ON THE TOPOLOGIES ON IND-VARIETIES AND RELATED IRREDUCIBILITY QUESTIONS

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ABSTRACT. In the literature there are two ways of endowing an affine ind-variety with a topology. One possibility is due to SHAFAREVICH and the other to KAMBAYASHI. In this paper we specify a large class of affine ind-varieties where these two topologies differ. We give an example of an affine ind-variety that is reducible with respect to SHAFAREVICH's topology, but irreducible with respect to KAMBAYASHI's topology. Moreover, we give a counter-example of a supposed irreducibility criterion given in [Sha81] which is different from the counter-example given by HOMMA in [Kam96]. We finish the paper with an irreducibility criterion similar to the one given by SHAFAREVICH.

0. **Introduction.** In the 1960s, in [Sha66], SHAFAREVICH introduced the notion of an infinite-dimensional variety and infinite-dimensional group. In this paper, we call them ind-variety and ind-group, respectively. His motivation was to explore some naturally occurring groups that allow a natural structure of an infinite-dimensional analogue to an algebraic group (such as the group of polynomial automorphisms of the affine space). More precisely, he defined an ind-variety as the successive limit of closed embeddings

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \dots$$

of ordinary algebraic varieties X_n and an ind-group as a group that carries the structure of an ind-variety compatible with the group structure. We denote the limit of $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ by $\varinjlim X_n$ and call $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ a *filtration*. If all X_n are affine, then $\varinjlim X_n$ is called *affine*. For example, one can define a filtration on the group of polynomial automorphisms of the affine space via the degree of an automorphism. Further examples of ind-groups are $\mathrm{GL}_n(k[t])$, $\mathrm{SL}_n(k[t])$, etc., where the filtrations are given via the degrees of the polynomial entries of the matrices (for properties of these filtrations in case $n = 2$ see [Sha04]). Fifteen years after his first paper [Sha66], SHAFAREVICH wrote another paper with the same title [Sha81], where he gave more detailed explanations of some statements of his first paper. Moreover, he endowed an ind-variety $\varinjlim X_n$ with the weak topology induced by the topological spaces $X_1 \subseteq X_2 \subseteq \dots$. Later KAMBAYASHI defined (affine) ind-varieties in [Kam96] and [Kam03] via a different approach. Namely, he defined an affine ind-variety as a certain spectrum of a so-called pro-affine algebra (see Section 1 for the definition). This pro-affine algebra is then the ring of regular functions on the affine ind-variety. With this approach KAMBAYASHI introduced a topology in a natural way on an affine ind-variety. Namely, a subset is closed

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if it is the zero-set of some regular functions on the affine ind-variety. In analogy to the Zariski topology defined on an ordinary affine variety, we call this topology again *Zariski topology*. In this paper, we call the weak topology on an affine ind-variety *ind-topology* to prevent confusion, as the weak topology is finer than the Zariski topology. The Zariski topology and the ind-topology differ in general. For example, it follows from Exercise 4.1.E, IV. in [Kum02] that these topologies differ on the infinite-dimensional affine space $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$ (see Example 1). The aim of this paper is to specify classes of affine ind-varieties where these topologies differ or coincide, and to study questions concerning the irreducibility of an affine ind-variety (with respect to these topologies).

This paper is organized as follows. We give some basic definitions and notations in Section 1. In the next section we describe a large class of ind-varieties where the two topologies differ. The main result of this paper is the following

Theorem A. *Let $X = \varinjlim X_n$ be an affine ind-variety. If there exists $x \in X$ such that X_n is normal or Cohen-Macaulay at x for infinitely many n, and the local dimension of X_n at x tends to infinity, then the ind-topology and the Zariski topology are different.*

This theorem follows from a more general statement given in Proposition 1 (see also Remark 1). As a corollary to this theorem we get

Corollary B. *Let $X = \varinjlim X_n$ be an affine ind-variety such that X_n is normal for infinitely many n. Then the ind-topology and the Zariski topology coincide if and only if for all $x \in X$ the local dimension of X_n at x is bounded for all n.*

This corollary follows from a more general statement given in Corollary 6. As a contrast to Theorem A, we show in Proposition 7 that the two topologies coincide if $X = \varinjlim X_n$ is "locally constant" with respect to the Zariski topology. More precisely we prove

Proposition C. *If $X = \varinjlim X_n$ is an affine ind-variety such that every point has a Zariski open neighbourhood U with $U \cap X_n = U \cap X_{n+1}$ for all sufficiently large n, then the ind-topology and the Zariski topology coincide.*

Section 3 contains an example of an affine ind-variety that is reducible with respect to the ind-topology, but irreducible with respect to the Zariski topology. This is the content of Example 4.

In the last section we give a counter-example to Proposition 1 in [Sha81] (see Example 5). The content of the proposition is: an ind-variety $X = \varinjlim X_n$ is irreducible with respect to the ind-topology if and only if the set of irreducible components of all X_n is directed under inclusion. One can see that the latter condition is equivalent to the existence of a filtration $X_1' \hookrightarrow X_2' \hookrightarrow \dots$ where each X_n' is irreducible and $\varinjlim X_n' = X$. In [Kam96], Homma gave a counter-example to that supposed irreducibility criterion. But in contrast to his counter-example, the number of irreducible components of X_n in our counter-example is bounded for all n. We finish the paper with the following irreducibility criterion. The proposition follows from Proposition 8.

Proposition D. *Let $X = \varinjlim X_n$ be an affine ind-variety where the number of irreducible components of X_n is bounded for all n. Then X is irreducible with respect to the ind-topology (Zariski topology) if and only if there exists a chain of irreducible subvarieties $X'_1 \subseteq X'_2 \subseteq \ldots$ in X (i.e., X'_n is an irreducible subvariety of some X_m) such that $\bigcup_n X'_n$ is dense in X with respect to the ind-topology (Zariski topology).*

1. **Definitions and notation.** Throughout this paper we work over an uncountable algebraically closed field k. We use the definitions and notation of Kambayashi in [Kam03] and Kumar in [Kum02]. Let us recall them briefly. A *pro-affine algebra* is a complete and separated commutative topological k-algebra such that 0 admits a countable base of open neighbourhoods consisting of ideals. Let A be a pro-affine algebra and let $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots$ be a base for $0 \in A$ as mentioned above. Let $A_n = A/\mathfrak{a}_n$ and let $\mathfrak{Spm}(A)$ be the set of closed maximal ideals of A. Then we have

$$A = \varprojlim A_n \quad \text{and} \quad \mathfrak{Spm}(A) = \bigcup_{n=1}^{\infty} \mathfrak{Spm}(A_n)$$

(cf. 1.1 and 1.2 in [Kam03]).

Definition 1. An *affine ind-variety* is a pair $(\mathfrak{Spm}(A), A)$ where A is a pro-affine algebra such that $A/\mathfrak{a}_n$ is reduced and finitely generated for some countable base of ideals $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots$ of $0 \in A$. We call A the *coordinate ring* of the affine ind-variety and the elements of A *regular functions*. Two ind-varieties are called *isomorphic* if the underlying pro-affine algebras are isomorphic as topological k-algebras. Such an isomorphism induces then a bijection of the spectra.

One can construct affine ind-varieties in the following way. Consider a *filtration of affine varieties*, i.e., a countable sequence of closed embeddings of affine varieties

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots .$$

Let $X = \bigcup_{n=1}^{\infty} X_n$ as a set and let $\mathcal{O}(X) := \varprojlim \mathcal{O}(X_n)$. We endow $\mathcal{O}(X)$ with the topology induced by the product topology of $\prod_n \mathcal{O}(X_n)$, where $\mathcal{O}(X_n)$ carries the discrete topology for all n. Then $(\mathfrak{Spm}(\mathcal{O}(X)), \mathcal{O}(X))$ is an affine ind-variety and there is a natural bijection $X \to \mathfrak{Spm}(\mathcal{O}(X))$ induced by the bijections $X_n \to \mathfrak{Spm}(\mathcal{O}(X_n))$. In the following, we denote this ind-variety by $\varinjlim X_n$. In fact, every affine ind-variety can be constructed in this way (up to isomorphy). Two filtrations $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ and $X'_1 \hookrightarrow X'_2 \hookrightarrow \ldots$ induce the same affine ind-variety (up to isomorphy) if and only if there exists a bijection

$$f \colon \bigcup_{n=1}^{\infty} X_n \to \bigcup_{n=1}^{\infty} X'_n$$

with the following property: for every i there exists j_i and for every j there exists i_j, such that $f|_{X_i} \colon X_i \to X_{j_i}$ and $f^{-1}|_{X_j} \colon X_j \to X_{i_j}$ are closed embeddings of affine varieties. Such filtrations are called *equivalent*.

2. **Topologies on affine ind-varieties.** So far we have not established any topology on the set $\mathfrak{Spm}(A)$ of an affine ind-variety $(\mathfrak{Spm}(A), A)$. As mentioned in the introduction there are two ways to introduce a topology on the set $\mathfrak{Spm}(A)$. The first possibility is due to Shafarevich [Sha66], [Sha81] and we call it the *ind-topology*. A subset $Y \subseteq \mathfrak{Spm}(A)$ is closed in this topology if and only if $A \cap \mathfrak{Spm}(A_n)$ is a closed subset of $\mathfrak{Spm}(A_n)$ for all n. One can easily check that this topology does

not depend on the choice of the ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$. The second possibility is due to KAMBAYASHI [Kam96], [Kam03] and we call it the *Zariski topology.* The closed subsets in this topology are the subsets of the form

$$V_{\mathfrak{Spm}(A)}(E) := \{\, \mathfrak{m} \in \mathfrak{Spm}(A) \mid \mathfrak{m} \supseteq E \,\},$$

where E is any subset of A. Clearly, the ind-topology is finer than the Zariski topology. But in general these two topologies on $\mathfrak{Spm}(A)$ differ. In the next proposition (which implies Theorem A) we specify a large class of affine ind-varieties where the two topologies differ.

Proposition 1. *We assume that* $\operatorname{char}(k) = 0$. *Let* $X = \varinjlim X_n$ *be an affine ind-variety. Assume that there exists* $x \in X$ *such that* $\mathcal{O}_{X_n,x}$ *satisfies* SERRE*'s condition* (S_2) *for infinitely many* n *and* $\dim_x X_n \to \infty$ *if* $n \to \infty$. *Then there exists a subset* $Y \subseteq X$ *such that*

i) Y *is closed in* X *with respect to the ind-topology,*
ii) Y *is not closed in* X *with respect to the Zariski topology.*

In particular, there exists no isomorphism $X \to X$ *of affine ind-varieties that is a homeomorphism if we endow the first* X *with the ind-topology and the second* X *with the Zariski topology.*

Remark 1. A Noetherian ring A satisfies SERRE's condition (S_2) if $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{\dim A_{\mathfrak{p}}, 2\}$ for all primes $\mathfrak{p} \subseteq A$. For example, this is satisfied if A is normal (and hence also if A is a unique factorization domain) or Cohen-Macaulay (and hence also if A is Gorenstein, locally a complete intersection or regular) (see Theorem 23.8 [Mat86]).

We will use the following lemmata to prove Proposition 1.

Lemma 2. *Let* Z *and* Y *be affine varieties and assume that there exists a closed embedding* $Z \hookrightarrow Y$. *If* $f\colon Z \twoheadrightarrow \mathbb{A}^{\dim Z}$ *is a finite surjective morphism, then there exists a finite surjective morphism* $g\colon Y \twoheadrightarrow \mathbb{A}^{\dim Y}$ *such that* $g|_Z = \iota \circ f$, *where* $\iota : \mathbb{A}^{\dim Z} \hookrightarrow \mathbb{A}^{\dim Y}$ *is given by* $\iota(v) = (v, 0)$.

Proof. Let $A := \mathcal{O}(Z)$, $B := \mathcal{O}(Y)$ and let $\psi\colon B \twoheadrightarrow A$ be the surjective homomorphism induced by $Z \hookrightarrow Y$. Further, let $f_1, \dots, f_n$ be the coordinate functions of f. By assumption $k[f_1, \dots, f_n] \subseteq A$ is an integral extension and $f_1, \dots, f_n$ are algebraically independent. Choose generators $b_1, \dots, b_l$ of the k-algebra B such that $\psi(b_i) = f_i$ for $i = 1, \dots, n$. For every $j = n+1, \dots, l$ there exists a monic polynomial $p_j \in k[b_1, \dots, b_n][T]$ such that $h_j := p_j(b_j) \in \ker(\psi)$, since $k[f_1, \dots, f_n] \subseteq A$ is integral. Thus,

$$k[b_1, \dots, b_n, h_{n+1}, \dots, h_l] \subseteq B$$

is an integral extension. If $b_1, \dots, b_n, h_{n+1}, \dots, h_l$ are algebraically independent, then we are done. Otherwise, there exists a non-zero polynomial $f(X_1, \dots, X_l)$ with coefficients in k such that $f(b_1, \dots, b_n, h_{n+1}, \dots, h_l) = 0$. Exactly the same as in the proof of Lemma 2, §33 [Mat86] one can see that there exist $c_1, \dots, c_{l-1} \in k$ such that h_l is integral over $k[b'_1, \dots, b'_n, h'_{n+1}, \dots, h'_{l-1}]$, where $b'_i := b_i - c_i h_l$ and $h'_i := h_i - c_i h_l$. Thus,

$$k[b'_1, \dots, b'_n, h'_{n+1}, \dots, h'_{l-1}] \subseteq B$$

is an integral extension. By induction there exists m with $n \leq m < l$ and algebraically independent elements $b''_1, \dots, b''_n, h''_{n+1}, \dots, h''_m \in B$ such that B is integral over $k[b''_1, \dots, b''_n, h''_{n+1}, \dots, h''_m]$ and $b''_i - b_i$, $h''_i \in \ker(\psi)$. This proves the lemma. ☐

Remark 2. From an iterative use of the lemma above we can deduce the following. For every affine ind-variety $X = \varinjlim X_n$ there exists a surjective map of the underlying sets $X \twoheadrightarrow \mathbb{A}^\infty$ such that the restriction to every X_n yields a finite surjective morphism $X_n \twoheadrightarrow \mathbb{A}^{\dim X_n}$.

Lemma 3. *We assume that* $\operatorname{char}(k) = 0$. *Let* Y *be an irreducible affine variety and let* X *be an affine scheme of finite type over* k *that is reduced in an open dense subset. If* $f \colon X \to Y$ *is a dominant morphism, then there exists an open dense subset* $U \subseteq Y$ *such that* $f^{-1}(u)$ *is reduced in an open dense subset for all* $u \in U$.

Proof. Without loss of generality, one can assume that f is flat and surjective (see Theorem 14.4 (Generic freeness) [Eis95]). Since X is reduced in an open dense subset, there exists an open dense subset $X' \subseteq X$ such that all fibres of $f|_{X'} \colon X' \to Y$ are reduced (see Corollary 10.7, Ch. III (Generic smoothness) and Theorem 10.2, Ch. III [Har77]; here we use $\operatorname{char}(k) = 0$). Let $K := X \setminus X'$ be endowed with the reduced induced closed subscheme structure of X and let $g := f|_K \colon K \to Y$. If g is not dominant, then the fibres of f over an open dense subset are reduced and we are done. Hence we can assume that g is dominant. Again according to Theorem 24.1 [Mat86] there exists an open dense subset $U \subseteq Y$ such that $g|_{g^{-1}(U)} \colon g^{-1}(U) \twoheadrightarrow U$ is flat and surjective. Thus, we have for all $u \in U$ and $x \in g^{-1}(u)$

$$\dim_x g^{-1}(u) = \dim_x g^{-1}(U) - \dim_u U < \dim_x X - \dim_u Y = \dim_x f^{-1}(u) \,.$$

It follows that $f^{-1}(u) \setminus g^{-1}(u)$ is a reduced open dense subscheme of $f^{-1}(u)$ for all $u \in U$. This implies the lemma. □

According to Ex. 11.10 [Eis95] we have the following criterion for reducedness of a Noetherian ring.

Lemma 4. *A Noetherian ring* A *is reduced if and only if*

(R_0) *the localization of* A *at each prime ideal of height* 0 *is regular,*
(S_1) A *has no embedded associated prime ideals.*

One can see that condition (R_0) is satisfied for a Noetherian ring A if $\operatorname{Spec}(A)$ is reduced in an open dense subset. Thus we get the following

Lemma 5. *Let* X *be a Noetherian affine scheme that is reduced in an open dense subset. If* $\mathcal{O}_{X,x}$ *satisfies* (S_1) *for a point* $x \in X$, *then* $\mathcal{O}_{X,x}$ *is reduced.*

Now we have the preliminary results to prove Proposition 1. The strategy is as follows. First we construct $0 \neq f_n \in \mathcal{O}(X_n)$ such that $f_n(x) = 0$, $f_n|_{X_{n-1}} = f_{n-1}^2$ and $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is reduced. The main part of the proof is devoted to showing the reducedness and for that matter we use the condition (S_2) of the local ring $\mathcal{O}_{X_n,x}$. Then we define $Y := \bigcup_n V_{X_n}(f_n)$. It follows that Y is closed in X with respect to the ind-topology. Afterwards, we prove that Y is not closed in X with respect to the Zariski topology. For that purpose, we take $\varphi = (\varphi_n) \in \mathcal{O}(X) = \varprojlim \mathcal{O}(X_n)$ that vanishes on Y, and we show that φ_n vanishes also on all irreducible components of X_n passing through x. The latter we deduce from the fact that

$$\varphi_n = \varphi_{n+i}|_{X_n} \in f_{n+i}|_{X_n}\mathcal{O}_{X_n,x} = f_n^{2^i}\mathcal{O}_{X_n,x}$$

for all $i \geq 0$ and Krull's Intersection Theorem.

Proof of Proposition 1. For the sake of simpler notation, we assume that $\mathcal{O}_{X_n,x}$ satisfies (S_2) and $\dim_x X_n = n$ for all n. Let X'_n be the union of all irreducible components of X_n containing x and let W_n be the union of all irreducible components of all X_i with $i \leq n$, not containing x and of strictly smaller dimension than n. Then, $X'_1 \cup W_1 \hookrightarrow X'_2 \cup W_2 \hookrightarrow \ldots$ is an equivalent filtration of X to $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$, since $\dim_x X_n \to \infty$. Thus, we can further impose that $\dim_x X_n = \dim X'_n = \dim X_n$ and $\dim_p X_n < \dim X_n$ for all $p \notin X'_n$. As $\mathcal{O}_{X_n,x}$ satisfies (S_2), it follows from Corollary 5.10.9 [Gro65] that X'_n is equidimensional.

Now, we construct the $0 \neq f_n \in \mathcal{O}(X_n)$. From Lemma 2 it follows that we can choose algebraically independent elements $x_1, \ldots, x_n \in \mathcal{O}(X_n)$ such that $\mathcal{O}(X_n)$ is finite over $k[x_1, \ldots, x_n]$ and x_n restricted to X_{n-1} is zero. We can assume that the finite morphism $X_n \twoheadrightarrow \mathbb{A}^n$ induced by $k[x_1, \ldots, x_n] \subseteq \mathcal{O}(X_n)$ sends x to $0 \in \mathbb{A}^n$. Since $\dim_p X_n < \dim X_n$ for all $p \notin X'_n$ and X'_n is equidimensional, it follows that

$$(*) \qquad k[x_1, \ldots, x_n] \hookrightarrow \mathcal{O}(X_n) \twoheadrightarrow \mathcal{O}(K) \quad \text{is injective}$$

for all irreducible components K of X'_n. Let us define

$$f_1 := c_1 x_1 \quad \textit{and} \quad f_{n+1} := f_n^2 + c_{n+1} x_{n+1} \, ,$$

where $c_1, c_2, \ldots \in k$, not all equal to zero. It follows that $f_n(x) = 0$ and $f_{n+1}|_{X_n} = f_n^2$. The aim is to prove that $c_1, c_2, \ldots \in k$ can be chosen such that not all are equal to zero and $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is reduced for $n > 1$. Consider the morphism

$$\psi_n \colon Z_n \longrightarrow \mathbb{A}^n \, ,$$

where Z_n is the affine scheme with coordinate ring

$$S_n := \mathcal{O}(X'_n)[c_1, \ldots, c_n]/(f_n)$$

and ψ_n is the restriction of the canonical projection $X'_n \times \mathbb{A}^n \twoheadrightarrow \mathbb{A}^n$ to the closed subscheme Z_n. If $(c_1, \ldots, c_n)$ is fixed, then $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is the local ring of the fibre $\psi_n^{-1}(c_1, \ldots, c_n)$ in the point $(x, c_1, \ldots, c_n) \in Z_n$. For that reason we will study the fibres of the morphism $\psi_n \colon Z_n \to \mathbb{A}^n$. We claim that Z_n is reduced in an open dense subset for $n > 1$. To prove this claim, we mention first that

$$(S_n)_{x_n} \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_n]/(f_{n-1}^2 + c_n x_n) \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_{n-1}]$$

is reduced. Let $R_n := k[x_1, \ldots, x_n][c_1, \ldots, c_n]/(f_n)$. It follows that the morphisms $\operatorname{Spec}(S_n) \twoheadrightarrow \operatorname{Spec}(R_n)$ and $\operatorname{Spec}(S_n/(x_n)) \twoheadrightarrow \operatorname{Spec}(R_n/(x_n))$ are both finite and surjective. As $\dim R_n/(x_n) < \dim R_n$ for $n > 1$ we get $\dim S_n/(x_n) < \dim S_n$. Since X'_n is equidimensional one can deduce from $(*)$ that Z_n is equidimensional. Hence, $\operatorname{Spec}((S_n)_{x_n}) \subseteq Z_n$ is an open dense reduced subscheme.

Since $\{x\} \times \mathbb{A}^n$ is contained in Z_n, it follows that ψ_n is surjective. For $n > 1$ there exists an open dense subset $U_n \subseteq \mathbb{A}^n$ such that

$$\psi_n|_{\psi_n^{-1}(U_n)} \colon \psi_n^{-1}(U_n) \twoheadrightarrow U_n$$

is surjective and flat, and every fibre is reduced in an open dense subset (see Lemma 3 and Theorem 24.1 [Mat86]). With the aid of $(*)$ it follows that f_n is an $\mathcal{O}_{X_n,x}$-regular sequence for every choice $(c_1, \ldots, c_n) \in U_n$. Since $\mathcal{O}_{X_n,x}$ satisfies (S_2), we get from Corollary 5.7.6 [Gro65] that $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ satisfies (S_1). But as $\psi_n^{-1}(c_1, \ldots, c_n)$ is reduced in an open dense subset, it follows from Lemma 5 that it is reduced in the point $(x, c_1, \ldots, c_n)$. Hence, for $n > 1$ it follows that $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$

is reduced if we choose $(c_1, \ldots, c_n) \in U_n$. For $i \geq n$ let $\pi_n^i : \mathbb{A}^i \twoheadrightarrow \mathbb{A}^n$ be the projection onto the first n components. As the field k is uncountable, one can choose inductively

$$0 \neq c_1 \in \bigcap_{i \geq 1} \pi_1^i(U_i)\,, \quad (c_1, \ldots, c_n, c_{n+1}) \in \bigcap_{i \geq n+1} \pi_{n+1}^i(U_i) \cap \{(c_1, \ldots, c_n)\} \times \mathbb{A}^1\,.$$

Hence, $(c_1, \ldots, c_n) \in U_n$ for all $n > 1$ and not all $c_1, c_2, \ldots$ are equal to zero. This finishes the construction of the f_n.

Let us define $Y := \bigcup_n V_{X_n}(f_n)$. Since $f_{n+1}|_{X_n} = f_n^2$ for all n, Y satisfies i). Take any $\varphi = (\varphi_n) \in \varprojlim \mathcal{O}(X_n)$ that vanishes on Y. We claim that $\varphi|_{X'} = 0$, where $X' := \bigcup_n X'_n$. It is enough to prove that $\varphi_n = 0$ in $\mathcal{O}_{X_n,x}$. Since $\varphi_m|_{Y_m} = 0$ and $\mathcal{O}_{X_m,x}/f_m\mathcal{O}_{X_m,x}$ is reduced, it follows that $\varphi_m \in f_m\mathcal{O}_{X_m,x}$. Using $f_{m+1}|_{X_m} = f_m^2$ again, we get by induction

$$\varphi_n = \varphi_{n+i}|_{X_n} \in f_n^{2^i}\mathcal{O}_{X_n,x} \quad \text{for all } i \geq 0\,,\ n > 1\,.$$

But according to Krull's Intersection Theorem (see Theorem 8.10 [Mat86]), we have $\bigcap_{i \geq 0} f_n^i \mathcal{O}_{X_n,x} = 0$, hence $\varphi_n = 0$ in $\mathcal{O}_{X_n,x}$. Since $f_n|_{X'_n} \neq 0$ (cf. ($*$)), we get $X' \cup Y \supsetneq Y$. Thus Y satisfies ii) according to the afore mentioned claim. $\square$

The following example is a special case of the construction in the proof of Proposition 1. We mention it here, since we will use it in future examples.

Example 1 (See Ex. 4.1.E, IV. in [Kum02])**.** Let $f_n \in k[x_1, \ldots, x_n] = \mathcal{O}(\mathbb{A}^n)$ be recursively defined as

$$f_1 := x_1\,, \quad f_{n+1} := f_n^2 + x_{n+1}\,.$$

Then $\bigcup_n V_{\mathbb{A}^n}(f_n)$ is a proper closed subset of the infinite-dimensional affine space $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$ with respect to the ind-topology, but it is dense in $\mathbb{A}^\infty$ with respect to the Zariski topology.

Let $\mathcal{G}$ be the group of polynomial automorphisms of the affine space $\mathbb{A}^n$, where n is a fixed number ≥ 2. We prove in the next example that the ind-topology and the Zariski topology on $\mathcal{G}$ differ if we consider $\mathcal{G}$ as an affine ind-variety via the filtration given by the degree of an automorphism.

Example 2. First, we define on $\mathcal{G}$ a filtration of affine varieties (via the degree). Let $\mathcal{E}$ be the set of polynomial endomorphisms of the affine space $\mathbb{A}^n$ and let $\mathcal{E}_d$ be the subset of all $\varphi \in \mathcal{E}$ of degree $\leq d$. Denote by $U_d \subseteq \mathcal{E}_d$ the subset of all $\varphi \in \mathcal{E}_d$ such that $\mathrm{Jac}(\varphi) \in k^*$. One can see that $U_d \subseteq \mathcal{E}_d$ is a locally closed subset and it inherits the structure of an affine variety from $\mathcal{E}_d$. With Corollary 0.2 [Kam79] and the estimate of the degree of the inverse of an automorphism due to Gabber (see Corollary 1.4 in [BCW82]) one can deduce that $\mathcal{G}_d \subseteq U_d$ is a closed subset. Thus $\mathcal{G}_d$ is locally closed in $\mathcal{E}_d$ and it inherits the structure of an affine variety from $\mathcal{E}_d$. Moreover, one can see that $\mathcal{G}_d$ is closed in $\mathcal{G}_{d+1}$. In the following, we consider $\mathcal{G}$ as an affine ind-variety via the filtration $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \ldots$ of affine varieties.

We claim that the ind-topology and the Zariski topology on $\mathcal{G}$ differ. Consider the subset

$$M := \{\, (x_1 + p, x_2, \ldots, x_n) \in \mathcal{G} \mid p \in k[x_n]\,\} \subseteq \mathcal{G}\,.$$

It is closed in $\mathcal{G}$ with respect to the ind-topology. We consider M as an affine ind-variety via $M := \varinjlim M \cap G_d$ and thus $M \simeq \mathbb{A}^\infty$ as affine ind-varieties. According to Example 1 there exists a proper subset $Y \subsetneq M$ that is closed with respect to the ind-topology, but it is dense in M with respect to the Zariski topology. Hence, every regular function on $\mathcal{G}$ vanishing on Y, vanishes also on M. This implies the claim.

Remark 3. A similar argument as in Example 2 shows that the ind-topology and the Zariski topology differ on $\mathrm{GL}_n(k[t])$ and also on $\mathrm{SL}_n(k[t])$.

Next, we give a corollary to Proposition 1 which implies Corollary B. Before we state the corollary, we introduce the following notation. For any ind-variety $X = \varinjlim X_n$ we choose connected components X_n^i of X_n, $i = 1, \ldots, k_n$, such that

$$X_n = \bigcup_{i=1}^{k_n} X_n^i \quad \text{and} \quad X_n^i \subseteq X_{n+1}^i \text{ for all } i = 1, \ldots, k_n$$

(it can be that $X_n^i = X_n^j$ for $i \neq j$). We remark that the decomposition of an ind-variety into connected components is the same for the ind-topology and the Zariski topology.

Corollary 6. *We assume that* $\mathrm{char}(k) = 0$. *Let* $X = \varinjlim X_n$ *be an affine ind-variety such that for i fixed, the number of irreducible components of* X_n^i *is bounded for all n. Moreover, assume that* $\mathcal{O}(X_n)$ *satisfies* (S_2) *for infinitely many n. Then the following statements are equivalent:*

i) *The ind-topology and the Zariski topology on* X *coincide.*
ii) *For all* $x \in X$ *the local dimension of* X_n *at* x *is bounded for all* n.
iii) *Every connected component of* X *is contained in some* X_n.

Proof. Every connected component of X is equal to some $X^i := \bigcup_n X_n^i$.
i) $\Rightarrow$ ii): This follows from Proposition 1.
ii) $\Rightarrow$ iii): As X_n^i satisfies (S_2) and is connected, X_n^i is equidimensional (see Corollary 5.10.9 [Gro65]). Thus, $X_n^i = X_{n+1}^i$ for n large enough, as the number of irreducible components of X_n^i is bounded for all n. Thus, $X^i \subseteq X_n$ for some n.
iii) $\Rightarrow$ i): This follows from the fact that every connected component of X is closed and open with respect to the Zariski topology. □

As a contrast to Proposition 1, the two topologies coincide if the affine ind-variety is "locally constant" with respect to the Zariski topology. The following proposition coincides with Proposition C.

Proposition 7. *Let* $X = \varinjlim X_n$ *be an affine ind-variety. Assume that every* $x \in X$ *has a Zariski open neighbourhood* $U_x \subseteq X$ *such that* $U_x \cap X_n = U_x \cap X_{n+1}$ *for all sufficiently large n. Then the two topologies on* X *coincide.*

Proof. Let $Y \subseteq X$ be a closed subset with respect to the ind-topology. One can see that $Y \cap U_x$ is closed in U_x with respect to the Zariski topology for all $x \in X$. This proves that Y is closed in X with respect to the Zariski topology. □

The following example is an application of the proposition above. We construct a proper ind-variety (i.e., it is not a variety) such that the ind-topology and the Zariski topology coincide and moreover, it is connected.

Example 3. Let L_n be defined as

$$L_n := V_{\mathbb{A}^n}(x_1 - 1, x_2 - 1, \ldots, x_{n-1} - 1) \subseteq \mathbb{A}^n .$$

Remark that $L_n \cap L_{n+1} = \{(1, \ldots, 1)\} \subseteq \mathbb{A}^n$ for all n and $L_n \cap L_m = \varnothing$ for all n, m with $|n - m| \geq 2$. Let $X := \varinjlim X_n$ where $X_n := L_1 \cup \ldots \cup L_n \subseteq \mathbb{A}^n$. It follows that $X \subseteq \mathbb{A}^\infty$ is a closed connected subset in the ind-topology. We claim that the ind-topology and the Zariski topology on X coincide. According to the proposition above it is enough to show that X is "locally constant" with respect to the Zariski topology. Let $x \in X$. Then there exists N such that $x \in L_N$, but $x \notin L_{N+1}$. Let $U_x := X \setminus V_{\mathbb{A}^\infty}(f_1, \ldots, f_N) \subseteq X$ where $f_i \in \mathcal{O}(\mathbb{A}^\infty)$ is given by

$$f_i|_{\mathbb{A}^n} = x_i - 1 \in k[x_1, \ldots, x_n] \quad \text{for all } n \geq N.$$

Thus, $U_x \subseteq X$ is a Zariski open neighbourhood of x. Moreover, for all $n > N$ we have $L_n \subseteq V_{\mathbb{A}^\infty}(f_1, \ldots, f_N)$. Hence we have $U_x \cap X_n = U_x \cap X_{n+1}$ for all $n \geq N$.

As remarked before Corollary 6, connectedness of an affine ind-variety is the same for both topologies. But this is no longer true for irreducibility as we will see in the next section (see Example 4).

3. **Irreducibility via the coordinate ring.**

It is well known that an affine variety X is irreducible if and only if the coordinate ring $\mathcal{O}(X)$ is an integral domain. This statement remains true for affine ind-varieties endowed with the Zariski topology. The proof is completely analogous to the proof for affine varieties. In the case of the ind-topology it is still true that $\mathcal{O}(X)$ is an integral domain if X is irreducible, as the ind-topology is finer than the Zariski topology. But the converse is in general false. In the following we give an example of an affine ind-variety X, which is reducible in the ind-topology, but its coordinate ring $\mathcal{O}(X)$ is an integral domain and thus it is irreducible in the Zariski topology.

Example 4. Throughout this example we work in the ind-topology. Let $g_n \in k[x_1, \ldots, x_n]$ be defined as

$$g_n := x_1 + \ldots + x_n \,,$$

and let f_n be defined as in Example 1. By construction, f_n and g_n are irreducible polynomials. The affine ind-variety $X := \varinjlim (V_{\mathbb{A}^n}(f_n) \cup V_{\mathbb{A}^n}(g_n))$ decomposes into the proper closed subsets $\bigcup_n V_{\mathbb{A}^n}(f_n)$ and $\bigcup_n V_{\mathbb{A}^n}(g_n))$ and thus X is reducible. We claim that $\mathcal{O}(X) = \varprojlim k[x_1, \ldots, x_n]/(f_n g_n)$ is an integral domain. Assume towards a contradiction that there exist $(\varphi_n), (\psi_n) \in \prod_{n=1}^\infty k[x_1, \ldots, x_n]$ such that (φ_n) and (ψ_n) define non-zero elements in $\mathcal{O}(X)$, but $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$. By definition, there exists $\alpha_n \in k[x_1, \ldots, x_n]$ such that

$$(*) \qquad \varphi_{n+1}(x_1, \ldots, x_n, 0) = \varphi_n + f_n g_n \alpha_n \quad \text{for all } n \,.$$

Since $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$, it follows that $f_n g_n$ divides $\varphi_n \psi_n$ for $n > 0$. Hence we can assume without loss of generality that f_n divides φ_n for infinitely many n. Eq. $(*)$ and the definition of f_{n+1} show that f_n divides φ_n for all n. Since $(\varphi_n) \neq 0$ in $\mathcal{O}(X)$ there exists $N > 1$ such that g_N does not divide φ_N. Let $\rho_n \in k[x_1, \ldots, x_n]$ such that $\varphi_n = f_n \rho_n$. It follows that g_N does not divide ρ_N, in particular $\rho_N \neq 0$. According to $(*)$ and the definition of f_{n+1} we have

$$(**) \qquad \rho_n = f_n \cdot \rho_{n+1}(x_1, \ldots, x_n, 0) - g_n \cdot \alpha_n \quad \text{for all } n \,.$$

Since g_N does not divide ρ_N it follows that there exists $p \in \mathbb{A}^N$ with $g_N(p) = 0$ and $\rho_N(p) \neq 0$. Let $\gamma_n \colon \mathbb{A}^1 \to \mathbb{A}^n$ be the curve defined by $\gamma_n(t) = (p, 0, \ldots, 0) +$

$(t, -t, 0, \dots, 0)$ for $n \geq N$. Since $g_n(\gamma_n(t)) = 0$ it follows from (∗∗) that $\rho_n(\gamma_n(t)) = f_n(\gamma_n(t))\rho_{n+1}(\gamma_{n+1}(t))$. This implies

$$0 \neq \rho_N(\gamma_N(t)) = \left(\prod_{i=N}^{n-1} f_i(\gamma_i(t))\right) \cdot \rho_n(\gamma_n(t)) \quad \text{for all } n \geq N\,.$$

Since $f_i(\gamma_i(t))$ is a polynomial of degree 2^{i-1} for all $i \geq N$, it follows that the polynomial $\rho_N(\gamma_N(t))$ is of unbounded degree, a contradiction.

4. **Irreducibility via the filtration.** One would like to give a criterion for connectedness or irreducibility in terms of the filtration $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ of the affine ind-variety. In the case of connectedness Shafarevich gave a nice description via the filtration (see Proposition 2 [Sha81]) and Kambayashi gave a proof for it (see Proposition 2.4 [Kam96]) (the proof works in both topologies, as connectedness of an affine ind-variety is the same for both topologies). In the case of irreducibility, things look different.

If we start with a filtration $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ of irreducible affine varieties, then one can see that $\varinjlim X_n$ is an irreducible affine ind-variety in both topologies. Likewise one can ask if every irreducible affine ind-variety is obtained from a filtration of irreducible affine varieties. One can see that the latter property is equivalent to the following condition: the set $\mathscr{K}$ of all irreducible components of all X_n is directed under inclusion for some (and hence every) filtration $X_1 \hookrightarrow X_2 \hookrightarrow \dots$. Shafarevich claims in [Sha81] that the latter condition is equivalent to the irreducibility of X in the ind-topology. But Homma gave in [Kam96] a counter-example X to this statement. For every filtration $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ of Homma's counter-example X the number of irreducible components of X_n tends to infinity if $n \to \infty$. Here we give another counter-example. Namely, we construct an irreducible affine ind-variety $X = \varinjlim X_n$ (irreducible with respect to both topologies) such that $\mathscr{K}$ is not directed, but X_n consists of exactly two irreducible components for $n > 1$.

Example 5. Let us define $g_n \in k[x_1, \dots, x_n]$ recursively by

$$g_1 := (x_1 - 1)\,, \qquad g_{n+1} := (x_1 - (n+1)) \cdot g_n - x_{n+1}\,.$$

By construction every g_n is an irreducible polynomial. Let $Y_n := V_{\mathbb{A}^n}(g_n) \subseteq \mathbb{A}^n$. It follows that $Y_n \subseteq Y_{n+1}$ for all n. Let further $Z_n := V_{\mathbb{A}^n}(x_2, \dots, x_n) \subseteq \mathbb{A}^n$ and $X_n := Y_n \cup Z_n$. It follows that $X_n \subseteq X_{n+1}$ is a closed subset for all n. Let $X := \varinjlim X_n$. We get

$$Y_n \cap Z_n = V_{\mathbb{A}^n}(g_n, x_2, \dots, x_n) = V_{\mathbb{A}^n}(\prod_{i=1}^{n}(x_1 - i), x_2, \dots, x_n) = \{e_1, 2e_1, \dots, ne_1\}\,,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{A}^n$. The set $\mathscr{K}$ defined above is not directed and X_n decomposes in two irreducible components for $n > 1$. It remains to show that X is irreducible with respect to the ind-topology, as in that case X is also irreducible in the Zariski topology. As Y_n is irreducible for all n, it follows that $Y = \bigcup_n Y_n$ is irreducible. Since

$$Z_m \subseteq \overline{\bigcup_{n=1}^{\infty} Y_n \cap Z_n} \subseteq \overline{Y} \subseteq X \qquad \text{for all } m\,,$$

we have $X = \overline{Y}$, where the closure is taken in the ind-topology. Since Y is irreducible, as a consequence X is also irreducible.

We conclude this paper with a criterion for the irreducibility of an affine ind-variety $X = \varinjlim X_n$ where the number of irreducible components of X_n is bounded for all n. Unfortunately we need for this criterion also information about the closure of a subset in the "global" object X and not only about the filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ itself. The following proposition implies Proposition D.

Proposition 8. *Let $X = \varinjlim X_n$ be an affine ind-variety such that the number of irreducible components of X_n is bounded by l for all n. Then X is irreducible in the ind-topology (Zariski topology) if and only if for all n there exists an irreducible component F_n of X_n such that $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_n F_n$ is dense in X with respect to the ind-topology (Zariski topology).*

Proof. One can read the proof either with respect to the ind-topology or with respect to the Zariski topology. Let $X = \varinjlim X_n$ be irreducible. For all n let us write $X_n = X_n^1 \cup \ldots \cup X_n^l$ where X_n^i is an irreducible component of X_n and for all n we have $X_n^i \subseteq X_{n+1}^i$ (it can be that $X_n^i = X_n^j$ for $i \neq j$). Thus, one gets

$$X = \overline{\bigcup_n X_n^1} \cup \ldots \cup \overline{\bigcup_n X_n^l} .$$

Since X is irreducible the claim follows. The converse of the statement is clear. $\square$

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