THE SHEETS OF A CLASSICAL LIE ALGEBRA

Inauguraldissertation

zur

Erlangung der Würde eines Doktors der Philosophie

vorgelegt der

Philosophisch-Naturwissenschaftlichen Fakultät

der Universität Basel

von

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aus

Basel (BS)

Basel, 2005

Genehmigt von der Philosophisch-Naturwissenschaftlichen Fakultät
auf Antrag der
Herren Professoren Dr. Hanspeter Kraft, Universität Basel, und
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Basel, den 8. Juli 2003

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Introduction

We consider the adjoint action of a connected complex semisimple group G on its Lie algebra $\mathfrak{g}$. A sheet of $\mathfrak{g}$ is a maximal irreducible subset of $\mathfrak{g}$ consisting of G-orbits of a fixed dimension. The Lie algebra $\mathfrak{g}$ is the finite union of its (not necessarily disjoint) sheets. It is known how sheets are classified, and how they intersect (see [2] for the whole story).

Let $\mathcal{S}$ be a sheet of $\mathfrak{g}$. A fundamental result says that $\mathcal{S}$ contains a unique nilpotent orbit. Let $\{e, h, f\}$ be a standard triple in $\mathfrak{g}$ such that e is contained in $\mathcal{S}$. Let $\mathfrak{g}^f$ be the centralizer of f in $\mathfrak{g}$ and define $X \subset \mathfrak{g}^f$ by $e + X = \mathcal{S} \cap (e + \mathfrak{g}^f)$. Katsylo then constructs in [9] a geometric quotient $\psi \colon \mathcal{S} \to (e + X)/A$ where A denotes the centralizer of the triple in G.

On the other hand, Borho and Kraft consider the categorical quotient $\pi_{\overline{\mathcal{S}}} \colon \overline{\mathcal{S}} \to \overline{\mathcal{S}}/\!/G$ and the normalization map of $\overline{\mathcal{S}}/\!/G$. They construct a homeomorphism from the normalization of $\overline{\mathcal{S}}/\!/G$ to the orbit space $\mathcal{S}/G$, which is equipped with the quotient topology. Suppose $\mathcal{S}$ were smooth (or normal). The restriction of $\pi_{\overline{\mathcal{S}}}$ to $\mathcal{S}$ then factors through the normalization of $\overline{\mathcal{S}}/\!/G$ and the induced map is a geometric quotient by a standard criterion of geometric invariant theory ([15], Proposition 0.2). We note that the induced map may be a geometric quotient without $\mathcal{S}$ being smooth (or normal).

The purpose of this work, however, is to investigate the smoothness of sheets. The main result is:

Theorem. *The sheets of classical Lie algebras are smooth.*

If $\mathfrak{g}$ is $\mathfrak{sl}_n$, this is a result of Kraft and Luna ([13]), and of Peterson ([17]) (see also [1] for a detailed proof). For the other classical Lie algebras a few partial results were obtained by Broer ([4]) and Panyushev ([16]). They both heavily use some additional symmetry. On the other hand, one of the sheets of G_2 is not normal (see [19]), the remaining ones being smooth. For most of the sheets of exceptional Lie algebras it is not known whether they are smooth or not.

This work is organized as follows:

In the first chapter, we recall the notions of decomposition class and of induced orbit, as well as their relevance to the theory of sheets. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{l}$ a nilpotent element. The G-conjugates of elements $y = z + x$ such that the centralizer of z is equal to $\mathfrak{l}$ form a decomposition class of $\mathfrak{g}$ ("similar Jordan decomposition"). The fact that every sheet contains a dense decomposition class leads to the classification of sheets by G-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ consisting of a Levi subalgebra of $\mathfrak{g}$ and a so called rigid orbit $\mathcal{O}_{\mathfrak{l}}$ in the derived algebra of $\mathfrak{l}$. A rigid orbit is a (nilpotent) orbit which itself is a sheet. The unique nilpotent orbit in the sheet corresponding to a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ as above is obtained by inducing $\mathcal{O}_{\mathfrak{l}}$ from $\mathfrak{l}$ to $\mathfrak{g}$: Let $\mathfrak{p}$ be any parabolic subalgebra of $\mathfrak{g}$ with Levi part $\mathfrak{l}$, and $\mathfrak{p}^u$ its unipotent radical. The induced orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$ is then defined as the unique orbit of maximal dimension in $G(\mathcal{O}_{\mathfrak{l}} + \mathfrak{p}^u)$.

In the second chapter, we explain Katsylo's results on sheets in detail. Let $\mathcal{S}$ be the sheet corresponding to a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ and let $\{e, h, f\}$ be a standard triple in $\mathfrak{g}$ such that

e is contained in $\mathcal{S}$. If the triple is suitably chosen the sheet $\mathcal{S}$ may be described as $G(e+\mathfrak{k})$ where $\mathfrak{k}$ denotes the center of $\mathfrak{l}$. We use the canonical isomorphism attached to the triple (2.1), and obtain a morphism $\varepsilon\colon e+\mathfrak{k}\to e+\mathfrak{g}^f$ such that $e+z$ and $\varepsilon(e+z)$ are G-conjugate for every $z\in\mathfrak{k}$. It turns out that $\varepsilon(e+\mathfrak{k})$ is an irreducible component of $e+X$, the intersection of $\mathcal{S}$ and $e+\mathfrak{g}^f$. Moreover, the centralizer of the triple in G acts transitively on the set of irreducible components of $e+X$, and its connected component acts trivially on $e+X$. Essentially by $\mathfrak{sl}_2$ theory, the two varieties $\mathcal{S}$ and $e+X$ are smoothly equivalent. This is the approach we use to investigate smoothness of sheets. At the end of the chapter, we apply these ideas to the regular sheet of $\mathfrak{g}$ and to admissible sheets of $\mathfrak{g}$. The regular sheet is the (very well known) open, dense subset consisting of the regular elements of $\mathfrak{g}$. It corresponds to the pair $(\mathfrak{h},0)$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. By Kostant, $e+\mathfrak{g}^f$ is contained in the regular sheet and every regular element is G-conjugate to a unique element of $e+\mathfrak{g}^f$. Hence ε maps $e+\mathfrak{h}$ onto $e+\mathfrak{g}^f$; it is the quotient by the Weyl group of G. The admissible sheets, in this context, are those coming nearest to the regular sheet.

In the remaining chapters, we deal with sheets in classical Lie algebras (in fact, our setting is slightly more general (3.1)). We prove that ε maps $e+\mathfrak{k}$ onto $e+X$; it turns out to be the quotient by some reflection group acting on $\mathfrak{k}$. Therefore $e+X$ is isomorphic to affine space and so $\mathcal{S}$ is smooth.

We first take a look at the linear group, that is, G is equal to $GL(V)$ for some complex vector space V. In this case, the sheets of $\mathfrak{g}$ are in one-to-one correspondence to the partitions of $\dim V$ (3.3). In order to make this explicit, we associate a partition to every $y\in\mathfrak{g}$ as follows: We decompose V as a $\mathbb{C}[y]$-module into a direct sum of cyclic submodules by successively cutting off cyclic submodules of maximal dimension. The dimensions of these direct summands define a partition of $\dim V$. The sheets of $\mathfrak{g}$ are then the sets $\mathcal{S}(\boldsymbol{l})$ consisting of elements $y\in\mathfrak{g}$ with fixed partition $\boldsymbol{l}$. The crucial observation is the fact that there is a decomposition of V into direct summands V_i which respects the setting of the second chapter in the following sense (Chapter 5): Let $\mathcal{S}$ be a sheet of $\mathfrak{g}$ described as $G(e+\mathfrak{k})$ and let $\varepsilon\colon e+\mathfrak{k}\to e+\mathfrak{g}^f$ be the corresponding map. For every $y\in e+\mathfrak{k}$, the $\mathbb{C}[y]$-module V decomposes into a direct sum of the *same* cyclic submodules V_i. We find elements e_i and subspaces $\mathfrak{k}_i$ of $\mathfrak{g}_i=\mathfrak{gl}(V_i)$ such that $G_i(e_i+\mathfrak{k}_i)$ is the regular sheet of $\mathfrak{g}_i$, and such that $e=\sum_i e_i$ and $\mathfrak{k}\subset\oplus_i\mathfrak{k}_i$. Let $\varepsilon_i\colon e_i+\mathfrak{k}_i\to e_i+\mathfrak{g}_i^{f_i}$ be the corresponding maps. Then ε is the restriction of $\sum_i\varepsilon_i$ to $\mathfrak{k}$. But we already know that ε_i is the quotient by the Weyl group of G_i. Finally, a straightforward calculation using basic invariants (power sums) shows that ε is the quotient by the normalizer of $\mathfrak{k}$ in the Weyl group of G, which in this case acts as reflection group on $\mathfrak{k}$. Since the centralizer of the triple $\{e,h,f\}$ in G is connected, the image of ε is equal to $e+X$.

The proof for the symplectic groups $Sp(V)$ and for the orthogonal groups $O(V)$ follows along the same lines. We begin with a classification of sheets in combinatorial terms (3.4). Then we use the combinatorial data to decompose V into a direct sum of subspaces V_i such that a proceeding similar to the linear case is possible (6.1). To be more precise, V decomposes as $\mathbb{C}[y]$-module into the direct sum of submodules V_i for every $y\in e+\mathfrak{k}$. These submodules may not be cyclic; however, they decompose into at most two cyclic submodules. The next step consists of identifying the maps $\varepsilon_i\colon e_i+\mathfrak{k}_i\to e_i+\mathfrak{g}_i^{f_i}$ as quotients by some reflection group acting on $\mathfrak{k}_i$. The case of V_i decomposing into two cyclic submodules of different dimension is the core of this work (6.3). It requires a lot of ad hoc calculation. The two other cases are readily reduced to the case of the regular sheet (6.2). At last, a calculation using basic invariants shows that ε is the quotient by some reflection group acting on $\mathfrak{k}$ (6.4).

Acknowledgments. I am grateful to HANSPETER KRAFT for arousing my interest in this subject, for all his valuable suggestions and support during the course of this work, and for making it possible to stay at the University of Michigan for a year. I got financial support from the Max Geldner Stiftung, Basel, during that year abroad. Many thanks go to PAVEL KATSYLO and BRAM BROER for sharing their ideas, to STEPHAN MOHRDIECK for his constant interest, and to JAN DRAISMA for numerous helpful conversations.

Chapter 1

Preliminaries

In the first section we recall the notion of sheet of an algebraic action. The remaining sections deal with the adjoint action on a reductive Lie algebra. We review some tools relevant to the investigation of sheets. This includes decomposition classes and induced orbits.

The ground field k is assumed to be algebraically closed and of characteristic 0. We identify an algebraic variety X with its k-points. We denote its regular functions by $k[X]$ and its rational functions by $k(X)$. By an algebraic group we mean a linear algebraic group. They are denoted by capital Roman letters, and their Lie algebras by the corresponding small Gothic letters.

1.1 Sheets of algebraic actions

Suppose an algebraic group G acts on an algebraic variety X. For any integer d we consider the set $X^{(d)} = \{\, x \in X \mid \dim Gx = d \,\}$. This set is G-stable and locally closed. Its G-irreducible components are called *sheets* of the action of G on X. Here we say that a G-variety is G-irreducible if it is not the union of two proper G-stable closed subsets. If G is connected, then G-irreducibility coincides with the usual notion of irreducibility.

For any subset Y of X we denote by Y^{reg} the set of regular elements of Y, i.e. those of maximal orbit dimension. If X is G-irreducible, then X^{reg} is a sheet called the *regular sheet* of X. Obviously, X is the finite union of its sheets. However, different sheets may have a non-empty intersection.

1.2 Decomposition classes and sheets of reductive Lie algebras

Let G be a reductive (not necessarily connected) algebraic group. We consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. In this section we give a description of the sheets using decomposition classes. The results are due to Borho and Kraft ([3]).

By means of its Jordan decomposition we associate with every $y = z + x \in \mathfrak{g}$ the pair $(\mathfrak{g}^z, x)$ consisting of the centralizer $\mathfrak{g}^z$ of the semisimple part z and the nilpotent part $x \in \mathfrak{g}^z$. Two elements of $\mathfrak{g}$ are said to be in the same *decomposition class* of $\mathfrak{g}$ if their associated pairs, the decomposition data, are G-conjugate.

Centralizers of semisimple elements in $\mathfrak{g}$ are usually called Levi subalgebras of $\mathfrak{g}$. Recall that a Levi subalgebra $\mathfrak{l}$ is reductive, that its centralizer $\mathfrak{k}$ in $\mathfrak{g}$ equals the center of $\mathfrak{l}$, and that $\mathfrak{l}$ is recovered as the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$. Furthermore, the centralizers of $\mathfrak{l}$ and $\mathfrak{k}$ in G coincide. The same is true for the normalizers.

We easily see that the decomposition class containing y is given by $G(\mathfrak{k}^{reg} + x)$ where $\mathfrak{k}$ denotes the double centralizer of z in $\mathfrak{g}$. A decomposition class is therefore G-irreducible. By the classification of Levi subalgebras and of nilpotent elements in reductive Lie algebras, there are only finitely many G-conjugacy classes of pairs $(\mathfrak{l}, x)$ consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a nilpotent element x in $\mathfrak{l}$. We deduce that $\mathfrak{g}$ is the finite disjoint union of its decomposition classes. This implies:

1.1 Lemma. *Every sheet $\mathcal{S}$ of $\mathfrak{g}$ contains a dense decomposition class $\mathcal{D}$, i.e. $\mathcal{S} = \overline{\mathcal{D}}^{reg}$.*

A *decomposition variety* is the closure of a decomposition class. Let $\mathcal{D}$ be the decomposition class corresponding to a pair $(\mathfrak{l}, x)$. We choose a parabolic Q of G° with Levi factor L and denote its unipotent radical by Q^u. Following Kostant, there is a parabolic subgroup P' of L and a nilpotent ideal $\mathfrak{n}'$ of $\mathfrak{p}'$ such that $P'x$ is dense in $\mathfrak{n}'$. Then we set $P = P'Q^u$ and $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{q}^u$. Further we set $\mathfrak{r} = \mathfrak{k} \oplus \mathfrak{n}$. We observe that $\mathfrak{n}$ is the nilradical of $\mathfrak{r}$ and that $\mathfrak{r}$ is an ideal of $\mathfrak{p}$. This construction eventually leads to a proof of one of the main results in [3]:

1.2 Theorem. ([3], 5.4. Theorem) *The decomposition variety $\overline{\mathcal{D}}$ is equal to $G\mathfrak{r}$ for a solvable ideal $\mathfrak{r}$ of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$.*

1.3 Corollary. (i) *The nilcone of $\overline{\mathcal{D}}$ is $G\mathfrak{n}$. It contains a dense orbit $\mathcal{O}$, which is the unique nilpotent orbit in $\overline{\mathcal{D}}^{reg}$.*

(ii) *$\overline{k^*Gy}$ contains $\mathcal{O}$ for every $y \in \overline{\mathcal{D}}^{reg}$.*

(iii) *$\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}^{reg}$ are finite unions of decomposition classes.*

(iv) *$\mathcal{D}$ is locally closed.*

(v) *$\overline{\mathcal{D}}^{reg} \cap (y + \mathfrak{n})$ is nonempty for every $y \in \mathfrak{r}$.*

(vi) *$\dim \overline{\mathcal{D}}^{reg} = \dim \mathfrak{k} + \dim G\mathfrak{n}$.*

The following statements are easy consequences of the proof of the theorem:

1.4 Lemma. *Let $y \in \mathfrak{r}^{reg}$. Then:*

(i) *$\dim G_y = \dim \mathfrak{p} - \dim \mathfrak{n}$.*

(ii) *$[\mathfrak{p}, y] = \mathfrak{n}$ and $\overline{Py} = y + \mathfrak{n}$.*

(iii) *$\mathfrak{g}^y \subset \mathfrak{p}$ and $G_y^\circ \subset P$.*

1.3 Induced orbits and sheets

The results in this section are taken from [2]. They allow us to explicitly determine the unique nilpotent orbit contained in the regular sheet of a decomposition variety.

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{l} \oplus \mathfrak{q}^u$. Let $y \in \mathfrak{l}$ be arbitrary. In this section L denotes the centralizer of $\mathfrak{k}$ in G.

1.5 Proposition-Definition. ([2], 2.1.) *The unique dense G-orbit in $G(y + \mathfrak{q}^u)$ is said to be induced from Ly (using $\mathfrak{q}$). It is denoted by $\operatorname{Ind}_{\mathfrak{l},\mathfrak{q}}^{\mathfrak{g}} Ly$.*

We list a few basic properties in a series of lemmas.

1.6 Lemma. *Let $x \in \mathfrak{l}$ be nilpotent and $z \in \mathfrak{k}$. Then $\operatorname{Ind}_{\mathfrak{g}^z,\mathfrak{q}+\mathfrak{g}^z}^{\mathfrak{g}}(z + G_z x) = G(z + x)$ and $\operatorname{Ind}_{\mathfrak{l},\mathfrak{q}^z}^{\mathfrak{g}^z}(z + Lx) = z + \operatorname{Ind}_{\mathfrak{l},\mathfrak{q}^z}^{\mathfrak{g}^z} Lx$.*

1.7 Lemma. *Let* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{q}^u$ *and* $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{q}'^u$ *be two parabolic subalgebras of* $\mathfrak{g}$*, and let* $\mathcal{O}_\mathfrak{l} \subset \mathfrak{l}$ *and* $\mathcal{O}_{\mathfrak{l}'} \subset \mathfrak{l}'$ *be two orbits. Then* $\operatorname{Ind}^\mathfrak{g}_{\mathfrak{l},\mathfrak{q}} \mathcal{O}_\mathfrak{l} = \operatorname{Ind}^\mathfrak{g}_{\mathfrak{l}',\mathfrak{q}'} \mathcal{O}_{\mathfrak{l}'}$*, if the pairs* $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ *and* $(\mathfrak{l}', \mathcal{O}_{\mathfrak{l}'})$ *are G-conjugate.*

In particular, the induced orbit does not depend on the choice of parabolic subalgebra with fixed Levi part. From now on we simply write $\operatorname{Ind}^\mathfrak{g}_\mathfrak{l} Ly$. The next lemma says that the induction procedure is transitive.

1.8 Lemma. *Let* $\mathfrak{l}$ *and* $\mathfrak{m}$ *be two Levi subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{l} \subset \mathfrak{m} \subset \mathfrak{g}$*. Then we have* $\operatorname{Ind}^\mathfrak{g}_\mathfrak{m} \operatorname{Ind}^\mathfrak{m}_\mathfrak{l} \mathcal{O}_\mathfrak{l} = \operatorname{Ind}^\mathfrak{g}_\mathfrak{l} \mathcal{O}_\mathfrak{l}$.

The link to the preceding section is made by the following result:

1.9 Proposition. ([2], 3.1. Satz a)) *Let* $\mathcal{O}_\mathfrak{l}$ *be a nilpotent orbit in* $\mathfrak{l}$ *and* $\mathcal{D}$ *the decomposition class with data* $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$*. Then* $\overline{\mathcal{D}}^{reg}$ *is the union of induced orbits* $\operatorname{Ind}^\mathfrak{g}_\mathfrak{l}(z + \mathcal{O}_\mathfrak{l})$ *for all* $z \in \mathfrak{k}$*. In particular, the unique nilpotent orbit* $\mathcal{O}$ *in* $\overline{\mathcal{D}}^{reg}$ *is* $\operatorname{Ind}^\mathfrak{g}_\mathfrak{l} \mathcal{O}_\mathfrak{l}$.

Proof. Let $y \in \mathfrak{r}^{reg}$. Then $\overline{Py} = y + \mathfrak{n} = z + \mathfrak{n}' + \mathfrak{q}^u$ where $z \in \mathfrak{k}$. Since $\mathfrak{n}' = \overline{P'x}$ with $\mathcal{O}_\mathfrak{l} = Lx$ we get $z + \mathfrak{n}' + \mathfrak{q}^u = z + \overline{P'x} + \mathfrak{q}^u$, and so $\overline{Gy} = \overline{G(z + x + \mathfrak{q}^u)}$. Hence $Gy = \operatorname{Ind}^\mathfrak{g}_\mathfrak{l}(z + \mathcal{O}_\mathfrak{l})$. The converse follows from part (iv) of Corollary 1.3. □

1.10 Corollary. ([2], 3.6.) *Let* $\mathcal{D}'$ *be a decomposition class with data* $(\mathfrak{l}', \mathcal{O}_{\mathfrak{l}'})$ *such that* $\mathfrak{l}'$ *contains* $\mathfrak{l}$*. Then* $\mathcal{D}'$ *is contained in* $\overline{\mathcal{D}}^{reg}$ *if and only if* $\mathcal{O}_{\mathfrak{l}'}$ *is the induced orbit* $\operatorname{Ind}^{\mathfrak{l}'}_\mathfrak{l} \mathcal{O}_\mathfrak{l}$ *up to G-conjugacy.*

We conclude this section with the classification of sheets of $\mathfrak{g}$. We need to determine those decomposition classes which are dense in a sheet of $\mathfrak{g}$. A nilpotent orbit is said to be *rigid* if it is not properly induced from another orbit.

1.11 Proposition. ([2], 4.2.) $\mathcal{D}$ *is dense in a sheet of* $\mathfrak{g}$ *if and only if* $\mathcal{O}_\mathfrak{l}$ *is rigid in* $\mathfrak{l}$.

1.12 Corollary. ([2], 4.4.) *Sheets of* $\mathfrak{g}$ *are classified by G-conjugacy classes of pairs* $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ *consisting of a Levi subalgebra* $\mathfrak{l}$ *and a rigid orbit* $\mathcal{O}_\mathfrak{l}$ *in* $\mathfrak{l}$.

In particular, $\mathcal{D}$ itself is a sheet of $\mathfrak{g}$ if and only if $\mathfrak{l}$ is $\mathfrak{g}$ and $\mathcal{O}_\mathfrak{g}$ is rigid in $\mathfrak{g}$. In Chapter 3 we will carry out the classification for classical Lie algebras $\mathfrak{g}$. At the moment, we only mention the two extremal cases: The pair $(\mathfrak{g}, 0)$ produces the center of $\mathfrak{g}$ (= the 0-dimensional orbits). The pair $(\mathfrak{h}, 0)$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ corresponds to the regular sheet $\mathfrak{g}^{reg}$ of $\mathfrak{g}$.

Chapter 2

Regular sheet of a decomposition variety

In this chapter we mainly exhibit Katsylo's contribution to the theory of sheets (cf. [9]). We provide purely algebraic proofs of his results.

2.1 Construction with a standard triple

Let G be a reductive group with Lie algebra $\mathfrak{g}$. A *standard triple* $\{e, h, f\}$ is a set of elements in $\mathfrak{g}$ which generate a subalgebra isomorphic to $\mathfrak{sl}_2$ and satisfy the bracket relations:

$$[h, e] = 2e \ , \ [h, f] = -2f \ , \ [e, f] = h.$$

We consider the eigenspace decomposition with respect to the adjoint action of h:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \ , \ \mathfrak{g}(j) = \{ y \in \mathfrak{g} \mid [h, y] = jy \}.$$

Note that $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i + j)$ holds for any integers i, j. We further see that $e \in \mathfrak{g}(2)$ and $\mathfrak{g}^h = \mathfrak{g}(0)$. The k^*-action defined by $(t, y) \mapsto t^{-j}y$ for $y \in \mathfrak{g}(j)$ and $t \in k^*$ factors through a 1-parameter subgroup of G. Since this action commutes with the usual scalar action, we may define a k^*-action $\delta : k^* \to \mathrm{GL}(\mathfrak{g})$ by $\delta(t).y = t^{-j+2}y$ for $y \in \mathfrak{g}(j)$ and $t \in k^*$.

2.1 Lemma. (i) *We have $\delta(k^*) \subset k^* \operatorname{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$. Therefore, δ maps G-orbits isomorphically onto G-orbits. It also stabilizes decomposition classes.*

(ii) *The map δ defines an attractive k^*-action on $e + \bigoplus_{i \leq 1} \mathfrak{g}(i)$ with isolated fixed point e.*

(iii) *If $y, y_0 \in \mathfrak{g}$ such that $\lim_{t \to 0} \delta(t).y = y_0$, then $\dim Gy_0 \leq \dim Gy$.*

Next, we consider the affine subspace $e + \mathfrak{g}^f$ in $\mathfrak{g}$. Since $[\mathfrak{g}, e] \oplus \mathfrak{g}^f = \mathfrak{g}$ by $\mathfrak{sl}_2$ theory, the morphism $G \times (e + \mathfrak{g}^f) \to \mathfrak{g}$ (given by the adjoint action) is smooth in $(1_G, e)$. The subvariety $e + \mathfrak{g}^f$ is said to be a *transversal slice* in $\mathfrak{g}$ to the orbit Ge at the point e. The map δ also defines an attractive k^*-action on $e + \mathfrak{g}^f$ with isolated fixed point e.

2.2 Lemma. *The morphism $G \times (e + \mathfrak{g}^f) \to \mathfrak{g}$ is smooth of relative dimension $\dim \mathfrak{g}^f$.*

Proof. The morphism $G \times (e + \mathfrak{g}^f) \to \mathfrak{g}$ is $\delta(k^*)$-equivariant with respect to the action defined by $\delta(t).(g, e + y) = (\delta(t)g\delta(t)^{-1}, \delta(t).(e + y))$. The smooth points of the morphism are therefore stable under both G and $\delta(k^*)$. Now the claim follows because the morphism is smooth in $(1_G, e)$. □

Let $U \subset G$ be the unipotent group corresponding to $\bigoplus_{i \leq -1} \mathfrak{g}(i)$. This group stabilizes $\bigoplus_{i \leq i_0} \mathfrak{g}(i)$ for all i_0. In particular, it acts on $e + \bigoplus_{i \leq 1} \mathfrak{g}(i)$.

2.3 Proposition. *The map* $\gamma\colon U \times (e + \mathfrak{g}^f) \to e + \bigoplus_{i\leq 1} \mathfrak{g}(i)$ *is an isomorphism.*

Proof. Since $\delta(k^*)$ normalizes U, the morphism γ is $\delta(k^*)$-equivariant with respect to the action defined in the proof of the previous lemma. The weights of $\delta(k^*)$ are all strictly positive. Since $\dim \mathfrak{g}^f = \dim \mathfrak{g}(1) + \dim \mathfrak{g}(0) = \dim \bigoplus_{i\leq 1} \mathfrak{g}(i) - \dim U$, both sides are affine spaces of the same dimension. The differential of γ at $(1, e)$ is injective (even bijective). Now the claim follows from [19], p. 121, Lemma 1. □

The inverse map of γ induces morphisms

$$\Gamma\colon e + \bigoplus_{i\leq 1} \mathfrak{g}(i) \to U$$

and

$$\mathcal{E}\colon e + \bigoplus_{i\leq 1} \mathfrak{g}(i) \to e + \mathfrak{g}^f$$

such that $\Gamma(e+y).\mathcal{E}(e+y) = e+y$ for $y \in \bigoplus_{i\leq 1} \mathfrak{g}(i)$. Obviously, $\mathcal{E}^{-1}(\mathcal{E}(e+y)) = U(e+y)$ and $\Gamma(g(e+y)) = g\Gamma(e+y)$ for every $y \in \bigoplus_{i\leq 1} \mathfrak{g}(i)$ and $g \in U$.

Let $\psi\colon \mathfrak{g} \to \overline{\mathfrak{g}}$ be a homomorphism of reductive Lie algebras such that $\{\overline{e} = \psi(e), \overline{h} = \psi(h), \overline{f} = \psi(f)\}$ is a standard triple. This triple produces the map $\overline{\gamma}$ in the following commutative diagram:

$$\begin{array}{ccc} U \times (e + \mathfrak{g}^f) & \xrightarrow{\gamma} & e + \bigoplus_{i\leq 1} \mathfrak{g}(i) \\ \psi \downarrow & & \downarrow \psi \\ \overline{U} \times (\overline{e} + \overline{\mathfrak{g}}^{\overline{f}}) & \xrightarrow{\overline{\gamma}} & \overline{e} + \bigoplus_{i\leq 1} \overline{\mathfrak{g}}(i)\ . \end{array}$$

We also obtain commutative diagrams for $\mathcal{E}$ and Γ similar to the one above.

Let A be the centralizer of $\{e, h, f\}$ (or $\{e, h\}$) in G. This group normalizes U, and it stabilizes $e + \mathfrak{g}^f$ and $e + \bigoplus_{i\leq 1} \mathfrak{g}(i)$. The maps γ, Γ, and $\mathcal{E}$ are equivariant with respect to A. Since $\delta(k^*)$ normalizes U, these maps are equivariant also with respect to the δ-action. For later reference we mention the following standard result ([6], Lemma 5.4).

2.4 Lemma. *The inclusion* $A \subset G_e$ *induces an isomorphism* $A/A^\circ \to G_e/G_e^\circ$ *of component groups.*

The centralizers A and their component groups A/A° (for simple groups) are listed, for instance, in [5], p. 398 seq.

2.2 Section of a sheet

Let $\mathcal{D}$ be the decomposition class with decomposition data $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$. The notation involving $\mathcal{D}$ is taken from (1.2). We are interested in $\overline{\mathcal{D}}^{reg}$, the regular sheet of $\overline{\mathcal{D}}$. Recall that $\overline{\mathcal{D}}^{reg}$ can be written as $G\mathfrak{r}^{reg}$, and its unique nilpotent orbit $\mathcal{O}$ as $G\mathfrak{n}^{reg}$.

2.5 Lemma. ([9], Lemma 3.1) *There is a standard triple* $\{e, h, f\}$ *in* $\mathfrak{g}$ *such that* $e \in \mathfrak{n}^{reg}$ *and* $h \in \mathfrak{l}$.

Proof. Let $\{\overline{e}, \overline{h}, \overline{f}\}$ be any standard triple with $\overline{e} \in \mathfrak{n}^{reg}$. We have $[\overline{h}, \overline{e}] = 2\overline{e} \in \mathfrak{n}$. From Lemma 1.4 (ii) and (iii) it follows that $\overline{h} \in \mathfrak{p}$. Choose a Levi factor $\mathfrak{m}$ of $\mathfrak{p}$ which is contained in $\mathfrak{l}$. Clearly, we can find $g \in P$ such that $g.\overline{h} \in \mathfrak{m}$. Then we define $e = g.\overline{e} \in \mathfrak{n}^{reg}$ and $h = g.\overline{h} \in \mathfrak{m} \subset \mathfrak{l}$. □

2.6 Proposition. ([9], Lemma 3.2) *Let $\{e, h, f\}$ be a standard triple such that $e \in \mathfrak{n}^{reg}$ and $h \in \mathfrak{l}$. Then $\overline{\mathcal{D}}^{reg} = G(e + \mathfrak{k})$.*

Proof. We first note that $\mathfrak{k}$ is contained in $\mathfrak{g}(0)$. For every $z \in \mathfrak{k}$, we may apply Lemma 2.1(ii) to $\lim_{t \to 0} \delta(t).(e+z) = e$ and obtain $e+z \in \mathfrak{r}^{reg}$. Conversely, let $y = z+x \in \mathfrak{k} \oplus \mathfrak{n} = \mathfrak{r}$ be regular element of $\mathfrak{r}$. Then Lemma 1.4 (ii) implies

$$\overline{Py} = y + \mathfrak{n} = z + \mathfrak{n} = e + z + \mathfrak{n} = \overline{P(e+z)}.$$

The last equality uses the first part of the proof. We deduce that $y \in P(e+z)$. □

The following corollary establishes a link to the theory of induced orbits (1.3).

2.7 Corollary. *The G-orbit of $z + e$ is induced from the L-orbit $z + \mathcal{O}_{\mathfrak{l}}$ for any $z \in \mathfrak{k}$, in short, $G(z + e) = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(z + \mathcal{O}_{\mathfrak{l}})$.*

Proof. Recall that $e \in \mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{q}^u$. Define $e' \in \mathfrak{n}'$ by $e \in e' + \mathfrak{q}^u$. From $\overline{Pe} = \mathfrak{n}$ it follows easily that $\overline{P'e'} = \mathfrak{n}'$. Hence $e' \in \mathcal{O}_{\mathfrak{l}}$. Since $z + e \in \mathfrak{r}^{reg}$, the unique dense G-orbit in $G(z + e' + \mathfrak{q}^u)^{reg}$ is equal to $G(z + e)$. □

We continue the investigation of $\overline{\mathcal{D}}^{reg}$, using now the standard triple $\{e, h, f\}$ from the previous proposition. Consider the schematic intersection $e + X = (e + \mathfrak{g}^f) \cap \overline{\mathcal{D}}^{reg}$. By base change, the morphism $G \times (e + X) \to \overline{\mathcal{D}}^{reg}$ is smooth of relative dimension $\dim G_e$ ($= \dim \mathfrak{g}^f$). It is even surjective because every $e + z \in e + \mathfrak{k} \subset e + \mathfrak{g}^h$ is conjugate to an element in $e + \mathfrak{g}^f$ by Proposition 2.3. Hence, geometric properties of $\overline{\mathcal{D}}^{reg}$ are reflected in $e + X$. The two varieties are said to be *smoothly equivalent*. We call $e + X$ the *section* of $\overline{\mathcal{D}}^{reg}$.

First, we collect some simple properties of $e + X$.

2.8 Lemma. *Let $e + X$ be the schematic intersection $(e + \mathfrak{g}^f) \cap \overline{\mathcal{D}}^{reg}$.*

(i) *The subscheme $e + X$ is reduced. Its dimension is equal to the dimension of $\mathfrak{k}$. Each orbit of $\overline{\mathcal{D}}^{reg}$ intersects $e + X$ in a finite number of points.*

(ii) *The action of A on $\mathfrak{g}$ stabilizes $e + X$. The action of A° is trivial on $e + X$.*

(iii) *The δ-action on $\mathfrak{g}$ stabilizes $e+X$. Therefore, $e+X$ is closed in $e+\mathfrak{g}^f$ and connected, its irreducible components contain e, and the nilpotent orbit of $\overline{\mathcal{D}}^{reg}$ intersects $e+X$ in e.*

Proof. (i) Use that $G \times (e + X) \to \overline{\mathcal{D}}^{reg}$ is smooth of relative dimension $\dim G_e$. Note that $\dim \overline{\mathcal{D}}^{reg} = \dim \mathfrak{k} + \dim Ge$ (1.3(v)).

(ii) The action of A stabilizes both $\overline{\mathcal{D}}^{reg}$ and $e + \mathfrak{g}^f$, hence $e + X$ as well. The A°-orbits in $e + X$ are connected and consist of a finite number of points by (i).

(iii) The δ-action stabilizes both $\overline{\mathcal{D}}^{reg}$ and $e + \mathfrak{g}^f$, hence $e + X$ as well. The remaining claims are all proved using the fact that the δ-action on $\overline{e + X}$ is attractive with fixed point e. For instance, since $\overline{e + X}$ is contained in $e + \mathfrak{g}^f$, part (iii) of Lemma 2.1 implies that all orbits in $\overline{e + X}$ have dimension $\geq \dim Ge$. But $\overline{e + X}$ is also contained in $\overline{\mathcal{D}}$. This implies that $e + X$ is closed in $e + \mathfrak{g}^f$.

□

Let $\varepsilon \colon e + \mathfrak{k} \to e + \mathfrak{g}^f$ be the restriction of $\mathcal{E} \colon e + \bigoplus_{i \leq 1} \mathfrak{g}(i) \to e + \mathfrak{g}^f$ to $e + \mathfrak{k}$. We define Y by $e + Y = \varepsilon(e + \mathfrak{k})$. Obviously, we have $e + Y \subset e + X$ and $G(e + Y) = \overline{\mathcal{D}}^{reg}$.

2.9 Lemma. *The subset $e + Y$ is an irreducible component of $e + X$.*

Proof. Using part (iii) of the previous lemma and Proposition 2.3, we deduce that the unique nilpotent element in $e + \mathfrak{k}$ is e, and so $\varepsilon^{-1}(e) = e$. Since ε is equivariant with respect to the δ-action, we may apply [12], p. 144, obtaining that ε is a finite map. Therefore, $e + Y$ is closed in $e + X$, and $\dim(e + Y) = \dim \mathfrak{k}$. But $\dim(e + X) = \dim \mathfrak{k}$ as well. □

2.10 Theorem. *The component group of A acts transitively on the set of irreducible components of $e + X$.*

Proof. Let $\{ e + X_i \mid i \in I \}$ be the set of irreducible components of $e + X$. We relate this set to the fibre $\nu^{-1}(e)$ where ν denotes the normalization map of $\overline{\mathcal{D}}^{reg}$.

STEP 1: The component group of G_e acts transitively on $\nu^{-1}(e)$: Consider the collapsing map

$$\Phi \colon G \times_P \mathfrak{r}^{reg} \to G\mathfrak{r}^{reg} = \overline{\mathcal{D}}^{reg}.$$

Since $Ge \cap \mathfrak{r}^{reg} = Ge \cap \mathfrak{n}^{reg} = Pe$ and $G_e^\circ \subset P$ by Lemma 1.4, the fibre $\Phi^{-1}(e)$ is isomorphic to the G_e/G_e°-orbit G_e/P_e. Because $G \times_P \mathfrak{r}^{reg}$ is smooth, hence normal, the map Φ factors through ν. Therefore, the component group of G_e acts transitively on $\nu^{-1}(e)$.

STEP 2: The component group of A acts transitively on $\nu^{-1}(e)$: This follows from Lemma 2.4.

STEP 3: We set $W = \overline{\mathcal{D}}^{reg}$, $Z = e + X$, $\widetilde{Z} = \nu^{-1}(e + X)$, and $\mu = \nu|_{\widetilde{Z}}$. Then the following diagram is cartesian (see [19], p. 62, Lemma 2):

$$\begin{array}{ccc} G \times \widetilde{Z} & \xrightarrow{\rho'} & \widetilde{W} \\ {\scriptstyle \nu' = \mathrm{id}_G \times \mu} \big\downarrow & & \big\downarrow {\scriptstyle \nu} \\ G \times Z & \xrightarrow{\rho} & W . \end{array}$$

Since ρ is smooth and surjective, we deduce that ν' is the normalization map of $G \times Z$. (Here a variety is called normal if its connected components are irreducible and normal.) It follows immediately that $\widetilde{Z}$ is the disjoint union of its irreducible components, and that the restriction of ν to such a component is the normalization map of some $e + X_i$ with $i \in I$. Therefore, each component of $\widetilde{Z}$ contains an element which maps onto e. Since A/A° permutes the components of $\widetilde{Z}$, the claim follows from the second step.

Remark. Lifting the k^*-action (with all its properties) to (the components of) $\widetilde{Z}$ we notice that the cardinality of $\nu^{-1}(e)$ is equal to $\#I$. Moreover, every irreducible component of $\widetilde{Z}$ contains exactly one element of $\nu^{-1}(e)$.

□

The following theorem is an important step in Katsylo's construction of a geometric quotient for the action of G on $\overline{\mathcal{D}}^{reg}$.

2.11 Theorem. ([9], Theorem 0.3) *Let $x, x' \in X$. The elements $e+x$ and $e+x'$ are G-conjugate if and only if they are A-conjugate.*

Proof. First, we state a claim similar to the previous theorem. We consider the product $(e+X)\times(e+Y)$ and define an action of A on this variety by $a.(e+x, e+y) = (e+ax, e+y)$.
Claim: The set

$$Z = \{\, (e+x, e+y) \in (e+X)\times(e+Y) \mid G(e+x) = G(e+y) \,\}$$

is closed in $(e+X)\times(e+Y)$ and A-stable. Moreover, the component group of A acts transitively on the set of irreducible components of Z.

We prove the claim. The arguments are essentially the same as in the proof of the previous theorem. We introduce a (G-irreducible) G-variety W and a surjective G-morphism $\Psi\colon W \to \overline{\mathcal{D}}^{reg}$ such that Z is the inverse image of $e+X$ as schemes. Again, we obtain a cartesian diagram

$$\begin{array}{ccc} G\times Z & \longrightarrow & W \\ \downarrow & & \downarrow \Psi \\ G\times(e+X) & \xrightarrow{\ \rho\ } & \overline{\mathcal{D}}^{reg}. \end{array}$$

Before being able to define W and Ψ, we need some preparation. Let

$$\beta\colon \mathfrak{r} = \mathfrak{k} + \mathfrak{n} \to \mathfrak{k} \to e + \mathfrak{k} \to e + Y.$$

be given by $\beta(z) = \varepsilon(e + \mathrm{pr}_{\mathfrak{k}}(z))$ for any $z \in \mathfrak{r}$. This map is P-equivariant because $Pz \subset z + \mathfrak{n}$ for any $z \in \mathfrak{r}$. For $z \in \mathfrak{r}^{reg}$, the P-orbits of z and $e + \mathrm{pr}_{\mathfrak{k}}(z)$ as well as the G-orbits of z and $\beta(z)$ coincide.

Let $\Gamma(\beta)$ be the graph of β. This is a closed, P-invariant subvariety of $\mathfrak{r}\times(e+Y)$. Therefore, $G\Gamma(\beta)$ is closed in $\overline{\mathcal{D}}\times(e+Y)$. We now define W to be $G\Gamma(\beta)^{reg}$. Let Ψ be the projection map from W onto $\overline{\mathcal{D}}^{reg}$. We observe that

$$\begin{aligned} W &= G\Gamma(\beta) \cap (\overline{\mathcal{D}}^{reg} \times (e+Y)) \\ &= \{\, (gz, e+y) \mid g\in G\,,\ z\in \mathfrak{r}^{reg}\,,\ y \in Y \text{ such that } Gz = G(e+y) \,\} \\ &= \{\, (g(e+x), e+y) \mid g\in G\,,\ x\in X\,,\ y\in Y \text{ such that } G(e+x) = G(e+y) \,\}. \end{aligned}$$

Here, the second equality follows from $G\beta(z) = Gz$ for $z \in \mathfrak{r}^{reg}$. This description of W implies that Z equals $\Psi^{-1}(e+X)$ as a subvariety. Using the smooth base change morphism ρ, we see that $\Psi^{-1}(e+X)$ is reduced. Moreover, the induced morphism $G\times Z \to W$ is smooth and surjective.

Finally, we proceed without difficulties through the three steps of the proof of the previous theorem.

It remains to deduce the theorem from the claim. Suppose x and x' are two elements of X such that $G(e+x) = G(e+x')$. By the previous theorem we may assume that $x' \in Y$, in other words, that $(e+x, e+x') \in Z$. One of the irreducible components of Z is given by $\{ (e+y, e+y) \mid y \in Y \}$. Indeed, this is a closed irreducible subset of Z, and its dimension is maximal ($= \dim Z$). The claim then implies that some A-conjugate of $(e+x, e+x')$ is contained in this particular component of Z. □

Now we turn to the proof of the main result in [9].

2.12 Theorem. ([9], Theorem 0.4) *There is an open morphism $\psi\colon \overline{\mathcal{D}}^{reg} \to (e+X)/A$ such that the fibres of ψ are the orbits of G, and such that ψ induces isomorphisms $k[U] \to k[\psi^{-1}(U)]^G$ for every open set U in $(e+X)/A$. In brief, the morphism ψ is a geometric quotient.*

Proof. We have a geometric quotient $\chi\colon e+X \to (e+X)/A$ of affine varieties. Theorem 2.10 says that $(e+X)/A$ is irreducible. We extend χ, using Theorem 2.11, G-invariantly to a map $\psi\colon \overline{\mathcal{D}}^{reg} \to (e+X)/A$. Obviously, the fibres of ψ are G-orbits. The map is in fact a morphism. In order to prove this and the remaining statements, we consider the following commutative diagram:

$$\begin{array}{ccc} G \times (e+X) & \xrightarrow{\ \rho\ } & \overline{\mathcal{D}}^{reg} \\ \big\downarrow & & \big\downarrow \psi \\ e+X & \xrightarrow{\ \chi\ } & (e+X)/A\,. \end{array}$$

Since ρ is smooth and surjective, it follows from the lemma below that ψ is a morphism. It is easy to see that ψ is open.

By definition, ψ induces an embedding of $k[U]$ into $k[\psi^{-1}(U)]^G$. On the other hand, a G-invariant function on $\psi^{-1}(U)$ is determined by its restriction to $\chi^{-1}(U)$, the intersection of $\psi^{-1}(U)$ and $e+X$. But χ induces an isomorphism of $k[U]$ onto $k[\chi^{-1}(U)]$, and so ψ induces an isomorphism $k[U]$ onto $k[\psi^{-1}(U)]^G$. □

2.13 Lemma. *Let $\rho\colon X \to Y$ and $\varphi\colon X \to Z$ be morphisms of varieties. Let $\psi\colon Y \to Z$ be a map such that $\psi\rho = \varphi$. If ρ is smooth and surjective, then ψ is a morphism.*

Proof. Consider the graph $\Gamma(\psi) \subset Y \times Z$ of the map ψ and the graph $\Gamma(\varphi) \subset X \times Z$ of the morphism φ. We know that $\Gamma(\varphi)$ is closed in $X \times Z$ and that the projection onto X induces an isomorphism $\Gamma(\varphi) \to X$. We want the respective properties to hold for $\Gamma(\psi)$. The following cartesian diagram will be useful:

$$\begin{array}{ccc} \Gamma(\varphi)\ \subset X \times Z & \xrightarrow{\ \mathrm{pr}_X\ } & X \\ {\scriptstyle \rho_Z = \rho \times \mathrm{id}_Z} \big\downarrow & & \big\downarrow \rho \\ \Gamma(\psi)\ \subset Y \times Z & \xrightarrow{\ \mathrm{pr}_Y\ } & Y\,. \end{array}$$

We first note that $\Gamma(\varphi) = \rho_Z^{-1}(\Gamma(\psi))$. Since ρ is smooth, hence open, and surjective, the same holds for ρ_Z, and so $\Gamma(\psi)$ is closed in $Y \times Z$. We therefore obtain a cartesian diagram

$$\begin{array}{ccc} \Gamma(\varphi) & \longrightarrow & X \\ {\scriptstyle \rho_Z}\downarrow & & \downarrow{\scriptstyle \rho} \\ \Gamma(\psi) & \longrightarrow & Y\ . \end{array}$$

Since ρ is smooth and surjective, hence faithfully flat, it follows from [7], IV 2.7.1 (viii) that $\Gamma(\psi) \to Y$ is an isomorphism. □

2.3 Regular sheet of a reductive Lie algebra

Let G be a *connected* reductive group and let e be a regular nilpotent element of $\mathfrak{g}$. Then we may choose a principal standard triple $\{e, h, f\}$ and a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g}^{reg} = G(e + \mathfrak{h})$. Kostant obtains in [11] the following results: First, the slice $e + \mathfrak{g}^f$ is contained in $\mathfrak{g}^{reg}$. It is therefore the section of the regular sheet. Second, the adjoint quotient $\pi\colon \mathfrak{g} \to \mathfrak{h}/W$ restricts to an isomorphism $\delta\colon e + \mathfrak{g}^f \to \mathfrak{h}/W$ where W denotes the Weyl group of G. Third, define $\tau_e\colon \mathfrak{h} \to e + \mathfrak{h}$ by $z \mapsto e + z$ for $z \in \mathfrak{h}$ and let W act on $e + \mathfrak{h}$ such that τ_e is equivariant. Then the restriction of π to $e + \mathfrak{h}$ is a quotient by W.

2.14 Theorem. *The morphism $\varepsilon\colon e + \mathfrak{h} \to e + \mathfrak{g}^f$ is surjective; it is a quotient map with respect to W.*

Proof. We have to show that $\delta \circ \varepsilon = \pi$ on $e + \mathfrak{h}$. By definition $\varepsilon(e + z)$ is contained in the G-orbit of $e + z$, and so $\delta(\varepsilon(e + z)) = \pi(e + z)$ for $z \in \mathfrak{h}$. □

2.4 Admissible sheets of a reductive Lie algebra

Let $\mathfrak{g}'$ be an admissible subalgebra of $\mathfrak{g}$ in the sense of Rubenthaler (see [18]). Then there exists a unique (up to G-conjugacy) even standard triple $\{e, h, f\}$ in $\mathfrak{g}$ such that $\{e, h, f\}$ is contained in $\mathfrak{g}'$ and the double centralizer of h in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}'$. Since $\{e, h, f\}$ is even in $\mathfrak{g}'$ as well, it has to be principal in $\mathfrak{g}'$. Let $\mathfrak{l}$ be the centralizer of h in $\mathfrak{g}$, and $\mathcal{S}$ the sheet in $\mathfrak{g}$ corresponding to $(\mathfrak{l}, 0)$. Using that e is even in $\mathfrak{g}$ we show that e is contained in $\mathcal{S}$. Then we obtain that $\mathcal{S}$ is equal to $G(e + \mathfrak{k})$. But now we note that $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}'$. Therefore the regular sheet of $\mathfrak{g}'$ is contained in $\mathcal{S}$; it is equal to $G'(e + \mathfrak{k})$. In this situation, the following theorem is easy to prove.

2.15 Theorem. *The morphism $\varepsilon\colon e + \mathfrak{k} \to e + \mathfrak{g}^f$ is a quotient map with respect to the normalizer of $\mathfrak{k}$ in W. Its image, $e + \mathfrak{g}'^f$, is equal to $e + X$.*

Proof. The morphism ε coincides with $\varepsilon'\colon e + \mathfrak{k} \to e + \mathfrak{g}'^f$. But ε' is surjective, moreover, it is the quotient by the Weyl group W' of G'. In Proposition 2.5 of [18], it is shown that the images of W' and $N_W(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$ are the same. Since A stabilizes $e + \mathfrak{k}$ and ε is A-equivariant, we obtain that $e + X$ is equal to $e + \mathfrak{g}'^f$. □

Every sheet $\mathcal{S}$ which comes up as above is called an *admissible* sheet of $\mathfrak{g}$.

2.16 Corollary. *The admissible sheets of $\mathfrak{g}$ are smooth.*

Chapter 3

Very stable decomposition varieties in classical Lie algebras

In this chapter the Lie algebra $\mathfrak{g}$ is classical in the following sense: it is a sum of general linear, symplectic, and orthogonal Lie algebras. Let G be the corresponding product of general linear, symplectic, and full orthogonal groups. This group acts on its Lie algebra by conjugation.

3.1 Results

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. We note that every Levi subalgebra of a classical Lie algebra is again classical. Let L be the corresponding (sub)group (of G) and $\mathcal{O}_{\mathfrak{l}}$ a nilpotent L-orbit in $\mathfrak{l}$. We denote by $\mathfrak{k}$ the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$ and by N the image of $N_G(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$.

Definition. We call the decomposition class given by the pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ *very stable* if $\mathcal{O}_{\mathfrak{l}}$ is trivial in every direct summand of $\mathfrak{l}$ of general linear type.

By Corollary 1.12 a sheet of $\mathfrak{g}$ is determined by a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ such that $\mathcal{O}_{\mathfrak{l}}$ is rigid in $\mathfrak{l}$. We will see that such a pair defines a very stable decomposition class. We obtain the following results for very stable decomposition classes:

Parametrization Theorem. *Let $\mathcal{D}$ be a very stable decomposition class and $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ its data. Then the map $\mathfrak{k} \to \overline{\mathcal{D}}^{reg}/G$ given by $z \mapsto \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(z + \mathcal{O}_{\mathfrak{l}})$ induces a bijection of orbit spaces $\mathfrak{k}/N \to \overline{\mathcal{D}}^{reg}/G$.*

We give a proof of the Parametrization Theorem in Chapter 4.

Main Theorem. *Let $\mathcal{D}$ be a very stable decomposition class. Then $\overline{\mathcal{D}}^{reg}$ is a smooth variety.*

The idea behind the proof of the Main Theorem is to use the results of (2.2). The section of $\overline{\mathcal{D}}^{reg}$ turns out to be isomorphic to the quotient space of $\mathfrak{k}$ by some reflection group acting on $\mathfrak{k}$ (Theorem 5.2 and Theorem 6.2). But this is isomorphic to affine space, and so $\overline{\mathcal{D}}^{reg}$ is smooth. In Chapter 5 we prove the Main Theorem for general linear groups, and in Chapter 6 for symplectic and orthogonal groups.

In the following sections we explicitly classify the relevant objects (cf. [10], §1,2,3,5). It is sufficient to do this for the "simple" case, i.e. for the general linear, the symplectic, and the orthogonal groups. It is also not difficult to see that the proof of the Main Theorem reduces to the "simple" case.

We also consider the case if the orthogonal group is replaced by its identity component, the special orthogonal group. In the last section we show how the Main Theorem follows in that case.

Clearly, all results hold if the general linear group is replaced by the special linear group.

3.2 Combinatorial conventions

Let $\mathbb{N}$ be the set of natural numbers $1, 2, 3, \ldots$. We define a *partition* $\boldsymbol{l}$ to be a finite subset of $\mathbb{N}^2$ such that if $(q,p) \in \boldsymbol{l}$ and $j \leq q$ and $i \leq p$ then $(j,i) \in \boldsymbol{l}$. So we identify a partition with its Young diagram. If $\boldsymbol{l}$ is a partition the length of the i-th row is $l_i = \#\{j \mid (j,i) \in \boldsymbol{l}\}$. The length of the j-th column is $l^j = \#\{i \mid (j,i) \in \boldsymbol{l}\}$. Obviously, $\boldsymbol{l}$ is determined by each of the non-increasing sequences $(l_1, l_2, \ldots)$ and $(l^1, l^2, \ldots)$. If the cardinality of $\boldsymbol{l}$ is N we say that $\boldsymbol{l}$ is a *partition of* N and write $\boldsymbol{l} \in \mathcal{P}(N)$.

3.3 General linear group

Let V be a vector space over k of dimension N. Denote by $GL(V)$ the group of automorphisms of V and by $\mathfrak{gl}(V)$ its Lie algebra. We recall the classification of nilpotent elements and of Levi subalgebras.

Let x be a nilpotent element of $\mathfrak{gl}(V)$ and consider its Jordan normal form. The sizes of the Jordan blocks define a partition $\boldsymbol{l} = (l_i) \in \mathcal{P}(N)$ after a possible renumbering. We then denote the orbit of x by $\mathcal{O}(\boldsymbol{l})$.

3.1 Lemma. *Nilpotent orbits in $\mathfrak{gl}(V)$ correspond bijectively to partitions of N by the Jordan normal form.*

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{gl}(V)$. There exists a decomposition $V = \bigoplus_j V_j$ such that $\mathfrak{l}$ is equal to $\bigoplus_j \mathfrak{gl}(V_j)$ and L is equal to $\prod_j GL(V_j)$. After a possible renumbering, we define a partition $\boldsymbol{l} \in \mathcal{P}(N)$ by $l^j = \dim V_j$. We say that $\mathfrak{l}$ is of type $\boldsymbol{l}$.

3.2 Lemma. *Conjugacy classes of Levi subalgebras of $\mathfrak{gl}(V)$ correspond bijectively to partitions of N by the type.*

The normalizer of $\mathfrak{l}$ in $GL(V)$ is generated by L and elements interchanging subspaces V_j of the same dimension.

3.3 Proposition. *Let $\mathfrak{l}$ be a Levi subalgebra of type $\boldsymbol{l}$. Then*

$$\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{gl}(V)}(0) = \mathcal{O}(\boldsymbol{l}).$$

In particular, the zero orbit is the unique rigid orbit in $\mathfrak{gl}(V)$. By definition, the very stable decomposition classes are given by pairs $(\mathfrak{l}, 0)$ where $\mathfrak{l}$ is any Levi subalgebra of $\mathfrak{gl}(V)$. For $\boldsymbol{l} \in \mathcal{P}(N)$, we denote by $\mathcal{D}(\boldsymbol{l})$ the decomposition class given by the pair $(\mathfrak{l}, 0)$ with $\mathfrak{l}$ of type $\boldsymbol{l}$ and by $\mathcal{S}(\boldsymbol{l})$ the regular sheet of $\overline{\mathcal{D}(\boldsymbol{l})}$. From Propositions 1.9 and 3.3 it follows that $\mathcal{O}(\boldsymbol{l})$ is the unique nilpotent orbit in $\mathcal{S}(\boldsymbol{l})$. This proves the following result:

3.4 Theorem. *Sheets (and very stable decomposition classes) of $\mathfrak{gl}(V)$ are in one-to-one correspondence with the partitions of N. The sheets are disjoint.*

3.4 Symplectic and orthogonal group

Fix a number ε equal to 0 or 1. Let V be a vector space over k of dimension N. We consider an *ε-form* on V, that is, a nondegenerate bilinear form $(\cdot,\cdot)$ on V such that $(v_1, v_2) = (-1)^\varepsilon (v_2, v_1)$ for all $v_1, v_2 \in V$. If $\varepsilon = 1$ the form is symplectic, if $\varepsilon = 0$ it is orthogonal. We define

$$\begin{array}{lllll} G & = & G_\varepsilon(V) & = & \{\, g \in GL(V) \mid (gv_1, gv_2) = (v_1, v_2) \text{ for all } v_1, v_2 \in V \,\} \\ \mathfrak{g} & = & \mathfrak{g}_\varepsilon(V) & = & \{\, x \in \mathfrak{gl}(V) \mid (xv_1, v_2) + (v_1, xv_2) = 0 \text{ for all } v_1, v_2 \in V \,\}. \end{array}$$

Thus, G is $Sp(V)$ for $\varepsilon = 1$ (for even N), and G is $O(V)$ for $\varepsilon = 0$.

The set of *ε-partitions* of N is defined by

$$\mathcal{P}_\varepsilon(N) = \{\, \boldsymbol{l} \in \mathcal{P}(N) \mid \#\{\, j \mid l_j = k \,\} \text{ is even for all } k \equiv \varepsilon \ (2) \,\}.$$

3.5 Lemma. *Nilpotent G-orbits in $\mathfrak{g}$ correspond bijectively to ε-partitions of N.*

If $\boldsymbol{l} \in \mathcal{P}_\varepsilon(N)$, then $\mathcal{O}(\boldsymbol{l}) \cap \mathfrak{g}$ is the corresponding orbit in $\mathfrak{g}$. We denote it by $\mathcal{O}_\varepsilon(\boldsymbol{l})$.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. Then, there exists a decomposition

$$V \;=\; \textstyle\bigoplus_j (V_j \oplus V_j') \;\oplus\; V_0$$

such that

$$\mathfrak{l} \;=\; \textstyle\bigoplus_j \mathfrak{gl}(V_j) \;\oplus\; \mathfrak{g}_\varepsilon(V_0)$$

and

$$L \;=\; \textstyle\prod_j GL(V_j) \;\times\; G_\varepsilon(V_0).$$

Here, the dimensions of $V_j' = \{\, v \in V \mid (v,u) \neq 0 \text{ for all } u \in V_j \,\}$ and of V_j are the same, and $(\cdot,\cdot)$ induces an ε-form on V_0. We set $R = \dim V_0$ and define, after a possible renumbering, a partition $\boldsymbol{s} \in \mathcal{P}(S)$ by $s^j = \dim V_j$. Note that $2S + R = N$. We say that $\mathfrak{l}$ is of type $(\boldsymbol{s}, R)$. Then we define

$$\mathcal{P}_\varepsilon^{Levi}(N) = \{\, (\boldsymbol{s}, R) \in \mathcal{P}(S) \times \mathbb{N}_{\geq 0} \mid 2S + R = N,\ R, S \geq 0 \text{ and } R \neq 2 \text{ if } \varepsilon = 0 \,\}.$$

3.6 Lemma. *G-conjugacy classes of Levi subalgebras of $\mathfrak{g}$ correspond bijectively to elements of $\mathcal{P}_\varepsilon^{Levi}(N)$.*

Let $N_G(\mathfrak{l})$ be the normalizer of $\mathfrak{l}$ in G. It is generated by L and elements interchanging subspaces V_j, V_j' of the same dimension.

Let $\mathfrak{l}$ be a Levi subalgebra of type $(\boldsymbol{s}, R)$. Recall that a decomposition class with data $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ is very stable if $\mathcal{O}_\mathfrak{l}$ is of the form $0 \oplus \mathcal{O}_\varepsilon(\boldsymbol{r})$ with $\boldsymbol{r} \in \mathcal{P}_\varepsilon(R)$. We denote by $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ the decomposition class given by the pair $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$. We define

$$\mathcal{P}_\varepsilon^{vs}(N) = \{\, (\boldsymbol{s}, \boldsymbol{r}) \in \mathcal{P}(S) \times \mathcal{P}_\varepsilon(R) \mid 2S + R = N,\ R, S \geq 0 \text{ and } R \neq 2 \text{ if } \varepsilon = 0 \,\}.$$

Observing that $N_G(\mathfrak{l})$ stabilizes $\mathcal{O}_\mathfrak{l}$ we obtain the following result:

3.7 Theorem. *Very stable G-decomposition classes of $\mathfrak{g}$ are in one-to-one correspondence with the elements of $\mathcal{P}_\varepsilon^{vs}(N)$.*

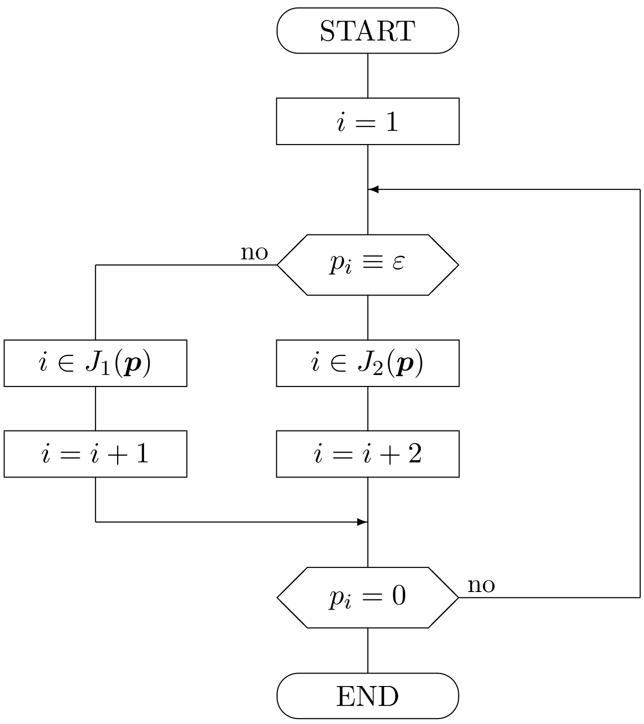
Given $(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}_\varepsilon^{vs}(N)$, we now determine $\boldsymbol{l} \in \mathcal{P}_\varepsilon(N)$ such that $\mathcal{O}_\varepsilon(\boldsymbol{l})$ is the unique nilpotent orbit in $\overline{\mathcal{D}(\boldsymbol{s},\boldsymbol{r})^{reg}}$. We define a partition $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}(N)$ by $p_i = 2s_i + r_i$. The unique largest partition in $\mathcal{P}_\varepsilon(N)$ dominated by $\boldsymbol{p}$ is called the *ε-collapse* of $\boldsymbol{p}$. We will give an explicit definition below.

3.8 Proposition. *Let $\mathfrak{l}$ be a Levi subalgebra of type $(\boldsymbol{s}, R)$ and $\mathcal{O}_\mathfrak{l} = 0 \oplus \mathcal{O}_\varepsilon(\boldsymbol{r})$ a nilpotent orbit in $\mathfrak{l}$. Denote by $\boldsymbol{l}$ the ε-collapse of $\boldsymbol{p}(\boldsymbol{s},\boldsymbol{r})$. Then*

$$\operatorname{Ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{l} = \mathcal{O}_\varepsilon(\boldsymbol{l}).$$

Combining this result with Proposition 1.9 we see that $\mathcal{O}_\varepsilon(\boldsymbol{l})$ is the unique nilpotent orbit in $\overline{\mathcal{D}(\boldsymbol{s},\boldsymbol{r})^{reg}}$.

We continue this section with some combinatorial definitions derived from an element of $\mathcal{P}_\varepsilon^{vs}(N)$. Let $(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}_\varepsilon^{vs}(N)$ and $\boldsymbol{p} \in \mathcal{P}(N)$ be as above. We define sets $J_1 = J_1(\boldsymbol{p})$, $J_2 = J_2(\boldsymbol{p})$ and $J = J_1 \cup J_2$ iteratively:



Using $I(\boldsymbol{p}) = \{\, i \in J_2(\boldsymbol{p}) \mid p_i \geq p_{i+1} + 2 \,\}$ we obtain a partition $\boldsymbol{l} \in \mathcal{P}_\varepsilon(N)$ by setting

$$l_i = \begin{cases} p_i - 1 & i \in I(\boldsymbol{p}), \\ p_i + 1 & i - 1 \in I(\boldsymbol{p}), \\ p_i & \text{otherwise.} \end{cases}$$

Then it is known that $\boldsymbol{l}$ is the ε-collapse of $\boldsymbol{p}$.

Further we define partitions by

	$\boldsymbol{p}_i \in \mathcal{P}(N_i)$	$\boldsymbol{l}_i \in \mathcal{P}(N_i)$	$\boldsymbol{s}_i \in \mathcal{P}(S_i)$	$\boldsymbol{r}_i \in \mathcal{P}(R_i)$
$i \in J_1(\boldsymbol{p})$	(p_i)	(l_i)	(s_i)	(r_i)
$i \in J_2(\boldsymbol{p})$	(p_i, p_{i+1})	(l_i, l_{i+1})	(s_i, s_{i+1})	(r_i, r_{i+1}).

Clearly $N = \sum_{i \in J} N_i$, $S = \sum_{i \in J} S_i$, $R = \sum_{i \in J} R_i$, and $N_i = 2S_i + R_i$. Furthermore, $\boldsymbol{p}_i$ equals $\boldsymbol{p}(\boldsymbol{s}_i, \boldsymbol{r}_i)$ and $\boldsymbol{l}_i \in \mathcal{P}_\varepsilon(N_i)$ is the ε-collapse of $\boldsymbol{p}_i$. Finally, we observe that $(\boldsymbol{s}_i, \boldsymbol{r}_i) \in \mathcal{P}_\varepsilon^{vs}(N_i)$, that is, $(\boldsymbol{s}_i, \boldsymbol{r}_i)$ determines a very stable decomposition class in $\mathfrak{g}_\varepsilon(V^{(i)})$ with $\dim V^{(i)} = N_i$.

We conclude this section with the following lemma.

3.9 Lemma. *Let $\boldsymbol{l}$ and $\boldsymbol{p}$ be two partitions such that $l^1 = p^1$. Denote by $\boldsymbol{l}'$ and $\boldsymbol{p}'$ the partitions obtained from $\boldsymbol{l}$ and $\boldsymbol{p}$ by removing their first column, respectively. If $\boldsymbol{l}$ is the ε-collapse of $\boldsymbol{p}$, then $\boldsymbol{l}'$ is the ε-collapse of $\boldsymbol{p}'$.*

3.5 Special orthogonal group

In this section we consider the connected component of the identity in an orthogonal or symplectic group G. It consists of the elements of determinant 1. We point out the differences which appear when objects in $\mathfrak{g}$ are classified with respect to G° instead of G. Obviously, we may assume that G is the orthogonal group of an even dimensional vector space V. Then G° is the special orthogonal group $SO(V)$. It is clear that the G-orbit of some object is either already a G°-orbit or splits into two G°-orbits. We keep all notation from the previous section.

We begin with the nilpotent elements of $\mathfrak{g}$.

3.10 Lemma. *Let $\boldsymbol{l} \in \mathcal{P}_0(N)$. Then the G-orbit $\mathcal{O}_0(\boldsymbol{l})$ splits if and only if $\boldsymbol{l}$ is very even, i.e. all l_j are even.*

The next lemma deals with the Levi subalgebras of $\mathfrak{g}$.

3.11 Lemma. *Let $(\boldsymbol{s}, R) \in \mathcal{P}_0^{Levi}(N)$. Then the G-conjugacy class of a Levi subalgebra of type $(\boldsymbol{s}, R)$ splits if and only if it is of very even type, i.e. all s^j are even and R is zero.*

We need the following general result .

3.12 Lemma. *Let x be a nilpotent element of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Then the G-orbit of the pair $(\mathfrak{l}, x)$ splits if and only if either the G-conjugacy class of $\mathfrak{l}$ or the $N_G(\mathfrak{l})$-orbit of x splits.*

Let $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ be a very stable decomposition class of $\mathfrak{g}$. Then, $\mathfrak{l}$ is of type $(\boldsymbol{s}, R)$ and $\mathcal{O}_\mathfrak{l} = Lx$ where $x = (0, \dots, 0, x_0)$ with $x_0 \in \mathcal{O}_0(\boldsymbol{r})$.

3.13 Lemma. *The $N_G(\mathfrak{l})$-orbit of x splits if and only if all s^j are even and $\boldsymbol{r}$ is very even.*

3.14 Theorem. *The very stable decomposition class $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ splits if and only if $\boldsymbol{p}$ is very even if and only if $\mathcal{O}_0(\boldsymbol{l})$ splits.*

Proof. Putting together the results of this section we see that the left hand side holds if and only if all s^j are even and $\boldsymbol{r}$ is very even or empty. But this is equivalent to the second statement. If $\boldsymbol{p}$ is very even, then $\boldsymbol{l}$ equals $\boldsymbol{p}$ and so $\mathcal{O}_0(\boldsymbol{l})$ splits. Finally, Lemma 1.7 implies that $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ splits if $\mathcal{O}_0(\boldsymbol{l})$ splits. □

3.15 Corollary. *If the very stable decomposition class $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ splits, then $\overline{\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})}^{reg}$ is the disjoint union of two irreducible subsets. These are regular sheets of very stable decomposition varieties with respect to the action of G°.*

3.16 Corollary. (Main Theorem) *The regular sheets of very stable decomposition varieties of $\mathfrak{g}$ with respect to the action of G° are smooth varieties.*

Chapter 4

Parametrization Theorem

Let $\mathfrak{g}$ be a classical Lie algebra and G the corresponding group (in the sense of Chapter 3).

4.1 Theorem. (Parametrization Theorem (cf. [2], 5.6. Satz)) *Let $\mathcal{D}$ be a very stable decomposition class with data $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$. Then, the map $\mathfrak{k} \to \overline{\mathcal{D}}^{reg}/G$ given by $z \mapsto \operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z + \mathcal{O}_\mathfrak{l})$ induces a bijection of orbit spaces $\mathfrak{k}/N \to \overline{\mathcal{D}}^{reg}/G$.*

Proof. In a first lemma we show that the induced map is in fact well defined. It is surjective by Proposition 1.9. In a second lemma we show that it is injective, thus completing the proof of the theorem. □

4.2 Lemma. *If $w \in N$ and $z \in \mathfrak{k}$, then $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z + \mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(wz + \mathcal{O}_\mathfrak{l})$.*

Proof. In Section 3.4 we already observed that $N_G(\mathfrak{l})\mathcal{O}_\mathfrak{l} = \mathcal{O}_\mathfrak{l}$. Therefore $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z + \mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_{w\mathfrak{l}}^\mathfrak{g} w(z + \mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(wz + w\mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(wz + \mathcal{O}_\mathfrak{l})$. □

4.3 Lemma. *If $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z + \mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z' + \mathcal{O}_\mathfrak{l})$ for $z, z' \in \mathfrak{k}$, then there exists $w \in N$ such that $wz = z'$.*

Proof. It follows from Lemma 1.6 that there exist nilpotent elements $x \in \mathfrak{g}^z$ and $x' \in \mathfrak{g}^{z'}$ such that $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z + \mathcal{O}_\mathfrak{l}) = G(z + x)$ and $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(z' + \mathcal{O}_\mathfrak{l}) = G(z' + x')$. By assumption, there is a $g \in G$ such that $z' + x' = g.(z + x)$, whence $z' = g.z$ and $x' = g.x$. Therefore $\mathfrak{g}^z$ contains both $\mathfrak{l}$ and $g^{-1}\mathfrak{l}$. Using again Lemma 1.6 (and the transitivity of induction) we see that $\operatorname{Ind}_\mathfrak{l}^{\mathfrak{g}^z}(\mathcal{O}_\mathfrak{l}) = \operatorname{Ind}_{g^{-1}\mathfrak{l}}^{\mathfrak{g}^z}(g^{-1}\mathcal{O}_\mathfrak{l})$. In this situation the next lemma produces an element $\tilde{g} \in G_z$ such that $\tilde{g}\mathfrak{l} = g^{-1}\mathfrak{l}$. Since $g\tilde{g}$ normalizes $\mathfrak{l}$, it induces an element $w \in N$ such that $wz = z'$. □

4.4 Lemma. *Let $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{l}$ be Levi subalgebras of $\mathfrak{g}$, and let $\mathcal{O}_{1,2} \subset \mathfrak{m}_{1,2}$ be very stable nilpotent orbits. If $\operatorname{Ind}_{\mathfrak{m}_1}^\mathfrak{l} \mathcal{O}_1 = \operatorname{Ind}_{\mathfrak{m}_2}^\mathfrak{l} \mathcal{O}_2$ and if $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are conjugate in $\mathfrak{g}$, then $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are conjugate in $\mathfrak{l}$.*

Proof. We may assume that $\mathfrak{g}$ is “simple”. Then, $\mathfrak{l}$ is a direct sum $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_r$ with $\mathfrak{l}_j$ of type $\mathfrak{gl}$ for $j = 1, \ldots, r$ and $\mathfrak{l}_0$ of type $\mathfrak{g}$ or equal to 0. Setting $\mathfrak{m}_{ij} = \mathfrak{m}_i \cap \mathfrak{l}_j$ for $i = 1, 2$ and $j = 0, \ldots, r$ we obtain $\mathfrak{m}_i = (\mathfrak{k} \cap \mathfrak{m}_i) \oplus \mathfrak{m}_{i0} \oplus \cdots \oplus \mathfrak{m}_{ir}$. In particular, $\mathcal{O}_i = \sum_{j=0}^r \mathcal{O}_{ij}$ with $\mathcal{O}_{ij} = 0$ for $j = 1, \ldots, r$, and

$$\operatorname{Ind}_{\mathfrak{m}_i}^\mathfrak{l} \mathcal{O}_i = \sum_{j=0}^r \operatorname{Ind}_{\mathfrak{m}_{ij}}^{\mathfrak{l}_j} \mathcal{O}_{ij}$$

for $i = 1, 2$. By the first assumption we have

$$\mathrm{Ind}_{\mathfrak{m}_{1j}}^{\mathfrak{l}_j}(0) = \mathrm{Ind}_{\mathfrak{m}_{2j}}^{\mathfrak{l}_j}(0)$$

for $j = 1, \ldots, r$. By Proposition 3.3 this implies that $\mathfrak{m}_{1j}$ and $\mathfrak{m}_{2j}$ are conjugate in $\mathfrak{l}_j$ for $j = 1, \ldots, r$. It remains to show that $\mathfrak{m}_{1,0}$ and $\mathfrak{m}_{2,0}$ are conjugate in $\mathfrak{l}_0$. But this follows immediately from the classification of Levi subalgebras in symplectic and orthogonal Lie algebras (Lemma 3.6). □

Remarks. (a) In the case of $\mathfrak{sl}_n$, this is proved by Kraft in [13].

(b) If G is a connected semisimple group and $\overline{\mathcal{D}}^{reg}$ a sheet of $\mathfrak{g}$, this is proved by Borho in [2].

Combining the Parametrization Theorem with Corollary 2.7 and Theorem 2.11 we obtain the following two results:

4.5 Corollary. *The map $\mathfrak{k} \to \overline{\mathcal{D}}^{reg}/G$ given by $z \mapsto G(e+z)$ induces a bijection of orbit spaces $\mathfrak{k}/N \to \overline{\mathcal{D}}^{reg}/G$.*

4.6 Corollary. *The map $\mathfrak{k} \to (e+X)/A$ given by $z \mapsto A\varepsilon(e+z)$ induces a bijective morphism of quotient spaces $\mathfrak{k}/N \to (e+X)/A$.*

Chapter 5

Main Theorem for general linear groups

Let V be a vector space of dimension N. We write G for the general linear group $GL(V)$. Let $\boldsymbol{l} \in \mathcal{P}(N)$ be a partition of N. We construct a standard triple $\{e, h, f\}$ in $\mathfrak{g}$ and a subalgebra $\mathfrak{k}$ contained in $\mathfrak{g}^h$, such that $\mathcal{O}(\boldsymbol{l}) = Ge$ and $\mathcal{S}(\boldsymbol{l}) = G(e + \mathfrak{k})$ as in (2.2).

First, we decompose V into a direct sum of subspaces $V^{(i)}$ of dimension $\dim V^{(i)} = l_i = N_i$ and choose an adapted basis $\{ {v_j}^{(i)} \mid i \geq 1 \ , \ j = 1, \ldots, N_i \}$. Define $e = \sum_i e_i$ with $e_i \in \mathfrak{g}_i = \mathfrak{gl}(V^{(i)})$ by

$$e_i.{v_j}^{(i)} = \begin{cases} {v_{j-1}}^{(i)} & j = 2, \ldots, N_i, \\ 0 & j = 1. \end{cases}$$

Then, we get $\mathcal{O}(\boldsymbol{l}) = Ge$ and $\mathcal{O}(\boldsymbol{l}_i) = G_i e_i$ with $\boldsymbol{l}_i = (l_i) \in \mathcal{P}(N_i)$ for all i.

Let $\mathfrak{h}_\mathfrak{i} \subset \mathfrak{g}_i$ and $\mathfrak{h} \subset \mathfrak{g}$ be Cartan subalgebras such that the ${v_j}^{(i)}$ are weight vectors with corresponding weight ${\omega_j}^{(i)}$. We define $h = \sum_i h_i$ in $\mathfrak{h}$ by

$${\omega_j}^{(i)}(h_i) = N_i + 1 - 2j \quad \text{for } j = 1, \ldots, N_i.$$

After adding the missing elements, we obtain standard triples $\{e, h, f\}$ in $\mathfrak{g}$ and $\{e_i, h_i, f_i\}$ in $\mathfrak{g}_\mathfrak{i}$ with $f = \sum_i f_i$. Finally, we define $\mathfrak{k}$ in $\mathfrak{h}$ by

$${\omega_j}^{(i)} = {\omega_j}^{(i')} \quad \text{for all } i \geq i' \geq 1.$$

Its centralizer $\mathfrak{l}$ in $\mathfrak{g}$ is a Levi subalgebra of type $\boldsymbol{l}$. We check that $\mathfrak{k}$ commutes with h, and that $\mathcal{S}(\boldsymbol{l}) = G(e + \mathfrak{k})$. We also see that $\mathfrak{h}_i$ is of type $\boldsymbol{l}_i$, that $\mathfrak{h}_i$ commutes with h_i, and that $\mathcal{S}(\boldsymbol{l}_i) = G_i(e_i + \mathfrak{h}_i)$ is the regular sheet of $\mathfrak{g}_i$.

Next, we consider the maps

$$\varepsilon\colon e + \mathfrak{k} \to \varepsilon(e + \mathfrak{k}) = e + Y \subset e + X = (e + \mathfrak{g}^f) \cap \mathcal{S}(\boldsymbol{l})$$

and

$$\varepsilon_i\colon e_i + \mathfrak{k}_\mathfrak{i} \to \varepsilon_i(e_i + \mathfrak{k}_\mathfrak{i}) = e_i + Y_i \subset e_i + X_i = (e_i + {\mathfrak{g}_\mathfrak{i}}^{f_i}) \cap \mathcal{S}(\boldsymbol{l}_i)$$

as defined in (2.2) . Further we set

$$\phi\colon \mathfrak{k} \to Y \subset X \subset \mathfrak{g}^f \ , \ \phi(z) = \varepsilon(e + z) - e \ , \ z \in \mathfrak{k},$$

and

$$\phi_i\colon \mathfrak{k}_\mathfrak{i} \to Y_i \subset X_i \subset {\mathfrak{g}_\mathfrak{i}}^{f_i} \ , \ \phi_i(z) = \varepsilon_i(e_i + z) - e_i \ , \ z \in \mathfrak{k}_\mathfrak{i}.$$

Using Proposition 2.3 we see that $\phi = (\sum_i \phi_i)|_\mathfrak{k}$. This fact allows us to first investigate the maps ϕ_i and then try to understand ϕ using the ϕ_i.

The first task is settled quickly. We denote by W_i the Weyl group of $\mathfrak{g}_i$. Its elements permute the weights ${\omega_j}^{(i)}$, $j = 1, \ldots, N_i$.

5.1 Theorem. *The morphism $\phi_i \colon \mathfrak{h}_i \to \mathfrak{g}_i^{f_i}$ is surjective; it is a quotient map with respect to W_i.*

Proof. This is Theorem 2.14. □

Recall that N denotes the image of $N_G(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$. This group is isomorphic to the product of symmetric groups $S_{N_i - N_{i+1}}$ over all i. The elements of the factor $S_{N_i-N_{i+1}}$ act on $\mathfrak{k}^*$ by permuting the weights ${\omega_j}^{(i)}$, $j = N_{i+1}+1, \ldots, N_i$. It is important to note that, on $\mathfrak{k}$, we have ${\omega_j}^{(i)} = {\omega_j}^{(i')}$ for $i \geq i' \geq 1$ and $j = 1, \ldots, N_i$. We also need to know the algebra of invariants $k[\mathfrak{k}]^N$ explicitly. The following notation for power sums is used: $PS_{m,n;s}(x) = x_m^s + x_{m+1}^s + \ldots + x_{n-1}^s + x_n^s$ for $1 \leq m \leq n$ and $s \geq 0$. Then, it is easy to see that

$$k[\mathfrak{k}]^N = \bigotimes_i k[PS_{N_{i+1}+1,N_i;s}(\omega^{(i)}), s = 1, \ldots, N_i - N_{i+1}].$$

We further need the group homomorphism

$$N \to \prod_i W_i\ ,\ w \mapsto (w^{(i)})_i$$

which makes the inclusion $\mathfrak{k} \subset \mathfrak{h} = \bigoplus_i \mathfrak{h}_i$ equivariant, i.e.

$$wz = \sum_i w^{(i)} z_i\ ,\ z = \sum_i z_i \in \mathfrak{k}\ ,\ w \in N.$$

5.2 Theorem. *The morphism $\phi \colon \mathfrak{k} \to \mathfrak{g}^f$ is a quotient map with respect to the reflection group N. The image of ϕ is X.*

Proof. We divide the proof into three steps:

STEP 1: The map ϕ is constant on orbits of N by Theorem 5.1 using the preceeding remarks: We have

$$\phi(wz) = \sum_i \phi_i(w^{(i)} z_i) = \sum_i \phi_i(z_i) = \phi(z)$$

for $w \in N$ and $z = \sum_i z_i \in \mathfrak{k}$.

STEP 2: Since G_e is connected, the image of ϕ is X by Theorem 2.10 and Lemma 2.4.

STEP 3: Let $\mathrm{res} \colon k[\mathfrak{h}] \to k[\mathfrak{k}]$ be the comorphism of the inclusion $\mathfrak{k} \subset \mathfrak{h} = \bigoplus_i \mathfrak{h}_i$. The first theorem yields that

$$\phi^* k[\mathfrak{g}^f] = \mathrm{res}\,(\textstyle\sum_i \phi_i)^* k[\bigoplus_i \mathfrak{g}_i^{f_i}] = \mathrm{res}\, k[\bigoplus_i \mathfrak{h}_i]^{\prod W_i}.$$

Therefore, it suffices to show that

$$\mathrm{res}\, k[\textstyle\bigoplus_i \mathfrak{h}_i]^{\prod W_i} = k[\mathfrak{k}]^N.$$

We obviously have

$$k[\textstyle\bigoplus_i \mathfrak{h}_i]^{\prod_i W_i} = \bigotimes_i k[PS_{1,N_i;s}(\omega^{(i)}), s = 1, \ldots, N_i].$$

Restricting the right hand side to $\mathfrak{k}$ and using

$$PS_{1,N_i;s}(x) = PS_{1,N_{i+1};s}(x) + PS_{N_{i+1}+1,N_i;s}(x)$$

we obtain

$$\bigotimes_i k[PS_{N_{i+1}+1,N_i;s}(\omega^{(i)}), s = 1, \ldots, N_i - N_{i+1}],$$

which is $k[\mathfrak{k}]^N$ described as above.

□

Chapter 6

Main Theorem for symplectic and orthogonal groups

The idea of the proof is the same as for the general linear groups. But some of the arguments turn out to be much more elaborate.

6.1 Strategy and basic construction

Let V be a vector space of dimension N and $(\cdot,\cdot)$ an ε-form on V. We write G for $G_\varepsilon(V)$. Let $(\boldsymbol{s},\boldsymbol{r})$ be an element of $\mathcal{P}_\varepsilon^{vs}(N)$. Recall the definition of $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}(N)$, of the index set $J = J(\boldsymbol{p})$, and of the ε-collapse $\boldsymbol{l}$ of $\boldsymbol{p}$. We construct a standard triple $\{e,h,f\}$ in $\mathfrak{g}$ and a subalgebra $\mathfrak{k}$ contained in $\mathfrak{g}^h$, such that $\mathcal{O}_\varepsilon(\boldsymbol{l}) = Ge$ and $\overline{\mathcal{D}(\boldsymbol{s},\boldsymbol{r})}^{reg} = G(e+\mathfrak{k})$ as in (2.2).

First, we decompose V orthogonally into a direct sum of subspaces $V^{(i)}$ of dimension $\dim V^{(i)} = N_i$ and choose an adapted basis $\{\, {v_j}^{(i)} \mid i \in J\ ,\ j = 1,\dots N_i \,\}$. By this we mean the following:

- $V^{(i)} = \langle\, {v_j}^{(i)} \mid j = 1,\dots N_i \,\rangle$ for $i \in J$.
- We obtain an ε-form on $V^{(i)}$ by restricting $(\cdot,\cdot)$.
- We have $({v_j}^{(i)}, {v_k}^{(i')}) \neq 0$ if and only if $i = i'$ and $j + k = N_i + 1$.

We write $\mathfrak{g}_i$ for $\mathfrak{g}_\varepsilon(V^{(i)})$, and G_i for $G_\varepsilon(V^{(i)})$. It is convenient to set $t_i = s_i - s_{i+1}$ for all $i \geq 1$. We define $e = e_J = \sum_{i\in J} e_i$ with $e_i \in \mathfrak{g}_i$ by

$$i \in J_1: \qquad e_i.{v_j}^{(i)} = \begin{cases} {v_{j-1}}^{(i)} & j = 2,\dots,N_i, \\ 0 & j = 1, \end{cases}$$

$$i \in J_2\ (t_i = 0,1): \qquad e_i.{v_j}^{(i)} = \begin{cases} {v_{j-2}}^{(i)} & j = 3,\dots,N_i, \\ 0 & j = 1,2, \end{cases}$$

$$i \in J_2\ (t_i \geq 2): \qquad e_i.{v_j}^{(i)} = \begin{cases} {v_{j-1}}^{(i)} & j = N_i - t_i + 3,\dots,N_i, \\ {v_{j-2}}^{(i)} + {v_{j-1}}^{(i)} & j = N_i - t_i + 2, \\ {v_{j-2}}^{(i)} & j = t_i + 1,\dots,N_i - t_i + 1, \\ {v_{j-1}}^{(i)} & j = 2,\dots,t_i, \\ 0 & j = 1. \end{cases}$$

Then, we get $\mathcal{O}_\varepsilon(\boldsymbol{l}) = Ge$, and $\mathcal{O}_\varepsilon(\boldsymbol{l}_i) = G_i e_i$ with $\boldsymbol{l}_i \in \mathcal{P}_\varepsilon(N_i)$ for all $i \in J$.

For any $i \in J$, let $\mathfrak{h}_i \subset \mathfrak{g}_i$ be Cartan subalgebras such that the ${v_j}^{(i)}$ are weight vectors with corresponding weight ${\omega_j}^{(i)}$. The following relations hold in $\mathfrak{h}_i^*$:

$${\omega_j}^{(i)} + {\omega_{N_i+1-j}}^{(i)} = 0 \quad , \quad {\omega_{M_i+1}}^{(i)} = 0 \quad (\text{if } N_i = 2M_i + 1)$$

We define $h = h_J = \sum_{i \in J} h_i$ with $h_i \in \mathfrak{h}_i$ by

$$\begin{array}{llcll} i \in J_1\colon & {\omega_j}^{(i)}(h_i) & = & N_i + 1 - 2j & j = 1, \dots, N_i, \\ i \in J_2\ (t_i = 0)\colon & {\omega_j}^{(i)}(h_i) & = & l_i + 1 - 2k & j = 2k-1 \text{ and } 2k\ ,\ k \geq 1, \\ i \in J_2\ (t_i \geq 1)\colon & {\omega_j}^{(i)}(h_i) & = & \begin{cases} l_i + 1 - 2j & j = 1, \dots, t_i - 1 \\ l_i + 1 - 2(t_i + k) & j = t_i + 2k \text{ and } t_i + 2k + 1\ ,\ k \geq 0. \end{cases} \end{array}$$

After adding the missing elements, we obtain standard triples $\{e, h, f\}$ in $\mathfrak{g}$ and $\{e_i, h_i, f_i\}$ in $\mathfrak{g}_i$ such that $f = \sum_{i \in J} f_i$.

If $i \in J_2$ and $t_i \geq 1$, we denote by a_i the element in G_i interchanging ω_{t+2j} and ω_{t+2j+1} for $j = 0, 1, \dots, M - t$. This element centralizes the triple $\{e_i, h_i, f_i\}$. It will play an important role in this chapter.

The definition of $\mathfrak{k}$ is a bit more involved. We define $\mathfrak{k}$ in $\bigoplus_{i \in J} \mathfrak{h}_i$ by

$$\begin{array}{lcll} {\omega_j}^{(i)} & = & 0 & i \in J\ ,\ j = S_i + 1, \dots, N_i - S_i, \\ {\omega_j}^{(i)} & = & {\omega_k}^{(i')} & i, i' \in J\ ,\ j \text{ and } k \text{ such that } \boldsymbol{s}(i,j) = \boldsymbol{s}(i',k), \end{array}$$

and $\mathfrak{k}_i$ in $\mathfrak{h}_i$ for every $i \in J$ by

$$\begin{array}{lcll} {\omega_j}^{(i)} & = & 0 & j = S_i + 1, \dots, N_i - S_i, \\ {\omega_j}^{(i)} & = & {\omega_k}^{(i)} & j \text{ and } k \text{ such that } \boldsymbol{s}(i,j) = \boldsymbol{s}(i,k). \end{array}$$

Here $\boldsymbol{s}(i,j)$ is given by

$$\begin{array}{llcll} i \in J_1\colon & \boldsymbol{s}(i,j) & = & S_i + 1 - j & j = 1, \dots, S_i, \\ i \in J_2\colon & \boldsymbol{s}(i,j) & = & \begin{cases} s_i + 1 - j & j = 1, \dots, t_i, \\ s_i + 1 - t_i - k & j = t_i + 2k - 1 \text{ and } t_i + 2k\ ,\ k = 1, \dots, s_i - t_i. \end{cases} \end{array}$$

Let $\mathfrak{l}$ be the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$. This is a Levi subalgebra of type $(\boldsymbol{s}, R)$. Similarily, we let $\mathfrak{l}_i$ be the centralizer of $\mathfrak{k}_i$ in $\mathfrak{g}_i$, which is a Levi subalgebra of type $(\boldsymbol{s}_i, R_i)$.

Finally, we define an element $e' = \sum_{i \in J} e'_i \in \mathfrak{l}$ with $e'_i \in \mathfrak{l}_i$ by

$$\begin{array}{llcl} i \in J_1\colon & e'_i.{v_j}^{(i)} & = & \begin{cases} {v_{j-1}}^{(i)} & j = s_i + 2, \dots, N_i - s_i, \\ 0 & \text{otherwise}, \end{cases} \\ i \in J_2\colon & e'_i.{v_j}^{(i)} & = & \begin{cases} {v_{j-2}}^{(i)} & j = s_i + 3, \dots, N_i - s_i, \\ 0 & \text{otherwise}. \end{cases} \end{array}$$

Then, we get $\mathcal{O}_{\mathfrak{l}} = 0 \oplus \mathcal{O}_\varepsilon(\boldsymbol{r}) = Le'$, and $\mathcal{O}_{\mathfrak{l}_i} = 0 \oplus \mathcal{O}_\varepsilon(\boldsymbol{r}_i) = L_i e'_i$ with $\boldsymbol{r}_i \in \mathcal{P}_\varepsilon(R_i)$ for all $i \in J$. We check that $\mathfrak{k}_i$ commutes with h_i, and $\mathfrak{k}$ with h as well. Setting $\mathcal{D} = \mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ and $\mathcal{D}_i = \mathcal{D}(\boldsymbol{s}_i, \boldsymbol{r}_i)$ we obtain that $\overline{\mathcal{D}}^{reg} = G(e + \mathfrak{k})$ and that $\overline{\mathcal{D}_i}^{reg} = G_i(e_i + \mathfrak{k}_i)$ for all $i \in J$.

Next, we consider the maps

$$\varepsilon\colon e + \mathfrak{k} \to \varepsilon(e + \mathfrak{k}) = e + Y \subset e + X = (e + \mathfrak{g}^f) \cap \overline{\mathcal{D}}^{reg}$$

and

$$\varepsilon_i\colon e_i + \mathfrak{k}_i \to \varepsilon_i(e_i + \mathfrak{k}_i) = e_i + Y_i \subset e_i + X_i = (e_i + \mathfrak{g}_i^{f_i}) \cap \overline{\mathcal{D}_i}^{reg}$$

as defined in (2.2). We set

$$\phi\colon \mathfrak{k} \to Y \subset X \subset \mathfrak{g}^f \ , \ \phi(z) = \varepsilon(e+z) - e \ , \ z \in \mathfrak{k},$$

and

$$\phi_i\colon \mathfrak{k}_i \to Y_i \subset X_i \subset \mathfrak{g}_i^{f_i} \ , \ \phi_i(z) = \varepsilon_i(e_i+z) - e_i \ , \ z \in \mathfrak{k}_i.$$

Using Proposition 2.3 we see that $\phi = (\sum_i \phi_i)|_{\mathfrak{k}}$.

Let N_i be the image of $N_{G_i}(\mathfrak{k}_i)$ in $\mathrm{Aut}(\mathfrak{k}_i)$. This group is isomorphic to

$$\begin{array}{ll} i \in J_1: & B_{s_i}, \\ i \in J_2: & B_{t_i} \times B_{s_{i+1}}. \end{array}$$

For $i \in J_1$, it acts on $\mathfrak{k}_i^*$ as the group of permutations and sign changes on the set of weights $\{\omega_j{}^{(i)}\}_{j=1,\dots,S_i}$. For $i \in J_2$, the first and second factor act as the group of permutations and sign changes on the set of weights $\{\omega_j{}^{(i)}\}_{j=1,\dots,t_i}$ and $\{\omega_{t_i+2k}{}^{(i)}\}_{k=1,\dots,s_{i+1}}$, respectively. Note that, on $\mathfrak{k}_i$, we have $\omega_{t_i+2k}{}^{(i)} = \omega_{t_i+2k-1}{}^{(i)}$ for $k = 1, \dots, s_{i+1}$.

6.1 Theorem. *The morphism $\phi_i\colon \mathfrak{k}_i \to \mathfrak{g}_i^{f_i}$ is a quotient map with respect to a suitable reflection subgroup $(N_i)_0$ of N_i. The image of ϕ_i is X_i.*

The definition of $(N_i)_0$ and the proof of the theorem will be given in Sections 2 and 3. We seperate the three cases: $i \in J_1$, $i \in J_2$ with $t_i = 0$, and $i \in J_2$ with $t_i \geq 1$. They are referred to as the elementary cases of type I, II, and III, respectively.

Recall that N denotes the image of $N_G(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$. This group is isomorphic to

$$\prod_{i\geq 1} B_{t_i} = \prod_{i\in J_1} B_{t_i} \times \prod_{i \in J_2} (B_{t_i} \times B_{t_{i+1}}).$$

The factor B_{t_i}, for $i \in J$, and the factor $B_{t_{i+1}}$, for $i \in J_2$, act on $\mathfrak{k}^*$ as the group of permutations and sign changes on the set of weights $\{\omega_j{}^{(i)}\}_{j=1,\dots,t_i}$ and $\{\omega_{t_i+2k}{}^{(i)}\}_{k=1,\dots,t_{i+1}}$, respectively. Note that, on $\mathfrak{k}$, we have $\omega_j{}^{(i)} = \omega_k{}^{(i')}$ for $i, i' \in J$ and $\boldsymbol{s}(i,j) = \boldsymbol{s}(i',k)$. We further need the group homomorphism

$$N \to \prod_{i \in J} N_i \ , \ w \mapsto (w^{(i)})_{i\in J}$$

which makes the inclusion $\mathfrak{k} \subset \mathfrak{k}_J = \bigoplus_{i\in J} \mathfrak{k}_i$ equivariant, i.e.

$$wz = \textstyle\sum_i w^{(i)} z_i \ , \ z = \sum_i z_i \in \mathfrak{k} \ , \ w \in N.$$

6.2 Theorem. *The morphism $\phi\colon \mathfrak{k} \to \mathfrak{g}^f$ is a quotient map with respect to a suitable reflection subgroup N_0 of N. The image of ϕ is X.*

The definition of N_0 and the proof of the theorem will be given in Section 4.

6.2 Elementary cases of type I and II

The case of type I is a slight generalization of the regular sheet. Let $(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}_\varepsilon^{vs}(N)$ determine the very stable decomposition class $\mathcal{D}$ in $\mathfrak{g}$. We have $J(\boldsymbol{p}) = J_1(\boldsymbol{p}) = \{1\}$ for $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}(N)$. We begin with listing the combinatorial data:

$$N = \begin{cases} 2M+1 & \text{if } \varepsilon = 0 \\ 2M & \text{if } \varepsilon = 1 \end{cases} \quad , \quad R = \begin{cases} 2Q+1 & \text{if } \varepsilon = 0 \\ 2Q & \text{if } \varepsilon = 1 \end{cases} \quad , \quad M = S + Q \quad ,$$

$$\boldsymbol{s} = (s) \in \mathcal{P}(S) \quad , \quad \boldsymbol{r} = (r) \in \mathcal{P}_\varepsilon(R) \quad , \quad \boldsymbol{p} = (2s+r) = \boldsymbol{l} \in \mathcal{P}_\varepsilon(N).$$

We note that $\mathcal{O}(\boldsymbol{l})$ is the regular nilpotent orbit of $\mathfrak{g}$. Furthermore, $\overline{\mathcal{D}}^{reg}$ is the regular sheet of $\mathfrak{g}$ if $Q = 0$.

Recall that A denotes the centralizer of $\{e,h,f\}$ in G. Here this group is central. We have already seen that N is isomorphic to B_s, which acts as the group of permutations and sign changes on the set of weights $\{\omega_j\}_{j=1,\dots,s}$. The algebra of invariants is given by $k[\mathfrak{k}]^N = k[PS_{1,s;2j}(\boldsymbol{\omega}^{(i)}), j = 1,\dots,s]$.

Now we are ready to prove Theorem 6.1.

Theorem. *The morphism $\phi\colon \mathfrak{k} \to \mathfrak{g}^f$ is a quotient map with respect to N. The image of ϕ is X.*

Proof. We reduce the proof of the theorem to the case of the regular sheet of $\mathfrak{g}$. We denote by ϕ_{reg} the map $\mathfrak{h} \to \mathfrak{g}^f$ corresponding to the regular sheet of $\mathfrak{g}$. Theorem 2.14 says that this is the quotient by the Weyl group. We may embed the action of N on $\mathfrak{k}$ into the Weyl group action on $\mathfrak{h}$ in an obvious way. Since ϕ is the restriction of ϕ_{reg} to $\mathfrak{k}$, the first claim follows straightforwardly using the description of the invariants as given above. Because A is central, the image of ϕ is X by Theorem 2.10. □

The case of type II turns out to be a slight generalization of an admissible sheet. Let $(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}_\varepsilon^{vs}(N)$ determine the very stable decomposition class $\mathcal{D}$ in $\mathfrak{g}$. Here we have $J(\boldsymbol{p}) = J_2(\boldsymbol{p}) = \{1\}$ and $I(\boldsymbol{p}) = \emptyset$ for $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}(N)$. We list the combinatorial data:

$$N = 2M = 2S + R \quad , \quad R = 2Q \quad , \quad Q \equiv \varepsilon \quad ,$$

$$\boldsymbol{s} = (s,s) \in \mathcal{P}(S) \quad , \quad \boldsymbol{r} = (r,r) \in \mathcal{P}_\varepsilon(R) \quad , \quad \boldsymbol{p} = (2s+r, 2s+r) = \boldsymbol{l} \in \mathcal{P}_\varepsilon(N).$$

We easily check that $\overline{\mathcal{D}}^{reg}$ is an admissible sheet of $\mathfrak{g}$ if $Q = \varepsilon$. The admissible subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ (see 2.4) is $\mathfrak{g}_{\varepsilon'}(M)$ with $\varepsilon' \in \{0,1\}$ such that $\varepsilon + \varepsilon' = 1$. The proof of the theorem reduces to Theorem 2.15 in exactly the same way as in the case of type I. We only need to note that A is connected.

6.3 Elementary case of type III

Let $(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}_\varepsilon^{vs}(N)$ determine the very stable decomposition class $\mathcal{D}$ in $\mathfrak{g}$. Here we have $J(\boldsymbol{p}) = J_2(\boldsymbol{p}) = I(\boldsymbol{p}) = \{1\}$ for $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s},\boldsymbol{r}) \in \mathcal{P}(N)$. We list the combinatorial data:

$$N = 2M = 2S + R \quad , \quad R = 2Q \quad , \quad Q \equiv \varepsilon \quad ,$$

$$\boldsymbol{s} = (s, s-t) \in \mathcal{P}(S) \text{ with } t \geq 1 \quad , \quad \boldsymbol{r} = (r,r) \in \mathcal{P}_\varepsilon(R),$$

$$\boldsymbol{p} = (2s + r, 2(s - t) + r) \in \mathcal{P}(N),$$

$$\boldsymbol{l} = (2s + r - 1, 2(s - t) + r + 1) \in \mathcal{P}_\varepsilon(N).$$

The definition of the subgroup N_0 of N is as follows: Recall that N is isomorphic to $B_t \times B_{s-t}$, which acts as the group of permutations and sign changes on the set of weights $\{\omega_j\}_{j=1,\dots,t}$ and $\{\omega_{t+2k}\}_{k=1,\dots,s-t}$. Let D_t be the subgroup of B_t consisting of elements changing an even number of signs. We define N_0 to be the subgroup in N corresponding to $D_t \times B_{s-t}$ in $B_t \times B_{s-t}$. We are now ready to prove Theorem 6.1.

Theorem. *The morphism $\phi\colon \mathfrak{k} \to \mathfrak{g}^f$ is a quotient map with respect to N_0. The image of ϕ is X.*

Proof. We divide the proof into three steps: The first two steps imply that ϕ^* induces an inclusion of $k[X]$ into $k[\mathfrak{k}]^{N_0}$. In the third step we show that ϕ^* maps $k[\mathfrak{g}^f]$ onto $k[\mathfrak{k}]^{N_0}$.

STEP 1: We show that ϕ is constant on N_0-orbits: For $z \in \mathfrak{k}$ and $w \in N_0$ we find $g_{w,z} \in U$ such that $e + wz = g_{w,z}.(e + z)$. This will be done in the Appendix for a set of generators of N_0.

STEP 2: Let a represent the non-central element of the component group of A (see (6.4) for the structure of the component group). We show that a stabilizes the image of ϕ forcing the image to be equal to X: For $z \in \mathfrak{k}$ we find $g_{a,z} \in U$ such that $a.(e+z) = g_{a,z}.(e+w_a z)$ for some w_a representing the non-trivial element in N/N_0. This will also be done in the Appendix.

INTERMEZZO: At this point we mention an argument which turns out to be surprisingly useful. In order to complete the proof of the theorem, we could try to apply the well known quotient map criterion ([12], p. 107). We first claim that the generic fibre of ϕ is an orbit of N_0: Let $z \in \mathfrak{k}^{reg}$ and $z' \in \mathfrak{k}$ be in the same fibre. The parametrization theorem says that $z' \in Nz$. Assume $z' \notin N_0 z$. Then, $z' \in (N_0 w_a)z$ and $\phi(z) = \phi(z') = \phi(w_a z) = a.\phi(z)$ by the first two steps. On the other hand, we have $z + e' \in P(z + e)$ and $G_{z+e'} = L_{e'} \subset P$, and so $G_{z+e} \subset P$. But we may easily check that P and aU are disjoint. Contradiction! Now suppose $\overline{\mathcal{D}}^{reg}$ were smooth (or normal). Then the theorem would follow using the quotient map criterion.

STEP 3: We show that $\phi^*\colon k[\mathfrak{g}^f] \to k[\mathfrak{k}]^{N_0}$ is surjective: Here, we separate the two cases $\varepsilon = 1$ and $\varepsilon = 0$. We first deal with the case $\varepsilon = 1$. We consider the two decomposition classes $\mathcal{D}_A$ and $\mathcal{D}_B$ in $\overline{\mathcal{D}}^{reg}$:

Let $\mathcal{D}_A = \mathcal{D}(\boldsymbol{s}_A, \boldsymbol{r}_A)$ where $\boldsymbol{s}_A = (s-1, s-t)$ and $\boldsymbol{r}_A = (r+1, r+1)$. Then $\boldsymbol{p}_A = \boldsymbol{p}(\boldsymbol{s}_A, \boldsymbol{r}_A) = \boldsymbol{l}$ and $J_A = J(\boldsymbol{p}_A) = J_1(\boldsymbol{p}_A) = \{1, 2\}$.

Lemma A. *Let $\mathfrak{k}_A \subset \mathfrak{k}$ be defined by $\{\omega_t = 0\}$.*

(i) *Then $\overline{\mathcal{D}_A}^{reg}$ is contained in $\overline{\mathcal{D}}^{reg}$, and so it is equal to $G(e + \mathfrak{k}_A)$.*

(ii) *The map $\phi_A^*\colon k[\mathfrak{g}^f] \to k[\mathfrak{k}_A]^{N_A}$ is surjective.*

Proof. We only mention that the second statement follows from the general case using only cases of type I (6.4). □

Let $\mathcal{D}_B = \mathcal{D}(\boldsymbol{s}_B, \boldsymbol{r}_B)$ where $\boldsymbol{s}_B = (t)$ and $\boldsymbol{r}_A = (2(s-t)+r, 2(s-t)+r)$. Then $\boldsymbol{p}_B = \boldsymbol{p}(\boldsymbol{s}_B, \boldsymbol{r}_B)$ and $\boldsymbol{p}$ are equal.

Lemma B. *Let $\mathfrak{k}_B \subset \mathfrak{k}$ be defined by $\{\, \omega_{t+2k} = 0 \mid k = 1, \dots, s-t \,\}$.*

(i) *Then $\overline{\mathcal{D}_B}^{reg}$ is contained in $\overline{\mathcal{D}}^{reg}$, and so it is equal to $G(e + \mathfrak{k}_B)$.*

(ii) *The map $\phi_B^* \colon k[\mathfrak{g}^f] \to k[\mathfrak{k}_B]^{(N_B)_0}$ is surjective.*

Proof. This is still a case of type III. For the second statement we use the argument made in the intermezzo, that is, we attempt to prove that $\overline{\mathcal{D}_B}^{reg}$ is smooth. By Theorem 6.13 at the end of the chapter, this task is reduced to the study of $\overline{\mathcal{D}(\boldsymbol{s}_B, \emptyset)}^{reg} \subset \mathfrak{g}_0(2t)$. But as regular sheet of $\mathfrak{g}_0(2t)$ this is certainly smooth. □

Now we put the results of Lemma A and Lemma B together. We use the following notation for coordinates on $\mathfrak{k}$:

$$X_j = \omega_j \ , \ j = 1, \dots, t \quad , \quad Y_k = \omega_{t+2k} \ , \ k = 1, \dots, s-t.$$

Then $\mathfrak{k}_A \subset \mathfrak{k}$ is defined by $\{X_t = 0\}$, and $\mathfrak{k}_B \subset \mathfrak{k}$ by $\{Y_1 = \ldots = Y_{s-t} = 0\}$. Moreover,

$$k[\mathfrak{k}]^N = k[X_1, \dots, X_t]^{B_t} \otimes k[Y_1, \dots, Y_{s-t}]^{B_{s-t}}$$

and

$$k[\mathfrak{k}]^{N_0} = k[X_1, \dots, X_t]^{D_t} \otimes k[Y_1, \dots, Y_{s-t}]^{B_{s-t}}.$$

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{k}/N & \xrightarrow{\bar\phi} & \mathfrak{g}^f/A \\
\uparrow & & \uparrow \\
\mathfrak{k} & \xrightarrow{\phi} & \mathfrak{g}^f \\
\uparrow \cup & & \uparrow \cup \\
\mathfrak{k}_A & \xrightarrow{\phi_A} & (\mathfrak{g}^f)^A \ .
\end{array}$$

The upper morphism $\bar\phi$ exists because $a\phi(z) = \phi(w_a z)$ for all $z \in \mathfrak{k}$. From this we get the following diagram of coordinate rings and comorphisms, where we use the fact that ϕ is constant on N_0-orbits and ϕ_A constant on N_A-orbits:

$$\begin{array}{ccc}
k[\mathfrak{k}]^N & \xleftarrow{\bar\phi^*} & k[\mathfrak{g}^f]^A \\
\downarrow \cap & & \downarrow \cap \\
k[\mathfrak{k}]^{N_0} & \xleftarrow{\phi^*} & k[\mathfrak{g}^f] \\
\downarrow p & & \downarrow \text{res} \\
k[\mathfrak{k}_A]^{N_A} & \xleftarrow{\phi_A^*} & k[(\mathfrak{g}^f)^A] \ .
\end{array}$$

We want to show that ϕ^* is surjective. We first note that the composition of the right vertical maps is surjective. It is also easy to see from the description of the generators of the invariants that the composition of the left vertical maps, $\tilde p \colon k[\mathfrak{k}]^N \to k[\mathfrak{k}_A]^{N_A}$, is surjective. Its kernel is given by

$$\operatorname{Ker} \tilde p = (X_1 k[\mathfrak{k}])^N = (X_1 X_2 \cdots X_t)^2 k[\mathfrak{k}]^N$$

(If an N-invariant f is divisible by X_1 then it is divisible by X_i^2 for all i.) Similarly, we see that $\operatorname{Ker} p = (X_1 k[\mathfrak{k}])^{N_0} = (X_1 X_2 \cdots X_t) k[\mathfrak{k}]^{N_0}$.

By Lemma A the homomorphism ϕ_A^* is surjective. Hence

$$k[\mathfrak{k}]^N = \operatorname{Im} \bar{\phi}^* + \operatorname{Ker} \tilde{p} = \phi^*(k[\mathfrak{g}^f]^A) + (X_1 X_2 \cdots X_t)^2 k[\mathfrak{k}]^N$$

and

$$k[\mathfrak{k}]^{N_0} = \operatorname{Im} \phi^* + \operatorname{Ker} p = \phi^*(k[\mathfrak{g}^f]) + (X_1 X_2 \cdots X_t)\, k[\mathfrak{k}]^{N_0}.$$

Since all algebras in the diagram above are naturally graded and all morphisms homogeneous the two preceding statements imply that

$$k[\mathfrak{k}]_d^N \subset \phi^*(k[\mathfrak{g}^f]^A) \quad \text{for } d < 2t$$

and that

$$k[\mathfrak{k}]_t^{N_0} = \phi^*(k[\mathfrak{g}^f]_t) + k \cdot X_1 X_2 \cdots X_t,$$

respectively.

Now we use diagrams involving $\mathfrak{k}_B$ instead of $\mathfrak{k}_A$:

$$\begin{array}{ccc} \mathfrak{k} & \xrightarrow{\phi} & \mathfrak{g}^f \\ \uparrow \cup & & \uparrow = \\ \mathfrak{k}_B & \xrightarrow{\phi_B} & \mathfrak{g}^f \end{array} \quad , \qquad \begin{array}{ccc} k[\mathfrak{k}]^{N_0} & \xleftarrow{\phi^*} & k[\mathfrak{g}^f] \\ \downarrow q & & \downarrow = \\ k[\mathfrak{k}_B]^{N_{B0}} & \xleftarrow{\phi_B^*} & k[\mathfrak{g}^f] \end{array} .$$

By Lemma B the homomorphism ϕ_B^* is surjective, and so

$$k[\mathfrak{k}]^{N_0} = \phi^*(k[\mathfrak{g}^f]) + \operatorname{Ker} q = \phi^*(k[\mathfrak{g}^f]) + (Y_1, Y_2, \ldots, Y_{s-t})^{N_0}.$$

From the description of the invariants we see that $k[\mathfrak{k}]_d^{N_0} = k[\mathfrak{k}]_d^N$ for $d < t$ and that $k[\mathfrak{k}]_t^{N_0} = k[\mathfrak{k}]_t^N \oplus k \cdot X_1 X_2 \cdots X_t$. This implies

$$(Y_1, Y_2, \ldots, Y_{s-t})_t^{N_0} = (Y_1, Y_2, \ldots, Y_{s-t})_t^N$$

because $(Y_1, Y_2, \ldots, Y_{s-t})$ is stable under N and $X_1 X_2 \cdots X_t \notin (Y_1, Y_2, \ldots, Y_{s-t})$. Since $k[\mathfrak{k}]_t^N \subset \phi^*(k[\mathfrak{g}^f])$ as shown above, we get $k[\mathfrak{k}]_t^{N_0} = \phi^*(k[\mathfrak{g}^f]_t)$, hence

$$X_1 X_2 \cdots X_t \in \phi^*(k[\mathfrak{g}^f]).$$

Finally, an easy induction on degrees implies that $k[\mathfrak{k}]^{N_0} = \phi^*(k[\mathfrak{g}^f])$, thus completing the proof of the theorem for $\varepsilon = 1$.

The case $\varepsilon = 0$ follows from the case $\varepsilon = 1$: Keeping in mind the argument made in the intermezzo we show that $\overline{\mathcal{D}}^{reg} = \overline{\mathcal{D}(\boldsymbol{s}, (r, r))}^{reg} \subset \mathfrak{g}_0(N) = \mathfrak{g}$ is smooth. Having just seen that $\overline{\mathcal{D}(\boldsymbol{s}, (r+1, r+1))}^{reg} \subset \mathfrak{g}_1(N+2)$ is smooth we may again apply Theorem 6.13, and obtain the desired result.

□

6.4 General case

Recall the structure of N defined in the first section. We define N_0 to be the subgroup of N corresponding to

$$\prod_{i\in J_1} B_{t_i} \times \prod_{i\in J_2} (D_{t_i} \times B_{t_{i+1}})$$

in

$$\prod_{i\in J_1} B_{t_i} \times \prod_{i\in J_2} (B_{t_i} \times B_{t_{i+1}}).$$

Using

$$N \to \prod_{i\in J} N_i \quad , \; w \mapsto (w^{(i)})_{i\in J}$$

we obtain

$$N_0 \to \prod_{i\in J_1} N_i \times \prod_{i\in J_2} (N_i)_0$$

and an isomorphism

$$N/N_0 \to \prod_{i\in J_2} N_i/(N_i)_0.$$

We also need a link between the component group of G_e and those of $(G_i)_{e_i}$ for $i \in J$. Following Hesselink [8] we define $B(\boldsymbol{l}) = \{\, j \mid l_j > l_{j+1} \;,\; l_j \not\equiv \varepsilon \,\}$. For $i \in J$ we set

$$B_i(\boldsymbol{l}) = \begin{cases} \{i\} & i \in J_1, \\ \{i, i+1\} & i \in J_2 \;,\; t_i \geq 2, \\ \{i+1\} & i \in J_2 \;,\; t_i = 1, \\ \emptyset & i \in J_2 \;,\; t_i = 0. \end{cases}$$

This is more or less the same as $B(\boldsymbol{l}_i)$. We have

$$\bigcup_{i\in J} B_i(\boldsymbol{l}) = B(\boldsymbol{l}) \cup \{\, i \in J_1 \mid i \notin B(\boldsymbol{l}) \,\}.$$

The component groups of centralizers of nilpotent elements in classical groups are isomorphic to vector spaces over $\mathbb{F}_2$. We give a basis for the component group of G_e using bases for the component groups of $(G_i)_{e_i}$ for $i \in J$.

For $i \in J$ with $B_i(\boldsymbol{l}) \neq \emptyset$ and $j \in B_i(\boldsymbol{l})$ we set

$$r_i(j) = \left\{ \begin{array}{ll} -1 & j = i \\ a_i & j = i+1 \end{array} \right\} \in (G_i)_{e_i}.$$

Here, a_i is the element defined in (6.1). Then we obtain

$$(G_i)_{e_i}/(G_i)^\circ_{e_i} = \langle\, \bar{r}_i(j) \mid j \in B_i(\boldsymbol{l}) \,\rangle_{\mathbb{F}_2}.$$

Furthermore, we set

$$r(j) = \left\{ \begin{array}{ll} r_j(j) & j \in J \\ r_{j-1}(j) & j \notin J \end{array} \right\} \in G_e.$$

Finally, we obtain

$$G_e/G_e^\circ = \langle\, \bar r(j) \mid j \in B(\boldsymbol{l})\,\rangle_{\mathbb{F}_2}.$$

There is an obvious surjection

$$\prod_{i\in J} (G_i)_{e_i}/(G_i)_{e_i}^\circ \to G_e/G_e^\circ$$

given by $\bar r_i(j) \mapsto \bar r(j)$. The kernel of this map, being $\langle\, \bar r_i(i) \mid i \in J_1 \;,\; i \notin B(\boldsymbol{l})\,\rangle_{\mathbb{F}_2}$, is central.

We are now ready to prove Theorem 6.2.

Theorem. *The morphism $\phi\colon \mathfrak{k} \to \mathfrak{g}^f$ is a quotient map with respect to N_0. Its image is X.*

Proof. We again proceed in three steps:

STEP 1: The map ϕ is constant on orbits of N_0 by Theorem 6.1: We have

$$\phi(wz) = \textstyle\sum_i \phi_i(w^{(i)} z_i) = \sum_i \phi_i(z_i) = \phi(z)$$

for $w \in N_0$ and $z = \sum_i z_i \in \mathfrak{k}$.

STEP 2: The image of ϕ equals X: We show that it is stabilized by A. By what we said above and by Lemma 2.4, it suffices to do this for the elements a_i. By the second step in the proof of the theorem in (6.3), there is an element $w_{a_i} \in N_i \setminus (N_i)_0$ such that $a_i.\phi_i(z_i) = \phi_i(w_{a_i} z_i)$ for $z_i \in \mathfrak{k}_i$. Let w be an element in N such that $w^{(i)} = w_{a_i}$ and $w^{(k)} \in (N_k)_0$ for all $k \in J \setminus \{i\}$. Then, we obtain

$$a.\phi(z) = a_i.\phi_i(z_i) + \textstyle\sum_{k\in J\setminus\{i\}} \phi_k(z_k) = \phi_i(w^{(i)} z_i) + \sum_{k \in J\setminus\{i\}} \phi_k(w^{(k)} z_k) = \phi(wz)$$

for $z \in \mathfrak{k}$.

STEP 3: We set $\mathfrak{k}_J = \bigoplus_{i\in J} \mathfrak{k}_i$ and $(N_J)_0 = \prod_{i\in J_1} N_i \times \prod_{i \in J_2} (N_i)_0$. Let $\operatorname{res}\colon k[\mathfrak{k}_J] \to k[\mathfrak{k}]$ be the comorphism of the inclusion $\mathfrak{k} \subset \mathfrak{k}_J$. Theorem 6.1 implies that

$$\phi^* k[\mathfrak{g}^f] = \operatorname{res} k[\mathfrak{k}_J]^{(N_J)_0}.$$

Therefore, it suffices to show that

$$\operatorname{res} k[\mathfrak{k}_J]^{(N_J)_0} = k[\mathfrak{k}]^{N_0}.$$

We denote by $Pr_{m,n}(x)$ the product $x_m x_{m+1} \cdots x_n$ for $1 \le m \le n$. Using the same notation for power sums as before we obtain

$$\begin{aligned} k[\mathfrak{k}_J]^{(N_J)_0} &= \textstyle\bigotimes_{i\in J} k[\mathfrak{k}_i]^{(N_i)_0} \\ &= \textstyle\bigotimes_{i\in J_1} k[PS_{1,S_i;2j}(\omega^{(i)}), j = 1,\dots,S_i] \\ &\quad \otimes \textstyle\bigotimes_{i \in J_2} \big(k[PS_{1,t_i;2j}(\omega^{(i)}), j = 1,\dots,t_i - 1][Pr_{1,t_i}(\omega^{(i)})] \\ &\qquad \otimes k[PS_{t_i+1,S_i;2j}(\omega^{(i)}), j = 1,\dots,s_{i+1}]\big) \end{aligned}$$

and

$$\begin{aligned} k[\mathfrak{k}]^{N_0} &= \textstyle\bigotimes_{i\in J_1} k[PS_{1,t_i;2j}(\omega^{(i)}), j = 1,\dots,t_i] \\ &\quad \otimes \textstyle\bigotimes_{i\in J_2} \big(k[PS_{1,t_i;2j}(\omega^{(i)}), j = 1,\dots,t_i - 1][Pr_{1,t_i}(\omega^{(i)})] \;. \\ &\qquad \otimes k[PS_{t_i+1,t_i+2t_{i+1};2j}(\omega^{(i)}), j = 1,\dots,t_{i+1}]\big) \end{aligned}$$

Now the claim follows straightforwardly.

□

6.5 A little help from invariant theory

In this section we consider a construction due to Kraft and Procesi (see [14] Sections 1,4, and 11.

Let V and V' be vector spaces of dimension N and N', respectively, and assume $N' \leq N$. Let $\varepsilon, \varepsilon' \in \{0,1\}$ be such that $\varepsilon + \varepsilon' = 1$. Choose an ε-form $(\cdot,\cdot)_V$ on V and an ε'-form $(\cdot,\cdot)_{V'}$ on V'. Given $X \in \mathrm{Hom}(V,V')$ its adjoint $X^* \in \mathrm{Hom}(V',V)$ is defined by $(Xv,v')_{V'} = (v, X^*v')_V$, where $v \in V$ and $v' \in V'$. Writing L for $\mathrm{Hom}(V,V')$ we consider the diagram

$$\begin{array}{ccc} L & \xrightarrow{\pi} & \mathfrak{g}_{\varepsilon'}(V') \\ {\scriptstyle\rho}\big\downarrow & & \\ \mathfrak{g}_{\varepsilon}(V) & & \end{array}$$

defined by $\pi(X) = XX^*$ and $\rho(X) = X^*X$ for $X \in L$. The group $G_{\varepsilon'}(V') \times G_\varepsilon(V)$ acts on L by $(g',g).X = g'Xg^{-1}$, and π and ρ are equivariant with respect to this action and the adjoint action of $G_{\varepsilon'}(V')$ and $G_\varepsilon(V)$ on $\mathfrak{g}_{\varepsilon'}(V')$ and $\mathfrak{g}_\varepsilon(V)$, respectively. The maps π and ρ are equivariant also with respect to suitable k^*-actions on $\mathfrak{g}_{\varepsilon'}(V')$, $\mathfrak{g}_\varepsilon(V)$ and L.

6.3 Theorem. (i) *π is the quotient map with respect to $G_\varepsilon(V)$.*

(ii) *$\rho\colon L \to \rho(L) = \{\, D \in \mathfrak{g}_\varepsilon(V) \mid \mathrm{rk}\, D \leq N' \,\}$ is the quotient map with respect to $G_{\varepsilon'}(V')$.*

We denote by L° the subset of surjective maps in L.

6.4 Proposition. ([14], 11.1 Proposition) (i) $\pi(L^\circ) = \{\, D' \in \mathfrak{g}_{\varepsilon'}(V') \mid \mathrm{rk}\, D' \geq 2N' - N \,\}$ *and $\pi|_{L^\circ}$ is a smooth morphism .*

(ii) $\rho(L^\circ) = \{\, D \in \mathfrak{g}_\varepsilon(V) \mid \mathrm{rk}\, D = N' \,\}$ *and $\rho|_{L^\circ}$ is a principal bundle with structure group $G_{\varepsilon'}(V')$ (in the etale topology).*

6.5 Corollary. *Let $X \in L^\circ$, $D = \rho(X)$, and $D' = \pi(X)$. Then:*

$$\rho^{-1}(G_\varepsilon(V)D) \cap L^\circ = (G_{\varepsilon'}(V') \times G_\varepsilon(V))X = \pi^{-1}(G_{\varepsilon'}(V')D') \cap L^\circ.$$

Proof. The first equality follows from part (ii) of the previous proposition. It is clear from the construction that $(G_{\varepsilon'}(V') \times G_\varepsilon(V))X \subseteq \pi^{-1}(G_{\varepsilon'}(V')D') \cap L^\circ$. We want to show that $\rho(\pi^{-1}(D') \cap L^\circ) \subset G_\varepsilon(V)D$. For any $Y \in L^\circ$ with $\pi(Y) = D'$ there exists a $g \in \mathrm{GL}(V)$ such that $Y = Xg$ and $Y^* = g^{-1}X^*$ because Y is surjective and Y^* is injective (see [12], II.4.1, Satz 2.a)). This implies $\rho(Y) = g^{-1}Dg \in \mathfrak{g}_\varepsilon(V)$. It then follows from [20], IV.2.19, that $\rho(Y)$ belongs to $G_\varepsilon(V)D$. □

6.6 Corollary. *The morphisms π and ρ induce bijections of orbit spaces*

$$\begin{array}{ccc} L^\circ/(G_{\varepsilon'}(V') \times G_\varepsilon(V)) & \longrightarrow & \pi(L^\circ)/G_{\varepsilon'}(V') \\ \big\downarrow & & \\ \rho(L^\circ)/G_\varepsilon(V) & & \end{array}$$

In particular, we have a bijection

$$\Psi\colon \rho(L^\circ)/G_\varepsilon(V) \longrightarrow \pi(L^\circ)/G_{\varepsilon'}(V')$$

given by $\mathcal{O} \mapsto \pi(\rho^{-1}(\mathcal{O}) \cap L^\circ)$. Its inverse Ψ^{-1} is given by $\mathcal{O}' \mapsto \rho(\pi^{-1}(\mathcal{O}') \cap L^\circ)$.

6.7 Corollary. *For any $G_\varepsilon(V)$-stable locally closed subset Z in $\rho(L^\circ)$ we set $L^\circ_Z = \rho^{-1}(Z) \cap L^\circ$. Then we obtain a diagram*

$$\begin{array}{ccc} L^\circ_Z & \xrightarrow{\pi_Z} & \pi(L^\circ_Z) \\ {\scriptstyle\rho_Z}\downarrow & & \\ Z & & \end{array}$$

where ρ_Z is a principal bundle with structure group $G_{\varepsilon'}(V')$, $L^\circ_Z = \pi^{-1}(\pi(L^\circ_Z)) \cap L^\circ$ and π_Z is a smooth morphism. Moreover, $\Psi(Z/G_\varepsilon(V)) = \pi(L^\circ_Z)/G_{\varepsilon'}(V')$.

We will now give a direct and more algebraic description of the map Ψ. Let $D \in \mathfrak{g}_\varepsilon(V)$ be arbitrary. The form $(\cdot, D(\cdot))_V$ on V induces an ε-form on the image $D(V)$ of D. Denote by $X \colon V \to D(V)$ the map given by $v \mapsto D(v)$. By definition, the adjoint map X^* of X is the inclusion $I \colon D(V) \to V$. We have $D = IX = X^*X$ and $D|_{D(V)} = XI = XX^* \in \mathfrak{g}_{\varepsilon'}(D(V))$. Assume that $\operatorname{rk} D = \dim V'$, i.e. $D \in \rho(L^\circ)$. Choose an isomorphism $\mu \colon D(V) \to V'$ respecting the ε'-forms on $D(V)$ and V', and define $Y \colon V \to V'$ by $v \mapsto \mu(D(v))$. We see that $Y \in L^\circ$. Using $\mu^* = \mu^{-1}$ we obtain that $\rho(Y) = Y^*Y = X^*\mu^*\mu X = X^*X = D$ and $\pi(Y) = YY^* = \mu XX^*\mu^* = \mu D|_{D(V)}\mu^{-1}$. If $\eta \colon D(V) \to V'$ is another such isomorphism, then $\pi(\eta D) = \eta\mu^{-1}\pi(\mu D)\mu\eta^{-1} \in G_{\varepsilon'}(V')\pi(Y)$. Thus we have proved:

6.8 Proposition. *For any $D \in \rho(L^\circ)$ the bijection Ψ maps the orbit of D to the orbit of $D|_{D(V)}$. For a nilpotent orbit $\mathcal{O}_\varepsilon(\boldsymbol{l})$, the image is $\mathcal{O}_{\varepsilon'}(\boldsymbol{l}')$ where $\boldsymbol{l}'$ is obtained from $\boldsymbol{l}$ by removing its first column.*

In order to apply the Kraft-Procesi construction in the theory of sheets, the following lemma is crucial.

6.9 Lemma. *Let $\mathcal{D}$ be the very stable decomposition class $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ of $\mathfrak{g}_\varepsilon(V)$ and $\mathcal{O}$ the nilpotent orbit $\mathcal{O}_\varepsilon(\boldsymbol{l})$ in $\overline{\mathcal{D}}^{reg}$. Then:*

(i) *For all elements of $\mathcal{D}$ the rank is equal to* $\dim V - r^1$.

(ii) *The rank of an element of $\mathcal{O}$ is equal to* $\dim V - l^1$.

(iii) *If (and only if) l^1 is equal to r^1, then all elements of $\overline{\mathcal{D}}^{reg}$ have the same rank.*

Proof. (i) Recall that $\mathcal{D}$ is given by $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ where $\mathfrak{l} = \bigoplus_j \mathfrak{gl}(V_j) \oplus \mathfrak{g}_\varepsilon(V_0)$ with $\dim V_j = s^j$ and $\mathcal{O}_\mathfrak{l} = 0 \oplus \mathcal{O}_\varepsilon(\boldsymbol{r})$. Every element of $\mathcal{D}$ is conjugate to one of the form $y = z + x$ where $z \in \mathfrak{k}^{reg}$ and $x \in \mathcal{O}_\varepsilon(\boldsymbol{r})$. Since $\operatorname{Ker} z = V_0$ it follows that $y(V) = \bigoplus_j V_j \oplus x(V_0)$ and $\operatorname{Ker} y = \operatorname{Ker} x|_{V_0} \subset V_0$. But $\dim \operatorname{Ker} x|_{V_0} = r^1$.

(ii) This is well known.

(iii) All elements of $\overline{\mathcal{D}}$ have rank at most $\dim V - r^1$. We consider the subset of $\overline{\mathcal{D}}^{reg}$ consisting of elements with rank strictly less than $\dim V - r^1$. This subset is closed, and it is G- and k^*-stable. Assume that it were non-empty. Then, it would contain $\mathcal{O}$ by (ii) of Corollary 1.3. However, this contradicts $l^1 = r^1$ and part (ii) of the lemma.

□

6.10 Proposition. *Assume that $\mathcal{D}$ belongs to $\rho(L^\circ)$. We denote by $\mathcal{D}'$ the very stable decomposition class $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r}')$ of $\mathfrak{g}_{\varepsilon'}(V')$ where $\boldsymbol{r}'$ is obtained from $\boldsymbol{r}$ by removing its first column. Then Ψ induces a bijection of orbit spaces*

$$\mathcal{D}/G_\varepsilon(V) \longrightarrow \mathcal{D}'/G_{\varepsilon'}(V').$$

In particular, $\pi(\rho^{-1}(\mathcal{D}) \cap L^\circ) = \mathcal{D}'$.

Proof. We need to look at $y|_{y(V)}$ for $y \in \mathcal{D}$. By what we said in the proof of part (i) of the previous lemma we see that $y|_{y(V)} = z|_{y(V)} + x|_{y(V)}$ where $z|_{y(V)}$ is a regular element of the centralizer of $\mathfrak{l}' = \bigoplus_j \mathfrak{gl}(V_j) \oplus \mathfrak{g}_{\varepsilon'}(x(V_0))$ and $x|_{y(V)} = x|_{x(V_0)}$ generates the nilpotent orbit $\mathcal{O}_{\varepsilon'}(\boldsymbol{r}')$ of $\mathfrak{g}_{\varepsilon'}(x(V_0))$. Note that this decomposition is indeed the Jordan decomposition. But $(\mathfrak{l}', 0 \oplus \mathcal{O}_{\varepsilon'}(\boldsymbol{r}'))$ are the data of $\mathcal{D}'$, and so it follows that $y|_{y(V)}$ is contained in $\mathcal{D}'$. It is also clear that every orbit of $\mathcal{D}'$ has a representative of the form $y|_{y(V)}$ for some $y \in \mathcal{D}$. □

6.11 Proposition. *Assume that $\overline{\mathcal{D}}^{reg}$ belongs to $\rho(L^\circ)$. We denote by $\mathcal{O}'$ the nilpotent orbit $\mathcal{O}_{\varepsilon'}(\boldsymbol{l}')$ where $\boldsymbol{l}'$ is obtained from $\boldsymbol{l}$ by removing its first column. Then:*

(i) *$\mathcal{O}'$ is the nilpotent orbit in $\overline{\mathcal{D}'}^{reg}$.*

(ii) *$\overline{\mathcal{D}'}^{reg}$ belongs to $\pi(L^\circ)$.*

(iii) *Ψ induces a bijection of orbit spaces*

$$\overline{\mathcal{D}}^{reg}/G_\varepsilon(V) \longrightarrow \overline{\mathcal{D}'}^{reg}/G_{\varepsilon'}(V').$$

In particular, $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg}) \cap L^\circ) = \overline{\mathcal{D}'}^{reg}$.

Proof. (i) We have $l^1 = r^1$ by assumption. Recall how we defined the partition $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s}, \boldsymbol{r})$. It is easy to see that here p^1 is equal to both r^1 and l^1. Let $\boldsymbol{p}'$ be the partition obtained from $\boldsymbol{p}$ by removing its first column. From $p^1 = r^1$ it follows that $\boldsymbol{p}(\boldsymbol{s}, \boldsymbol{r}') = \boldsymbol{p}'$. Since $l^1 = p^1$ we may then use Lemma 3.9.

(ii) This is similar to the proof of part (iii) in Lemma 6.9.

(iii) For simplicity we denote by π and ρ their respective restrictions to L°. Both, π and ρ are open morphisms. For a $G_{\varepsilon'}(V') \times G_\varepsilon(V)$-stable subset S of L° we have $\pi^{-1}(\pi(S)) = S$ and $\rho^{-1}(\rho(S)) = S$. If S is closed, then $\pi(S)$ and $\rho(S)$ are also closed. We need to show that $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg})) = \overline{\mathcal{D}'}^{reg}$. Since $\overline{\mathcal{D}}^{reg} \subset \rho(L^\circ)$ we may regard $\overline{\mathcal{D}}$ as closure with respect to $\rho(L^\circ)$ and similarily for $\mathcal{D}'$ in $\pi(L^\circ)$. Using the properties of π and ρ mentioned above we obtain from $\pi(\rho^{-1}(\mathcal{D})) = \mathcal{D}'$ that $\pi(\rho^{-1}(\overline{\mathcal{D}})) = \overline{\mathcal{D}'}$. We now consider the subset of $\overline{\mathcal{D}'}^{reg}$ consisting of elements not contained in $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg}))$. This subset is closed because $\overline{\mathcal{D}}^{reg}$ is open in $\overline{\mathcal{D}}$ and π is an open morphism. It is also $G_{\varepsilon'}(V')$- and k^*-stable. Assume it were non-empty. By part (ii) of Corollary 1.3, it would contain the unique nilpotent orbit of $\overline{\mathcal{D}'}^{reg}$, which is $\mathcal{O}'$ by part (i) above. But this contradicts the fact that $\mathcal{O}'$, being equal to $\pi(\rho^{-1}(\mathcal{O}))$, belongs to $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg}))$. Hence, $\overline{\mathcal{D}'}^{reg}$ is contained in $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg}))$. The same kind of argument in $\overline{\mathcal{D}}$ shows that $\overline{\mathcal{D}}^{reg}$ is contained in $\rho(\pi^{-1}(\overline{\mathcal{D}'}^{reg}))$. It follows that $\pi(\rho^{-1}(\overline{\mathcal{D}}^{reg})) = \overline{\mathcal{D}'}^{reg}$.

□

The next result is an immediate consequence of the previous proposition and Corollary 6.7.

6.12 Corollary. *Assume that* $\overline{\mathcal{D}}^{reg}$ *belongs to* $\rho(L^\circ)$. *Then* $\pi(L^0_{\overline{\mathcal{D}}^{reg}}) = \overline{\mathcal{D}'}^{reg}$, *and we obtain a diagram*

$$\begin{array}{ccc} L^0_{\overline{\mathcal{D}}^{reg}} & \xrightarrow{\pi_{\overline{\mathcal{D}}^{reg}}} & \overline{\mathcal{D}'}^{reg} \\ {\scriptstyle \rho_{\overline{\mathcal{D}}^{reg}}}\downarrow & & \\ \overline{\mathcal{D}}^{reg} & & \end{array}$$

where $\rho_{\overline{\mathcal{D}}^{reg}}$ *is a principal bundle and* $\pi_{\overline{\mathcal{D}}^{reg}}$ *is a smooth morphism. In particular,* $\overline{\mathcal{D}}^{reg}$ *is smooth if and only if* $\overline{\mathcal{D}'}^{reg}$ *is smooth.*

Finally, we apply the corollary to a decomposition class $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ of type III. We assume that $\boldsymbol{r} = (r, r)$ with $r \geq 1$. In this case it follows that $\overline{\mathcal{D}(\boldsymbol{s}, (r, r))}^{reg}$ is smooth if and only if $\overline{\mathcal{D}(\boldsymbol{s}, (r-1, r-1))}^{reg}$ is smooth. Note that $\mathcal{D}(\boldsymbol{s}, (r-1, r-1))$ is again of type III. The result we used in Section 6.3 now follows by induction.

6.13 Theorem. *Let* $\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})$ *and* $\mathcal{D}(\boldsymbol{s}, \overline{\boldsymbol{r}})$ *be two decomposition classes of type III. Then,* $\overline{\mathcal{D}(\boldsymbol{s}, \boldsymbol{r})}^{reg}$ *is smooth if and only if* $\overline{\mathcal{D}(\boldsymbol{s}, \overline{\boldsymbol{r}})}^{reg}$ *is smooth.*

Appendix

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a root system Φ with respect to $\mathfrak{h}$ and a basis Δ of Φ. Let $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha = [e_\alpha, f_\alpha]$ be a standard triple of $\mathfrak{g}$ for any $\alpha \in \Phi^+$, and let s_α be the corresponding reflection in the Weyl group W. The identity resulting from the following calculation is the key tool in this appendix: For $z \in \mathfrak{h}$ we have

$$\begin{aligned}
\exp \operatorname{ad}(\alpha(z) f_\alpha).(e_\alpha + z) &= e_\alpha + z + \alpha(z)[f_\alpha, e_\alpha] + \alpha(z)[f_\alpha, z] + \tfrac{1}{2}\alpha(z)^2 [f_\alpha, [f_\alpha, e_\alpha]] \\
&= e_\alpha + z - \alpha(z) h_\alpha + \alpha(z)^2 f_\alpha + \tfrac{1}{2}\alpha(z)^2(-2 f_\alpha) \\
&= e_\alpha + s_\alpha z \, .
\end{aligned}$$

Let $\beta \in \Phi^+$, $\beta \neq \alpha$. Then

$$\exp \operatorname{ad}(t f_\alpha).e_\beta = e_\beta + t[f_\alpha, e_\beta] + \dots .$$

This reduces to

$$\exp \operatorname{ad}(t f_\alpha).e_\beta = e_\beta$$

if $\beta - \alpha$ is not a root.

Example. Let $e = \sum_{\beta \in \Delta} e_\beta$ be the regular nilpotent element of $\mathfrak{g}$. If $\alpha \in \Delta$ we get

$$\exp \operatorname{ad}(\alpha(z) f_\alpha).(e + z) = e + s_\alpha z$$

for $z \in \mathfrak{h}$.

The symplectic group ($\varepsilon = 1$)

Let $G = Sp_{2M}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We fix a set of simple roots $\{\alpha_1, \dots, \alpha_M\}$ in $\mathfrak{h}^*$ by

$$\begin{aligned}
\alpha_i &= \omega_i - \omega_{i+1} \quad \text{for } i = 1, \dots, M-1, \\
\alpha_M &= 2\omega_M
\end{aligned}$$

The image of $N_G(\mathfrak{h})$ in $\operatorname{Aut}(\mathfrak{h})$ is generated by reflections $s_{\alpha_1}, \dots, s_{\alpha_M}$ which act as follows:

$$\begin{aligned}
s_{\alpha_i} &: \quad \omega_i \longleftrightarrow \omega_{i+1} \\
s_{\alpha_M} &: \quad \omega_M \longleftrightarrow -\omega_M
\end{aligned}$$

Type III

We first reformulate the definitions made in (6.1) now using roots. Let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ given by the set

$$\{\alpha_{t+2k-1}\}_{k=1,\dots,s-t} \cup \{\alpha_{2s-t+j}\}_{j=1,\dots,Q}$$

of simple roots and let $\mathfrak{l}$ be its Levi part. The parabolic subalgebra $\mathfrak{p}$ contained in $\mathfrak{q}$ is defined by the set

$$\{\alpha_{t+2k-1}\}_{k=1,\dots,1/2(M-t-1)}$$

of simple roots. We may write the nilpotent elements e and e' as sums of root vectors:

$$
\begin{aligned}
e &= \sum_{i=1}^{t-1} e_{\alpha_i} + \sum_{i=t-1}^{M-1} e_{\alpha_i+\alpha_{i+1}} \quad \text{if } t \geq 2,\\
e &= \sum_{i=1}^{M-1} e_{\alpha_i+\alpha_{i+1}} \quad \text{if } t = 1,\\
e' &= \sum_{i=2s-t+1}^{M-1} e_{\alpha_i+\alpha_{i+1}}.
\end{aligned}
$$

The center of $\mathfrak{l}$ is given by

$$\mathfrak{k} = \{\, z \in \mathfrak{h} \mid \alpha_{t+2k-1}(z) = 0 \ , \ k = 1, \dots, s-t \ , \ \alpha_{2s-t+j}(z) = 0 \ , \ j = 1, 2, \dots, Q \,\}.$$

The image N of $N_G(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$ is isomorphic to $B_t \times B_{s-t}$. It is generated by the following reflections:

$$
\begin{array}{rcccc}
s_{\alpha_j} \ (j=1,\dots,t-1) & : & \omega_j & \longleftrightarrow & \omega_{j+1}\\
s_{2\omega_t} & : & \omega_t & \longleftrightarrow & -\omega_t\\
s_{\alpha_{t+2k-1}+\alpha_{t+2k}} \cdot s_{\alpha_{t+2k}+\alpha_{t+2k+1}} \ (k=1,\dots,s-t-1) & : & \omega_{t+2k} & \longleftrightarrow & \omega_{t+2k+2}\\
s_{\omega_{2s-t-1}+\omega_{2s-t}} & : & \omega_{2s-t} & \longleftrightarrow & -\omega_{2s-t}
\end{array}
$$

If $s_{2\omega_t}$ is replaced by

$$s_{\omega_{t-1}+\omega_t} \quad : \quad (\omega_{t-1}, \omega_t) \longleftrightarrow (-\omega_{t-1}, -\omega_t)$$

the new set of reflections generates the subgroup N_0 of N isomorphic to $D_t \times B_{s-t}$.

The Lie algebra $\mathfrak{u}$ derived from the element h is the unipotent radical of the parabolic subalgebra given by the set

$$\{-\alpha_{t+2(j-1)}\}_{j=1,\dots,1/2(M-t-1)}$$

of simple roots. Finally, the non-central element of the component group of A is induced by

$$a = s_{\alpha_t} \cdot s_{\alpha_{t+2}} \cdot \dots \cdot s_{\alpha_{M-1}}.$$

The following technical lemma will be useful:

Lemma. *Let $\widetilde{W}$ be the subgroup of W generated by*

$$s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_{t-2}}, s_{\alpha_t+\alpha_{t+1}}, s_{\alpha_{t+1}+\alpha_{t+2}}, \dots, s_{\alpha_{M-1}+\alpha_M}.$$

If $w \in \widetilde{W}$, then there exists $g_w \colon \mathfrak{h} \to U$ such that

$$e + wz = g_w(z).(e + z)$$

for $z \in \mathfrak{h}$.

Proof. Using the identities given at the beginning of the appendix we easily check that $g_{s_{\alpha_j}}$ defined by

$$g_{s_{\alpha_j}}(z) = \exp \operatorname{ad} \alpha_j(z) f_{\alpha_j}$$

for $j = 1, \ldots, t-2$, and $g_{s_{\alpha_{j-1}+\alpha_j}}$ defined by

$$g_{s_{\alpha_{j-1}+\alpha_j}}(z) = \exp \operatorname{ad}(\alpha_{j-1} + \alpha_j)(z) f_{\alpha_{j-1}+\alpha_j}$$

for $j = t+1, \ldots, M$ have the claimed property. Then define

$$g_{w_2 w_1}(z) = g_{w_2}(w_1 z) g_{w_1}(z)$$

for any $w_1, w_2 \in \widetilde{W}$. □

Now we give the details missing in the first two steps of the proof of the theorem in (6.3).

Lemma. *For $w \in N$ there exists $g_w \colon \mathfrak{k} \to U$ such that*

$$e + wz = \begin{cases} g_w(z).(e+z) & \text{if } w \in N_0 \\ a.g_w(z).(e+z) & \text{if } w \notin N_0 \end{cases}$$

for $z \in \mathfrak{k}$.

Proof. Evidently, it suffices to consider the generators of N_0 and N as described above.

s_{α_j} $(j=1,\ldots,t-2)$: These elements are contained in $\widetilde{W}$.

$s_{\alpha_{t-1}}$: We claim that $g_{s_{\alpha_{t-1}}}$ defined by

$$g_{s_{\alpha_{t-1}}}(z) = \exp \operatorname{ad} \alpha_{t-1}(z)(f_{\alpha_{t-1}} + f_{\alpha_{t+1}} + \ldots + f_{\alpha_M})$$

has the desired property. The main point in the calculation is to check that

$$[f_{\alpha_{t-1}} + f_{\alpha_{t+1}} + \ldots + f_{\alpha_M}, e] = -h_{\alpha_{t-1}}.$$

Then use that $z \in \mathfrak{k}$.

$s_{2\omega_t}$: There exists $w_1 \in \widetilde{W}$ such that $s_{2\omega_t}$ coincides with aw_1 on $\mathfrak{k}$.

$s_{\omega_{t-1}+\omega_t}$: Since this element is equal to $s_{2\omega_t} s_{\alpha_{t-1}} s_{2\omega_t}$, we can find $g_{s_{\omega_{t-1}+\omega_t}}$ using $g_{s_{2\omega_t}}$ and $g_{s_{\alpha_{t-1}}}$ (note that a normalizes U).

$s_{\alpha_{t+2k-1}+\alpha_{t+2k}} \cdot s_{\alpha_{t+2k}+\alpha_{t+2k+1}}$ $(k=1,\ldots,s-t-1)$: These elements are contained in $\widetilde{W}$.

$s_{\omega_{2s-t-1}+\omega_{2s-t}}$: This element is equal to $w_2 s_{\omega_{M-2}+\omega_{M-1}} w_2^{-1}$ where w_2 is a product of elements in $\widetilde{W}$ of the form $s_{\alpha_{t+2k-1}+\alpha_{t+2k}} \cdot s_{\alpha_{t+2k}+\alpha_{t+2k+1}}$. Therefore, it suffices to find $g_{s_{\omega_{M-2}+\omega_{M-1}}}$ when $\mathfrak{k}$ is given by

$$\mathfrak{k} = \{ z \in \mathfrak{h} \mid \alpha_{t+1}(z) = \alpha_{t+3}(z) = \cdots \alpha_{M-2}(z) = \alpha_M(z) = 0 \}.$$

We claim that $g_{s_{\omega_{M-2}+\omega_{M-1}}}$ defined by

$$g_{s_{\omega_{M-2}+\omega_{M-1}}}(z) = \exp \operatorname{ad}(\omega_{M-2} + \omega_{M-1})(z)(f_{\alpha_{M-2}+\alpha_{M-1}} + f_{\alpha_{M-1}+\alpha_M})$$

has the desired property.

□

The orthogonal group ($\varepsilon = 0$)

Let $G = O_{2M}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We fix a set of simple roots $\{\alpha_1, \dots, \alpha_M\}$ in $\mathfrak{h}^*$ by

$$\begin{aligned} \alpha_i &= \omega_i - \omega_{i+1} \quad \text{for } i = 1, \cdots, M-1, \\ \alpha_M &= \omega_{M-1} + \omega_M \end{aligned}$$

The image of $N_G(\mathfrak{h})$ in $\mathrm{Aut}(\mathfrak{h})$ is generated by reflections $s_{\alpha_1}, \dots, s_{\alpha_{M-1}}$ and $t_{2\omega_M}$ which act as follows:

$$\begin{aligned} s_{\alpha_i} &: \quad \omega_i \longleftrightarrow \omega_{i+1} \\ t_{2\omega_M} &: \quad \omega_M \longleftrightarrow -\omega_M \end{aligned}$$

Type III

Let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ given by the set

$$\{\alpha_{t+2k-1}\}_{k=1,\dots,s-t} \cup \{\alpha_{2s-t+j}\}_{j=1,\dots,Q}$$

of simple roots and let $\mathfrak{l}$ be its Levi part. The parabolic subalgebra $\mathfrak{p}$ contained in $\mathfrak{q}$ is defined by the set

$$\{\alpha_{t+2k-1}\}_{k=1,\dots,1/2(M-t)}$$

of simple roots. We may write the nilpotent elements e and e' as sums of root vectors:

$$\begin{aligned} e &= \sum_{i=1}^{t-1} e_{\alpha_i} + \sum_{i=t-1}^{M-2} e_{\alpha_i+\alpha_{i+1}} + e_{\alpha_M} \quad \text{if } t \geq 2, \\ e &= \sum_{i=1}^{M-2} e_{\alpha_i+\alpha_{i+1}} + e_{\alpha_M} \quad \text{if } t = 1, \\ e' &= \sum_{i=2s-t+1}^{M-2} e_{\alpha_i+\alpha_{i+1}} + e_{\alpha_M}. \end{aligned}$$

The center of $\mathfrak{l}$ is given by

$$\mathfrak{k} = \{\, z \in \mathfrak{h} \mid \alpha_{t+2k-1}(z) = 0\ ,\ k = 1, \dots, s-t\ ,\ \omega_{M-j+1}(z) = 0\ ,\ j = 1, 2, \dots, Q \,\}.$$

The image N of $N_G(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k})$ is generated by the following reflections:

$$\begin{aligned} s_{\alpha_i}\ {\scriptstyle (j=1,\dots,t-1)} &: \quad \omega_j \longleftrightarrow \omega_{j+1} \\ t_{2\omega_t} &: \quad \omega_t \longleftrightarrow -\omega_t \\ s_{\alpha_{t+2k-1}+\alpha_{t+2k}} \cdot s_{\alpha_{t+2k}+\alpha_{t+2k+1}}\ {\scriptstyle (k=1,\dots,s-t-1)} &: \quad \omega_{t+2k} \longleftrightarrow \omega_{t+2k+2} \\ s_{\omega_{2s-t-1}+\omega_{2s-t}} &: \quad \omega_{2s-t} \longleftrightarrow -\omega_{2s-t} \end{aligned}$$

If $t_{2\omega_t}$ is replaced by

$$s_{\omega_{t-1}+\omega_t} \quad : \quad (\omega_{t-1}, \omega_t) \longleftrightarrow (-\omega_{t-1}, -\omega_t)$$

this new set of reflections generates the subgroup N_0 isomorphic to $D_t \times B_{s-t}$.

The Lie algebra $\mathfrak{u}$ derived from the element h is the unipotent radical of the parabolic subalgebra given by the set

$$\{-\alpha_{t+2(j-1)}\}_{j=1,\ldots,1/2(M-t)}$$

of simple roots. The non-central element of the component group of A is induced by

$$a = s_{\alpha_t} \cdot s_{\alpha_{t+2}} \cdot \ldots \cdot s_{\alpha_{M-2}} \cdot t_{2\omega_M}.$$

The following technical lemma will be useful:

Lemma. *Let $\widetilde{W}$ be the subgroup of W generated by*

$$s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_{t-2}}, s_{\alpha_t+\alpha_{t+1}}, s_{\alpha_{t+1}+\alpha_{t+2}}, \ldots, s_{\alpha_{M-2}+\alpha_{M-1}}, s_{\alpha_M}.$$

If $w \in \widetilde{W}$, then there exists $g_w \colon \mathfrak{h} \to U$ such that

$$e + wz = g_w(z).(e+z)$$

for $z \in \mathfrak{h}$.

Proof. Using the identities given at the beginning of the appendix we easily check that $g_{s_{\alpha_j}}$ defined by

$$g_{s_{\alpha_j}}(z) = \exp \operatorname{ad} \alpha_j(z) f_{\alpha_j}$$

for $j = 1, \ldots, t-2, M$, and $g_{s_{\alpha_{j-1}+\alpha_j}}$ defined by

$$g_{s_{\alpha_{j-1}+\alpha_j}}(z) = \exp \operatorname{ad}(\alpha_{j-1}+\alpha_j)(z) f_{\alpha_{j-1}+\alpha_j}$$

for $j = t+1, \ldots, M-1$ have the claimed property. □

Now we give the details in the first two steps of the proof of the theorem in (6.3).

Lemma. *For $w \in N$ there exists $g_w \colon \mathfrak{k} \to U$ such that*

$$e + wz = \begin{cases} g_w(z).(e+z) & \text{if } w \in N_0 \\ a.g_w(z).(e+z) & \text{if } w \notin N_0 \end{cases}$$

for $z \in \mathfrak{k}$.

Proof. Evidently, it suffices to consider the generators of N_0 and N as described above.

s_{α_j} $(j=1,\ldots,t-2)$: These elements are contained in $\widetilde{W}$.

$s_{\alpha_{t-1}}$: We claim that $g_{s_{\alpha_{t-1}}}$ defined by

$$g_{s_{\alpha_{t-1}}}(z) = \exp \operatorname{ad} \alpha_{t-1}(z)(f_{\alpha_{t-1}} + f_{\alpha_{t+1}} + \ldots + f_{\alpha_{M-1}})$$

has the desired property.

$t_{2\omega_t}$: There exists $\tilde{w} \in \widetilde{W}$ such that $s_{2\omega_t}$ coincides with $a\tilde{w}$ on $\mathfrak{k}$.

$s_{\omega_{t-1}+\omega_t}$: Since this element is equal to $t_{2\omega_t} s_{\alpha_{t-1}} t_{2\omega_t}$, we can find $g_{s_{\omega_{t-1}+\omega_t}}$ using $g_{t_{2\omega_t}}$ and $g_{s_{\alpha_{t-1}}}$ (note that a normalizes U).

$s_{\alpha_{t+2k-1}+\alpha_{t+2k}} \cdot s_{\alpha_{t+2k}+\alpha_{t+2k+1}}$ $(k=1,\ldots,s-t-1)$: These elements are contained in $\widetilde{W}$.

$s_{\omega_{2s-t-1}+\omega_{2s-t}}$: This element is a $\widetilde{W}$-conjugate of s_{α_M}, which is contained in $\widetilde{W}$.

□

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