Notes on Quasipositivity and Combinatorial Knot Invariants

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Introduction

A characteristic property of a knot is a criterion that allows us to recognize this knot. For example, the trivial knot is the only knot which bounds a disk, whence it is characterized by this property. This does not mean it is easy to recognize the trivial knot, since it might still be difficult to tell whether a given knot bounds a disk or not. In general, characteristic properties of knots are hard to handle. Therefore we usually content ourselves looking at weaker properties of knots, in particular at knot invariants, that allow us to distinguish certain knots from others. Kurt Reidemeister's diagrammatical formulation of knot theory in his famous book *Knotentheorie* ([**39**]) gave rise to a variety of combinatorial knot properties, such as the minimal crossing number, alternation or the Jones polynomial. Some of them do also have a topological interpretation, notably the fundamental group. In the present doctoral thesis I discuss relations between various knot properties, with a special emphasis on quasipositivity.

A link is called quasipositive if it has a special braid diagram, namely a product of conjugates of the positive standard generators of the braid group. If this product contains words of the form

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1}$$

only then we call the link strongly quasipositive. Here σ_i is the i-th positive standard generator of the braid group. The concept of quasipositive links is due to Lee Rudolph. He showed that every quasipositive link is a transverse $\mathbb{C}$-link, i.e. a transverse intersection of a complex plane curve with the standard sphere $S^3 \subset \mathbb{C}^2$. Recently, M. Boileau and S. Orevkov proved the converse, namely, that every transverse $\mathbb{C}$-link is a quasipositive link. However, most of the considerations in my thesis are based upon a diagrammatical point of view.

In the first chapter I introduce the class of track knots and, at the same time, a method to construct knots with prescribed unknotting numbers. Track knots are closely related to the class of divide knots, which were introduced by Norbert A'Campo and served as a starting point for my studies in knot theory. Furthermore, track knots share a basic property with quasipositive knots. Actually they are quasipositive, as we shall see in the second chapter. For this purpose I introduce a new diagrammatical description of quasipositive knots.

In the third chapter I prove that a knot is positive if and only if it is homogeneous and strongly quasipositive. This result follows quite easily from three famous inequalities due to D. Bennequin, H. Morton and P. Cromwell. The fourth chapter contains the main result of my thesis: any finite number of Vassiliev invariants of a knot can be realized by a quasipositive knot. This is closely related to Lee Rudolph's result which says that any Alexander polynomial of a knot can be realized by a quasipositive knot.

Crossing changes play an important part in my thesis, two. They give rise not only to the notion of unknotting numbers, but also to the Gordian complex of knots. The Gordian complex of knots is a simplicial complex whose vertex set consists of all the isotopy classes of smooth oriented knots in S^3. An edge of the Gordian complex is a pair of knots of Gordian distance one, i.e. a pair of knots which differ by one crossing change. Similarly, an n-simplex is a set of $(n+1)$ knots whose pairwise Gordian distance is one. In the fifth chapter I prove that every n-simplex of the Gordian complex of knots is contained in an $(n+1)$-simplex. This is a generalization of M. Hirasawa and Y. Uchida's result, who showed that every edge of the Gordian complex is contained in a simplex of infinite dimension. Further we shall see that a knot of unknotting number two can be unknotted via infinitely many different knots of unknotting number one.

The sixth chapter is devoted to the study of a special class of knots, namely arborescent knots arising from plumbing positive and negative Hopf bands along any tree. In particular, I determine the minimal crossing numbers of these knots. This is kind of a counterpart to a result of W. B. R. Lickorish and M. B. Thistlethwaite on the minimal crossing number of Montesinos links, i.e. links associated with star-shaped trees.

At last, some problems appear in the seventh and last chapter. Appendix A contains a table of all quasipositive knots up to ten crossings. Some examples of track knots are presented in Appendix B, while Appendix C contains a table of all special fibered arborescent knots up to ten crossings.

Altough the succession of the chapters follows a certain logical and chronological order, most of them can be read independently, notably chapters 3 and 6. However, chapter 4 relies upon the diagrammatical description of quasipositive knots presented in chapter 2.

The material of the first two chapters is published in the Osaka Journal of Mathematics. A short article with the contents of chapter 3 is accepted for publication in the Mathematical Proceedings of the Cambridge Philosophical Society. Further, a text on the theme of chapter 6 is accepted for publication in Commentarii Mathematici Helvetici, and a short article on unknotting sequences (Theorem 11) is accepted for publication in the Quarterly Journal of Mathematics.

Contents

CHAPTER 1

Combinatorial Invariants of Track Knots

1. Track Knots

A plane curve is a generic immersion of a circle into the plane. We may interpret a plane curve as the shadow of a knot in the 3-space. Naturally, we cannot reconstruct a knot from its shadow, unless we happen to know some additional information about it. For instance, a Legendrian knot in respect of the standard contact structure $dz - ydx$ on R^3 is reconstructible from its shadow in the x-y plane. Moreover, every knot can be brought into a Legendrian position by an isotopy of the ambient space R^3. There is another way of assigning a link to a plane curve, more precisely, to a divide.

A divide is the intersection of a plane curve with the unit disk in $\mathbb{R}^2$, provided the plane curve is transverse to the unit circle. The link of a divide P is the set

$$L(P) = \{(x, u) \in T(\mathbb{R}^2) \mid x \in P,\ u \in T_xP,\ |x|^2 + |u|^2 = 1\},$$

where the space of tangent vectors to the plane is identified with $\mathbb{R}^2 \times \mathbb{R}^2$. We remark that $L(P)$ lies in the three-dimensional sphere

$$ST(\mathbb{R}^2) = \{(x, u) \in T(\mathbb{R}^2) \mid |x|^2 + |u|^2 = 1\}.$$

The concept of links associated with divides is due to N. A'Campo and emerged from the study of isolated singularities of complex plane curves (see [**1**]). In [**2**] and [**3**], A'Campo specified some properties of divide knots, including fiberedness and a Gordian number result. These strong results are at the expense of the size of the class of divide knots. In fact, only 8 knots up to ten crossings are divide knots; this follows from the classification in M. Ishikawa's doctoral thesis ([**21**], Appendix B). A large extension of the class of divide links was introduced by W. Gibson and M. Ishikawa [**15**]. They dropped the relativity condition $\partial P \subset \{x \in \mathbb{R}^2 \mid |x| = 1\}$ for divides and kept A'Campo's result on the Gordian number. T. Kawamura [**25**] and Ishikawa independently proved the quasipositivity of these links of free divides.

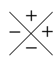
Borrowing from all these, we propose a new construction of knots associated with labelled generic immersions of intervals into the plane. In spirit, this construction is based upon M. Hirasawa's algorithm for drawing diagrams of divide links (see [**18**]).

Let C be the image of a generic immersion of the interval $[0,1]$ into the plane. In particular, C has no multiple points apart from a finite number of transversal double points, none of which is the image of 0 or 1. Further we enrich C, as follows (see Figure 1 for an illustration).

(i) A small disk around each double point of C is cut into four regions by C. Label each of these regions by a sign, such that the sum of the four signs is non-negative. There are four types of patterns of signs around a double point, called a, b, c and d. They are shown in Figure 2. If the tangent space T_pC at a double point p of C is the set $\{(x,y) \in \mathbb{R}^2 \mid (y - y(p))^2 = (x - x(p))^2\}$, then we may represent patterns of four signs at p by one of the following symbols:

a, a_1, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d.

An index i at a symbol means that the corresponding pattern has to be turned counter-clockwise by the angle $i\frac{\pi}{2}$.

For example, b_1  stands for the pattern  .

Henceforth we shall use these symbols.

(ii) Specify a finite number of different points $p_1, p_2, \dots, p_r$ on the edges of C (i.e. on the connected components of $C - \{\text{double points}\}$, such that $C - \{p_1, p_2, \dots, p_r\}$ is simply connected, but not necessarily connected. r is greater than or equal to the number of double points of C.

A labelled generically immersed interval in the plane will always be denoted by C_λ.

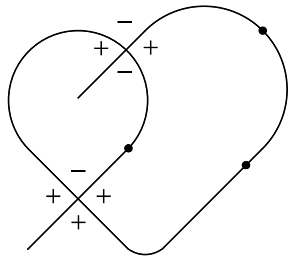


FIGURE 1

The following algorithm associates a knot diagram, hence a knot in the 3-space, to a labelled generically immersed interval C_λ.

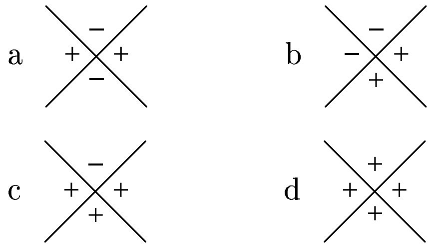


FIGURE 2. Patterns of signs

(1) Draw a parallel companion of C_λ. In other words, replace C_λ by the boundary of a small band following C_λ. Join the two strands with an arc at both end points of C_λ and orient the resulting plane curve clockwise, in regard of the small band (see Figure 3).

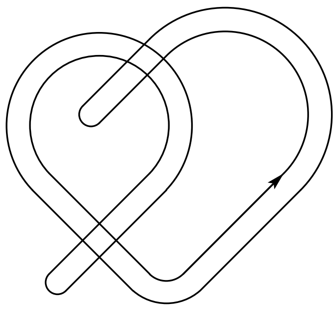


FIGURE 3

(2) At each double point of C_λ, place over- and under-crossings according to the signs of the four regions, as shown in Figure 4.

The characters a, b, c and d stand for 'above', 'between', 'conventional' and 'double', respectively. 'Conventional' crossings appear in the visualization of links of divides, see Hirasawa [**18**].

(3) Add a full twist to the band at each specified point of C_λ, in a manner that gives rise to two positive crossings (see Figure 4).

The knot diagram arising from C_λ by these three steps will be denoted by $D(C_\lambda)$, the corresponding knot by $K(C_\lambda)$.

DEFINITION 1. A *track knot* is a knot which can be realized as a knot associated with a labelled generically immersed interval C_λ. If it can be realized without any double point of type b, then we call it a *special track knot*.

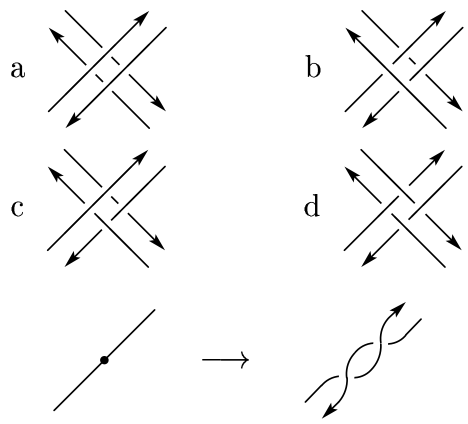


FIGURE 4

REMARK. We observe that the classes of track knots and special track knots are closed under connected sum. The connected sum operation corresponds to the gluing of two labelled immersed intervals along end points. This is not true for knots of free divides; the connected sum of the free divide knot 5_2 (in Rolfsen's numbering [**40**]) with itself is not a free divide knot.

2. Slice and Gordian Numbers

Let L be an oriented link with n components in $S^3 = \partial B^4$. The slice number $\chi_s(L)$ of L is the maximal Euler characteristic among all smooth, oriented surfaces in B^4 which are bounded by L and have no closed components. The surfaces in consideration need not be connected. If K is a knot then its *4-genus* is defined as

$$g^*(K) = \frac{1}{2}(1 - \chi_s(K)).$$

The *clasp number* $c_s(L)$ of a link L is the minimal number of transversal double points of n generically immersed disks in B^4 with boundary L. We will also be concerned with the *Gordian unknotting number* $u(L)$, which is the minimal number of crossing changes needed to transform L into the trivial link with n components. Here a crossing change is a strand passage operation along an embedded disk (see Figure 5 for an illustration). The notion of unknotting numbers appears in an article of H. Wendt ([**54**]), where he calls it *Überschneidungszahl.*

The following two inequalities relate the numbers defined above:

$$u(L) \geqslant c_s(L) \geqslant \frac{1}{2}(1 - \chi_s(L)). \tag{1}$$

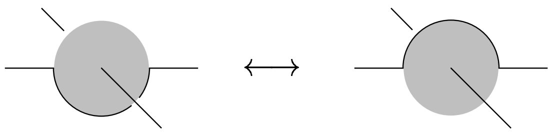


FIGURE 5. A crossing change along a disk

They can be shown by purely geometrical arguments, see Kawamura [**24**].

Gordian unknotting numbers and 4-dimensional invariants of track knots are easy to determine. Let K be a track knot associated with a labelled generically immersed interval C_λ. Further let A, B, C and D be the numbers of double points of C_λ with patterns of signs of type a, b, c and d, respectively.

THEOREM 1 ([**4**]). *The clasp number and the 4-genus of K equal $C + 2D$. If B is zero, then the Gordian unknotting number and the ordinary genus of K equal $C + 2D$, too.*

COROLLARY 1. *Both the clasp number and the 4-genus are additive under connected sum of track knots. Moreover, the Gordian unknotting number is additive under connected sum of special track knots.*

REMARK. The connected sum of a knot with its mirror image always bounds an embedded disk in B^4, thus the clasp number and the 4-genus are not additive under connected sum of knots in general. It is still a conjecture that the Gordian unknotting number is additive under connected sum of knots (see M. Boileau and C. Weber [**11**]).

PROOF OF THEOREM 1. We first show that the 4-genus of K does not exceed $C+2D$. If $C = D = 0$, then K is clearly slice, i.e. K bounds a disk in B^4. Indeed, the band following C_λ provides an immersed disk in S^3 with boundary K. At each double point of type b we may push a part of the band into $\mathring{B}^4$ to get an embedded disk. But then, at each double point of type c, we add one handle to the band, as Figure 6(c) suggests. Similarly, we add two handles to the band at each double point of type d, see Figure 6(d). This creates an embedded surface in B^4 of genus $C+2D$ with boundary K. If $B = 0$, then it is an embedded surface in S^3.

We remark that the spots where we add handles to the band can be interpreted as clasp singularities of the immersed band. Therefore the clasp number of K does not exceed $C + 2D$, either. Next, we show that the unknotting number of K does not exceed $C + 2D$, provided B is zero. If $C = D = 0$, then K is the unknot since it bounds an embedded disk in S^3. On a knot diagram level, double points of type

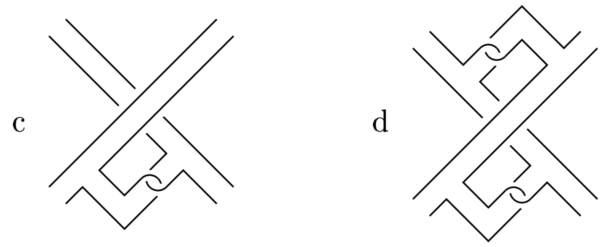


FIGURE 6

c differ from double points of type a only by one crossing change, see Figure 4. Similarly, double points of type d differ from double points of type a by two crossing changes. Hence we conclude $u(K) \leqslant C+2D$.

We still have to prove that $C+2D$ is a lower bound for the four numbers in question. If we prove $g^*(K) \geqslant C+2D$, then we are done, thanks to (1). For this purpose we need the slice-Bennequin inequality. Let D_L be the diagram of an oriented link L. The writhe $w(D_L)$ is the number of positive minus the number of negative crossings of the diagram D_L. Smoothing D_L at all crossings produces a union of Seifert circles. Let $s(D_L)$ be their number.

SLICE-BENNEQUIN INEQUALITY. $\chi_s(L) \leqslant s(D_L) - w(D_L)$.

The slice-Bennequin inequality was first established for closed braid diagrams by Lee Rudolph [**44**]; the proof of the general case can be found in Rudolph [**46**]. In a recent paper ([**38**]), J. Rasmussen determines the 4-genera of positive knots by using the theory of Khovanov homology of knots. His work provides the foundation for a combinatorial proof of the slice-Bennequin inequality, see [**48**]. Originally, a '3-dimensional' version of the inequality (concerning Seifert surfaces) was proved by D. Bennequin [**9**].

Now let us compute $w(D(C_\lambda))$ and $s(D(C_\lambda))$ for the knot diagram of the labelled generically immersed interval C_λ.

(1) $w(D(C_\lambda)) = 2C+4D+2r$, where r is the number of specified points on C_λ.

(2) Each double point and each specified point of C_λ gives rise to a small Seifert circle, see Figure 7. Moreover, each connected component of $C_\lambda - \{p_1, p_2, \dots, p_r\}$ gives one Seifert circle. The number of connected components of $C_\lambda - \{p_1, p_2, \dots, p_r\}$ being $1+r-(A+B+C+D)$, we conclude

$$s(D(C_\lambda)) = A+B+C+D+r+1+r-(A+B+C+D) = 2r+1.$$

Thus the slice-Bennequin inequality yields $\chi_s(K) \leqslant 1-2C-4D$ and $g^*(K) = \frac{1}{2}(1-\chi_s(K)) \geqslant C+2D$. $\square$

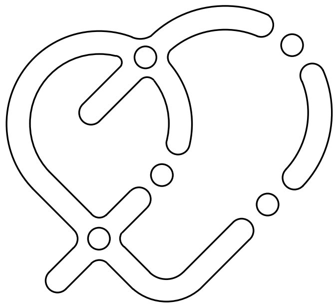


FIGURE 7

REMARKS.

(i) If we renounce twisting the band at some specified points, then the statements of Theorem 1 are no longer true. The labelled immersed interval (without specified points) of Figure 8 has one double point of type c and gives the unknot.

(ii) The statement of Theorem 1 about the unknotting number can be extended for track knots with $B = 1$. However, if $B \geqslant 2$, then the unknotting number may be greater than $C+2D$. E.g. the knots 9_{46} and 10_{140} are slice track knots (see Appendix B) and their unknotting numbers are certainly not zero.

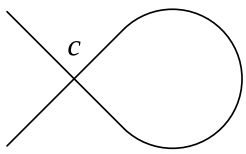


FIGURE 8

3. The HOMFLY Polynomial

In this section we present an inequality for track knots which involves the HOMFLY polynomial. The HOMFLY polynomial $P_L(a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ of an oriented link L is a Laurent polynomial in two variables a and z which is normalized to one for the trivial knot and satisfies the following relation (see [**13**]):

$$\frac{1}{a}P_{\times}(a,z) - aP_{\times}(a,z) = zP_{)(}(a,z). \qquad (2)$$

Writing $P_L(a,z) = \sum_{k=e(L)}^{E(L)} a_k(z)a^k$, with $a_{e(L)}(z)$, $a_{E(L)}(z) \neq 0$, as a Laurent polynomial in one variable a, we define its range in a as

$[e(L), E(L)]$. H. R. Morton gave some bounds for $e(L)$ and $E(L)$ in terms of the writhe and the number of Seifert circles of a diagram of L.

THEOREM 2 (H. R. Morton [**29**]). *For any diagram D_L of an oriented link L*

$$w(D_L) - (s(D_L) - 1) \leqslant e(L) \leqslant E(L) \leqslant w(D_L) + (s(D_L) - 1).$$

The first inequality is tailor-made for track knots.

THEOREM 3 ([**4**]). $2g^*(K) \leqslant e(K)$ *for any track knot K.*

PROOF. Choose a track knot diagram D of K. The proof of Theorem 1 tells us that $g^*(K) = \frac{1}{2}(1 - s(D) + w(D))$, which is exactly half the lower bound in Morton's theorem. □

The slice-Bennequin inequality being an equality for closed quasipositive braid diagrams (see [**44**]), we observe that Theorem 3 is true both for track knots and quasipositive knots. In fact, the classification of quasipositive knots up to ten crossings in Appendix A is based upon the inequality of Theorem 3 . This is a strong indication of the quasipositivity of track knots. In the next chapter, we shall prove the quasipositivity of track knots.

CHAPTER 2

Quasipositivity

A quasipositive braid is a product of conjugates of a positive standard generator of the braid group. If a link can be realized as the closure of a quasipositive braid then we call it quasipositive. When Lee Rudolph introduced quasipositive links (in [**42**]), he showed that they could be realized as transverse $\mathbb{C}$-links, i.e. as transverse intersections of complex plane curves with the standard sphere $S^3 \subset \mathbb{C}^2$. Here a complex plane curve is any set $f^{-1}(0) \subset \mathbb{C}^2$, where $f(z, w) \in \mathbb{C}[z, w]$ is a non-constant polynomial. Conversely, every transverse $\mathbb{C}$-link is a quasipositive link, as was recently proved by M. Boileau and S. Orevkov (in [**10**]). For a thorough introduction into this subject, we refer the reader to Rudolph's text on the knot theory of complex plane curves [**47**]. As the name suggests, the notion of quasipositivity generalizes the notion of positivity (see [**33**] or [**46**]). However, this is not quite obvious. In the following, we propose a description of quasipositive knots which is based upon Seifert diagrams rather than braid diagrams.

Any planar knot diagram gives rise to a system of Seifert circles with signed arcs, where each arc stands for a crossing joining two Seifert circles, as shown in Figure 1. The sign of an arc tells us whether the crossing is positive or negative.

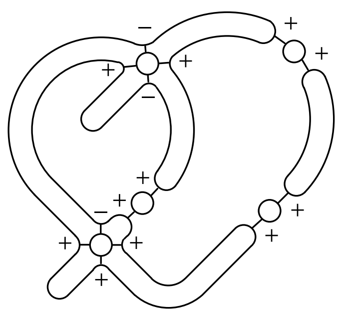


FIGURE 1. A system of Seifert circles

DEFINITION 2. A knot diagram is *quasipositive* if its set of crossings can be partitioned into single crossings and pairs of crossings, such that the following three conditions are satisfied.

(1) Each single crossing is positive.
(2) Each pair of crossings consists of one positive and one negative crossing joining the same two Seifert circles.
(3) A pair of crossings does not separate other pairs of crossings. More precisely, going from one crossing of a pair to its opposite counterpart along a Seifert circle, one cannot meet only one crossing of a pair.

EXAMPLES.

(i) Positive knot diagrams are obviously quasipositive.

(ii) Track knot diagrams are quasipositive: negative arcs are incident with a small Seifert circle corresponding to a double point of type a, b or c. They can be paired with neighbouring positive crossings of the same small Seifert circle (see Figure 2). At this point, it is essential that $C_\lambda - \{p_1, p_2, \ldots, p_r\}$ is simply connected. This guarantees that pairs of crossings do not get entangled (see Figure 1).

(iii) Quasipositive braid diagrams are quasipositive.

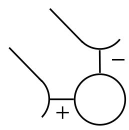


FIGURE 2. A pair of crossings

THEOREM 4 ([**4**]). *A quasipositive knot diagram represents a quasipositive knot.*

Together with Example (ii), Theorem 4 proves the quasipositivity of track knots.

THEOREM 5 ([**4**]). *Track knots are quasipositive.*

In order to prove Theorem 4, we adopt the pattern of Takuji Nakamura's proof of strong quasipositivity of positive links (see [**33**]).

PROOF OF THEOREM 4. Any link diagram can be deformed into a braid representation, i.e. a system of concentric Seifert circles, by a finite sequence of bunching operations or concentric deformations of two types, without changing the writhe and the number of Seifert circles of the link diagram. This algorithm is due to Shuji Yamada, see [**55**]. We shall explain these two deformations and their effect on quasipositive knot diagrams.

First of all, we may consider only knot diagrams which have an outermost Seifert circle S_1, i.e. one that contains all the other Seifert circles. This corresponds to choosing a point on the sphere S^2 appropriately.

If S_1 contains a maximal Seifert circle S_2 with the opposite orientation of S_1, then we apply a concentric deformation of type I to S_2, as shown in Figure 3.

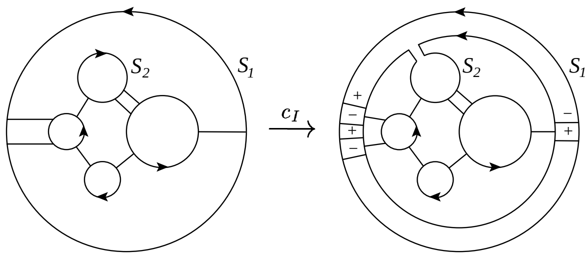


FIGURE 3. A concentric deformation of type I

If S_1 contains maximal Seifert circles with the same orientation as S_1 only, then we apply a concentric deformation of type II to any of these maximal Seifert circles, say to S_2, as shown in Figure 4.

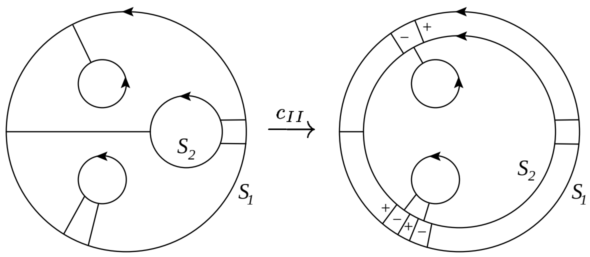


FIGURE 4. A concentric deformation of type II

In the next step, we consider maximal Seifert circles inside S_2, and so on. This algorithm clearly ends in a braid representation. Now we observe that concentric deformations of both types preserve the quasipositivity of knot diagrams in the above sense. They merely introduce new pairs of crossings, which do not get entangled. Figures 5 and 6 show how a positive crossing (or a pair of crossings, respectively) gets more 'conjugated' by new pairs of crossings after a concentric deformation.

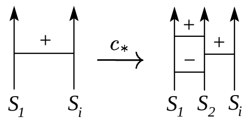


FIGURE 5

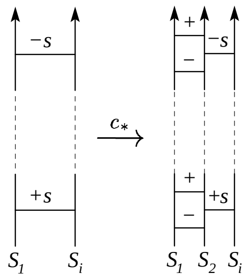


FIGURE 6

Thus, starting with a quasipositive knot diagram, we end up with a quasipositive braid diagram, which clearly represents a quasipositive knot.

□

CHAPTER 3

Quasipositivity and Homogeneity

1. A Characterization of Positive Knots

Quasipositive links have special braid diagrams, namely products of conjugates of the positive standard generators of the braid group. If such a product contains words of the form

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1}$$

only, then we call the corresponding link strongly quasipositive. Here σ_i is the i-th positive standard generator of the braid group. The word $\sigma_{i,j}$ is often called a positive band, see Figure 1, where the corresponding section of its Seifert diagram is drawn, too.

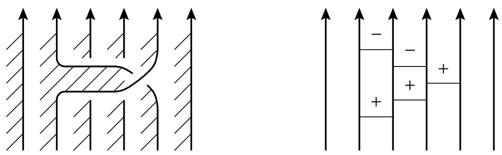


FIGURE 1. A positive band: $\sigma_{2,5}$

A geometric characterization of strongly quasipositive links was given by Lee Rudolph ([**43**]). In [**46**], Rudolph proved that positive knots are strongly quasipositive and asked whether positive knots could be characterized as strongly quasipositive knots which satisfy some extra geometric conditions. It turns out that the appropriate extra condition is homogeneity. Homogeneous links were introduced by P. R. Cromwell ([**12**]) as a generalization of alternating links and positive links. A link diagram is homogeneous around a Seifert circle S if the following two conditions are met:

(1) all the crossings inside S, which are adjacent to S, have the same sign.
(2) all the crossings outside S, which are adjacent to S, have the same sign.

A link is homogeneous if it has a diagram which is homogeneous around each Seifert circle. A homogeneous link diagram is shown in Figure 2, together with its Seifert diagram.

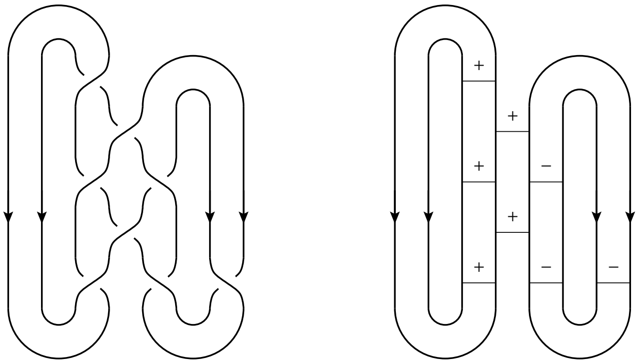


FIGURE 2. A homogeneous diagram

We observe that a braid diagram is strongly quasipositive and homogeneous if and only if it is a positive braid diagram. This suggests that the two classes of strongly quasipositive knots and homogeneous knots have a small intersection. We shall make this statement more precise.

THEOREM 6 ([**5**]). *A knot is positive if and only if it is homogeneous and strongly quasipositive.*

The proof of Theorem 6 is based upon three famous inequalities. One of them is Morton's inequality ([**29**]), which we have already encountered in the first chapter. One more inequality involves the HOMFLY polynomial $P_K(a, z)$. Again, we denote its minimal degree in the variable a by $e(K)$, its maximal degree by $E(K)$. The minimal genus among all orientable surfaces spanning a knot K will be denoted by $g(K)$.

THEOREM 7 (D. Bennequin [**9**]). *Let D be a closed braid diagram with $s(D)$ strings of a knot K. Let $w(D)$ be the writhe of D, i.e. the number of positive minus the number of negative crossings of D. Then*

$$g(K) \geqslant \frac{1}{2}(1 + w(D) - s(D)).$$

THEOREM 8 (P. R. Cromwell [**12**]). *For any homogeneous knot K,*

$$e(K) \leqslant 2g(K).$$

Moreover, equality holds if and only if K is positive.

PROOF OF THEOREM 6. First we observe that a positive knot is homogeneous and strongly quasipositive (see Lee Rudolph [**46**], T. Nakamura [**33**] or chapter 2 of the present text). Conversely, let K be a homogeneous strongly quasipositive knot. K has a closed braid diagram

D which is a product of $\sigma_{i,j}$'s. From D we can construct a natural Seifert surface of K. It is a union of $s(D)$ disks and $w(D)$ bands with a positive half-twist. Such a band is depicted on the left side of Figure 1. The Euler characteristic of this Seifert surface equals $s(D) - w(D)$; its genus is $\frac{1}{2}(1+w(D)-s(D))$. By Bennequin's inequality, it is a minimal genus Seifert surface of K, i.e.

$$g(K) = \frac{1}{2}(1 + w(D) - s(D)).$$

Now Morton's inequality tells us that

$$2g(K) \leqslant e(K),$$

whereas Cromwell's inequality tells us that

$$e(K) \leqslant 2g(K).$$

We conclude that equality holds, and hence, by Cromwell's Theorem, that K is positive. □

2. Construction of Non-Homogeneous Knots

Theorem 6 enables us to construct many non-homogeneous knots. For instance, it has been shown by Lee Rudolph that any Alexander polynomial of a knot can be realized as the Alexander polynomial of a strongly quasipositive knot (see [**41**]). On the other hand, the coefficients of the Alexander polynomial of a positive knot satisfy strong conditions. In this way, we can find many non-homogeneous knots.

We discuss one application, concerning positive Hopf plumbings. Let K be a knot which can be obtained from the unknot by iterated plumbing of positive Hopf bands. By a result of Lee Rudolph ([**45**]), K is strongly quasipositive. Hence K is either positive or non-homogeneous.

Example. The knot 10_{145} (in Rolfsen's notation [**40**]) is a non-positive divide knot (see N. A'Campo [**2**] for a definition of divide knots). Divide knots bound a unique Seifert surface which is a plumbing of positive Hopf bands (see M. Ishikawa [**20**]). Hence the knot 10_{145} is non-homogeneous.

CHAPTER 4

Quasipositivity and Vassiliev Invariants

It has been known that any Alexander polynomial of a knot can be realized by a strongly quasipositive knot, as we mentioned in the previous chapter. Rewriting the Alexander polynomial in Conway's variable, we get a polynomial whose coefficients are certain Vassiliev invariants of finite order. Thus it seems reasonable to ask whether any finite number of Vassiliev invariants can be realized by a quasipositive knot. This question was posed by A. Stoimenow in [**51**]; we shall answer it in this chapter.

THEOREM 9 ([**8**]). *For any oriented knot K and any natural number n there exists a quasipositive knot Q whose Vassiliev invariants of order less than or equal to n coincide with those of K.*

The following corollary is an immediate consequence of Theorem 9.

COROLLARY 2. *It is impossible to decide whether a given knot in the standard sphere in $\mathbb{C}^2$ is isotopic to a transverse intersection of a complex plane curve with this sphere by means of finitely many Vassiliev invariants.*

The proof of Theorem 9 is based upon a construction of Y. Ohyama, who showed that any finite number of Vassiliev invariants can be realized by an unknotting number one knot (see [**35**]). His construction involves certain C_n-moves, which were defined by K. Habiro in [**17**], see also [**36**]. A special C_n-move is defined diagrammatically in Figure 1. It takes place in a section with $2(n+1)$ endpoints or $(n+1)$ strands, respectively. The strands are numbered from 1 to $n+1$ and are all connected outside the indicated section, since they belong to the same knot, say K. Going along K according to its orientation, starting at the first strand, we encounter the other strands in a certain order which depends on how the strands are connected outside the indicated section. This order defines a permutation, say $\sigma \in S_n$, of the numbers 2, 3, ..., n+1.

In [**37**], Y. Ohyama and T. Tsukamoto explain the effect of a C_n-move on Vassiliev invariants of order n. Their result ([**37**], Theorem 1.2) implies the following:

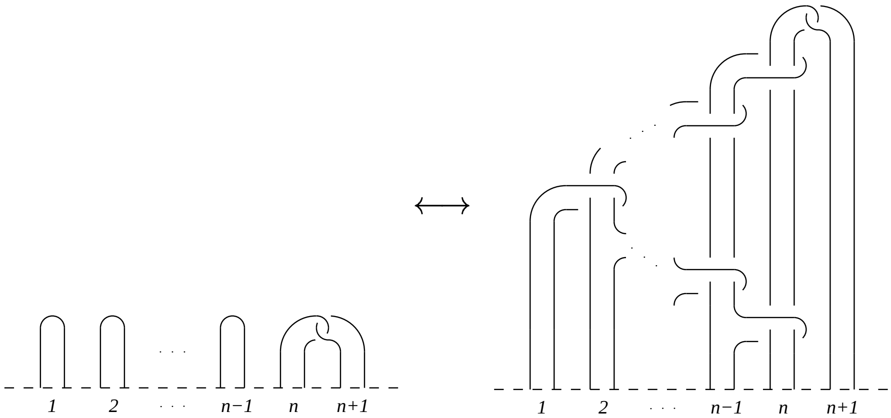


FIGURE 1

(1) A C_n-move does not change the values of Vassiliev invariants of order less than n.

(2) Let K and $\widetilde{K}$ be two knots which differ by one C_n-move, and v_n any Vassiliev invariant of order n. Then $|v_n(K) - v_n(\widetilde{K})|$ depends only on the permutation $\sigma \in S_n$ defined by the cyclic order of the $(n+1)$ strands of the section where the C_n-move takes place.

PROOF OF THEOREM 9. Starting from the diagram of the positive twist knot 5_2 shown in Figure 2, we construct a quasipositive knot Q with the desired properties by applying several C_n-moves, $2 \leqslant i \leqslant n$, step by step.

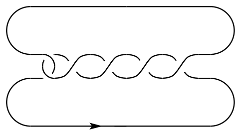


FIGURE 2

In the first step, we construct a quasipositive knot Q_1 whose Vassiliev invariants of order two (the Casson invariant) equals that of K. Choose natural numbers a and b, such that $v_2(K) = 2 + a - b$. Here $v_2(K)$ is the Casson invariant of K. Using these two numbers, we define a knot Q_1 diagrammatically, as shown in Figure 3.

By construction, we have

$$v_2(Q_1) = 2 + a - b = v_2(K).$$

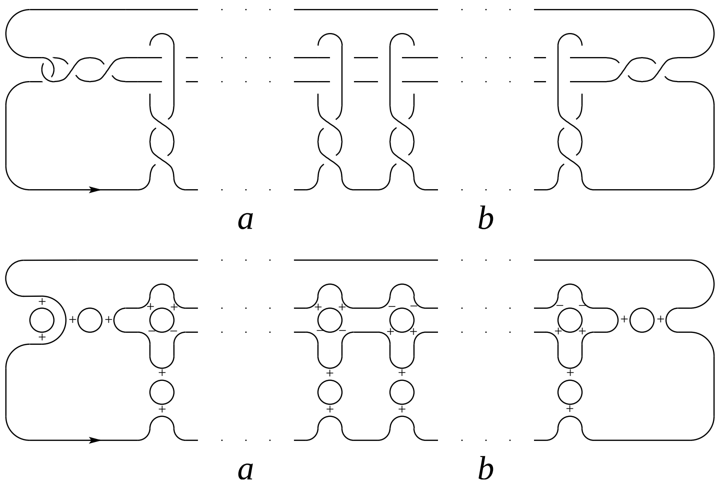


FIGURE 3

This follows easily by one application of the following relation for the Casson invariant of knots:

$$v_2(\times) - v_2(\times) = lk()\,()).$$

Indeed, a crossing change at the clasp on the left side of the diagram of Q_1 produces a trivial knot, and the linking number lk of the corresponding link equals $2 + a - b$. Moreover, the Seifert diagram of Q_1 at the bottom of Figure 3 is quasipositive, whence Q_1 is a quasipositive knot (see chapter 2).

In the second step, we arrange the Vassiliev invariants of order three. Since 'all' the Vassiliev invariants of order less than or equal to two of Q_1 and K coincide (i.e. $v_2(Q_1) = v_2(K)$), we conclude that Q_1 and K are related by a sequence of C_3-moves. This is K. Habiro's result for $n = 2$ (see [**17**]). Let $K_1 = Q_1$, K_2, …, $K_l = K$ be a sequence of knots, such that two succeedding knots are related by a C_3-move. Our aim is to replace this sequence of knots by a sequence of quasipositive knots $\widetilde{K}_1 = Q_1$, $\widetilde{K}_2$, …, $\widetilde{K}_l$, such that

$$v_3(\widetilde{K}_{i+1}) - v_3(\widetilde{K}_i) = v_3(K_{i+1}) - v_3(K_i),$$

$1 \leqslant i \leqslant l - 1$. By Ohyama and Tsukamoto's result, $|v_3(K_2) - v_3(Q_1)|$ depends only on the permutation $\sigma \in S_3$ defined by the cyclic order of the four strands of the section where the C_3-move takes place, as explained above. From this viewpoint, i.e. if we are only interested in the change of the Vassiliev invariants of order three, there are only finitely many combinatorial patterns of C_3-moves. A 'standard' pattern of a C_3-move can be applied inside a local box on the right side of the diagram of Q_1, as shown in Figure 4.

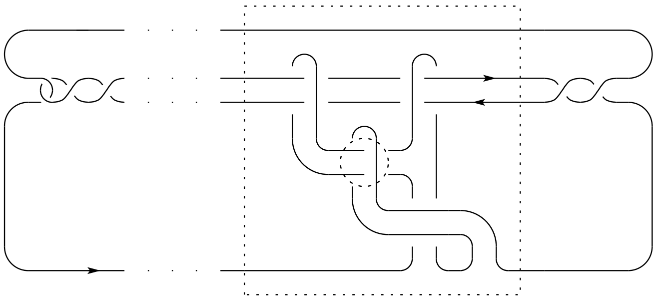


FIGURE 4

Moreover, we can choose a quasipositive representative for this pattern, i.e. a representative whose Seifert diagram (inside the box) is quasipositive, see Figure 5. Here we remark that the two segments above and below the cross-shaped Seifert circle belong to the same Seifert circle since they are connected outside the local box.

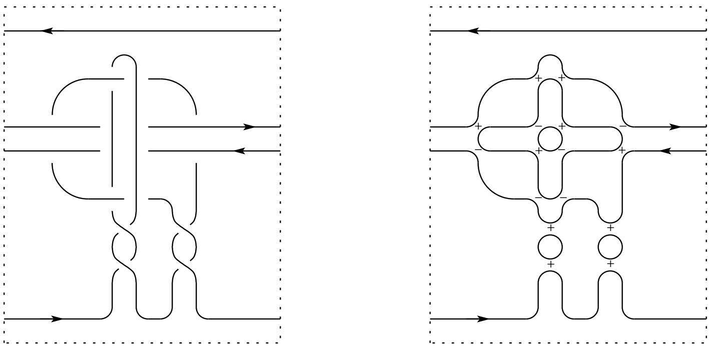


FIGURE 5

However, this standard pattern corresponds to one specific permutation $\sigma \in S_3$. In order to get patterns corresponding to other permutations, we have to permute the strands inside the local box, as shown by two examples on the left side of Figure 6. We observe that all these patterns have quasipositive representatives. They are depicted on the right side of Figure 6, together with their Seifert diagrams.

Thus we can replace the knot K_2 by a quasipositive knot $\widetilde{K}_2$, such that

$$v_3(\widetilde{K}_2) - v_3(Q_1) = \pm(v_3(K_2) - v_3(Q_1)).$$

If the sign of this difference is wrong (i.e. '$-$'), we may arrange it to be '$+$' by changing four crossings between two strands inside the local box, see Figure 4, where the four crossings are encircled. This modified pattern has the inverse effect on Vassiliev invariants of order

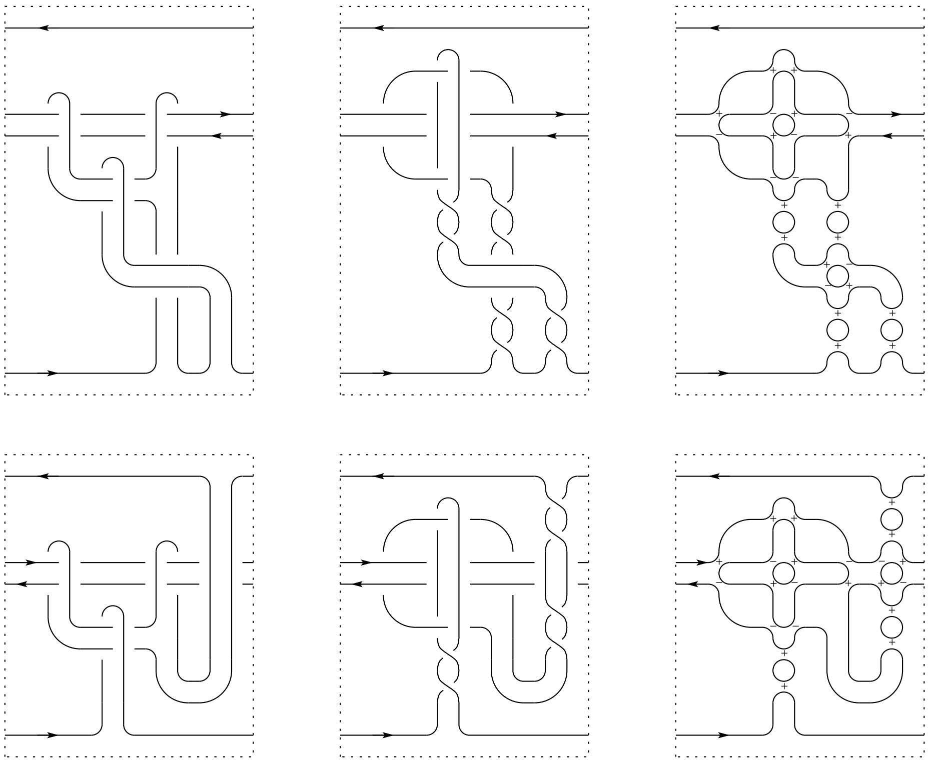


FIGURE 6

three, as follows from Ohyama and Tsukamoto's calculation ([**37**], proof of theorem 1.2).

Likewise, we can replace all C_3-moves of the sequence $K_1 = Q_1$, K_2, …, $K_l = K$ by C_3-moves that take place in a clearly arranged box and preserve the quasipositivity of the knot Q_1. In this way, we obtain a sequence of quasipositive knots $\widetilde{K}_1 = Q_1$, $\widetilde{K}_2$, …, $\widetilde{K}_l$ and end up with a quasipositive knot $Q_2 := \widetilde{K}_l$ whose Vassiliev invariants of order two and three coincide with those of K.

At this point we merely sketch how the process continues: in the i-th step, we arrange the Vassliev invariants of order $i+1$ and define a quasipositive knot Q_i whose Vassiliev invariants of order less than or equal to $i+1$ coincide with those of K. For this purpose, we need only observe that every combinatorial pattern of a C_{i+1}-move has a quasipositive representative with i^2+i conjugating pairs of crossings, i.e. pairs of crossings satisfying the conditions (2) and (3) of quasipositive knot diagrams (see chapter 2). The heart of such a quasipositive representative for a C_4-move is shown in Figure 7, together with its Seifert diagram. Here the 12 negative crossings can be paired with positive ones along vertical lines. In addition, we have already seen that it is easy to permute two strands inside the local box without losing quasipositivity.

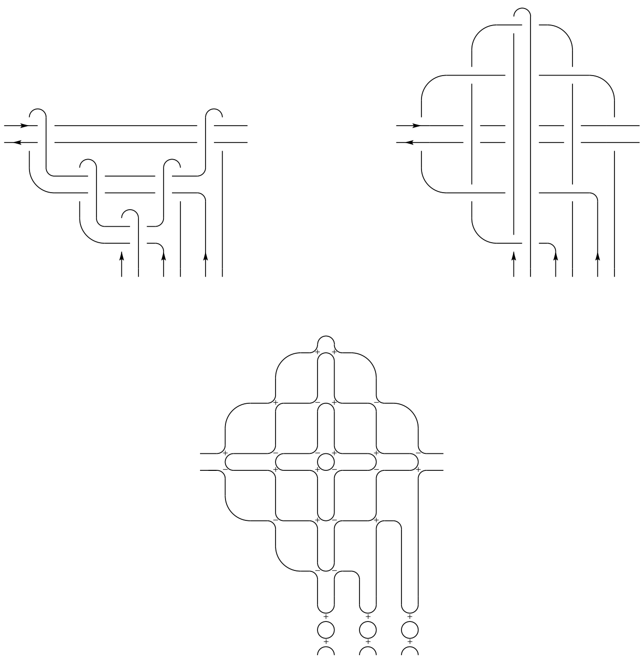


FIGURE 7

At last, the quasipositive knot $Q := Q_{n-1}$ has the required properties. □

REMARKS.

(1) All the quasipositive knots Q_i can be unknotted by a single crossing change at the clasp that appears on the left side of their defining diagram (see e.g. Figure 3). In particular, the unknotting number of Q is one, unless Q happens to be the trivial knot.
(2) By Theorem 9 and Habiro's result, we conclude that every knot can be transformed into a quasipositive knot by a finite sequence of C_n-moves, for any fixed natural number n. It would be interesting to have a direct proof for this fact, which in turn implies Theorem 9. This would possibly simplify the construction of the desired quasipositive knots.
(3) The knot Q might even be chosen to be strongly quasipositive. However, we do not know how to prove that.

CHAPTER 5

Vassiliev Invariants and the Gordian Complex of Knots

1. The Gordian Complex of Knots

In this chapter, we will focus on a certain neighbourhood relation on the set of knots induced by the crossing change operation. The Gordian complex of knots is a simplicial complex whose vertex set consists of all the isotopy classes of smooth oriented knots in S^3. An edge of the Gordian complex is a pair of knots of Gordian distance one, i.e. a pair of knots which differ by one crossing change (see section 2, chapter 1). Similarly, an n-simplex is a set of $(n+1)$ knots whose pairwise Gordian distance is one. Recently, M. Hirasawa and Y. Uchida proved that every edge of the Gordian complex is contained in a simplex of infinite dimension (see [**19**]) and asked whether every n-simplex is contained in a simplex of infinite dimension, for arbitrary $n \in \mathbb{N}$. The next theorem gives a positive answer to their question.

THEOREM 10. *Every n-simplex of the Gordian complex of knots is contained in an $(n+1)$-simplex.*

Furthermore, we show that if two knots are not connected by an edge, then there are infinitely many shortest paths between these two knots in the 1-skeleton of the Gordian complex of knots.

THEOREM 11 ([**7**]). *For any pair of knots K and $\widetilde{K}$ of Gordian distance two there exist infinitely many non-equivalent knots whose Gordian distance to K and $\widetilde{K}$ is one.*

Here the Gordian distance between two knots is the minimal number of crossing changes needed to transform one knot into the other (see [**19**] and [**30**]). Theorem 11 has an interesting special case. We shall state it as a corollary.

COROLLARY 3. *A knot of unknotting number two can be unknotted via infinitely many different knots of unknotting number one.*

Once again, the proof of Theorem 10 involves C_n-moves and Vassiliev invariants; it is written in the second section. For technical reasons, we have to assume that all knots are endowed with an orientation.

Nevertheless, Theorems 10 and 11 are true both for oriented knots and non-oriented knots. The third section contains the proof of Theorem 11 and can be read independently. We conclude this section with a remark on the set-up of Theorem 10.

Let Δ be an n-simplex of the Gordian complex of knots ($n \in \mathbb{N}$, $n \geqslant 1$). The vertices K_0, K_1,..., K_n of Δ have pairwise Gordian distance one. In particular, K_0 and K_i are connected by a strand passage operation along an embedded disk D_i, $1 \leqslant i \leqslant n$. We observe that we may move a disk D_i by an isotopy $\{D_i(t)\}_{t\in[0,1]}$ that preserves the following two conditions:

(1) The boundary of the disk $\partial D_i(t)$ meets K_0 in an interval, for all $t \in [0, 1]$.

(2) The interior of the disk $int(D_i(t))$ meets K_0 transversaly in one point, for all $t \in [0, 1]$.

Such an isotopy of strand passage disks does not affect the knot type of K_i. More precisely, the isotopy $\{D_i(t)\}_{t\in[0,1]}$ of disks defines an isotopy of corresponding knots $\{K_i(t)\}_{t\in[0,1]}$. In this way we can arrange that all the disks D_i, $1 \leqslant i \leqslant n$ are pairwise disjoint. Thus K_0 has a diagram with a section as shown in Figure 1, with n distinguished spots A_1, A_2,..., A_n, where crossing changes should take place to obtain K_1, K_2,..., K_n, respectively. This section has $2n$ strands labelled s_1, s_2,..., s_{2n}. Going along the knot K_0 according to its orientation, starting at s_1, we encounter the strands s_i ($2 \leqslant i \leqslant 2n$) in a certain order which depends on how the strands are connected outside the indicated section. This order defines a permutation, say $\sigma \in S_{2n-1}$, of the numbers 2, 3,..., 2n. We claim that we can easily change the order of two succeeding strands. Indeed, let s_1, s_2 be two succeeding strands, as shown in Figure 2. By applying regular isotopy only, we can transform the indicated section into the section of Figure 3. We observe that the order of the strands s_1, s_2 is reversed, now. Moreover, a crossing change at the clasps involving s_1 or s_2 yields the same knots as before.

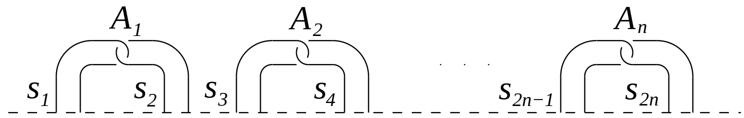


FIGURE 1

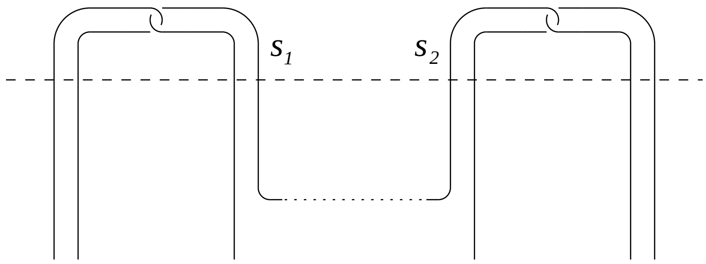


FIGURE 2

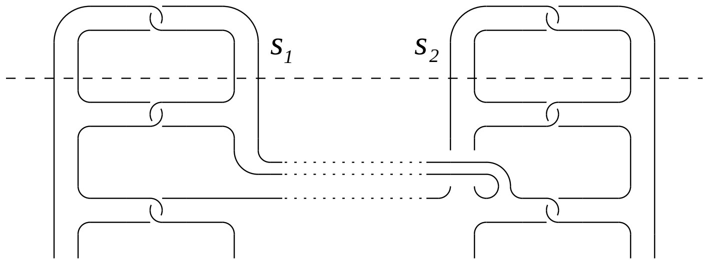


FIGURE 3

Hence we may assume that we encounter the strands s_i of Figure 1 in an order corresponding to a fixed prescribed permutation $\sigma \in S_{2n-1}$. This will be important in the proof of Theorem 10.

2. Constructing simplices by C_n-moves

Given an n-simplex Δ of the Gordian complex of knots, we shall construct a knot that spans an $(n+1)$-simplex, together with Δ.

Let K_0, K_1,..., K_n be the vertices of Δ. K_0 has a diagram with a section as shown in Figure 1, such that a crossing change at A_i transforms K_0 into K_i, $1 \leqslant i \leqslant n$. Using the first Reidemeister move, we may introduce an extra band with a clasp, such that a crossing change at that clasp does not change the knot type. So K_0 has a diagram with a section as shown in Figure 4, where A_{n+1} is the additional clasp.

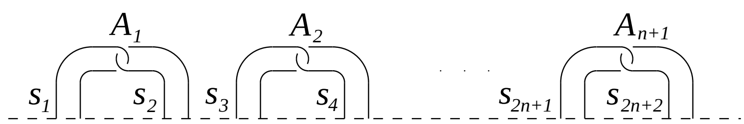


FIGURE 4

Now we are ready to define a knot K_0'. Weave the n bands with clasps on the left of Figure 4 around the rightmost band, using the weaving pattern of the C_{n+1}-move shown in Figure 1, chapter 4, to obtain a knot K_0'. We observe that K_0' is a neighbour (in the Gordian complex) of all the knots K_0, K_1,..., K_n, since the weaving pattern of the C_{n+1}-move shown in Figure 1, chapter 4, is part of a Brunnian link (see [**40**]). In order to establish Theorem 10, we have to prove that K_0' is different from K_0, K_1,..., K_n. For this purpose, we first investigate how K_0' and K_0 are related. K_0' and K_0 differ by an n-times iterated double of a C_{n+1}-move, which is a C_{2n+1}-move (see [**36**]). By the result of Y. Ohyama and T. Tsukamoto, we know that $|v_{2n+1}(K_0') - v_{2n+1}(K_0)|$ depends only on the permutation $\sigma \in S_{2n-1}$ defined by the cyclic order of the $(2n+2)$ strands of the section of Figure 4. Further, there always exists an order of the $(2n+2)$ strands, such that this difference is not zero (see e.g. [**35**], where Y. Ohyama constructs an infinite family of knots by applying C_n-moves which change some Vassiliev invariants of order n). According to the discussion at the end of the first section, we may assume that we have already arranged such an order in the diagram of Figure 4. Thus K_0' and K_0 are non-equivalent. However, K_0' could still coincide with one of the knots K_i, $1 \leqslant i \leqslant n$. Suppose K_0' coincides with K_1. In the above construction of K_0', we have inserted an extra band with a seemingly useless clasp, such that a crossing change at that clasp did not change the knot type. We may as well insert two such extra bands with clasps. Then we may apply a similar construction as above, using a C_{2n+3}-move instead of a C_{2n+1}-move to obtain a knot K_0''. As before, K_0'' and K_0 are distinguished by a Vassiliev invariant of order $(2n+3)$. Moreover, K_0'' and K_1 are distinguished by a Vassiliev invariant of order $(2n+1)$. Indeed, K_0'' and K_0 have the same Vassiliev invariants up to order $(2n+2)$, since they differ by one C_{2n+3}-move, whereas K_0 and $K_0' = K_1$ are distinguished by a Vassiliev invariant of order $(2n+1)$. If K_0'' happens to coincide with one of the knots K_2, K_3,..., K_n, we may insert one more band with a clasp to the diagram of K_0 and apply a C_{2n+5}-move to obtain a knot K_0'''. Repeating this process $(n+1)$-times, at most, we end up with a knot whose Gordian distance to K_i is one, $i = 0, 1, \ldots, n$.

3. An infinite family of shortest paths

In this section we show that there are infinitely many knots between two knots of Gordian distance two, as stated in Theorem 11. Although we could use an analogous construction as in the second section, we prefer another construction which is much simpler.

The two knots K and $\widetilde{K}$ having Gordian distance two, there exists a knot K_0 which differs from K and $\widetilde{K}$ by one crossing change. K_0 has a diagram with a section as shown in Figure 1, for $n = 2$, such that a crossing change at A_1 (or A_2) transforms K_0 into K (or $\widetilde{K}$, respectively). Since we can slide one of the bands along K_0, we may assume that K_0 has a diagram with a section as shown in Figure 5, such that a crossing change at A (or B) transforms K_0 into K (or $\widetilde{K}$, respectively).

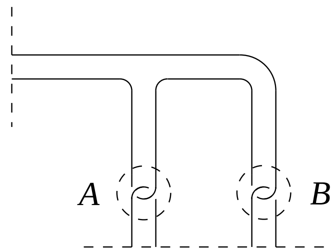


FIGURE 5

Again, we are ready to define a family of knots $\{K_n\}_{n\in\mathbb{N}}$ diagrammatically. A diagram of the knot K_n is shown in Figure 6. It is understood that it coincides with the diagram of Figure 5 outside the indicated section.

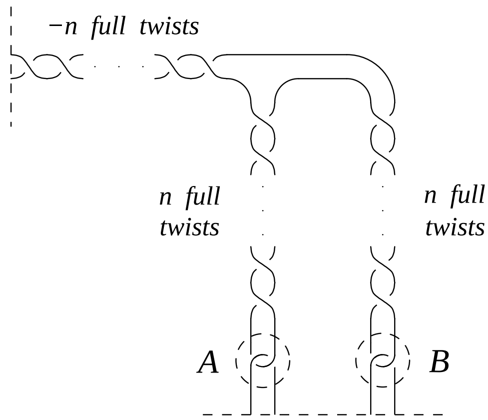


FIGURE 6. A family of knots $\{K_n\}_{n\in\mathbb{N}}$

The horizontal band is twisted $-n$ times, in a manner that gives rise to $2n$ negative crossings. The two vertical bands are both twisted n times. We observe that all the knots K_n have the two required properties: changing a crossing at A transforms K_n into K. Similarly, a crossing change at B transforms K_n into $\widetilde{K}$.

However, we have to prove that the family $\{K_n\}_{n\in\mathbb{N}}$ contains infinitely many different knots. For this purpose we consider the Alexander polynomial, written in Conway's notation (see [**22**]). This polynomial in one variable z is normalized to one for the trivial knot and satisfies the following relation:

$$P_{\times}(z) - P_{\times}(z) = zP_{)(}(z)\,.$$

We remark that the Conway polynomial corresponds to the evaluation $P_L(1, z)$ of the HOMFLY polynomial. As we shall apply this skein relation several times, it is convenient to introduce a concise notation for the knots arising from crossing changes and smoothings at different spots.

NOTATION. For x, y, z $\in \mathbb{Z} \cup \{\infty\}$, $K_{x\,y\,z}$ denotes the knot with x full twists in the horizontal band at the top left of the section shown in Figures 5 & 6, y full twists in the left vertical band and z full twists in the right vertical band. The special case where x (y or z) is ∞ means that we have to smooth the diagram (in an oriented manner) at the corresponding place. As an example, $K_{\infty\,0\,\infty}$ is shown in Figure 7.

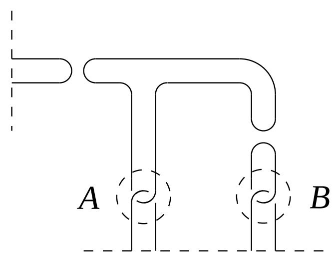


FIGURE 7. $K_{\infty\,0\,\infty}$

In particular, $K_{-n\,n\,n}$ denotes K_n ($n \in \mathbb{N}$).

LEMMA 1.

$$\lim_{n\to\infty} \frac{1}{n^2} P(K_n\,; z) = -z^2 P(K_{\infty\,0\,\infty}\,; z)$$

Before we prove Lemma 1, let us complete the proof of Theorem 11. Suppose the family $\{K_n\}_{n\in\mathbb{N}}$ contained finitely many different knots, only, then the limit of Lemma 1 would clearly be zero. However, this is not the case, since $P(K_{\infty\,0\,\infty}\,; z)$ is not zero. Indeed, $K_{\infty\,0\,\infty}$ is a connected sum of a knot and two Hopf links. Therefore, its Conway polynomial $P(K_{\infty\,0\,\infty}; z)$ is a product of a Conway polynomial of a knot and $\pm z^2$, which can certainly not be zero.

PROOF OF LEMMA 1. Applying the skein relation of the Conway polynomial n times at the crossings of the left vertical band of $K_{-n\,n\,n}$, we get

$$P(K_{-n\,n\,n}\,;z) = P(K_{-n\,0\,n}\,;z) + nzP(K_{0\,\infty\,0}\,;z).$$

Continuing at the crossings of the right vertical band of $P(K_{-n\,0\,n}\,;z)$, we get

$$P(K_{-n\,n\,n}\,;z) = P(K_{-n\,0\,0}\,;z) + nzP(K_{-n\,0\,\infty}\,;z) + nzP(K_{0\,\infty\,0}\,;z).$$

Finally, we apply the skein relation at the crossings of the horizontal bands of $K_{-n\,0\,0}$ and $K_{-n\,0\,\infty}$ to obtain

$$\begin{aligned} P(K_{-n\,n\,n}\,;z) = {} & P(K_{0\,0\,0}\,;z) - nzP(K_{\infty\,0\,0}\,;z) \\ & + nz(P(K_{0\,0\,\infty}\,;z) - nzP(K_{\infty\,0\,\infty}\,;z)) \\ & + nzP(K_{0\,\infty\,0}\,;z). \end{aligned}$$

□

CHAPTER 6

On Minimal Diagrams of certain Fibered Knots

1. Hopf Plumbing and Minimal Diagrams

One of the most striking applications of the Jones polynomial concerns crossing numbers of knots. Indeed, several proofs of the first Tait conjecture were announced shortly after the discovery of the Jones polynomial (see [**23**], [**31**] and [**52**]). Yet there is another bound for the crossing number of knots coming from the HOMFLY polynomial (see [**32**] and [**16**]). In this chapter we show that the latter estimate works especially well for fibered knots. More precisely, we exhibit a large class of fibered arborescent knots and find minimal diagrams for them. This class contains 50 knots of Rolfsen's table ([**40**]) and is not contained in the class of Montesinos links, for which minimal diagrams are known, already (see [**28**]).

Let Γ be a planar tree with signed vertices, as shown in Figure 1.

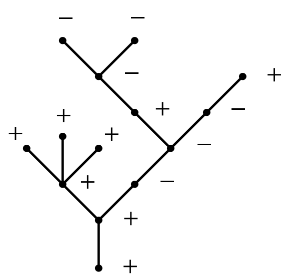


FIGURE 1. A planar tree with signs

There is a well-known procedure, called Hopf plumbing, which associates a link $K(\Gamma)$ to a planar tree with signs Γ:

(1) draw a positive or negative Hopf band (see Figure 2) at each vertex of Γ, according to its sign.
(2) plumb the Hopf bands together along the edges of Γ, as shown in Figure 3.

We shall determine the minimal crossing number and draw minimal diagrams of knots associated with trees with signs. For this purpose we have to introduce two quantities of trees with signs Γ. Deleting all

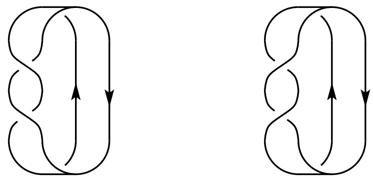


FIGURE 2. A positive and a negative Hopf band

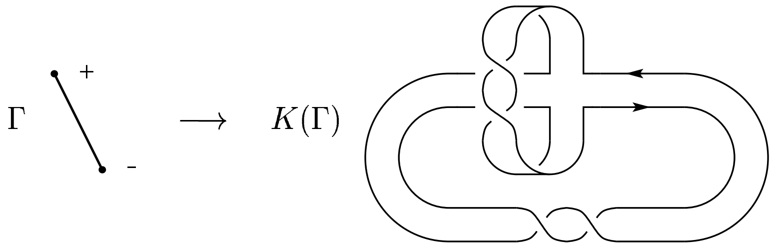


FIGURE 3. Plumbing along an edge

negative vertices of Γ we get a forest Γ_+, i.e. a union of trees (with positive signs). An analogous procedure (deleting all positive vertices of Γ) yields Γ_-. We set

$$P(\Gamma) = \sum_{T \in \Gamma_+} s(T),$$

$$N(\Gamma) = \sum_{T \in \Gamma_-} s(T),$$

where the sums run over all trees T of the forests Γ_+ or Γ_-, respectively, and the function s is defined as follows:

$$s(T) = 1 + \min\{\#S | S \subset E(T),\ T - S \text{ has no vertices of valency greater than two}\}$$

Here $E(T)$ is the set of edges of the tree T. We remark that s depends on the abstract structure of a tree only, whereas P and N depend on the abstract structure and the signs of a tree. Now we are ready to state our main result about the minimal crossing number $c(K(\Gamma))$.

THEOREM 12 ([**6**]). *Let Γ be a tree with signs, such that $K(\Gamma)$ is a knot with one component. Then*

$$c(K(\Gamma)) = V(\Gamma) + P(\Gamma) + N(\Gamma).$$

Here $V(\Gamma)$ is the number of vertices of Γ.

REMARK. The class of knots associated with trees with signs contains the class of slalom divide knots (see N. A'Campo [**3**] for a definition of slalom divide knots). Indeed, according to M. Hirasawa ([**18**], see also M. Ishikawa [**20**]), we know that slalom divide knots correspond to knots of certain slalom trees with positive signs only. Thus Theorem 12 proves a conjecture of M. Ishikawa ([**21**]) about the minimal crossing number of slalom divide knots.

It is well-known that every knot can be represented as the closure of a braid. The minimal number of strands among all the braids representing K is called the braid index $b(K)$ of the knot K.

THEOREM 13 ([**6**]). *Let* Γ *be a tree with signs, such that* $K(\Gamma)$ *is a knot with one component. Then*

$$b(K(\Gamma)) = P(\Gamma) + N(\Gamma) + 1.$$

In the following section we prove a recursive formula for the function s for trees, which we shall use in the proof of Theorem 12. In the third section we present a lower bound for the minimal crossing number of knots and prove the inequality

$$c(K(\Gamma)) \geqslant V(\Gamma) + P(\Gamma) + N(\Gamma).$$

The proofs of Theorems 12 and 13 will be accomplished in the fourth section.

2. The Function s for Trees

Let T be a tree, $E(T)$ its set of edges. We call $S \subset E(T)$ a *grinding subset* for T, if $T - S$ has no vertex of valency greater than two. Thus the function $s(T)$ equals one plus the minimal cardinality among all grinding subsets for T.

Let k be an outer edge of T. Figure 4 shows a section of T near k.

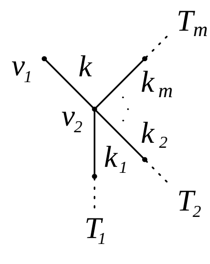


FIGURE 4. A section of T

Two vertices of T are incident with k; they are labelled v_1 and v_2. The trees of the forest $T - \{v_1, v_2\}$ are labelled T_1, T_2, ..., T_m. They are attached to the vertex v_2 by the edges k_1, k_2, ..., k_m.

PROPOSITION 1.

$$s(T) = \max\{s(T - v_1),\ \sum_{i=1}^{m} s(T_i)\}$$

PROOF. First we observe that

$$s(T - v_1) \leqslant s(T) \leqslant 1 + s(T - v_1).$$

Further, a careful inspection of the definition of s shows that

$$\sum_{i=1}^{m} s(T_i) \leqslant s(T) \leqslant 1 + \sum_{i=1}^{m} s(T_i).$$

Indeed, if S is a grinding subset for T, then $S - \{k, k_1, k_2, \ldots k_m\}$ is a grinding subset for the forest $T - \{v_1, v_2\}$. This implies the first inequality, since $\#S \cap \{k, k_1, k_2, \ldots k_m\} \geqslant m - 1$. Conversely, if S is a grinding subset for the forest $T - \{v_1, v_2\}$, then $S \cup \{k_1, k_2, \ldots k_m\}$ is a grinding subset for T. This implies the second inequality.

In order to prove Proposition 1, we have to exclude the case

$$s(T) = 1 + s(T - v_1) = 1 + \sum_{i=1}^{m} s(T_i).$$

Suppose $s(T) = 1 + s(T - v_1)$. Let S be a minimal grinding subset for $T - v_1$, i.e. a grinding subset for $T - v_1$ of minimal cardinality. Since $s(T) = 1 + s(T - v_1)$, the vertex v_2 must have valency two in $T - v_1 - S$; say k_1, $k_2 \in E(T - v_1 - S)$. We claim that $S \cap E(T_i)$ is a minimal grinding subset for T_i, $1 \leqslant i \leqslant m$. If $i \neq 1, 2$, this is obvious, since then $k_i \notin E(T - v_1 - S)$. The case $i = 1$ (and, analogously, the case $i = 2$) needs a special consideration. Suppose T_1 had a grinding subset $S_1 \subset E(T_1)$ of smaller cardinality than $S \cap E(T_1)$. Then the subset

$$\widetilde{S} = S - (S \cap E(T_1)) \cup S_1 \cup \{k_1\} \subset E(T - v_1)$$

were a minimal grinding subset for $T - v_1$. Furthermore, the vertex v_2 would have valency one in $T - v_1 - \widetilde{S}$, which is a contradiction to the assumption $s(T) = 1 + s(T - v_1)$. At last, $S \cup \{k\}$ is a minimal grinding subset for T with

$$\#S \cup \{k\} = m - 1 + \sum_{i=1}^{m} \#S \cap E(T_i) = m - 1 + \sum_{i=1}^{m} (s(T_i) - 1).$$

Hence

$$s(T) = \sum_{i=1}^{m} s(T_i).$$

□

3. The HOMFLY Polynomial and Crossing Numbers

We have already seen that the HOMFLY polynomial $P_K(a,z)$ provides a tool for detecting quasipositive knots, via Morton's inequality (see section 2, chapter 2 and appendix A). In this chapter we shall present another application of Morton's inequality, concerning minimal crossing numbers of knots. As usual, we denote the minimal degree in the variable a of $P_K(a,z)$ by $e(K)$, the maximal degree by $E(K)$. The following Theorem is mainly due to K. Murasugi ([**32**]); an explicit formulation can be found in H. Gruber (see Lemma 3.2. and Corollary 6.1. in [**16**]).

THEOREM 14.
(i) $c(K) \geqslant \frac{1}{2}(E(K)-e(K))+2g(K)$
(ii) *If* $c(K) = \frac{1}{2}(E(K)-e(K))+2g(K)$*, then*

$$b(K) = \frac{1}{2}(E(K)-e(K))+1.$$

Here $g(K)$ is the minimal genus among all orientable surfaces spanning the knot K.

PROOF OF THEOREM 14. Let D be a minimal diagram of the knot K. There is a natural Seifert surface $S(D)$ associated with D. Its Euler characteristic $\chi(S(D))$ equals $s(D)-c(D)$, where $s(D)$ and $c(D)$ denote the number of Seifert circles and the number of crossings of the diagram D, respectively. We conclude

$$c(K) = c(D) = s(D) - \chi(S(D)) \geqslant b(K) + 2g(K) - 1.$$

The last inequality follows from a Theorem of S. Yamada ([**55**]) which asserts that the number of Seifert circles in any diagram of a knot K cannot be smaller than the braid index of K. An alternative simple proof of this fact is due to P. Vogel ([**53**]). Now both statements of Theorem 14 follow immediately by Morton's inequality (see [**29**]; a thorough discussion on braid index criteria can be found in A. Stoimenow [**49**]):

$$b(K) \geqslant \frac{1}{2}(E(K)-e(K))+1.$$

□

We observe that the plumbing construction of Hopf bands along a tree provides a natural fiber surface of minimal genus (see [**14**]). Therefore

$$2g(K(\Gamma)) = V(\Gamma),$$

in case $K(\Gamma)$ is a knot. Comparing the first statement of Theorem 14 and Theorem 12, we see that it remains to settle the equation

$$E(K(\Gamma)) - e(K(\Gamma)) = 2P(\Gamma) + 2N(\Gamma)$$

to establish the desired lower bound for the crossing number of Theorem 12. It turns out that the range in the variable a of $P_K(a, z)$ is not affected by the specialization $z = 1$. Therefore we restrict our computations to the polynomial in one variable

$$O_K(a) = P_K(a, 1).$$

Further we redefine $e(K)$ and $E(K)$ as the minimal and maximal degree of $O_K(a)$, respectively. In order to compute $e(K)$ and $E(K)$, we have to keep book on the signs of the extremal coefficients very carefully. Thus let us write $\sigma_e(K)$ and $\sigma_E(K)$ for the signs of the minimal and maximal coefficients of $O_K(a)$, respectively.

LEMMA 2. *Let Γ be a tree with signs (here $K(\Gamma)$ may have several components).*

(i) $e(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) - 2N(\Gamma)$
(ii) $E(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) + 2P(\Gamma)$
(iii) $\sigma_e(K(\Gamma)) = (-1)^{V(\Gamma_-)+N(\Gamma)}$
(iv) $O_{K(\Gamma)}(a)$ *is either even or odd and alternating, i.e. there are natural numbers $c_k \neq 0$, $e \leqslant k \leqslant E$, such that*

$$O_{K(\Gamma)}(a) = (-1)^{\sigma_e} \sum_{l=0}^{\frac{1}{2}(E-e)} (-1)^l c_{e+2l}\, a^{e+2l}.$$

In particular, $\sigma_E(K(\Gamma)) = (-1)^{V(\Gamma_-)+P(\Gamma)}$.

PROOF OF LEMMA 2. First we observe that Lemma 2 is true if Γ has one vertex only. Indeed,

$$O_{K(\cdot\ +)} = 2a - a^3,$$

$$O_{K(\cdot\ -)} = a^{-3} - 2a^{-1}.$$

Now let us assume that all the statements of Lemma 2 are true for all trees Γ with $n(\geqslant 1)$ vertices, at most. We have to verify the statements for a tree Γ with $n+1(\geqslant 2)$ vertices. Let k be an outer edge of Γ, as shown in Figure 4. As before, two vertices (v_1 and v_2) are incident with k. In addition, each vertex carries a sign. Now let us remember the defining relation for $O_K(a)$:

$$\frac{1}{a}O_{\times}(a) - aO_{\times}(a) = O_{)(}(a). \tag{3}$$

We shall apply this relation to a crossing of the Hopf band at the vertex v_1. For this purpose, we have to understand the effect of the following two operations:

(1) smoothing a crossing at v_1 (in an oriented manner) causes a small collapse; the Hopf band at v_1 disappears. At this point, it may be helpful to look at Figure 3.

$$K(\Gamma) \longrightarrow K(\Gamma - \{v_1\})$$

(2) a crossing change at v_1 causes a big collapse; the Hopf bands at v_1 and v_2 disappear, and we end up with a connected sum of all the arborescent links corresponding to the trees of the forest $\Gamma - \{v_1, v_2\}$. We notice that these trees T_1, T_2, ... , T_m are trees with signs.

$$K(\Gamma) \longrightarrow K(T_1)\# \ldots \#K(T_m),\ T_i \in \Gamma - \{v_1, v_2\}$$

After these preparations, we are in a position to carry out the inductive step. However, we have to consider four cases corresponding to the signs of the vertices v_1 and v_2, separately.

Case 1. v_1 and v_2 carry negative signs. Relation (3) for $K(\Gamma)$ at v_1 reads

$$\frac{1}{a} O_{K(T_1)}(a) \cdots O_{K(T_m)}(a) - a O_{K(\Gamma)}(a) = O_{K(\Gamma - \{v_1\})}(a),$$

since the HOMFLY polynomial is multiplicative under the connected sum operation. Thus

$$O_{K(\Gamma)}(a) = \frac{1}{a^2} O_{K(T_1)}(a) \cdots O_{K(T_m)}(a) - \frac{1}{a} O_{K(\Gamma - \{v_1\})}(a). \qquad (4)$$

We remark that

$$P(\Gamma) = P(\Gamma - \{v_1\}) = \sum_{i=1}^{m} P(T_i).$$

Therefore, the maximal degrees and the signs of the maximal coefficients of $\frac{1}{a^2} O_{K(T_1)}(a) \cdots O_{K(T_m)}(a)$ and $-\frac{1}{a} O_{K(\Gamma - \{v_1\})}(a)$ agree, and we conclude that $O_{K(\Gamma)}(a)$ is either even or odd and alternating. Furthermore,

$$E(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) + 2P(\Gamma),$$
$$\sigma_E(K(\Gamma)) = (-1)^{V(\Gamma_-) + P(\Gamma)}.$$

As to the minimal degree, we need the statement of Proposition 1 in the second section:

$$N(\Gamma) = \max\{N(\Gamma - \{v_1\}), \sum_{i=1}^{m} N(T_i)\}.$$

In any case, we conclude

$$e(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) - 2N(\Gamma).$$

Case 2. v_1 carries a negative sign, v_2 carries a positive sign. As in the first case, equation (4) holds for $O_{K(\Gamma)}(a)$. The crucial quantities are

$$P(\Gamma) = P(\Gamma - \{v_1\}),$$

$$|P(\Gamma) - \sum_{i=1}^{m} P(T_i)| \leqslant 1,$$

$$N(\Gamma) = 1 + N(\Gamma - \{v_1,\}) = 1 + \sum_{i=1}^{m} N(T_i).$$

Thus $\frac{1}{a^2}O_{K(T_1)}(a)\cdots O_{K(T_m)}(a)$ contributes to the required minimal degree of $O_{K(\Gamma)}(a)$ with the correct sign, whereas $-\frac{1}{a}O_{K(\Gamma-\{v_1\})}(a)$ contributes to the required maximal degree of $O_{K(\Gamma)}(a)$ with the correct sign. In particular, $O_{K(\Gamma)}(a)$ is either even or odd and alternating.

The remaining two cases (i.e. v_1 carries a positive sign) can be treated analogously; no new phenomena occur. Alternatively, we may consider the mirror image of $K(\Gamma)$ and replace the variable a of $O_{K(\Gamma)}(a)$ by $-a^{-1}$.

□

As an immediate consequence of Lemma 2, the width of $O_{K(\Gamma)}(a)$ equals $2P(\Gamma)+2N(\Gamma)$. However, the width of $P_{K(\Gamma)}(a,z)$ in the variable a could still be greater than $2P(\Gamma)+2N(\Gamma)$. In any case, due to the first statement of Theorem 14, we get the required lower bound for the number of crossings:

$$c(K(\Gamma)) \geqslant V(\Gamma) + P(\Gamma) + N(\Gamma),$$

in case $K(\Gamma)$ is a knot.

4. Construction of Minimal Diagrams

In this section, we construct minimal diagrams for knots associated with trees with signs and accomplish the proofs of Theorems 12 and 13. First we remark that a knot $K(\Gamma)$ has a natural diagram with $2V(\Gamma)$ crossings. This fact is illustrated in Figure 5, for the knot $K($$)$.

If the signs of the vertices of Γ are distributed in an alternating pattern, then $P(\Gamma)+N(\Gamma) = V(\Gamma)$, and Theorem 12 is true. This is no surprise, since in this case, the natural diagram of $K(\Gamma)$ is alternating. Now suppose Γ contains an edge k with two positive vertices. Then we can change the natural diagram of $K(\Gamma)$ in order to eliminate one

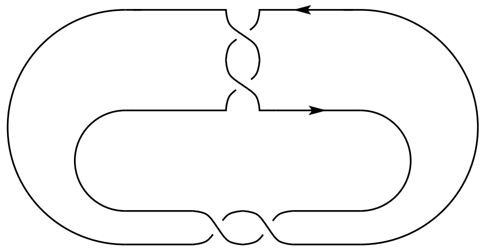


FIGURE 5. A natural diagram of $K(\Gamma)$

crossing, as shown in Figure 6. The two squares contain the parts of the diagram of $K(\Gamma)$ corresponding to $\Gamma - k$; we may possibly have to flype a square through the corresponding twisted band before we can apply such a reduction move.

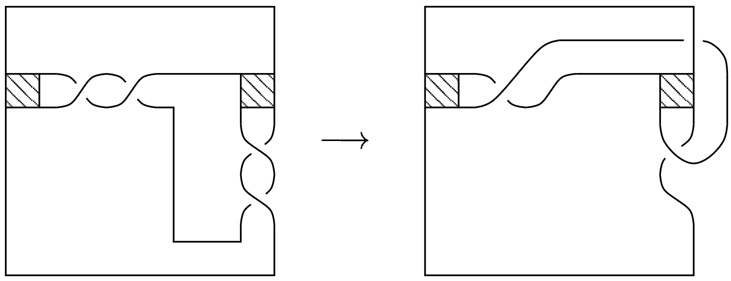


FIGURE 6. Eliminating a crossing at a positive edge

However, the Hopf bands of the two involved vertices are damaged by this process; they both 'lose one crossing'. Hence we cannot apply this procedure to more than two edges containing the same vertex. If T is a subtree of Γ_+, then we can choose $V(T) - P(T)$ edges of T, such that their union does not contain any vertex of valency greater than 2 (by definition of the function P). Thus we can eliminate $V(T) - P(T)$ of the crossings associated with T, even if T is a single point (since then $V(T) - P(T) = 1 - 1 = 0$). The same is true for subtrees of Γ_-, of course, replacing $P(T)$ by $N(T)$. Altogether, we can eliminate $V(\Gamma) - P(\Gamma) - N(\Gamma)$ crossings of the natural diagram of $K(\Gamma)$, ending up with

$$2V(\Gamma) - (V(\Gamma) - P(\Gamma) - N(\Gamma)) = V(\Gamma) + P(\Gamma) + N(\Gamma)$$

crossings. As a consequence, Theorem 12 is true. At last, Theorem 13 is a consequence of the second statement of Theorem 14.

CHAPTER 7

Problems

PROBLEM 1. Does there exist a quasipositive knot which is not a track knot?

The free divide knots 9_{16}, 10_{124}, 10_{152} and 10_{154} might be good candidates. However, we do not know any criterion for distinguishing track knots from quasipositive knots.

PROBLEM 2. Is it true that a knot is strongly quasipositive if and only if it is quasipositive and its 4-genus equals its ordinary genus? In particular, is it true that special track knots are strongly quasipositive?

PROBLEM 3. Do alternating quasipositive knots have positive diagrams? More generally, is it true that a knot is positive if and only if it is homogeneous and quasipositive.

Up to ten crossings, the 'statements' of Problems 2 and 3 are true, as we can see from the classification of quasipositive knots in Appendix A.

PROBLEM 4. Can every knot be transformed into a strongly quasipositive knot by a finite sequence of C_n-moves, for any fixed natural number n? Equivalently, can any finite number of Vassiliev invariants of a knot be realized by a strongly quasipositive knot?

The results of chapter 6 say nothing about the topology of the Gordian complex of knots. This awaits exploration. We continue with some questions and problems related to the Gordian complex of knots.

PROBLEM 5. Does the Gordian complex of knots have non-trivial homology groups, or fundamental group?

PROBLEM 6. Given a knot of unknotting number two, is it possible to unknot it via a knot of arbitrary high bridge number?

This question was posed by Y. Uchida. We found that the answer is affirmative for the torus knot 5_1, but we cannot prove a general result.

Y. Nakanishi suggested restricting ourselves to the class of fibered knots.

PROBLEM 7. Is it possible to unknot a fibered knot of unknotting number two via infinitely many different fibered knots of unknotting number one?

We may ask the same question for other classes of knots, e.g. alternating knots or quasipositive knots.

PROBLEM 8. Find similar results for the Δ-unknotting operation or the #-unknotting operation (For a report on unknotting operations, see Y. Nakanishi [**34**]).

PROBLEM 9. Determine the unknotting number of the knot $3_1\#!5_1$. Here '#' stands for the connected sum operation and '!' denotes the mirror image operation.

The knot $3_1\#!5_1$ can be transformed into several ten crossing knots by one crossing change: 10_{48}, 10_{125}, 10_{126}, 10_{148}, 10_{153}. The unknotting number of the latter knots is conjectured to be two (see [**26**]). Apparently, there is no simple proof for that fact, whence it is interesting to prove that the unknotting number of the knot $3_1\#!5_1$ is three.

PROBLEM 10. More generally, prove the additivity of the unknotting number under connected sum of knots.

APPENDIX A

Table of Quasipositive Knots

This appendix contains a complete list of quasipositive and strongly quasipositive knots up to ten crossings. As we mentioned in the first chapter, this classification relies on the inequality of Theorem 3:

$$2g^*(K) \leqslant e(K),$$

for any quasipositive knot K.

Looking at Kawauchi's table of knots [**26**] (see [**27**] for an updated version), we observe that 60 of 249 prime knots up to 10 crossings satisfy this inequality. Among these 60 knots, 42 have positive diagrams, whence they are strongly quasipositive:

3_1, 5_1, 5_2, 7_1, 7_2, 7_3, 7_4, 7_5, 8_{15}, 8_{19}, 9_1, 9_2, 9_3, 9_4, 9_5, 9_6, 9_7, 9_9, 9_{10}, 9_{13}, 9_{16}, 9_{18}, 9_{23}, 9_{35}, 9_{38}, 9_{49}, 10_{49}, 10_{53}, 10_{55}, 10_{63}, 10_{66}, 10_{80}, 10_{101}, 10_{120}, 10_{124}, 10_{128}, 10_{134}, 10_{139}, 10_{142}, 10_{152}, 10_{154}, 10_{161}.

REMARKS.

(i) Since knots are always listed up to mirror image, we must be more precise: 'a knot K satisfies the inequality ...' means 'either K or its mirror image $!K$ satisfies the inequality ...'.

(ii) The 4-genus of the knot 10_{51} is not known. However, it is known not to be slice, hence the inequality $2g^*(10_{51}) \leqslant e(10_{51}) = 0$ is not satisfied.

The remaining 18 knots are listed in Table 1, except for the knot 10_{132}, which is not quasipositive. A. Stoimenow already pointed out that the quasipositivity of the knot 10_{132} would imply the quasipositivity of its untwisted 2-cable link, together with a violation of Morton's inequality, which is a contradiction (see [**50**]). Table 1 contains one strongly quasipositive, non-positive knot: 10_{145}. It is non-positive since it is non-homogeneous (see P. R. Cromwell [**12**]). The other 16 knots are not strongly quasipositive since their 4-genus is smaller than their genus. In particular, they are non-positive. So in Table 1 we list all quasipositive, non-positive prime knots up to 10 crossings in Rolfsen's numbering, together with a quasipositive braid representation, the 4-genus g^* and the ordinary genus g. In the second column a, b, ... and

Knot	quasipositive braid representation	g^*	g
8_{20}	(abAbaBA)(baB)	0	1
8_{21}	(abA)b(Abba)	1	2
9_{45}	a(Bcb)b(bacB)	1	2
9_{46}	(abbcBBA)(bacB)	0	1
10_{126}	aa(aaabAAA)b	1	3
10_{127}	abbb(bAbbaB)	2	3
10_{131}	a(aaBCbdBcbAA)(BcbdcBCb)d(Bcb)	1	2
10_{133}	aab(bDCbcdB)(bCBcACbcdCBcaCbcB)(bCBcaCbcB)	1	2
10_{140}	(abbbcBBBA)b(Cbc)	0	2
10_{143}	a(BBBaaabbb)	1	3
10_{145}	(abA)cd(abA)(bcB)(bcdCB)(cdC)b	2	2
10_{148}	ab(bbacBB)(cbC)	1	3
10_{149}	a(bbCbccBB)a(bcccB)	2	3
10_{155}	(abA)(ABcbCba)(bcB)	0	3
10_{157}	a(Baab)b(baaB)	2	3
10_{159}	a(BBaabb)(baB)	1	3
10_{166}	(abcBA)(acbA)(Bcb)(Aba)	1	2

TABLE 1. Quasipositive, non-positive prime knots up to 10 crossings

A, B, ... stand for σ_1, σ_2, ... and σ_1^{-1}, σ_2^{-1}, ... Parentheses should help to recognize positive bands. The braid of the knot 10_{145} is strongly quasipositive.

This classification of quasipositive and strongly quasipositive knots gives us an interesting criterion for detecting strongly quasipositive knots.

PROPOSITION 2. *A knot with 10 crossings at most is strongly quasipositive if and only if it is quasipositive and its 4-genus equals its ordinary genus.*

APPENDIX B

Examples of Track Knots

In this appendix we present all track knots associated with the labelled immersed interval shown in Figure 1. It has two double points and two specified points.

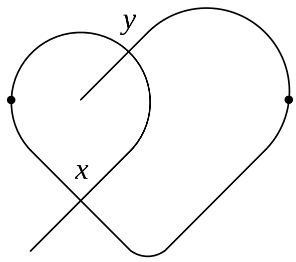


FIGURE 1

There are 11^2 patterns of signs, represented by a symbol (x and y) at each double point (see chapter 1). Knots associated with different patterns of signs need not be different. It is still remarkable that we obtain 24 different prime knots in this way. They are listed in Table 1. The second and third column of Table 1 show the Dowker-Thistlethwaite numbering and the Rolfsen numbering, respectively. The fourth column tells us whether the knot is a free divide knot or not. In [**15**], Gibson and Ishikawa listed knots of free divides. Up to 10 crossings, their list is complete. We add the 4-genus in the fifth column. It equals the clasp number and, except for the knots 9_{46}, 10_{140} and $11n139$, also the unknotting number.

(x, y)	DT numbering	Rolfsen numbering	free divide	g^*
(b, c)	$7a4$	7_2	No	1
(b, c_1)	$5a1$	5_2	Yes	1
(b, d)	$7a5$	7_3	Yes	2
(b_1, b_1)	$9n5$	9_{46}	No	0
(b_1, b_3)	$10n29$	10_{140}	No	0
(b_1, c)	$12n121$	–	No	1
(b_1, c_1)	$3a1$	3_1	Yes	1
(b_1, d)	$10n14$	10_{145}	Yes	2
(b_3, b_3)	$11n139$	–	No	0
(b_3, c_3)	$10n4$	10_{133}	No	1
(c, c_3)	$8a2$	8_{15}	No	2
(c, d)	$10n30$	10_{142}	No	3
(c_1, b_1)	$8n2$	8_{21}	No	1
(c_1, b_3)	$9n2$	9_{45}	No	1
(c_1, c_1)	$5a2$	5_1	Yes	2
(c_1, c_3)	$7a3$	7_5	Yes	2
(c_1, d)	$10n31$	10_{161}	Yes	3
(c_3, b_3)	$10n19$	10_{131}	No	1
(c_3, d)	$10n22$	10_{128}	No	3
(d, b_1)	$11n118$	–	No	2
(d, b_3)	$12n407$	–	No	2
(d, c_1)	$7a7$	7_1	Yes	3
(d, c_3)	$10n6$	10_{134}	No	3
(d, d)	$12n591$	–	?	4

TABLE 1. Knots associated with a special immersed interval

APPENDIX C

Table of Special Fibered Knots

The subsequent table lists all knots associated with trees with signs up to ten crossings. We need only consider eight types of trees, on account of Theorem 1. They are drawn in the first column of Table 9. The second column lists all knots with 10 or fewer crossings that arise from the trees of the first column, depending on different choices of signs. We use Rolfsen's notation ([**40**]).

We may ask ourselves for which knots the inequality

$$c(K) \geqslant \frac{1}{2}(E(K) - e(K)) + 2g(K)$$

is actually an equality. A result of K. Murasugi says that equality holds for alternating fibered knots (see [**32**], Theorem A and Corollary 2). It turns out that equality holds for 86 knots, up to ten crossings, namely the 50 knots of Table 1, and 36 more knots:

8_{16}, 8_{17}, 8_{18}, 9_{29}, 9_{32}, 9_{33}, 9_{34}, 9_{40}, 9_{47}, 10_5, 10_9, 10_{17}, 10_{45}, 10_{69}, 10_{75}, 10_{81}, 10_{82}, 10_{85}, 10_{88}, 10_{89}, 10_{91}, 10_{94}, 10_{96}, 10_{99}, 10_{100}, 10_{104}, 10_{105}, 10_{106}, 10_{107}, 10_{109}, 10_{110}, 10_{112}, 10_{115}, 10_{116}, 10_{118}, 10_{123}.

We remark that all these knots are fibered. However, there exist non-fibered knots whose minimal crossing number equals $\frac{1}{2}(E(K) - e(K)) + 2g(K)$, e.g. the knots $11a263$ and $14n6302$ (here we use the Dowker-Thistlethwaite numbering). These examples were found by A. Stoimenow.

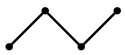
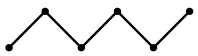
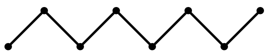
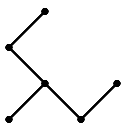
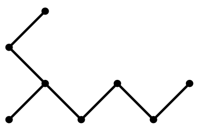
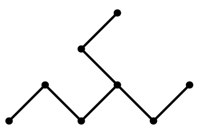
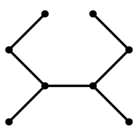
tree	knots
	3_1, 4_1
	5_1, 6_2, 6_3, 7_6, 7_7, 8_{12}
	7_1, 8_2, 8_7, 8_9, 9_{11}, 9_{17}, 9_{20}, 9_{26}, 9_{27}, 9_{31}, 10_{29}, 10_{41}, 10_{42}, 10_{43}, 10_{44}
	9_1, 10_2
	8_5, 8_{10}, 8_{19}, 9_{22}, 9_{24}, 9_{28}, 9_{30}, 9_{36}, 9_{43}, 10_{59}, 10_{60}, 10_{70}, 10_{71}, 10_{73}, 10_{78}, 10_{138}
	10_{46}, 10_{47}, 10_{48}, 10_{124}
	10_{62}, 10_{64}, 10_{139}
	10_{79}, 10_{152}

TABLE 1. Trees and knots up to 10 crossings

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