^2 = I$ by assumption. Moreover, $J = IR_S$ and so $J^2 = (IR_S)(IR_S) \supset I^2 R_S = IR_S = J$, proving (1) by Theorem 3. Moreover, $R/I \subset R_S/J$ and $R_S/J = (R/I)_{\bar{S}}$ where $\bar{S}$ is the image of S in R/I. Now Lemma 3 below shows that $R/I \xrightarrow{\simeq} R_S/J$, proving (2).

For (3) we can assume, by induction, that $S = \{s^i \mid i = 0, 1, 2, \dots\}$. We already know from (1) that every finite dimensional representation of R_S is completely reducible. It remains so show that the intersection in R_S of the maximal two-sided ideals of finite codimension is zero. Put

$$\mathcal{M} := \{M \subset R \mid M \text{ a two-sided maximal ideal of finite codim}, s \notin M\}.$$

For every $M \in \mathcal{M}$ we have $R_S M R_S = R_S M = M R_S$. In fact, given $m \in M$ and $i \geq 0$ we have $s^i m = m' s^i$ and $m s^i = s^i m''$ for suitable $m', m'' \in R$, because s is normalizing. Since the image of s in R/M is non-zero and normalizing, we see that s is invertible modulo M and so $m', m'' \in M$. This implies that $m s^{-i} \in R_S M$ and $s^{-i} m \in M R_S$ which proves the claim. Moreover, it follows that $(R/M)_{\bar{S}} = R/M$ where $\bar{S}$ is the image of S in R/M, and so $R/M \xrightarrow{\simeq} R_S/R_S M R_S$.

We now claim that $\bigcap_{M \in \mathcal{M}} R_S M = (0)$. Let $a = r s^{-i}$ be an arbitrary element in $\bigcap_{M \in \mathcal{M}} R_S M$. Then $r \in R \cap (\bigcap_{M \in \mathcal{M}} R_S M) = \bigcap_{M \in \mathcal{M}} M$. Putting

$$\mathcal{M}' := \{M \subset R \mid M \text{ a two-sided maximal ideal of finite codim}, s \in M\}$$

we have, by assumption,

$$\bigcap_{M \in \mathcal{M}} M \cap \bigcap_{M \in \mathcal{M}'} M = (0).$$

Since $s \in \bigcap_{M \in \mathcal{M}'} M$ we see that $rs = 0$ for every $r \in \bigcap_{M \in \mathcal{M}} M$. Hence $a = r s^{-i} = (rs) s^{-i-1} = 0$. This completes our proof. □

Lemma 3. *Let A be a left or right Artinian algebra and $S \subset A$ an Ore set. Then the canonical map $A \to A_S$ is surjective.*

Proof. This is clear since a non 0-divisor in an Artinian algebra is invertible. □

Example 2. The assumptions of part (3) of Proposition 4 are necessary as shown be the following examples. Let $\mathfrak{g}$ be a simple (complex) Lie algebra and $\mathfrak{U}(\mathfrak{g})$ its enveloping algebra. If we choose for S the set of all non-zero central elements then S consists of normalizing elements, but $\mathfrak{U}(\mathfrak{g})_S$ is a simple ring, hence not FCR.

Putting $S = \{s^i \mid i = 0, 1, 2 \dots\}$ where s a nilpotent element of $\mathfrak{g}$ then S is an Ore set. (This was first observed by Lepowsky and follows from the fact that $\mathrm{ad}(s)$ is locally nilpotent in $\mathfrak{U}(\mathfrak{g})$.) But $\mathfrak{U}(\mathfrak{g})_S$ has no finite dimensional representation and so $\mathfrak{U}(\mathfrak{g})_S$ is not FCR.

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