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# Stable Marriages and Search Frictions*

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**Abstract:** We embed a two-sided matching market with non-transferable utility, a marriage market, into a random search model. We study steady-state equilibria and characterize the limit of the corresponding equilibrium matchings as exogenous search frictions become small. The central question is whether the set of such limit matchings coincides with the set of stable matchings for the underlying marriage market. We show that this is the case if and only if there is a unique stable matching. Otherwise, the set of limit matchings contains the set of all stable deterministic matchings, but also contains unstable random matchings. These unstable random matchings are Pareto dominated. Thus, vanishing frictions do not guarantee the efficiency of decentralized marriage markets.

**Keywords:** Marriage Market, Stable Matchings, Random Matchings, Search Frictions.

***JEL* Classification Numbers:** C78, D83.

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# 1 Introduction

This paper considers a two-sided matching market. Agents on each side of the market match with at most one agent from the other side of the market, and agents cannot make transfers. Following standard practice, we refer to such a market as a *marriage market* and use the corresponding terminology (Roth and Sotomayor, 1990). The central theoretical problem in a marriage market is to determine who will match with whom. The concept of stable matchings provides an answer (Gale and Shapley, 1962): A matching is stable if no matched agent prefers to be single and no pair of agents prefers each other to their assignment in the matching.

Most of the recent literature on matching considers questions of economic design in centralized markets. However, many of those markets that are analyzed using stability concepts, including the marriage market in its literal sense, are decentralized. Roth and Sotomayor (1990, p.22) conjecture that even in a decentralized market, we might expect matchings to be stable if frictions are negligible in the sense that "the agents have a very good idea of one another's preferences and have easy access to each other." The purpose of this paper is to test this conjecture. In particular, we assume complete information about preferences and ask whether such "easy access" to potential partners implies the approximate stability of matchings.

To investigate this question, we embed a marriage market in a search model with random meetings. In the underlying marriage market, men and women have strict preferences over mates and staying single; no further restrictions on preferences are imposed. In the search model, the rate at which men and women meet one another is determined by the size of the population of agents searching for a potential partner according to a continuous contact function. If a man and a woman meet, they decide whether to accept each other. If both accept, the agents leave as a matched pair. Otherwise, both continue searching. The opportunity cost of rejecting a partner and waiting for a better match is an exogenous risk that an agent will have abandon the search and remain single. Exogenously arriving unmatched men and women keep the stock of agents who are searching for partners from depleting. We study the matchings that result from steady-state equilibria. Due to the randomness inherent in the contact process and the possibility of exogenous exit from the market while still single, the equilibrium matchings arising in our model are random, that is, they correspond to lotteries over deterministic matchings. How difficult it is to access potential partners depends on the speed of the contact process. Thus, the speed of the contact process is a measure of frictions.

For any given speed of the contact process, all agents stay single with a strictly positive probability, precluding the stability of equilibrium matchings.

To investigate whether equilibrium matchings approximate stable matchings when search frictions are small, we thus study the limits of equilibrium matchings when the contact process becomes infinitely fast. Our first proposition provides a complete characterization of such limit matchings in terms of the "fundamentals" of the underlying marriage market. For deterministic matchings, it is immediate from the characterization that such a matching is a limit matching if and only if it is stable. This conclusion, however, does not extend to random limit matchings. Random limit matchings fail to satisfy a weak stability requirement (implied by all existing stability notions for random matchings) and are, in addition, Pareto dominated.

To obtain firm conclusions about the stability of limit matchings, we need, therefore, to address the existence of random limit matchings. Our second proposition shows that such matchings exist if and only if there is more than one stable matching in the underlying marriage market. The lattice-structure of the set of stable deterministic matchings is key to the proof of this result.

Taken together, our results imply a clean and somewhat unexpected dichotomy between those marriage markets that have a unique stable matching and those that do not. In the first case, all equilibrium matchings are approximately stable when frictions are small. In the second case, vanishing frictions do not imply approximate stability.

Three key features differentiate our model and results from related contributions investigating the limits of steady-state equilibria in search models for vanishing frictions.

First, utility is nontransferable. This is in contrast to the literature on convergence to competitive equilibria in dynamic matching and bargaining games, surveyed in Osborne and Rubinstein (1990) and Gale (2000). This literature may be viewed (see Lauermann, 2012) as investigating the convergence to stable outcomes in a simple version of the assignment problem (Shapley and Shubik, 1971) in which utility is assumed to be transferable.

The critical property of matching with nontransferable utility is that agents may disagree about whether or not to form a match.[1] This inherent potential for disagreement implied by nontransferability plays an essential role in our construction of sequences of equilibrium matchings converging to random limit matchings. In particular, we exploit the fact that if, say, a woman strictly prefers to match with the man she has met and the man is indifferent as to either accepting the match or continuing to search, there is no inducement the

[1]In contrast, if transfers are possible, agents always agree whether a match should be formed, because the available surplus from the match is either positive or not. Smith (2006) uses the observation that disagreements about matchings in social settings are common as an argument for the nontransferable utility model.

woman can offer to the man to break his indifference.[2]

Second, the composition and size of the stock of searching agents are endogenously determined by agents' acceptance decisions and, in addition, depend on the speed of the contact process. Consequently, whether an agent has "easy access" to partners he or she finds attractive depends not only on the search frictions per se, but also on the equilibrium behavior of all other agents and the properties of the contact function. Taking these effects into account is not only of technical but also of economic interest.[3] Doing so also differentiates our work from the two existing papers (Eeckhout, 1999; Adachi, 2003) that investigate convergence to stable matchings in frictional search models with nontransferable utility. Both of these papers assume that the distribution of searching agents is exogenous to simplify their analysis. Indeed, dealing with the endogeneity of the stock is one of the main challenges in proving our characterization of limit matchings.

Third, we consider the general version of the marriage model as introduced in Gale and Shapley (1962), in which multiplicity of stable matchings is a common occurrence. This is in contrast to most of the literature studying frictional matching with nontransferable utility in which it is assumed that agents agree on the ranking of their potential partners.[4] In such models with vertical heterogeneity, there is a unique stable matching featuring positive assortative mating (Becker, 1973).[5] Our analysis not only shows that in models with vertical heterogeneity the convergence to a stable outcome is assured, but also demonstrates that this result is not robust. Clearly, to reach the latter conclusion, it is essential to consider more general preference structures.

The papers most closely related to ours are Eeckhout (1999) and Adachi (2003), because they also consider the relationship between stable matchings in a marriage market and the equilibrium matchings in a search model when

[2]Such indifferences may arise in equilibrium despite our assumption that preferences are strict because continuation payoffs are determined endogenously to equilibrate the market.

[3]Burdett and Coles (1997) and Shimer and Smith (2000) discuss the importance of taking into account the feedback effect from agents' decision to the steady-state distribution of types. In the context of the literature on convergence to competitive equilibria, Gale (1987) has pioneered the investigation of models in which the stock of searching agents is endogenously determined and offers extensive discussion. See Lauermann (2012) for further discussion.

[4]Burdett and Coles (1999) and Smith (2011) survey the literature. Notable contributions include McNamara and Collins (1990); Johnstone, Reynolds, and Deutsch (1996); Burdett and Coles (1997); Morgan (1998); Eeckhout (1999); Bloch and Ryder (2000); Smith (2006). Burdett and Wright (1998) and Burdett, Imai, and Wright (2004) also study search models with nontransferable utility, but consider a model with match-specific random shocks in which all agents on one side of the market are identical ex-ante.

[5]See Eeckhout (2000) and Clark (2006) for more general conditions ensuring uniqueness of stable matchings.

frictions become negligible. Both of these papers find convergence to stable matchings. This is consistent with our findings: The underlying marriage market in Eeckhout (1999) has a unique stable matching, and Adachi (2003), who considers general preferences as we do, eliminates random limit matchings from consideration by requiring an agent to accept a partner whenever the agent is indifferent as to accepting the partner or continuing to search. As we have already noted, Eeckhout (1999) and Adachi (2003) assume an exogenous stock of agents, whereas the stock is endogenous in our model.

Section 2 introduces the marriage market model, discusses stable matchings, and introduces the notions of coherent and regret-free matchings that play a key role in our analysis. Section 3 embeds the marriage market in a search model and notes some properties of equilibrium matchings. We present our characterization of limit matchings in Section 4. Section 5 concludes.

# 2 Marriage Market

## 2.1 The Model

There are two finite, disjoint sets of agents: the set $M$ of men and the set $W$ of women. Each agent has a complete, transitive, and strict preference ordering over the set of agents on the other side of the market and the prospect of remaining single. The extension of this preference to the set of corresponding lotteries has an expected utility representation. Hence, we represent the preferences of men by utility functions $u : M \times W \to \mathbb{R}$ and the preferences of women by utility functions $v : M \times W \to \mathbb{R}$ with the normalization that the utility of remaining single is 0 for each agent. As preferences are strict, we have $u(m, w) \neq 0$ and $v(m, w) \neq 0$ for all $(m, w) \in M \times W$ as well as $u(m, w) \neq u(m, w')$ and $v(m, w) \neq v(m', w)$ for all $(m, w) \in M \times W$, $w' \neq w \in W$, and $m' \neq m \in M$. To avoid trivialities, we assume that there is at least one pair of agents $(m, w)$ such that $(m, w)$ find each other mutually acceptable, that is, $u(m, w) > 0$ and $v(m, w) > 0$ holds. It is convenient to define $u(h, h) := 0$ for all $h \in M \cup W$.

We refer to a tuple $(M, W; u, v)$ satisfying the above assumptions as a *marriage market.*

A *deterministic matching* is given by a *matching function* $\mu : M \cup W \to M \cup W$ satisfying $\mu(m) \in W \cup \{m\}$, $\mu(w) \in M \cup \{w\}$, and $\mu^2(h) = h$ for all $m \in M$, $w \in W$, and $h \in M \cup W$. A *matching* (without the qualification) is a probability distribution over deterministic matchings. That is, matchings correspond to lotteries over deterministic matchings. We refer to non-deterministic matchings as *random matchings*.[6] Random Matchings play a key role in our analysis because search frictions naturally imply that random events play a decisive role in the process of partnership formation.

Throughout the following we identify each matching with its *assignment matrix* $x \in \mathbb{R}^{|M|\times|W|}$, where $x(m,w)$ is the probability that a match between man $m$ and woman $w$ is formed. The Birkhoff-von Neumann theorem[7] shows that $x$ represents a matching if and only if it satisfies

$$\sum_{w\in W} x(m,w) \leq 1 \text{ for all } m \in M, \tag{1}$$

$$\sum_{m\in M} x(m,w) \leq 1 \text{ for all } w \in W, \tag{2}$$

$$x(m,w) \geq 0 \text{ for all } (m,w) \in M \times W. \tag{3}$$

We use $x(m,m) := 1 - \sum_{w\in W} x(m,w)$ to denote the probability that man $m$ remains single in a matching; $x(w,w) := 1 - \sum_{m\in M} x(m,w)$ is the probability that woman $w$ remains unmatched. Observe that a matching is deterministic if and only if $x(m,w) \in \{0,1\}$ holds for all $(m,w) \in M \times W$; all other matchings are random. Given any matching $x$ we say that $h \in M \cup W$ is *fully matched* if $x(h,h) = 0$, *partially matched* if $0 < x(h,h) < 1$, and *unmatched* if $x(h,h) = 1$ holds. Types $(m,w) \in M \times W$ are *partners* if $x(m,w) > 0$ holds.

With every matching we associate the payoff vectors $U(x) \in \mathbb{R}^{|M|}$ and $V(x) \in \mathbb{R}^{|W|}$ given by

$$U(m;x) = \sum_{w\in W} x(m,w)u(m,w), \tag{4}$$

$$V(w;x) = \sum_{m\in M} x(m,w)v(m,w). \tag{5}$$

Recall that we have normalized the utility from staying single to zero. Hence, these payoffs correspond to the expected utility of the matching $x$.

## 2.2 Stable Matchings

A deterministic matching is *individually rational* if $u(m,\mu(m)) \geq 0$ holds for all $m$ and $v(\mu(w),w) \geq 0$ holds for all $w$. It is *pairwise stable* if there does not exist $(m,w) \in M \times W$ such that $u(m,w) > u(m,\mu(m))$ and $v(m,w) > v(\mu(w),w)$

[6]Random matchings are studied by Vande Vate (1989); Rothblum (1992); Roth, Rothblum, and Vande Vate (1993); Kesten and Ünver (2010); Echenique, Lee, Shum, and Yenmez (2011).

[7]See Budish, Che, Kojima, and Milgrom (2011) for an extensive discussion in the context of assignment problems.

holds. It is *stable* if it is individually rational and pairwise stable. Every marriage market has a stable deterministic matching (Gale and Shapley, 1962). Our assumption that there is a mutually acceptable pair implies that in every stable deterministic matching $\mu(h) \neq h$ holds for some $h \in M \cup W$.

The standard extension of stability to random matchings is the one introduced in Vande Vate (1989). For our purposes, it is convenient to work with an equivalent definition, which is formulated in terms of the assignment matrix: we extend the stability definition for deterministic matchings by saying that a matching $x$ is stable if and only if there exist stable deterministic matchings $x^1, \ldots, x^k$ such that $x$ is a convex combination of $x^1, \ldots, x^k$.[8] That is, stable matchings can be obtained as lotteries over stable deterministic matchings. Thus, the stability of a matching has a natural interpretation as an ex-post stability requirement.[9] The following observation about stable matchings proves useful below.

**Lemma 1.** *In a stable matching there are no partially matched agents.*

*Proof.* The set of agents who are unmatched is the same for all stable deterministic matches (Roth and Sotomayor, 1990, Theorem 2.22). As stable matchings are convex combinations of stable deterministic matchings, every stable matching thus satisfies $x(h,h) \in \{0,1\}$ for all $h \in M \cup W$. Hence, there are no partially matched agents in a stable matching. $\square$

### 2.3 Coherent Matchings

We extend the definitions of individual rationality and pairwise stability from deterministic to all matchings by saying that a matching $x$ is *individually rational* if

$$x(m,w) > 0 \Rightarrow u(m,w) \geq 0 \text{ and } v(m,w) \geq 0 \tag{6}$$

holds and that it is *pairwise stable* if there does not exist $(m,w) \in M \times W$ such that

$$u(m,w) > U(m;x) \text{ and } v(m,w) > V(w;x) \tag{7}$$

holds. We refer to a matching satisfying both (6) and (7) as *coherent.*

Observe that for deterministic matchings $x$ we have $U(m;x) = u(m,\mu(m))$ and $V(w;x) = v(\mu(w),w)$, so that for deterministic matchings the above definitions of individual rationality and pairwise stability are equivalent to the

[8]See Roth, Rothblum, and Vande Vate (1993) for the equivalence between this definition and the one given in Vande Vate (1989).

[9]As argued in Echenique, Lee, Shum, and Yenmez (2011), from an ex-ante perspective the notion of a strongly-stable matching from Roth, Rothblum, and Vande Vate (1993) is more appealing. All our results continue to hold when using this alternative definition for the stability of matchings.

ones given previously. In particular, a deterministic matching is stable if and only if it is coherent.

Stability of a random matching is, however, *not* equivalent to (6) and (7) being satisfied. In fact, coherency is neither necessary nor sufficient for stability of a random matching. First, in contrast to a stable random matching, a coherent random matching may have partially matched agents. In particular, condition (7) allows for the possibility that there exists a man and a woman who are both unmatched with strictly positive probability even though they are mutually acceptable. Second, (7) is an ex-ante rather than an ex-post condition. In particular, not every stable random matching is coherent. As these points are of central importance for our subsequent analysis, we illustrate them with the following two simple examples (to which we return when discussing our main results).

EXAMPLE 1: Consider a marriage market with two men, $M = \{m_1, m_2\}$, and two women, $W = \{w_1, w_2\}$. Preferences are described by the bi-matrix

| | $w_1$ | $w_2$ |
|---|---|---|
| $m_1$ | $2,1$ | $1,2$ |
| $m_2$ | $1,2$ | $2,1$ |

where the first entry in the cell corresponding to $(m_i, w_j)$ is the payoff $u(m_i, w_j)$ and the second entry is the payoff $v(m_i, w_j)$. In this example, all matchings are individually rational. There are two stable deterministic matchings given by the assignment matrices

$$x_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The set of stable random matchings is given by the set of all strict convex combinations of $x_1$ and $x_2$; that is, a random matching is stable if and only if it can be written as $x = ax_1 + bx_2$ with $a > 0$, $b > 0$ and $a + b = 1$. In this example, stability of a random matching implies its coherence, but many additional random matchings are coherent. For instance, it is easy to verify that matchings of the form $ax_1 + bx_2$ are coherent if and only if $a \geq 0$, $b \geq 0$, $2a + b \geq 1$, and $a + 2b \geq 1$ are satisfied. □

EXAMPLE 2: Consider a marriage market with three men, $M = \{m_1, m_2, m_3\}$, and three women, $W = \{w_1, w_2, w_3\}$. Preferences are described by the bi-matrix

| | $w_1$ | $w_2$ | $w_3$ |
|---|---|---|---|
| $m_1$ | $7,3$ | $6,6$ | $3,7$ |
| $m_2$ | $3,7$ | $7,3$ | $6,6$ |
| $m_3$ | $6,6$ | $3,7$ | $7,3$ |

As in the previous example, all matchings are individually rational. There are three stable deterministic matchings given by

$$x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In contrast to the situation in Example 1, not all stable random matchings are coherent. Consider, for instance, the stable random matching $\hat{x} = 0.5x_1 + 0.5x_3$. In this matching, we have $U(m_i; \hat{x}) = V(w_j; \hat{x}) = 5$ for $i, j = 1, 2$. As $u(m_1, w_2) = v(m_1, w_2) = 6 > 5$, this matching is not coherent. □

## 2.4 Regret-Free Matchings

We say that a matching is *regret-free* if it has the property that any pair of agents who are matched with a strictly positive probability in $x$ weakly prefer to match with each other, rather than participating in the lottery resulting in the matching $x$. Formally, the requirement is:

$$x(m, w) > 0 \Rightarrow u(m, w) \geq U(m; x) \text{ and } v(m, w) \geq V(m; x). \tag{8}$$

Obviously, every deterministic matching is regret-free (as the two inequalities in (8) hold as equalities if $x(m, w) = 1$). For random matchings, the situation is rather different, and the property of regret-free random matchings identified in the following lemma plays an important role in our subsequent analysis.

**Lemma 2.** *Let $x$ be a regret-free matching. Then every fully matched agent has a unique partner. In particular, $0 < x(m, w) < 1$ implies that $m$ and $w$ are partially matched.*

*Proof.* Suppose $x$ satisfies (8). Assume $m \in M$ is fully matched. Using $x(m, m) = 0$ and (4) we have

$$\sum_{w \in W} x(m, w) \left[u(m, w) - U(m; x)\right] = 0.$$

Because $x$ satisfies (8), all the summands on the left side of this equality are positive. Hence, all summands are equal to zero. Consequently, $x(m, w) = 0$ holds for all $w$ satisfying $u(m, w) - U(m; x) \neq 0$. Because preferences are strict, it follows that there is $w \in W$ satisfying $x(m, w) = 1$. Hence, $m$ has a unique partner. An analogous argument shows that every fully matched $w \in W$ has a unique partner. Consequently, $0 < x(m, w) < 1$ implies $0 < x(m, m) < 1$ as well as $0 < x(w, w) < 1$ (because neither $m$ nor $w$ can be unmatched or fully matched if $0 < x(m, w) < 1$ holds). □

The intuition behind the result in Lemma 2 is simple: if every partner of an agent provides a payoff at least as high as the average payoff an agent receives, then it must either be the case that the agent only has one partner or it must be the case that there is a strictly positive probability that the agent remains unmatched. After all, in contrast to the children in Lake Wobegon, not all partners can be better than average.

# 3 Search

## 3.1 The Model

We embed a marriage market $(M, W, u, v)$ in a continuous-time search model with a continuum of agents, similar to the search model studied by Burdett and Coles (1997).[10] We interpret $M$ and $W$ as the set of possible types for men and women, respectively. For each $m \in M$ and $w \in W$, new agents of the corresponding type are born at a constant flow rate that is equal to $\eta > 0$. Newborn agents start searching for partners and continue doing so until they either match with an agent of the opposite sex or exit the search process without having found a partner. Agents exit as unmatched singles at an exogenous rate $\delta > 0$.[11] There is no explicit discounting. Therefore, the expected payoff of an agent of type $m \in M$ is given as specified in (4) when $x(m, w)$ is the probability that he exits the search process in a match with an agent of type $w$. Similarly, (5) is the expected payoff of an agent of type $w \in W$ when $x(m, w)$ is the probability that she exits in a match with an agent of type $m$.

Let $f(m) > 0$ denote the mass of men of type $m$ searching for a partner and let $f$ denote the corresponding vector of masses. Define $\bar{f} = \sum_{m \in M} f(m)$. Similarly, let $g(w) > 0$ denote the mass of women of type $w$ searching for a partner, let $g$ be the corresponding vector, and define $\bar{g} = \sum_{w \in W} g(w)$. We suppress time indices because we consider steady states.

[10] In particular, our model of entry and exit is the one from Burdett and Coles (1997). As noted by Eeckhout (1999), assuming that there is (i) a fixed population of infinitely lived agents who (ii) exit the search process only if they have found a partner and (iii) enter the search process when their partnership dissolves due to exogenous separation, yields identical steady-state conditions (cf. Shimer and Smith, 2000; Smith, 2006).

[11] On an individual level, the exit rate $\delta$ acts like a discount rate and is a convenient way to model the opportunity costs of waiting for a better match. On the aggregate level, a strictly positive exit rate ensures that for every stationary strategy profile (as modeled by the matching probabilities that we introduce below) there is a corresponding steady-state. Therefore, the restriction to a steady-state does not by itself introduce restrictions on the set of admissible strategy profiles beyond stationary.

The mass of meetings between men and women that occur per unit time is given by $\lambda \cdot C(\bar{f}, \bar{g})$. We refer to $C : \mathbb{R}^2_+ \to \mathbb{R}_+$ as the *contact function* and to the parameter $\lambda > 0$ as the *speed* of the contact process. We assume that the contact process is random in the sense that the fraction of meetings involving a man of type $m$ and a woman of type $w$ is given by $f(m) \cdot g(w)/(\bar{f} \cdot \bar{g})$, so that all men in the market meet women of type $w$ at rate

$$\gamma(w; f, g) = \lambda \cdot c(\bar{f}, \bar{g}) \cdot g(w) > 0 \tag{9}$$

and all women in the market meet men of type $m$ at rate

$$\phi(m; f, g) = \lambda \cdot c(\bar{f}, \bar{g}) \cdot f(m) > 0. \tag{10}$$

The function $c : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ appearing in these expressions is defined by

$$c(\bar{f}, \bar{g}) = \frac{C(\bar{f}, \bar{g})}{\bar{f} \cdot \bar{g}}. \tag{11}$$

Only minimal assumptions are imposed on the contact function: (i) $C$ is continuous and (ii) $C$ is strictly positive if and only if both of its arguments are strictly positive. This general specification of the contact process contains the matching functions commonly considered in the labor market search literature (Petrongolo and Pissarides, 2001; Stevens, 2007) as special cases, except for the linear matching function (Diamond and Maskin, 1979).[12]

If two agents meet, they observe each other's type and simultaneously decide whether to accept each other or not. If both accept, the agents leave the search process as a matched pair. If at least one agent rejects the match, both return to search. Let $\alpha(m, w)$ denote the probability that a meeting between a man of type $m$ and a woman of type $w$ results in a match. Let $\alpha \in [0, 1]^{|M| \times |W|}$ denote the corresponding matrix of *matching probabilities.*

A *steady state* is described by a tuple $(f, g, \alpha)$ satisfying

$$\eta = f(m) \left[ \delta + \sum_{w \in W} \alpha(m, w) \gamma(w; f, g) \right] \text{ for all } m \in M, \tag{12}$$

$$\eta = g(w) \left[ \delta + \sum_{m \in M} \alpha(m, w) \phi(m; f, g) \right] \text{ for all } w \in W. \tag{13}$$

The left side of these equations represents the inflow of newborn agents of a given type. The right side is the corresponding outflow of single and matched agents.

[12]We refer to $C$ as a contact function rather than as a matching function, because in our model not every meeting between agents needs to result in a match.

We say that the steady state $(f, g, \alpha)$ *induces* the matching $x$ given by

$$x(m, w) = \frac{\lambda}{\eta}\alpha(m, w)c(\bar{f}, \bar{g})f(m)g(w) \geq 0. \tag{14}$$

The right side of (14) is the probability that a man of type $m$ exits from search in a match with a woman of type $w$ in the steady state $(f, g, \alpha)$.[13] A matching is a *steady-state matching* if it is induced by some steady state.

Let $x$ be a steady-state matching. Summing (14) over all $w$ and using (9), we obtain $x(m, m) = 1 - f(m)\sum_{w \in W} \alpha(m, w)\gamma(w; f, g)/\eta$. Using (12), this can be rewritten as

$$x(m, m) = \frac{\delta}{\eta} f(m) > 0. \tag{15}$$

Similarly, we obtain

$$x(w, w) = \frac{\delta}{\eta} g(w) > 0. \tag{16}$$

The strict inequalities in (15) and (16) indicate that due to the frictions inherent in the matching process there are no fully matched agents in a steady-state matching. From Lemma 1, this implies that steady-state matchings are unstable.

## 3.2 Equilibrium

We now turn to the determination of the matching probabilities $\alpha$, which so far we have treated as exogenous. As the consent of both agents in a meeting is needed to form a match, we require that $\alpha(m, w) = 0$ holds whenever one of the partners obtains a strictly higher expected payoff from continuing to search rather than matching. For equilibrium, we also require that $\alpha(m, w) = 1$ holds if the match provides both partners with a utility strictly higher than the utility from continued search.

[13] To see that, observe that (9) and (12) imply

$$\frac{\lambda}{\eta}\alpha(m, w)c(\bar{f}, \bar{g})f(m)g(w) = \frac{\alpha(m, w)\gamma(w; f, g)}{\delta + \sum_{w' \in W} \alpha(m, w')\gamma(w'; f, g)}.$$

As $\delta + \sum_{w' \in W} \alpha(m, w')\gamma(w'; f, g)$ is the exit rate for a man of type $m$ and $\alpha(m, w)\gamma(w; f, g)$ is the rate at which such an agent exits in a match with $w$, the desired conclusion follows. Of course, an analogous observation can be made for the probability that a woman of type $w$ exits in a match with a man of type $m$. Using (10) and (13) we have

$$\frac{\lambda}{\eta}\alpha(m, w)c(\bar{f}, \bar{g})f(m)g(w) = \frac{\alpha(m, w)\phi(m; f, g)}{\delta + \sum_{m' \in M} \alpha(m', w)\phi(m', f, g)}.$$

**Definition 1.** *A steady state $(f, g, \alpha)$ is an equilibrium if*

$$\alpha(m, w) = \begin{cases} 0 & \text{if } u(m, w) < U(m; x) \text{ or } v(m, w) < V(w; x), \\ 1 & \text{if } u(m, w) > U(m; x) \text{ and } v(m, w) > V(w; x) \end{cases} \tag{17}$$

*holds for all $(m, w) \in M \times W$ where $x$ is the matching induced by $(f, g, \alpha)$. A matching $x$ is an* equilibrium matching *if it is induced by some equilibrium.*

Observe that any specification of $\alpha(m, w) \in [0, 1]$ is consistent with equilibrium in the case that the inequalities $u(m, w) \geq U(m; x)$ and $v(m, w) \geq V(w; x)$ are both satisfied and (at least) one of them holds with equality. This may be interpreted as allowing an agent to randomize the acceptance decision in case he or she is indifferent as to whether or not to accept his or her current match.

The following lemma characterizes equilibrium matchings in terms of the parameters $(M, W, u, v)$ of the marriage market and the parameters $(\eta, \delta, \lambda, C)$ of the search model. The result is a straightforward implication of the "accounting identities" linking steady states and their induced matchings. Despite its simplicity, the result is useful because we can conduct the subsequent formal analysis without any explicit reference to the underlying steady states or equilibria. Instead, we can focus directly on the induced matchings. To simplify notation we let $\bar{x}^M = \sum_m x(m, m)$ and $\bar{x}^W = \sum_w x(w, w)$.

**Lemma 3.** *A matching $x$ is a steady-state matching if and only if*

$$x(m, w) \leq \frac{\eta\lambda}{\delta^2} c(\frac{\eta}{\delta}\bar{x}^M, \frac{\eta}{\delta}\bar{x}^W) x(m, m) x(w, w) \tag{18}$$

*holds for all $(m, w) \in M \times W$. It is an equilibrium matching if and only if, in addition,*

$$x(m, w) = \begin{cases} 0 & \text{if } u(m, w) < U(m; x) \text{ or } v(m, w) < V(w; x), \\ \frac{\lambda\eta}{\delta^2} c(\frac{\eta}{\delta}\bar{x}^M, \frac{\eta}{\delta}\bar{x}^W) x(m, m) x(w, w) & \text{if } u(m, w) > U(m; x) \text{ and } v(m, w) > V(w; x) \end{cases} \tag{19}$$

*holds.*

*Proof.* See Appendix A. □

The motivation for the equilibrium condition (17) suggests that every equilibrium matching is individually rational and regret-free. The proof of the following lemma (in Appendix A) shows how to obtain this result directly from the characterization of equilibrium matchings in Lemma 3.

**Lemma 4.** *Every equilibrium matching is individually rational and regret-free.*

# 4 Limit Matchings

In this section, we consider a fixed marriage market $(M, W, u, v)$ and investigate the limit as the search process becomes frictionless, in the sense that the speed of the contact process $\lambda$ converges to infinity. Throughout, the parameters $\delta > 0, \eta > 0$ and the contact function $C$ are kept fixed.

**Definition 2.** *A matching $x_*$ is a limit matching if there exists a sequence $(\lambda_k)$ converging to infinity and a sequence of matchings $(x_k)$ converging to $x_*$ such that for all $k$ the matching $x_k$ is an equilibrium matching for the search model with parameters $(\eta, \delta, \lambda_k, C)$.*

It is easy to see that the individual rationality and regret-freeness of equilibrium matchings is preserved in the limit. The following proposition establishes that strengthening individual rationality to coherency yields a necessary and sufficient condition for a matching to be a limit matching.

**Proposition 1.** *A matching is a limit matching if and only if it is coherent and regret-free.*

The proof of Proposition 1 is relegated to Appendix B. The intuition for the coherency of limit matchings is straightforward *if* the rate at which agents meet every type of potential partner converges to infinity: In the limit each type of agent will then only match with the best possible type of potential partner for whom he or she is strictly acceptable, implying that no pair of types satisfies (7). However, it is not obvious that meeting rates converge to infinity as the speed of the matching technology converges to infinity.[14]

The difficulty is that an increase in the speed at which agents match (and thus exit from search) implies a reduction in the steady-state masses of agents. Furthermore, if some types of agents match faster than others do, their share in the stock decreases as the speed of the contact processes increases. This generates the possibility that the meeting rate for some type of agent may converge to a finite limit. Indeed, sequences of steady-state matchings with this property are easily constructed. The main subtlety in proving the coherency of limit matchings is to exclude this possibility for sequences of equilibrium matchings. In doing so, we exploit the continuity of the contact function $C$ and make use of the characterization of regret-free matchings in Lemma 2.

[14]This is in contrast to a "cloning model", such as the one studied by Adachi (2003). In a cloning model, the steady-state masses $(f, g)$ are taken as given so that the population share of each type is strictly positive and constant along the sequence. Therefore, the matching rate necessarily converges to infinity as the speed of the contact process does so. Hence, in such a model, the counterpart to the necessary conditions for limit matchings in Proposition 1 is straightforward.

The main difficulty in proving the sufficiency of the conditions in Proposition 1 is in showing that every random matching $x_*$ with the stated properties can be obtained as a limit matching. In essence, we must construct a sequence of equilibria such that, for every pair of types $(m, w)$ satisfying $0 < x_*(m, w) < 1$, the associated sequence of matching probabilities converges to an interior limit. This in turn requires that along the sequence at least one of the types $m$ and $w$ is indifferent between accepting the match with his or her designated partner (whereas the other type (weakly) prefers to accept the match).

As a deterministic matching is stable if and only if it is coherent and every deterministic matching is regret-free, the following result is an immediate implication of Proposition 1.

**Corollary 1.** *A deterministic matching is a limit matching if and only if it is stable. In particular, limit matchings exist because stable deterministic matchings exist.*[15]

If one restricts attention to deterministic matchings, Corollary 1 indicates that the sets of limit matchings and stable matchings are identical. For random matchings, however, this equivalence breaks down, because regret-free random matchings are not stable.

**Corollary 2.** *Random limit matchings are unstable.*

*Proof.* Let $x$ be a random limit matching. Because $x$ is random, there exists a pair $(m, w) \in M \times W$ such that $0 < x(m, w) < 1$ holds. Because $x$ is also regret-free, Lemma 2 implies that $m$ and $w$ are partially matched in $x$. From Lemma 1, $x$ is unstable. □

As the definition of a stable random matching given in Section 2 may appear somewhat technical, we point out that the result in Corollary 2 does not hinge on this particular definition. Rather, the problem is more fundamental: In a random limit matching, there exists a pair $(m, w)$ such that both $m$ and $w$ stay single with strictly positive probability even though man $m$ and woman $w$ find each other acceptable. The existence of such a pair not only indicates that random limit matchings are unstable in a rather strong sense, it also implies that random limit matchings are (weakly) ex-ante Pareto dominated. To state this more precisely, define a matching $x$ to be Pareto dominated if there exists a matching $x'$ such that $U(m; x') \geq U(m; x)$ as well as $V(m; x') \geq V(m; x)$ holds for all $m$ and $w$ and at least one of the inequalities is strict.

[15] Observe that the existence of a limit matching implies that for sufficiently high $\lambda$ an equilibrium in the search model exists. Hence, for high contact speeds, Corollary 1 implies the existence of equilibria in the search model.

**Corollary 3.** *Random limit matchings are Pareto dominated.*

*Proof.* Let $x$ be a random limit matching. As in the proof of Corollary 2, we obtain the existence of a pair $(m, w)$ such that $0 < x(m, w) < 1$, $x(m, m) > 0$, and $x(w, w) > 0$ holds. Consider the matching $x'$ that is identical to $x$ except that $x'(m, w) = x(m, w) + \min\{x(m, m), x(w, w)\}$ holds. Because $x$ is individually rational, we then have $U(m; x') > U(m; x)$ and $U(w; x') > U(w; x)$, whereas the payoffs of all other men and women remain unchanged.[16] $\square$

To illustrate these Corollaries, we return to the examples introduced in Section 2.

Example 1 (Continued): We have already noted that the random matching

$$x_* = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

is coherent. As $U(m_i; x_*) = V(w_j; x_*) = 1$ holds for $i, j = 1, 2$, it is also regret-free and thus a random limit matching. As implied by Corollary 2, $x_*$ is not stable as each type of agent stays unmatched with probability $1/3$, whereas in a stable matching all types are matched for sure. $x_*$ is Pareto dominated by the two stable deterministic matchings and strictly Pareto dominated by the matching

$$x' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The proof of Proposition 2 shows $x_*$ is the unique random limit matching. $\square$

Example 2 (Continued): The three stable deterministic matchings, namely

$$x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

are limit matchings. In addition, the two random matchings

$$x' = \frac{18}{24}x_1 + \frac{3}{24}x_2 \text{ and } x'' = \frac{18}{24}x_3 + \frac{3}{24}x_2,$$

are coherent and regret-free. To see this, observe that the matching $x'$ results in the payoffs $U(m_i; x') = 6$ and $V(w_j; x') = 3$, whereas the matching $x''$ results

[16] A slight variation of this argument yields a stronger result, namely, that a strict increase in payoff can be achieved for all agents that are partially matched in $x$. Building on the constructions in the proof of Proposition 2, it can also be shown that every random limit matching is Pareto dominated by a stable deterministic matching.

in the payoffs $U(m_i; x'') = 3$ and $V(w_j; x'') = 6$ for $i, j = 1, 2, 3$. Hence, $x'$ and $x''$ are limit matchings. As $x(h, h) = 1/8$ holds for all agents $h$ and all partners are mutually acceptable, these random limit matchings are unstable and Pareto dominated. □

Corollary 2 implies that the stability of limit matchings can only be taken for granted if no random limit matchings exist. Examples 1 and 2 indicate that it is easy to construct examples in which random limit matchings exist. This prompts our next result, which states a necessary and sufficient condition for the existence of random limit matchings.

**Proposition 2.** *Random limit matchings exist if and only if there exists more than one stable matching.*

The proof of Proposition 2 is given in Appendix C. The key observation underlying the proof is the following: Given a random matching that is coherent and regret-free, it is possible to construct two distinct stable deterministic matchings. These two matchings support the original random matching, in the sense that the union of any agent's partners in the two deterministic matchings coincides with his or her set of partners in the random matching. Because we can construct two distinct stable deterministic matchings from any random limit matching, there is no random limit matching if there is a unique stable deterministic matching. Proving the other direction of the equivalence in the statement of the proposition is harder and relies on the lattice-structure of the set of stable deterministic matchings (see Roth and Sotomayor, 1990, Chapter 3). The difficulty is that not any pair of stable deterministic matchings supports a random limit matching. (For instance, in Example 2 there is no random limit matching supported by $x_1$ and $x_3$.) As we show in the proof of Proposition 2, any two stable deterministic matchings, which have the property that all men have the same preferences over these two matchings and that there is no stable matching "between" those two in the men's preference ordering, support a random limit matching. In fact, our proof shows more: namely, that for every pair of such consecutive stable deterministic matchings there is exactly one random limit matching supported by this pair. (This implies, in particular, that there are no further random limit matchings in Examples 1 and 2.) The proof of Proposition 2 is then completed by the observation that consecutive stable deterministic matchings exist whenever there is more than one stable deterministic matching.

As an immediate consequence of the preceding results, we have the result advertised in the Introduction, namely that uniqueness is necessary and sufficient to ensure the equivalence of stable matchings and limit matchings.

**Corollary 4.** *All limit matchings are stable if and only if there is a unique limit matching which then coincides with the unique stable deterministic matching.*

*Proof.* Suppose there is a unique stable deterministic matching. From Corollary 1 this matching is a limit matchings and there are no other deterministic limit matchings. Because of Proposition 2, there is no random limit matching. Therefore, the unique stable deterministic matching is also the unique limit matching.

Suppose that there is more than one stable deterministic matching. From Proposition 2, this implies that a random limit matching exists. From Corollary 2, it follows that an unstable limit matching exists.

This covers all cases that are possible because stable deterministic matchings exist. □

# 5 Concluding Remarks

In this paper, we have revisited the classical marriage problem and its solution concept, namely, stability, in a frictional decentralized environment. We have demonstrated that when frictions vanish, the set of equilibrium matchings converges to the set of stable matchings if and only if there is a unique stable matching. Otherwise, additional, unstable and Pareto dominated random matchings arise in the limit.

Many of the marriage markets considered in the economic literature possess a unique stable matching, as discussed in the Introduction. It is thus an important class of marriage markets for which our Corollary 4 confirms the conjecture by Roth and Sotomayor (1990) that stable matchings approximate equilibrium matchings in marriage markets with negligible frictions. However, our analysis reveals that the uniqueness of stable matchings plays an essential role in obtaining this conclusion.

# Appendix A

**Proof of Lemma 3:** Suppose that $x$ is a steady-state matching, and let $(f, g, \alpha)$ be a steady state that induces the matching. Using (15) and (16), we have

$$f(m) = \frac{\eta}{\delta} x(m, m), \quad g(w) = \frac{\eta}{\delta} x(w, w). \tag{20}$$

Substitution of these expressions for $f(m)$ and $g(w)$ into (14) and observing that $\alpha(m, w) \leq 1$ for all $(m, w)$, immediately implies (18), as claimed.

Suppose the matching $x$ satisfies (18) for all $(m, w) \in M \times W$. Observe that (18) implies $x(m, m) > 0$ and $x(w, w) > 0$ for all $m$ and $w$.[17] We may

[17]Suppose, for instance, $x(m, m) = 0$. Then (18) implies $x(m, w) = 0$ for all $w$ and, thus, $x(m, m) = 1 - \sum_w x(m, w) = 1$, a contradiction.

thus use (20) to define $(f, g)$ and set

$$\alpha(m,w) = \frac{\delta^2 x(m,w)}{\eta\lambda c(\frac{\eta}{\delta}\bar{x}^M, \frac{\eta}{\delta}\bar{x}^W)x(m,m)x(w,w)}. \tag{21}$$

We show that the so constructed $(f, g, \alpha)$ is a steady state and $(f, g, \alpha)$ induces $x$. Clearly, we have $f(m) > 0$ and $g(w) > 0$, and (18) implies $0 \leq \alpha(m, w) \leq 1$. From (20) and (21)

$$\begin{aligned}
\frac{f(m)}{\eta}\left[\delta + \lambda c(\bar{f},\bar{g})\sum_{w\in W}\alpha(m,w)g(w)\right] &= x(m,m) + \textstyle\sum_{w\in W} x(m,w), \\
\frac{g(w)}{\eta}\left[\delta + \lambda c(\bar{f},\bar{g})\sum_{m\in M}\alpha(m,w)f(m)\right] &= x(w,w) + \textstyle\sum_{m\in M} x(m,w),
\end{aligned}$$

holds for all $m \in M$ and $w \in W$. Because the right sides of these equations are equal to 1, it follows that (12) and (13) are satisfied. Hence, $(f, g, \alpha)$ is a steady state. Furthermore, using (20) and (21), it is easy to see that (14) is satisfied, so that $x$ is the steady-state matching induced by $(f, g, \alpha)$.

The above argument has established that (21) holds whenever a matching $x$ is induced by the steady state $(f, g, \alpha)$. Therefore, (17) implies that every equilibrium matching satisfies (19). Conversely, if a steady-state matching $x$ satisfies (19), then the steady state that induces it satisfies (17). Thus, $x$ is an equilibrium matching. □

**Proof of Lemma 4:** Let $x$ be an equilibrium matching. The first half of (19) can be rewritten as

$$x(m,w) > 0 \Rightarrow u(m,w) \geq U(m;x) \text{ and } v(m,w) \geq V(m;x),$$

coinciding with (8). Hence, $x$ is regret-free.

Using (4) and (5) we have

$$\begin{aligned}
x(m,m)U(m;x) &= \sum_{w\in W} x(m,w)\left[u(m,w) - U(m;x)\right], \\
x(w,w)V(w;x) &= \sum_{m\in M} x(m,w)\left[v(m,w) - V(w;x)\right].
\end{aligned}$$

Using (19) to infer

$$\begin{aligned}
x(m,w)\left[u(m,w) - U(m;x)\right] &= x(m,w)\max\left[0, u(m,w) - U(m;x)\right], \\
x(m,w)\left[v(m,w) - V(w;x)\right] &= x(m,w)\max\left[0, v(m,w) - V(w;x)\right].
\end{aligned}$$

We have

$$x(m,m)U(m;x) = \sum_{w \in W} x(m,w) \max[0, u(m,w) - U(m;x)] \geq 0,$$
$$x(w,w)V(w;x) = \sum_{m \in M} x(m,w) \max[0, v(m,w) - V(w;x)] \geq 0.$$

The strict inequalities in (15) and (16) imply that $x(m,m) > 0$ and $x(w,w) > 0$ holds for all $m$ and $w$. Therefore, the above equations imply $U(m;x) \geq 0$ and $V(m;x) \geq 0$. Using (19) again, $x(m,w) > 0$ implies $u(m,w) \geq 0$ and $v(m,w) \geq 0$. Consequently, (6) holds, that is, $x$ is individually rational. $\square$

# Appendix B: Proof of Proposition 1

The following two lemmas establish Proposition 1. Throughout the proofs, we set $\delta = 1$ and $\eta = 1$ to simplify notation. Doing so is without loss of generality. First, setting $\delta = 1$ just amounts to choosing the units in which time is measured in such a way that the rate of exogenous exit is equal to 1. Second, we may set $\eta = 1$ by subsuming the effect of $\eta$ in the contact function.[18]

**Lemma 5.** *Every limit matching is coherent and regret-free.*

*Proof.* Let $x_*$ be a limit matching, and let $(\lambda_k)$ and $(x_k)$ be the corresponding sequences satisfying the properties in Definition 2. For all $m \in M$ and $w \in W$ define the corresponding sequences of meeting rates $(\phi_k(m))$ and $(\gamma_k(w))$ by [19]

$$\phi_k(m) := \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m), \tag{22}$$
$$\gamma_k(w) := \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) x_k(w,w). \tag{23}$$

Throughout we assume that $\phi_k(m)$ and $\gamma_k(w)$ converge to $\phi_*(m) \in \mathbb{R} \cup \{\infty\}$, and $\gamma_*(w) \in \mathbb{R} \cup \{\infty\}$ for all $m$ and $w$, respectively.[20]

[18] If the original contact function is $\hat{C}$, define the new contact function as $C(\bar{f}, \bar{g}) = \hat{C}(\eta\bar{f}, \eta\bar{g})/\eta$. Then,

$$\eta\hat{c}(\eta\bar{f}, \eta\bar{g}) = \frac{\hat{C}(\eta\bar{f}, \eta\bar{g})}{\eta\bar{f}\bar{g}} = \frac{C(\bar{f}, \bar{g})}{\bar{f}\bar{g}} = c(\bar{f}, \bar{g}),$$

so that $\eta$ disappears from the conditions describing steady-state and equilibrium matchings.

[19] Let $(f_k, g_k, \alpha_k)$ be an equilibrium inducing $x_k$. Recall that we have set $\delta = \eta = 1$. From (15) and (16), we thus have $f_k(m) = x_k(m,m)$ and $g_k(w) = x_k(w,w)$. Substituting into (9) and (10), shows that the interpretation of the following expressions as meeting rates is appropriate.

[20] This assumption is without loss of generality: If $x_*$ is a limit matching with the sequence $x_k$ converging to it, then there exists a subsequence of $x_k$ such that the limits $\phi_*(m)$ and $\gamma_*(w)$ are well-defined.

We first show that $x_*$ is individually rational and regret free. Suppose that $x_*(m,w) > 0$ holds. As $x_k$ converges to $x_*$, there exists some number $K$, such that for all $k > K$ we have $x_k(m,w) > 0$. As $x_k$ is an equilibrium matching, Lemma 4 implies $u(m,w) \geq 0$ and $v(m,w) \geq 0$. Therefore, $x_*$ is individually rational. Similarly, Lemma 4 implies $u(m,w) \geq U(m; x_k)$ and $v(m,w) \geq V(w; x_k)$ for all $k > K$. As $U(m; \cdot)$ and $V(w, \cdot)$ are continuous in $x$, it follows that $u(m,w) \geq U(m; x_*)$ and $v(m,w) \geq V(w; x_*)$ hold. Hence, $x_*$ is regret-free.

It remains to show that $x_*$ is pairwise stable. Suppose not. Then there exists a pair $(m,w)$ for which

$$u(m,w) > U(m; x_*) \text{ and } v(m,w) > V(m; x_*). \tag{24}$$

Because $x_k$ converges to $x_*$, there exists a number $K$ such that $u(m,w) > U(m; x_k)$ and $v(m,w) > V(m; x_k)$ holds for all $k > K$. From (19), this implies

$$x_k(m,w) = \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m) x_k(w,w), \tag{25}$$

for such $k$. As in the proof of Lemma 4, we have

$$\begin{aligned} x_k(m,m)U(m; x_k) &= \sum_{w' \in W} x_k(m,w') \max\left[0, u(m,w') - U(m; x_k)\right], \\ x_k(w,w)V(w; x_k) &= \sum_{m' \in M} x_k(m',w) \max\left[0, v(m',w) - V(w; x_k)\right], \end{aligned}$$

and, thus,

$$\begin{aligned} x_k(m,m)U(m; x_k) &\geq x_k(m,w)\left[u(m,w) - U(m; x_k)\right], \\ x_k(w,w)V(w; x_k) &\geq x_k(m,w)\left[v(m,w) - V(w; x_k)\right]. \end{aligned}$$

Using (25), $x_k(h,h) > 0$ for all $h \in M \cup W$ (cf. (15) and (16)), and the definitions of $\phi_k(m)$ and $\gamma_k(w)$ in (22) and (23), this implies

$$\begin{aligned} U(m; x_k) &\geq \gamma_k(w)\left[u(m,w) - U(m; x_k)\right], \\ V(w; x_k) &\geq \phi_k(m)\left[v(m,w) - V(w; x_k)\right]. \end{aligned}$$

Because $U(m; x_k)$ and $V(w; x_k)$ converges to the finite limits satisfying (24), these inequalities imply

$$\phi_*(m) < \infty \text{ and } \gamma_*(w) < \infty. \tag{26}$$

It is immediate from (26) that either it is true that $x_*(m,m) = x_*(w,w) = 0$ or that $z_* = \lim_{k\to\infty} \lambda_k c(\bar{x}_k^M, \bar{x}_k^W)$ exists and is finite. The following argument excludes the second possibility. Finiteness of $z_*$ implies that $\phi_*(m') < \infty$ and $\gamma_*(w') < \infty$ holds for all $m' \in M$ and $w' \in W$. Using (18), this implies $x_*(m', w') = 0$ whenever $x_*(m', m') = 0$ or $x_*(w', w') = 0$ holds (because $x_*(m', w') \leq \phi_*(m')x_*(w', w')$ and $x_*(m', w') \leq \gamma_*(w')x_*(m', m')$). Consequently, $x_*(m', m') = 0$ implies $x_*(m', m') = 1 - \sum_{w' \in W} x_*(m', w') = 1$, which is a contradiction. Hence, finiteness of $z_*$ implies that $x_*(m', m') > 0$ holds for all $m' \in M$. Similarly, we obtain $x_*(w', w') > 0$ for all $w' \in W$. In particular, we have $\bar{x}_*^M = \sum_{m' \in M} x_*(m, m) > 0$ and $\bar{x}_*^W = \sum_{w' \in W} x_*(w', w') > 0$. Because the contact function $C$ is strictly positive whenever both of its arguments are strictly positive, we obtain $c(\bar{x}_*^M, \bar{x}_*^W) > 0$ from (11). Hence, $z_* = \lim_{k \to \infty} \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) = \infty$, which is a contradiction.

Hence, (26) implies $x_*(m, m) = 0$ and $x_*(w, w) = 0$. The same argument as in the preceding paragraph implies $x_*(m, w) = 0$. From Lemma 2, as $x_*$ is regret-free, $x_*(m, m) = 0$ and $x_*(w, w) = 0$ imply that there exist $w'$ and $m'$ such that $x_*(m, w') = 1$ and $x_*(m', w) = 1$. Using (18), we have $x_k(m, w') \leq \phi_k(m)x_k(w', w')$ for all $k$, so that $x_*(m, w') = 1$ and $x_*(w', w') = 0$ (which is implied by $x_*(m, w') = 1$) implies $\phi_*(m) = \infty$. (An analogous argument implies $\gamma_*(w) = \infty$.) This is a contradiction to (26). Hence, $x_*$ is pairwise stable. □

**Lemma 6.** *Every coherent and regret-free matching is a limit matching.*

*Proof.* Let $x_*$ be a coherent and regret-free matching. We begin by constructing a sequence of matchings $(x_k)$ converging to $x_*$ and then show that we can find a sequence $(\lambda_k)$ converging to infinity such that for all sufficiently large $k$ the matching $x_k$ is an equilibrium matching for the search model with parameters $(1, 1, \lambda_k, C)$. This suffices to establish that $x_*$ is a limit matching, because we may then take a subsequence of $(x_k, \lambda_k)$ that satisfies the conditions in Definition 2.

For $k \in \mathbb{N}$ define $x_k$ by

$$x_k(m, w) = \begin{cases} x_*(m, w) & \text{if } x_*(m, w) < 1, \\ (1 - \frac{1}{k+1}) & \text{if } x_*(m, w) = 1. \end{cases} \tag{27}$$

Because $x_*$ is a matching, $x_k$ satisfies (1) – (3). Thus, $x_k$ is a matching. Clearly, the sequence of matchings $(x_k)$ converges to $x_*$.

To construct the sequence $(\lambda_k)$, define $h_k : \mathbb{R}_+ \to \mathbb{R}$ by

$$h_k(\lambda) = \lambda \left( \frac{1}{k+1} \right)^2 c(\bar{x}_k^M, \bar{x}_k^W) - \left( 1 - \frac{1}{k+1} \right). \tag{28}$$

As $x_*$ is regret-free, Lemma 2 ensures that $\bar{x}_k^M > 0$ and $\bar{x}_k^W > 0$ holds for all

$k$,[21] ensuring that $h_k$ is well-defined for all $k$. In addition, $h_k$ is continuous in $\lambda$ and satisfies $h_k(0) < 0$, as well as $\lim_{\lambda\to\infty} h_k(\lambda) = \infty$. Hence, we may choose a sequence $(\lambda_k)$ satisfying

$$h_k(\lambda_k) = 0, \lambda_k > 0 \tag{29}$$

for all $k$. We now show that $(\lambda_k)$ converges to infinity. From (28) and (29), this follows if

$$\left(\frac{1}{k+1}\right)^2 c(\bar{x}_k^M, \bar{x}_k^W) \to 0. \tag{30}$$

Suppose $\bar{x}_*^M \cdot \bar{x}_*^W > 0$ holds, ensuring that $c(\bar{x}_*^M, \bar{x}_*^W)$ is well-defined. As $(x_k)$ converges to $x_*$ and $c$ is continuous, this implies (30). Suppose $\bar{x}_*^M = 0$. As $x_*$ is regret-free, Lemma 2 then implies that $x_*$ is deterministic. From (27), it then follows that $x_k(m,m) = 1/(k+1)$ holds for all $m \in M$ and $x_k(w,w) \in \{1/(k+1), 1\}$ holds for all $w \in W$. Hence, we have $\bar{x}_k^M = |M|/(k+1)$ and $\bar{x}_k^W \geq |W|/(k+1)$, implying

$$\left(\frac{1}{k+1}\right)^2 c(\bar{x}_k^M, \bar{x}_k^W) = \left(\frac{1}{k+1}\right)^2 \frac{C(\bar{x}_k^M, \bar{x}_k^W)}{\bar{x}_k^M \cdot \bar{x}_k^W} \leq \frac{1}{|M|\,|W|} C(\bar{x}_k^M, \bar{x}_k^W),$$

where we have used (11) for the equality. As the contact function $C$ is continuous, satisfies $C(0, \bar{x}_*^W) = 0$, and $x_k$ converges to $x_*$, this implies (30). An analogous argument establishes (30) for the case $\bar{x}_*^W = 0$, finishing the demonstration that $(\lambda_k)$ converges to infinity.

Let $K$ be such that $x_*(h,h) > 1/(K+1)$ holds for all $h \in M \cup W$ for which $x_*(h,h) > 0$. We now show that for all $k > K$ the matching $x_k$ is a steady-state matching for the search model with parameters $(1, 1, \lambda_k, C)$. Using Lemma 3, we have to show

$$x_k(m,w) \leq \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m) x_k(w,w) \tag{31}$$

for all $(m,w) \in M \times W$. Consider $(m,w)$ satisfying $x_*(m,w) = 1$. Then

$$\frac{x_k(m,w)}{\lambda_k \cdot c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m) x_k(w,w)} = \frac{1 - \frac{1}{k+1}}{\lambda_k \cdot c(\bar{x}_k^M, \bar{x}_k^W) \left(\frac{1}{k+1}\right)^2} = 1, \tag{32}$$

where the first equality uses (27) to infer $x_k(m,w) = 1 - 1/(k+1)$ as well as $x_k(m,m) = x_k(w,w) = 1/(k+1)$ from $x_*(m,w) = 1$, and the second

[21] If there exists $(m,w)$ such that $x_*(m,w) = 1$ holds, then we have $x_k(m,w) = 1 - 1/(k+1)$, $x_k(m,w') = 0$, and $x_k(m',w) = 0$ for all $w' \neq w$ and $m' \neq m$. Consequently, $x_k(m,m) = x_k(w,w) = 1/(k+1) > 0$ holds, implying $\bar{x}_k^M > 0$ and $\bar{x}_k^W > 0$. If there exists no $(m,w)$ such that $x_*(m,w) = 1$ holds, then (27) implies $x_k = x_*$. In addition, Lemma 2 implies that no agent is fully matched in $x_*$. Consequently, $\bar{x}_*^M = \bar{x}_k^M > 0$ and $\bar{x}_*^W = \bar{x}_k^W > 0$ hold.

equality is from (29). It follows that (31) holds with equality. Consider $(m,w)$ satisfying $0 < x_*(m,w) < 1$. As $x_*$ is regret-free, Lemma 2 and (27) then imply $x_*(m,m) = x_k(m,m) > 0$ and $x_*(w,w) = x_k(w,w) > 0$. Hence,

$$\begin{aligned}
\frac{x_k(m,w)}{\lambda_k \cdot c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m) x_k(w,w)} &= \frac{x_*(m,w)}{\lambda_k \cdot c(\bar{x}_k^M, \bar{x}_k^W) x_*(m,m) x_*(w,w)} \\
&< \frac{1 - \frac{1}{k+1}}{\lambda_k \cdot c(\bar{x}_k^M, \bar{x}_k^W) \left(\frac{1}{k+1}\right)^2} \\
&= 1,
\end{aligned}$$

where the inequality uses that $k > K$ implies $x_*(m,m) > 1/(k+1)$, $x_*(w,w) > 1/(k+1)$, and $x^*(m,w) < 1 - 1/(k+1)$. It follows that (31) holds with strict inequality. Finally, from (27), $x_*(m,w) = 0$ implies $x_k(m,w) = 0$, so that (31) trivially holds for all $(m,w)$ satisfying $x_*(m,w) = 0$.

Using Lemma 3 it remains to show that the condition

$$x_k(m,w) = \begin{cases} 0 & \text{if } u(m,w) < U(m;x_k) \text{ or } v(m,w) < V(w;x_k), \\ \lambda_k c(\bar{x}_k^M, \bar{x}_k^W) x_k(m,m) x_k(w,w) & \text{if } u(m,w) > U(m;x_k) \text{ and } v(m,w) > V(w;x_k) \end{cases} \tag{33}$$

is satisfied for sufficiently large $k$. From (27) the payoffs associated with $x_k$ are given by

$$U(m;x_k) = \begin{cases} \left(1 - \frac{1}{k+1}\right) U(m;x_*) & \text{if } x_*(m,m) = 0, \\ U(m;x_*) & \text{if } x_*(m,m) > 0, \end{cases} \tag{34}$$

and by

$$V(w;x_k) = \begin{cases} \left(1 - \frac{1}{k+1}\right) V(w;x_k) & \text{if } x_*(w,w) = 0, \\ V(w;x_*) & \text{if } x_*(w,w) > 0. \end{cases} \tag{35}$$

In both cases, Lemma 2 is used to exclude the possibility that $x^*(h,h) = 0$ holds when $h$ has more than one partner in $x^*$.

Consider $m$ satisfying $x_*(m,m) = 1$, so that $x_*(m,w) = x_k(m,w) = 0$ holds for all $w \in W$ and we have $U(m;x_*) = U(m;x_k) = 0$. If $u(m,w) < 0$, it is immediate that (33) holds. Hence, assume $u(m,w) > 0$. We have to show that $v(m,w) \leq V(w;x_k)$ holds for sufficiently large $k$ as this implies that $x_k(m,w) = 0$ is consistent with (33). If $x_*(w,w) > 0$, then the inequality $v(m,w) \leq V(w;x_k)$ is immediate from $V(w;x_k) = V(w;x_*)$ and the pairwise stability of $x_*$. If $x_*(w,w) = 0$, the pairwise stability of $x_*$ implies $v(m,w) < V(w;x_*)$. (To obtain the strict inequality here, we use the strictness of preferences together with the implication from Lemma 2 that there exists $m' \neq m$

such that $x_*(m', w) = 1$, implying $V(w; x_*) = v(m', w)$ for some $m' \neq m$.) As $V(w; x_k)$ converges to $V(w; x_*)$, the inequality $v(m, w) < V(w; x^*)$ implies $v(m, w) < V(w; x_k)$ for sufficiently large $k$, yielding the desired result. An analogous argument shows that (33) holds for all $(m, w)$ satisfying $x_*(w, w) = 1$.

Consider $m$ satisfying $x_*(m, m) = 0$. Lemma 2 then implies that there exists $w' \in W$ such that $x_*(m, w') = 1$ and, thus, $U(m; x_*) = u(m, w')$. and $V(w'; x_*) = v(m, w')$. By the individual rationality of $x_*$ and the strictness of preferences, we have $u(m, w') > 0$ and $v(m, w') > 0$. Hence, (34) implies $u(m, w') > U(m, x_k)$ for all $k$, and (35) implies $v(m, w') > V(w'; x_k)$ for all $k$. Consequently, (33) requires that (31) holds with equality for the pair $(m, w')$, which is ensured by construction of $\lambda_k$, cf. (32). Consider now $w \neq w'$. Because $x_*(m, w') = 1$ we have $x_*(m, w) = 0$. If $u(m, w) < U(m; x_*)$, then $u(m, w) < U(m; x_k)$ holds for all $k$ sufficiently large, implying that $x_k(m, w) = 0$ is consistent with (33). Hence, assume $u(m, w) > U(m; x_*)$. (The case of equality cannot arise because $U(m; x_*) = u(m, w')$, $w \neq w'$ and because preferences are strict.) We have to show that $v(m, w) \leq V(w; x_k)$ holds for sufficiently large $k$. This follows by the same argument as in the case $x_*(m, m) = 1$ discussed in the preceding paragraph. Hence, we conclude that (33) holds for all $(m, w)$ with $x_*(m, m) = 0$. An analogous argument shows that (33) holds for all $(m, w)$ satisfying $x_*(w, w) = 0$.

It remains to establish (33) for pairs $(m, w)$ for which $0 < x_*(m, m) < 1$ and $0 < x_*(w, w) < 1$. Using (27), (34), and (35), we have $x_k(m, w) = x_*(m, w)$, $U(m; x_k) = U(m, x_*)$, and $V(w; x_k) = V(w; x_*)$ for all such pairs. Pairwise stability of $x^*$ then implies that there does not exist such $(m, w)$ satisfying $u(m, w) > U(m; x_k)$ and $v(m, w) > V(m; x_k)$. Because $x_*$ is regret-free, $x_k(m, w) > 0$ also implies $u(m, w) \geq U(m; x_k)$ as well as $v(m, w) \geq V(w; x_k)$. Therefore, $x_k$ satisfies (33). $\square$

# Appendix C: Proof of Proposition 2

In the following, it is often more convenient to identify deterministic matchings with their matchings functions, instead of using the assignment matrices.

**Lemma 7.** *If there exists a random matching $x$ that is coherent and pairwise stable, then there exist two distinct stable deterministic matchings $\mu_1 \neq \mu_2$.*

*Proof.* Let $x$ be a random matching that is coherent and regret-free. Let

$$P_x(m) = \{w \in W : x(m, w) > 0\} \text{ and } P_x(w) = \{m \in M : x(m, w) > 0\}$$

denote the sets of partners. We now construct $\mu_1$ and $\mu_2$. Let $\mu_1(h) = h$ if $x(h, h) = 1$. For all other agents, let $\mu_1(m) = \text{argmax}_{w \in P_x(m)}\, u(m, w)$ and

$\mu_1(w) = \operatorname{argmin}_{m \in P_x(w)} v(m,w)$, respectively. By the strictness of preferences, the function $\mu_1 : M \cup W \to M \cup W$ is uniquely defined. Similarly, let $\mu_2(h) = h$ if $x(h,h) = 1$. For all other agents, let $\mu_2(m) = \operatorname{argmin}_{w \in P_x(m)} u(m,w)$ and $\mu_2(w) = \operatorname{argmax}_{m \in P_x(w)} v(m,w)$, respectively. We show in the following that $\mu_1$ and $\mu_2$ are stable deterministic matchings that satisfy $\mu_1 \neq \mu_2$.

We begin by verifying that $\mu_1$ and $\mu_2$ are deterministic matchings. It is immediate from the definitions that for $i = 1,2$ the conditions $\mu_i(m) \in W \cup \{m\}$ and $\mu_i(w) \in M \cup \{w\}$ are satisfied for all $m$ and $w$. Hence, our task is to verify that for $i = 1,2$ the condition $\mu_i^2(h) = h$ holds for all $h \in M \cup W$. For $h$ satisfying $x(h,h) = 1$ this is immediate. Consider $h$ satisfying $x(h,h) = 0$. Because $x$ is regret-free, Lemma 2 implies that every such agent has a unique partner $h' \neq h$ and $h'$ is in turn the unique partner of $h$. Consequently, we have $\mu_1^2(h) = \mu_2^2(h) = h$ for all such agents. It remains to consider $h$ satisfying $0 < x(h,h) < 1$, that is, the set of partially matched agents.

Let $M_x$ and $W_x$ denote the sets of partially matched men and women, respectively, in the matching $x$. As $x$ is regret-free, Lemma 2 implies that these sets are not empty. Because every agent in $M_x \cup W_x$ has at least one partner (otherwise the agent would be unmatched), the set of partners $P_x(h)$ is not empty for all $h \in M_x \cup W_x$. Furthermore, for all $h \in M_x \cup W_x$ we have $P_x(h) \subset M_x \cup W_x$. (If, say, $m \in M_x$ has a partner $w' \in W \setminus W_x$, then $w'$ must be fully matched. From Lemma 2, this implies $x(m,w') = 1$, implying $m \notin M_x$.) Hence, for $i = 1,2$ we have $\mu_i(M_x) \subseteq W_x$ and $\mu_i(W_x) \subseteq M_x$.

We use the individual rationality of $x$ for the first inequalities in the following displayed expressions,

$$u(m,\mu_1(m)) > (1 - x(m,m))u(m,\mu_1(m)) \geq \sum_{w \in P_x(m)} x(m,w)u(m,w) = U(m;x),$$

$$v(\mu_2(w),w) > (1 - x(w,w)v(\mu_2(w),w) \geq \sum_{m \in P_x(w)} x(m,w)v(m,w) = V(w;x),$$

for all $m \in M_x$ and $w \in W_x$. Hence, we have

$$u(m,\mu_1(m)) > U(m;x) \quad \text{and} \quad v(\mu_2(w),w) > V(w;x), \tag{36}$$

for all $m \in M_x$ and $w \in W_x$.

Because $x$ is pairwise stable, (36) implies

$$v(m,\mu_1(m)) \leq V(\mu_1(m);x) \text{ and } u(\mu_2(w),w) \leq U(\mu_2(w);x)$$

for all $m \in M_x$, respectively for all $w \in W_x$.

Because $x$ is regret free, these inequalities must, in fact, hold with equality, so that we obtain

$$v(m,\mu_1(m)) = V(\mu_1(m);x) \quad \text{and} \quad u(\mu_2(w),w) = U(\mu_2(w);x), \tag{37}$$

for all $m \in M_x$ and $w \in W_x$. From the first equality in (37), $\mu_1(m) = \mu_1(m') = w$ implies $v(m', w) = v(m, w)$ and thus, from strictness of preferences, $m = m'$. Hence, the restriction of $\mu_1$ to $M_x$, denoted by $\eta_1$ in the following, is an injection into $W_x$. An analogous argument using the second equality in (37) shows that the restriction of $\mu_2$ to $W_x$, denoted by $\eta_2$, is an injection into $M_x$. Because the sets $M_x$ and $W_x$ are finite it follows (as a trivial application of the Cantor-Bernstein-Schröder theorem) that $\eta_1$ and $\eta_2$ are bijections. Let $\eta_1^{-1} : W_x \to M_x$ and $\eta_2^{-1} : M_x \to W_x$ denote the corresponding inverses. To establish that $\mu_1$ and $\mu_2$ are deterministic matchings, it remains to show that $\eta_1^{-1}(w) = \mu_1(w)$ and $\eta_2^{-1}(m) = \mu_2(m)$ holds for all $w \in W_x$ and $m \in M_X$. Consider $w \in W_x$. From the first equality in (37), we have $v(\eta_1^{-1}(w), w) = V(w; x)$. Observing that $\eta_1^{-1}(w)$ is a partner of $w$ and that (because $x$ is regret-free and preferences are strict) all other partners $m$ of $w$ satisfy $v(m, w) > V(w; x)$, the desired conclusion $\eta_1^{-1}(w) = \mu_1(w)$ follows. An analogous argument yields $\eta_2^{-1}(m) = \mu_2(m)$ for all $m \in M_x$.

Consider any $m \in M_x$. Substituting $w = \mu_1(m)$ into the second inequality in (36), we have $v(\mu_2(\mu_1(m)), \mu_1(m)) > V(\mu_1(m); x)$. From the first equality in (37), this implies $\mu_2(\mu_1(m)) \neq m$. Because $M_x$ is not empty, it follows that $\mu_1$ and $\mu_2$ are different. To complete the proof of the lemma, it remains to show that $\mu_1$ and $\mu_2$ are stable. Let $x_i$ denote the assignment matrix corresponding to $\mu_i$. By construction, $x_i(m, w) = 1$ implies that $(m, w)$ are partners in $x$, that is, we have $x(m, w) > 0$. Because $x$ is individually rational, it follows that $\mu_i$ is individually rational. Next, by the construction of $\mu_1$ and $\mu_2$ we have

$$\begin{aligned} U(m; x_1) = U(m; x_2) = U(m; x) \text{ for all } m \notin M_x, \\ V(w; x_1) = V(w; x_2) = V(w; x) \text{ for all } w \notin W_x. \end{aligned}$$

From (36) and (37) we have:

$$\begin{aligned} U(m; x_1) > U(m; x_2) = U(m; x) \text{ for all } m \in M_x, \\ V(w; x_2) > V(w; x_1) = V(w; x) \text{ for all } w \in W_x. \end{aligned}$$

In particular, for $i = 1, 2$ we have $U(m; x_i) \geq U(m; x)$ for all $m \in M$ and $V(w; x_i) \geq V(w; x)$ for all $w \in W$. As $x$ satisfies (7) it follows that $x_1$ and $x_2$ satisfy (7). Hence, $\mu_1$ and $\mu_2$ are stable. $\square$

To prove the converse of Lemma 7 and, thus, to finish the proof of Proposition 2, we rely on some well-known results about the structure of the set of stable deterministic matchings. Given any two deterministic matchings $\mu_1$ and $\mu_2$, define $\mu_1 >_M \mu_2$ if $u(m, \mu_1(m)) \geq u(m, \mu_2(m))$ holds for all $m \in M$ and $u(m, \mu_1(m)) > u(m, \mu_2(m))$ for at least one $m$. Define $\mu_1 >_W \mu_2$ in an analogous way. If $\mu_1$ and $\mu_2$ are both stable, then (Roth and Sotomayor, 1990, Theorem 2.13)

$$\mu_1 >_M \mu_2 \Leftrightarrow \mu_2 >_W \mu_1. \tag{38}$$

Two stable deterministic matchings $\mu_1$ and $\mu_2$ are *consecutive* (Roth and Sotomayor, 1990, p. 61) if $\mu_1 >_M \mu_2$ holds and there does not exist a stable deterministic matching $\mu_3$ *between* $\mu_1$ and $\mu_2$, that is, satisfying $\mu_1 >_M \mu_3 >_M \mu_2$. Consecutive stable deterministic matchings exist if and only if there is more than one stable deterministic matching: If there is more than one stable deterministic matching, the M-optimal stable matching $\mu_M$ and the W-optimal stable matching $\mu_W$ (Roth and Sotomayor, 1990, Definition 2.11) satisfy $\mu_M >_M \mu_W$. The set of deterministic matchings is finite, therefore there exists a matching $\mu'$ such that $\mu_M$ and $\mu'$ are consecutive. To finish the proof of Proposition 2, it thus suffices to show random limit matchings exist if consecutive stable deterministic matchings exist. We do so through a sequence of lemmas.

**Lemma 8.** *Let $\mu_1$ and $\mu_2$ be consecutive stable deterministic matchings. Then there does not exist $(m, w) \in M \times W$ satisfying*

$$u(m, \mu_1(m)) > u(m, w) > u(m, \mu_2(m)) \text{ and } v(\mu_2(w), w) > v(m, w) > v(\mu_1(w), w). \tag{39}$$

*Proof.* Let $\mu_1$ and $\mu_2$ be stable deterministic matchings satisfying $\mu_1 >_M \mu_2$. Let $(m, w) \in M \times W$ satisfy (39). Because $\mu_1$ and $\mu_2$ are individually rational, it follows that $m$ and $w$ are mutually acceptable in the sense that $u(m, w) > 0$ and $v(m, w) > 0$ holds. In addition, it is immediate from (39) that for $\mu = \mu_1$ none of the conditions in property (5) on page 62 of Roth and Sotomayor (1990) is satisfied. Hence, $m$ and $w$ are on each other's lists under the *profile of reduced lists* $P(\mu_1)$ (Roth and Sotomayor, 1990, p. 61–62). Using the observation after the proof of Theorem 3.17 in Roth and Sotomayor (1990, p. 66), it follows that $u(m, \mu'(m)) \geq u(m, w)$ holds if $\mu_1$ and $\mu'$ are consecutive stable deterministic matchings. As (39) yields $u(m, w) > u(m, \mu_2(m))$, the matchings $\mu_1$ and $\mu_2$ are not consecutive. Hence, if $\mu_1$ and $\mu_2$ are consecutive stable deterministic matchings, then there does not exist a pair $(m, w) \in M \times W$ that satisfies (39). □

We say that the deterministic matchings $\mu_1$ and $\mu_2$ *support* the random matching $x$ if the assignment matrices $x_1$ and $x_2$ associated with $\mu_1$ and $\mu_2$ satisfy

$$x(m, w) = 0 \Leftrightarrow [x_1(m, w) = 0 \text{ and } x_2(m, w) = 0], \tag{40}$$

$$x(m, w) = 1 \Leftrightarrow [x_1(m, w) = 1 \text{ and } x_2(m, w) = 1], \tag{41}$$

for all $(m, w) \in M \times W$.

**Lemma 9.** *Suppose the random matching $x$ is supported by the consecutive stable deterministic matchings $\mu_1$ and $\mu_2$. If*

$$U(m;x) = u(m,\mu_2(m)) \text{ and } V(w;x) = v(\mu_1(w),w) \tag{42}$$

*holds for all $m \in M$ and $w \in W$, then $x$ is coherent and regret-free.*

*Proof.* Let $x$, $\mu_1$ and $\mu_2$ satisfy the conditions in the statement of the lemma, and let $x_1 \neq x_2$ denote the assignment matrices associated with $\mu_1$ and $\mu_2$.

Because $\mu_1$ and $\mu_2$ support $x$, (40) implies that $x(m,w) > 0$ only holds if either $x_1(m,w) = 1$ or $x_2(m,w) = 1$ (or both) hold. As $x_1$ and $x_2$ are individually rational this implies $u(m,w) > 0$. Hence, $x$ is individually rational.

From (41) for any pair $(m,w)$ that satisfies $x(m,w) = 1$, it must be that $U(m;x) = u(m,w)$ and $V(w;x) = v(m,w)$. Hence, condition (8) holds for such pairs. To verify that $x$ is regret-free, it remains to consider pairs $(m,w)$ for which $0 < x(m,w) < 1$. From (40) and (41), either $x_1(m,w) = 1$ or $x_2(m,w) = 1$ (but not both) hold. Consider the first of these cases. In this case, $u(m,w) = u(m,\mu_1(m)) \geq u(m,\mu_2(m))$ and $v(m,w) = v(\mu_1(w),w)$, where the inequality is from $\mu_1 >_M \mu_2$. Condition (42) then implies $u(m,w) \geq U(m;x)$ and $v(m,w) = V(w;x)$, establishing that (8) holds. In the second case, it must be that $u(m,w) = u(m,\mu_2(m))$ and $v(m,w) = v(\mu_2(w),w) \geq v(\mu_1(w),w)$, where the inequality is from $\mu_1 >_M \mu_2$ and (38). As in the previous case, (42) then implies (8). Hence, $x$ is regret-free.

It remains to show the pairwise stability of $x$, that is, to show there is no pair $(m,w)$ such that (7) holds. Suppose there is such a pair. Condition (42) then implies

$$u(m,w) > u(m,\mu_2(m)) \text{ and } v(m,w) > v(\mu_1(w),w). \tag{43}$$

By the stability of $\mu_2$, the first of these inequalities implies $v(\mu_2(w),w) \geq v(m,w)$, whereas from stability of $\mu_1$ the second inequality implies $u(m,\mu_1(m)) \geq u(m,w)$. From Lemma 8, one, at the most, one of the inequalities $v(\mu_2(w),w) \geq v(m,w)$ and $u(m,\mu_1(m)) \geq u(m,w)$ can be strict. Suppose the first inequality holds with equality. Because of the strictness of preferences, this requires $\mu_2(w) = m$, and, therefore, $U(m,\mu_2(m)) = u(m,w)$, which contradicts the first inequality in (43). Similarly, if $u(m,\mu_1(m)) = u(m,w)$ holds, then $v(m,w) = v(\mu_1(w),w)$, which contradicts the second inequality in (43). Therefore, there is no pair $(m,w)$ satisfying (43), proving that $x$ is pairwise stable. □

To conclude the proof of Proposition 2, it remains to show that if $\mu_1$ and $\mu_2$ are consecutive stable deterministic matchings, then there exists a random matching $x$ that is supported by $\mu_1$ and $\mu_2$ and satisfies (42). This is implied by the following lemma, which proves the stronger result that such a random matching $x$ exists whenever $\mu_1 >_M \mu_2$ holds.

**Lemma 10.** *Let $\mu_1$ and $\mu_2$ be stable deterministic matchings such that $\mu_1 >_M \mu_2$. Then there exists a feasible random matching $x$ that is supported by $\mu_1$ and $\mu_2$ and satisfies conditions (42).*

*Proof.* Let $\tilde{M} = \{m \in M \mid \mu_1(m) \neq \mu_2(m)\}$ and $\tilde{W} = \{w \in W \mid \mu_1(w) \neq \mu_2(w)\}$. As $\mu_1 \neq \mu_2$, we have $\tilde{M} \neq \emptyset$ and $\tilde{W} \neq \emptyset$. As $\mu_1$ and $\mu_2$ are both stable, the set of unmatched agents in these matchings is the same (cf. Roth and Sotomayor, 1990, Theorem 2.22). Therefore, $\mu_i(\tilde{M}) = \tilde{W}$ and $\mu_i(\tilde{W}) = \tilde{M}$ hold for $i = 1, 2$. Define an oriented graph whose nodes are $\tilde{M} \cup \tilde{W}$ as follows: (i) there is an arc from $m \in \tilde{M}$ to $w \in \tilde{W}$ if $\mu_1(m) = w$, and (ii) there is an arc from $w \in \tilde{W}$ to $m \in \tilde{M}$ if $\mu_2(w) = m$. Because every node in this finite graph has a unique direct successor and a unique direct predecessor, it follows that the graph is the union of a set of disjoint directed cycles. Let $m_1 w_1 m_2 w_2 \ldots m_\ell w_\ell m_{\ell+1}$ with $m_{\ell+1} = m_1$ be such a cycle and consider the set of equations

$$
\begin{aligned}
u(m_1, w_\ell) &= x(m_1, w_\ell)u(m_1, w_\ell) + x(m_1, w_1)u(m_1, w_1)\\
v(m_1, w_1) &= x(m_1, w_1)v(m_1, w_1) + x(m_2, w_1)v(m_2, w_1)\\
u(m_2, w_1) &= x(m_2, w_1)u(m_2, w_1) + x(m_2, w_2)u(m_2, w_2)\\
v(m_2, w_2) &= x(m_2, w_2)v(m_2, w_2) + x(m_3, w_2)v(m_3, w_2)\\
&\cdots\cdots\\
v(m_\ell, w_\ell) &= x(m_\ell, w_\ell)v(m_\ell, w_\ell) + x(m_{\ell+1}, w_\ell)v(m_{\ell+1}, w_\ell).
\end{aligned}
$$

Because the direct predecessor of each node is distinct from its direct successor, it must be that $\ell > 1$. For $i = 1, \cdots, \ell$ the following inequalities are satisfied:

$$u(m_i, w_i) > u(m_i, w_{i-1}) > 0, \tag{44}$$
$$v(m_{i+1}, w_i) > v(m_i, w_i) > 0, \tag{45}$$

where we set $w_0 = w_\ell$, and where the first inequality in (44) is from $\mu_1 >_M \mu_2$, the second inequality from the individual rationality of $\mu_2$, and in both cases the strictness of the inequality is from the strictness of preferences. Because of (38), an analogous argument yields (45). Using these properties, we now argue that the above systems of equations has a solution that satisfies

$$x^*(m_i, w_i) > 0,\ x^*(m_{i+1}, w_i) > 0, \tag{46}$$
$$x^*(m_i, w_i) + x^*(m_{i+1}, w_i) < 1,\ x^*(m_i, w_{i-1}) + x(m_1, w_i) < 1 \tag{47}$$

for all $i = 1, \cdots, \ell$.

We construct a solution as follows. For any $c \in [0,1]$, let $x(m_1, w_\ell) = c$. Using (44), the first equation determines $x(m_1, w_1) \in [0,1)$ as a strictly decreasing, affine function $h_1$ of $c$, which satisfies $h_1(1) = 0$. Substitute $x(m_1, w_1) = h_1(c)$ into the next equation, and note that (45) determines $x(m_2, w_1) \in (0,1)$ as a strictly increasing affine function $h_2$ of $c$. Proceeding in this fashion until the last equation is reached, this iterative procedure defines a strictly increasing and affine mapping from $[0,1]$ into $(0,1)$. Obviously, this function has a unique fixed point $0 < c^* < 1$. Let $x^*(m_{\ell+1}, w_\ell) = x^*(m_1, w_\ell) = c^*$ and use $h_1, h_2, \cdots$ to determine the remaining values. This yields a solution that satisfies (46). The inequalities in (47) are then implied by (44) and (45).

Apply this argument to all cycles; and complete the specification of $x^*$ by setting $x^*(m,w) = 0$ for $(m,w)$ satisfying $x_1(m,w) = x_2(m,w) = 0$ and by setting $x^*(m,w) = 1$ for $(m,w)$ satisfying $x_1(m,w) = x_2(m,w) = 1$, where $x_1$ and $x_2$ are the assignment matrices corresponding to $\mu_1$ and $\mu_2$. Because of its construction, $x^*$ satisfies (1) – (3) as well as (40) and (41). Thus, $x^*$ is a random matching supported by $\mu_1$ and $\mu_2$. For $m \notin \tilde{M}$ the matching $x^*$ satisfies $U(m; x^*) = u(m, \mu_1(m)) = u(m, \mu_2(m))$ so that (42) is satisfied. Similarly, (42) holds for $w \notin \tilde{W}$. For $m \in \tilde{M}$ we have (by the above construction of $x^*$):

$$u(m, \mu_2(m)) = x^*(m, \mu_1(m))u(m, \mu_1(m)) + x^*(m, \mu_2(m))u(m, \mu_2(m)).$$

As $U(m; x^*)$ is given by the right side of this equation, (42) holds. Similarly, for $w \in \tilde{W}$ we have

$$V(w; x) = v(\mu_1(w), w) = x^*(\mu_1(w), w)v(\mu_1(w)), w) + x^*(\mu_2(w), w)v(\mu_2(w), w),$$

yielding (42) for those types. □

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