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An effective "Theorem of André" for CM-points on a plane curve

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Abstract

It is a well known result of Y. André (a basic special case of the André-Oort conjecture) that an irreducible algebraic plane curve containing infinitely many points whose coordinates are CM-invariants is either a horizontal or vertical line, or a modular curve $Y_0(n)$. André's proof was partially ineffective, due to the use of (Siegel's) class-number estimates. Here we observe that his arguments may be modified to yield an effective proof. For example, with the diagonal line $X_1 + X_2 = 1$ or the hyperbola $X_1X_2 = 1$ it may be shown quite quickly that there are no imaginary quadratic τ_1, τ_2 with $j(\tau_1) + j(\tau_2) = 1$ or $j(\tau_1)j(\tau_2) = 1$, where j is the classical modular function. 2010 MSC codes 11G30, 11G15, 11G18.

In the paper [A], André established the most basic nontrivial special case of the André-Oort conjecture, by proving that *if an irreducible complex affine algebraic plane curve is not a horizontal or vertical line, then it contains infinitely many points* (x_1, x_2) *such that both* x_1, x_2 *are singular moduli if and only if it is a modular curve* $Y_0(n)$ *(see for example* [H, p. 207]*), for some* n.

His arguments involved, among other things, Siegel's lower bounds for class-numbers of imaginary quadratic orders, and so led to ineffectiveness; for instance, his theorem shows that there are at most finitely many imaginary quadratic τ_1, τ_2 such that $j(\tau_1) + j(\tau_2) = 1$, but did not allow the determination of all such pairs.

The purpose of this note is just to observe that in fact a modification of André's arguments leads to a completely effective result, and to work out an example. We have the following version of his theorem:

Effective theorem of André. Let X be an irreducible complex affine algebraic plane curve, which is neither a horizontal or vertical line nor a modular curve $Y_0(n)$. Then it contains at most finitely many points (x_1, x_2) such that both x_1, x_2 are singular moduli, and these points can be effectively found in terms of an effective presentation for X.

Example. There are no imaginary quadratic τ_1, τ_2 such that $j(\tau_1)j(\tau_2) = 1$.

While working on the first draft of this note (which included also the example $j(\tau_1) + j(\tau_2) = 1$), the authors learned that Lars Kühne (ETH) had five months earlier independently obtained an effective version of André's Theorem (see [K1,K2]). We feel that the concise exposition here will also be of value. Further he obtained a good explicit dependence on the height of the curve, and he also has some uniform estimates for the number of solutions which are independent of this height. He also handled $j(\tau_1) + j(\tau_2) = 1$, so we omit our own (slightly longer) argument.

For our proof let X be the plane curve in André's Theorem, which we suppose to be given by $f(X_1, X_2) = 0$, where f has "effectively known" algebraic coefficients. (The case when f is defined effectively over a field of positive transcendence degree may be immediately reduced to the case of algebraic coefficients.) Below, $c_1, c_2, \ldots$ shall denote strictly positive numbers which can be effectively determined in terms only of equations for X. Their effectivity is a standard affair to which we shall make no further reference.

By taking the union of X with its conjugates over $\mathbb{Q}$ we may assume that X is defined and irreducible over $\mathbb{Q}$. We let (x_1, x_2) run through the set of CM-points in X; that is, $x_i = j(\tau_i)$, where j is the modular function and τ_1, τ_2 are imaginary quadratic. We denote by D_i the discriminant of τ_i and by $\mathcal{O}_i = \mathbb{Z} + \mathbb{Z}((D_i + \sqrt{D_i})/2)$ its order (i.e. the set of α in $\mathbb{C}$ which stabilize the lattice $\mathbb{Z}\tau_i + \mathbb{Z}$). By symmetry, we may assume throughout that $|D_1| \geqslant |D_2|$.

André in [A, lemme 1] starts with an ingenious Galois argument, showing that for almost all points in this set we have $\mathbb{Q}(\sqrt{D_1}) = \mathbb{Q}(\sqrt{D_2})$; it is here that the ineffectivity arises, through the use of Siegel's class-number estimates. We shall entirely avoid this point of his proof.

As in [A], we use the fact that X is defined over $\mathbb{Q}$ to replace (x_1, x_2) by suitable conjugates. The conjugates of x_1 run through the values $j(\tau)$ of the elliptic modular function corresponding to lattices $\mathbb{Z}\tau + \mathbb{Z}$ whose stabilizers coincide with the order $\mathcal{O}_1$; see for example [**L**, theorem 5, p.133]. In particular, some conjugate corresponds to the full order $\mathcal{O}_1$, and we may thus assume that $x_1 = j((D_1 + \sqrt{D_1})/2)$.

Now the Fourier expansion $j(\tau) = q^{-1} + 744 + 196884q + \cdots$ for $q = \exp(2\pi i\tau)$ shows that

$$|\log|j(\tau)| - 2\pi y| \leqslant c_1 \exp(-2\pi y) \tag{1}$$

for the imaginary part y of τ. So in particular $|x_1| \to \infty$ as $|D_1| \to \infty$. In his paper, André now has a crucial "Lemme 2", asserting that for $|x_1| \to \infty$ we also have $|x_2| \to \infty$. We reproduce this argument in effective form; however we do not state it as a lemma, and glue the relevant conclusion with the rest of the argument.

Since no component of X is a vertical line, any x_1 determines at most finitely many x_2 with $f(x_1, x_2) = 0$. Further if $|x_1| \geqslant c_2$ these are given by finitely many convergent Puiseux series $x_2 = P(x_1)$. We choose such a P corresponding to singular moduli x_1, x_2.

Suppose first that $P(\infty)$ is a complex number l. Since no component of X is a horizontal line, there are at most finitely many (x_1, x_2) in X with $x_2 = l$, so we may assume $x_2 \neq l$. Note that then l is necessarily algebraic. We have $P(t) = l + \gamma t^{-\alpha} +$ lower order terms, for some complex $\gamma \neq 0$ and rational $\alpha > 0$, whence

$$\log|x_2 - l| \leqslant -\alpha\pi\sqrt{|D_1|} + c_3.$$

We may now pick a complex τ_2 in the standard modular fundamental domain $\mathcal{F}$ so that $j(\tau_2) = x_2$; note that τ_2 is imaginary quadratic over $\mathbb{Q}$. Since the restriction of j to $\mathcal{F}$ is a bijection, this implies that τ_2 is near to some ζ in $\mathbb{C}$, with $j(\zeta) = l$. More precisely, expanding the j function as a Taylor series around ζ, we get

$$\log|x_2 - l| \geqslant \kappa \log|\tau_2 - \zeta| - c_4,$$

where $\kappa = 1, 2, 3$ (depending on ζ, i.e. whether $l = j(\zeta)$ is a regular value of j, or whether $l = 0, 1728$ is a critical value of j). Hence

$$\log|\zeta - \tau_2| \leqslant -c_5\sqrt{|D_1|}.$$

However the second author has established in [M] (p.1) that for any ζ with algebraic $j(\zeta)$ and any algebraic $w \neq \zeta$ we have an inequality

$$\log|\zeta - w| > -C\max\{1, h(w)\}^{3+\epsilon},$$

where the positive constant $C > 0$ is effective and depends only on ζ, $[\mathbb{Q}(w) : \mathbb{Q}]$, $\epsilon > 0$ (and where $h(w)$ denotes as usual the logarithmic Weil height). Here $\zeta - w$ is essentially a linear form in elliptic periods. We can apply the result, with $\epsilon = 1$ for example, putting $w = \tau_2$ and recalling that τ_2 has been chosen in the standard fundamental domain and that it is quadratic over $\mathbb{Q}$. We easily find (on looking at a minimal equation for τ_2 over $\mathbb{Z}$) that $h(\tau_2) \leqslant c_6 \log|D_2|$. Recalling also $|D_2| \leqslant |D_1|$, we see that the last two displayed inequalities are inconsistent for $|D_1| \geqslant c_7$ and large enough c_7.

Hence in this case we have $P(\infty) = \infty$, and now we may write $P(t) = \gamma_0 t^{\beta} +$ lower order terms, for some complex $\gamma_0 \neq 0$ and rational $\beta > 0$.

Let us now choose both τ_1, τ_2 in the standard fundamental domain, so that $x_i = j(\tau_i)$. In view of our opening normalization on x_1, we may write

$$\tau_1 = \frac{c + \sqrt{D_1}}{2}, \qquad \tau_2 = \frac{b + \sqrt{D_2}}{2a},$$

where a, b, c are integers with $a \geqslant |b|$, $c = 0, 1$. Of course a and $-b$ are coefficients in the equation for τ_2.

Now the expansion $x_2 = P(x_1)$ shows that we have an inequality

$$\big|\log|x_2| - \log|\gamma_0| - \beta\log|x_1|\big| \leqslant c_8\exp(-c_9\sqrt{|D_1|}).$$

Then (1) yields at first

$$\left|\pi\frac{\sqrt{|D_2|}}{a} - \log|\gamma_0| - \beta\pi\sqrt{|D_1|}\right| \leqslant c_{10}\exp\left(-c_{11}\frac{\sqrt{|D_2|}}{a}\right). \tag{2}$$

Hence for $|D_1|$ large enough we get say $\sqrt{|D_2|}/a \geqslant (1/2)\beta\sqrt{|D_1|}$ and so $a \leqslant 2/\beta$; and then we get an inequality similar to (2) with $\sqrt{|D_1|}$ replacing $\frac{\sqrt{|D_2|}}{a}$ on the right. We can write the left as $|\Lambda|$ for $\Lambda = \delta\pi i - \log|\gamma_0|$ with $\delta = \sqrt{D_2}/a - \beta\sqrt{D_1}$; and then standard results on linear forms in logarithms show that, also for $|D_1|$ large enough, we must have $\Lambda = 0$. Thus the two logarithms πi, $\log|\gamma_0|$ must be linearly dependent over $\mathbb{Q}$, which forces $\log|\gamma_0| = 0$ and then $\delta = 0$.

We conclude that $D_1/D_2 = (a\beta)^{-2}$ is a rational square in a finite computable set, so (since $|a|, |b|, |c|$ are bounded by c_{16}) there are coprime integers $r, s, t \neq 0$ in $\mathbb{Z}$ satisfying $|r|, |s|, |t| \leqslant c_{17}$, such that $\tau_2 = (r + s\tau_1)/t$. But then the point (x_1, x_2) lies on the modular curve $Y_0(st)$.

Thus all of the relevant points either satisfy $\max(|D_1|, |D_2|) \leqslant c_{18}$ or lie in the union of finitely many curves $Y_0(n)$, $1 \leqslant n \leqslant c_{19}$, which may be effectively computed, and this is a rephrasing of the desired conclusion.

Now to the examples.

The diagonal $X_1 + X_2 = 1$ can be treated without elliptic periods, because there is only one Puiseux expansion $P(t) = -t + 1$. Moreover $\gamma_0 = -1$ so we end up with a linear form in only one logarithm which can be handled with a very simple Liouville-type estimate.

The hyperbola $X_1X_2 = 1$ also has only one Puiseux expansion, namely $P(t) = t^{-1}$ which goes to finite l, so it looks like elliptic periods may be needed. However $l = 0$ happens to be $j(\zeta)$ with algebraic $\zeta = \rho = (1 + \sqrt{-3})/2$. So now $\zeta - w$ can be handled again by Liouville. However to find all the points we need explicitly to invert j near its critical point ρ.

In this way we now show that there are no imaginary quadratic τ_1, τ_2 with

$$j(\tau_1)j(\tau_2) = 1. \qquad (3)$$

LEMMA 1. *If τ is in the standard fundamental domain with imaginary part y then*

$$||j(\tau)| - e^{2\pi y}| \leqslant 2079.$$

Proof. We have $j = q^{-1} + \sum_{n=0}^{\infty} c_n q^n$ with $c_n \geqslant 0$. As $y \geqslant \sqrt{3}/2$ we get

$$||j| - |q^{-1}|| \leqslant \sum_{n=0}^{\infty} c_n|q|^n \leqslant \sum_{n=0}^{\infty} c_n q_0^n$$

for $q_0 = e^{-\pi\sqrt{3}}$. On the other hand $q_0 = e^{2\pi i \tau_0}$ for $\tau_0 = \sqrt{-3}/2$, so the sum on the far right is $j(\tau_0) - q_0^{-1} = 2078.813\ldots$

LEMMA 2. *For any τ with $|\tau - \rho| \leqslant \sqrt{3}/4$ we have $|j(\tau)| \leqslant 30000$.*

Proof. We divide the disc $|\tau - \rho| \leqslant \sqrt{3}/4$ into six parts by means of the circles $|\tau| = 1$, $|\tau - 1| = 1$ and the vertical line through $1/2$. Then a calculation using the functions τ, $\tau - 1$, $1/(1-\tau)$, $\tau/(1-\tau)$, $(\tau-1)/\tau$, $-1/\tau$ applied going round the boundary of the disc clockwise shows that every τ in the disc is modular equivalent to a point of the subset of the fundamental domain with $y \leqslant y_0 = (16\sqrt{3} + \sqrt{183})/26 = 1.586\ldots$ It therefore suffices to consider the boundary of this subset. On the vertical and circular parts we get easily by monotonicity $|j| \leqslant \max\{1728, -j(1/2 + iy_0)\} < 20561$. On the horizontal part $|q| = q_0 = e^{-2\pi y_0}$ and now with $c_{-1} = 1$

$$|j| = |q|^{-1}\left|\sum_{n=0}^{\infty} c_{n-1}q^n\right| \leqslant q_0^{-1}\sum_{n=0}^{\infty} c_{n-1}q_0^n = j(iy_0) < 22049.$$

The next two estimates correspond to $\kappa = 3$ in the general proof above.

LEMMA 3. *If* $\tau = 1/2 + iy$ *is in the standard fundamental domain with imaginary part* y *and* $|j(\tau)| < \varepsilon < 1/100000$ *then*

$$\left| y - \frac{\sqrt{3}}{2} \right| < \frac{1}{34} |\varepsilon|^{1/3}.$$

Proof. For any real ζ with $0 \leqslant \zeta - \sqrt{3}/2 \leqslant 1/1000$ we have for the fourth derivative

$$j''''\left(\frac{1}{2} + i\zeta\right) = 24 \frac{1}{2\pi i} \int_{|\tau-\rho|=\frac{\sqrt{3}}{4}} \frac{j(\tau)}{(\tau - \frac{1}{2} - i\zeta)^5} d\tau.$$

Using Lemma 2 we get

$$\left| j''''\left(\frac{1}{2} + i\zeta\right) \right| \leqslant 24 \frac{\sqrt{3}}{4} \frac{30000}{(\frac{\sqrt{3}}{4} - \frac{1}{1000})^5} < 30000000. \tag{4}$$

Next define the real-valued function $f(y) = j(1/2 + iy)$ $(y > 0)$; we deduce the same bound (4) for $|f''''(\zeta)|$. For $0 \leqslant \eta - \sqrt{3}/2 \leqslant 1/1000$ the Mean Value Theorem gives

$$f'''(\eta) - f'''\left(\frac{\sqrt{3}}{2}\right) = \left(\eta - \frac{\sqrt{3}}{2}\right) f''''(\zeta) \qquad \left(\frac{\sqrt{3}}{2} < \zeta < \eta\right).$$

One checks $j'''(\rho) = -162 i \Gamma(\frac{1}{3})^{18}/\pi^9$ and so

$$\left| f'''\left(\frac{\sqrt{3}}{2}\right) \right| = |j'''(\rho)| > 270000.$$

Therefore

$$|f'''(\eta)| \geqslant 270000 - \frac{1}{1000} 30000000 = 240000.$$

Now $j(\rho) = j'(\rho) = j''(\rho) = 0$ so $f(\sqrt{3}/2) = f'(\sqrt{3}/2) = f''(\sqrt{3}/2) = 0$. With τ as in Lemma 3 we have by a Higher Mean Value Theorem

$$\varepsilon > |j(\tau)| = |f(y)| = \frac{1}{6} |f'''(\eta)| \left| y - \frac{\sqrt{3}}{2} \right|^3 \qquad \left(\frac{\sqrt{3}}{2} < \eta < y\right). \tag{5}$$

Here $y \leqslant \sqrt{3}/2 + 1/1000$ else the opposite would imply by monotonicity

$$j(\tau) < j\left(\rho + \frac{i}{1000}\right) < -\frac{4}{100000}$$

against a hypothesis. So also $\eta < \sqrt{3}/2 + 1/1000$ and by (5) we get $|y - \sqrt{3}/2| \leqslant (6\,\varepsilon/240000)^{1/3}$. This is slightly better than required.

LEMMA 4. *If* $\tau = e^{i\theta}$ *is in the standard fundamental domain with* $\theta \leqslant \pi/2$ *and* $|j(\tau)| < \varepsilon < 1/100000$ *then*

$$\left| \theta - \frac{\pi}{3} \right| < \frac{1}{33} |\varepsilon|^{1/3}.$$

Proof. For any real θ with $0 \leqslant \theta - \pi/3 \leqslant 1/1000$ we have for the fourth derivative

$$j''''(e^{i\theta}) = 24\frac{1}{2\pi i}\int_{|\tau-\rho|=\frac{\sqrt{3}}{4}} \frac{j(\tau)}{(\tau - e^{i\theta})^5}d\tau.$$

It follows as before that

$$|j''''(e^{i\theta})| \leqslant 24\frac{\sqrt{3}}{4}\frac{30000}{(\frac{\sqrt{3}}{4} - \frac{1}{1000})^5} < 30000000.$$

Similar arguments give

$$|j'(e^{i\theta})| < 70000, \quad |j''(e^{i\theta})| < 330000, \quad |j'''(e^{i\theta})| < 2300000.$$

Next for the real-valued function $g(\theta) = j(e^{i\theta})$ $(0 < \theta < \pi)$ we have

$$g''''(\theta) = e^{i\theta}j'(e^{i\theta}) + 7e^{2i\theta}j''(e^{i\theta}) + 6e^{3i\theta}j'''(e^{i\theta}) + e^{4i\theta}j''''(e^{i\theta}).$$

Thus for $0 \leqslant \theta - \pi/3 \leqslant 1/1000$ we conclude

$$|g''''(\theta)| \leqslant 70000 + 7\cdot 330000 + 6\cdot 2300000 + 30000000 = 46180000.$$

The Mean Value Theorem gives

$$g'''(\theta) - g'''\left(\frac{\pi}{3}\right) = \left(\theta - \frac{\pi}{3}\right)g''''(\phi) \qquad \left(\frac{\pi}{3} < \phi < \theta\right).$$

As before we find $g(\pi/3) = g'(\pi/3) = g''(\pi/3) = 0$. It follows that

$$\left|g'''\left(\frac{\pi}{3}\right)\right| = |j'''(\rho)| = 162\frac{\Gamma(\frac{1}{3})^{18}}{\pi^9} > 270000$$

and so

$$|g'''(\theta)| \geqslant 270000 - \frac{1}{1000}46180000 = 223820.$$

With τ as in Lemma 4 we have by a Higher Mean Value Theorem

$$\varepsilon > |j(\tau)| = |g(\theta)| = \frac{1}{6}|g'''(\phi)|\left|\theta - \frac{\pi}{3}\right|^3 \qquad \left(\frac{\pi}{3} < \phi < \theta\right). \tag{6}$$

Now $\theta \leqslant \pi/3 + 1/1000$ else the opposite would imply by monotonicity

$$j(\tau) > j(\rho e^{i/1000}) > \frac{4}{100000}$$

against a hypothesis. So also $\phi < \pi/3 + 1/1000$ and by (6) we get $|\theta - \pi/3| \leqslant (6\varepsilon/223820)^{1/3}$. This is slightly better than required.

Now in (3) let D_1, D_2 be the discriminants; we may take $\sqrt{|D_1|} \geqslant \sqrt{|D_2|}$. By conjugating we may take $\tau_1 = (D_1 + \sqrt{D_1})/2$.

First assume D_1 is odd. Then we can even take $\tau_1 = (1 + \sqrt{D_1})/2$ in the fundamental domain and τ_2 also in the fundamental domain. By Lemma 1

$$j(\tau_1) = -|j(\tau_1)| \leqslant -e^{\pi\sqrt{|D_1|}} + 2079 \leqslant -e^{\pi\sqrt{14}} + 2079 < -100000$$

provided $|D_1| \geqslant 14$. So $-1/100000 < j(\tau_2) < 0$. It follows that $\tau_2 = 1/2 + iy$ $(y > \sqrt{3}/2)$, because we can dispense with real part $-1/2$. Thus by Lemma 3

$$\left|y - \frac{\sqrt{3}}{2}\right| < \frac{1}{34}|j(\tau_2)|^{1/3} \leqslant E$$

with $E = \frac{1}{34}(e^{\pi\sqrt{|D_1|}} - 2079)^{-1/3}$. Also $\tau_2 = (a + \sqrt{D_2})/2a$ so

$$\left|\frac{\sqrt{|D_2|}}{2a} - \frac{\sqrt{3}}{2}\right| \leqslant E, \quad |\sqrt{|D_2|} - a\sqrt{3}| \leqslant 2aE \leqslant \frac{2\sqrt{|D_1|}}{\sqrt{3}}E < 1$$

provided $|D_1| \geqslant 14$. Then

$$||D_2| - 3a^2| \leqslant (1 + 2\sqrt{|D_1|})\frac{2\sqrt{|D_1|}}{\sqrt{3}}E < 1.$$

Thus $|D_2| = 3a^2$ giving $\tau_2 = \rho$, absurd.

So we may now assume D_1 is even. Then we can take $\tau_1 = \sqrt{D_1}/2$ and τ_2 also in the fundamental domain. By Lemma 1

$$j(\tau_1) = |j(\tau_1)| \geqslant e^{\pi\sqrt{|D_1|}} - 2079 \geqslant e^{\pi\sqrt{14}} - 2079 > 100000$$

provided $|D_1| \geqslant 14$. So $1/100000 > j(\tau_2) > 0$. It follows that $\tau_2 = e^{i\theta}$ $(\pi/3 < \theta < \pi/2)$, because we can dispense with $\theta \geqslant \pi/2$. Thus by Lemma 4

$$\left|\theta - \frac{\pi}{3}\right| < \frac{1}{33}|j(\tau_2)|^{1/3} \leqslant E$$

where now $E = 1/33(e^{\pi\sqrt{|D_1|}} - 2079)^{-1/3}$. Also $\tau_2 = (b + \sqrt{D_2})/2a$ so

$$\left|\frac{\sqrt{|D_2|}}{2a} - \frac{\sqrt{3}}{2}\right| \leqslant E, \quad |\sqrt{|D_2|} - a\sqrt{3}| \leqslant 2aE \leqslant \frac{2\sqrt{|D_1|}}{\sqrt{3}}E < 1$$

provided $|D_1| \geqslant 14$. Then

$$||D_2| - 3a^2| \leqslant (1 + 2\sqrt{|D_1|})\frac{2\sqrt{|D_1|}}{\sqrt{3}}E < 1.$$

Thus $|D_2| = 3a^2$, and since $1 = |\tau_2| = (b^2 - D_2)/4a^2$ we get again the absurd $\tau_2 = \rho$.

It remains to check all $j(\tau)$ with τ of discriminant with absolute value at most 13. But this means $|D| = 3, 4, 7, 8, 11, 12$. These all have class number one, with j respectively

$$0,\ 1728,\ -3375,\ 8000,\ -32768,\ 54000$$

and visibly no two of these multiply to 1.

This example begs the question: are there any τ with $j(\tau)$ a unit? Possibly this could be answered with the methods of Gross-Zagier [GZ] on the factorization of products of $j(\tau) - j(\tau')$ by taking $\tau' = \rho$ (at least when $D(\tau)$ is not divisible by 3). But Habegger has very recently shown that there are at most finitely many.

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