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Affine Surfaces With a Huge Group of Automorphisms

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We describe a family of rational affine surfaces S with huge groups of automorphisms in the following sense: the normal subgroup $\mathrm{Aut}(S)_{\mathrm{alg}}$ of $\mathrm{Aut}(S)$ generated by all algebraic subgroups of $\mathrm{Aut}(S)$ is not generated by any countable family of such subgroups, and the quotient $\mathrm{Aut}(S)/\mathrm{Aut}(S)_{\mathrm{alg}}$ cointains a free group over an uncountable set of generators.

1 Introduction

The automorphism group of an algebraic curve defined over a field k is always an algebraic group of dimension at most 3, the biggest possible group being $\mathrm{Aut}(\mathbb{P}^1_k) = \mathrm{PGL}(2, k)$. The situation is very different starting from dimension 2 even for complete or projective surfaces S: of course some groups such as $\mathrm{Aut}(\mathbb{P}^2_k) = \mathrm{PGL}(3, k)$ are still algebraic groups but in general $\mathrm{Aut}(S)$ only exists as a group scheme locally of finite type over k [10] and it may fail, for instance, to be an algebraic group in the usual sense because it has (countably) infinitely many connected components. This happens, for example, for the automorphism group of the blow-up $S_9 \to \mathbb{P}^2$ of the base-points of a general pencil of two cubics, which contains a finite index group isomorphic to $\mathbb{Z}^8$, acting on the pencil by translations. Note, however, that the existence of $\mathrm{Aut}(S)$ as a group scheme implies at least that S has a largest connected algebraic group of automorphisms: the identity component of $\mathrm{Aut}(S)$ equipped with its reduced structure.

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The picture tends to be much more complicated for noncomplete surfaces, in particular, affine ones. For instance, the group $\mathrm{Aut}(\mathbb{A}^2_k)$ of the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ contains algebraic groups of any dimension and hence is very far from being algebraic. In fact, the subgroups

$$T_n = \{(x, y) \mapsto (x, y + P(x)),\ P \in k[x], \deg P \leq n\} \simeq \mathbb{G}_{a,k}^{n+1}$$

of unipotent triangular automorphisms of degree at most n form an increasing family of connected subgroups of automorphism of $\mathbb{A}^2_k$ in the sense of [12] so that $\mathrm{Aut}(\mathbb{A}^2_k)$ does not admit of any largest connected algebraic group of automorphisms. It is interesting to observe, however, that as a consequence of Jung's Theorem [8], $\mathrm{Aut}(\mathbb{A}^2_k)$ is generated by a countable family of connected algebraic subgroups, namely $\mathrm{GL}(2, k)$ and the above triangular subgroups T_n, $n \geq 1$. A similar phenomenon turns out to hold for other classical families of rational affine surfaces with large groups of automorphisms: for instance, for the smooth affine quadric in $\mathbb{A}^3$ with equation $xy - z^2 + 1 = 0$ and more generally for all normal affine surfaces defined by an equation of the from $xy - P(z) = 0$, where $P(z)$ is a nonconstant polynomial, whose automorphism groups have been described explicitly first by Makar-Limanov [9] by purely algebraic methods and more recently by the authors [1] in terms of the birational geometry of suitable projective models.

All examples above share the common property that the normal subgroup $\mathrm{Aut}(S)_{\mathrm{Alg}} \subset \mathrm{Aut}(S)$ generated by all algebraic subgroups of $\mathrm{Aut}(S)$ is in fact generated by a countable family of such subgroups and that the quotient $\mathrm{Aut}(S)/\mathrm{Aut}(S)_{\mathrm{Alg}}$ is countable. So one may wonder whether such a property holds in general for quasi-projective surfaces. It turns out that there exist normal affine surfaces which have a much bigger group of automorphisms and the purpose of this article is to describe explicitly one family of such surfaces. Our main result can be summarized as follows:

Theorem 1.1. Let k be an uncountable field and let $P, Q \in k[w]$ be polynomials having at least 2 distinct roots in the algebraic closure $\bar{k}$ ok k and such that $P(0) \neq 0$. Then for the affine surface S in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the system of equations

$$yu = xP(x),$$

$$vx = uQ(u),$$

$$yv = P(x)Q(u),$$

the following hold:

(1) The normal subgroup $\mathrm{Aut}(S)_{\mathrm{Alg}} \subset \mathrm{Aut}(S)$ is not generated by a countable union of algebraic groups.
(2) The quotient $\mathrm{Aut}(S)/\mathrm{Aut}(S)_{\mathrm{Alg}}$ contains a free group over an uncountable set of generators. □

Note that the surfaces described in Theorem 1.1 can be chosen to be either singular or smooth, depending on the multiplicity of the roots of P and Q.

The result is obtained from a systematic use of the methods developed in [1] for the study of affine surfaces admitting of many $\mathbb{A}^1$-fibrations. By virtue of pioneering work of Gizatullin [7], the latter essentially coincide with surfaces admitting of normal projective completions X for which the boundary divisor is a so-called zigzag, that is, a chain B of smooth proper rational curves supported in the smooth locus of X. These have been extensively studied by Gizatullin and Danilov [2, 3, 7] during the seventies and more recently by the authors [1] (see also [6], in which such surfaces are called *Gizatullin surfaces*).

An important invariant of a zigzag is the sequence of self-intersections of its components, called its type, which in our context can be chosen to be of the form $(0, -1, -a_1, \ldots, -a_r)$, where the $a_i \geq 2$ are a possibly empty sequence of integers. In this setting, the simplest possible zigzag has type $(0, -1)$ and the corresponding affine surface is the affine plane $\mathbb{A}^2$ viewed as the complement in the Hirzebruch surface $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1$ of the union of a fiber of ρ_1, with self-intersection 0, and the exceptional section of ρ_1 with self-intersection -1. The next family in terms of the number of irreducible components in the boundary zigzag B consists of types $(0, -1, -a_1)$, $a_1 \geq 2$. These correspond to the normal hypersurfaces of $\mathbb{A}^3$ defined by equations of the form $xy - P(z) = 0$ with $\deg(P) = a_1$, which were studied in detail in [1].

The present article is in fact devoted to the study of the next family, that is, affine surfaces corresponding to zigzags of types $(0, -1, -a_1, a_2)$, where $a_1, a_2 \geq 2$. The surfaces displayed in Theorem 1.1 provide explicit realizations of general members of this family as subvarieties of $\mathbb{A}^4$, but we describe more generally isomorphism classes and automorphism groups of all surfaces in the family.

From this point of view, Theorem 1.1 above says that a relatively minor increase of the complexity of the boundary zigzag has very important consequences on the geometry and the automorphism group of the inner affine surface. The next cases, that is zigzags of type $(0, -1, -a_1, \ldots, -a_r)$ with $r \geq 3$, could be studied in exactly the same way as we do here; the amount of work needed would just be bigger.

The article is organized as follows: in the first section, we review the main techniques introduced in [1] to study normal affine surfaces completable by a zigzag in terms of the birational geometry of suitable projective models of them, called standard pairs. We also characterize the nature of algebraic subgroups of their automorphism groups in this framework (Proposition 2.7). The next two sections are devoted to the study of isomorphism classes of surfaces associated to zigzags of type $(0, -1, -a_1, -a_2)$ and the description of isomorphisms between these in terms of elementary birational links between the corresponding standard pairs. In the last section, we apply these intermediate results to obtain a theorem describing the structure of their automorphism groups and of their $\mathbb{A}^1$-fibrations. Theorem 1.1 is then a direct consequence of this precise description.

2 Recollection on Standard Pairs and their Birational Geometry

2.1 Standard pairs and associated rational fibrations

Recall that a *zigzag* on a normal projective surface X is a connected SNC-divisor B supported in the smooth locus of X, with irreducible components isomorphic to the projective line over k and whose dual graph is a chain. In what follows, we always assume that the irreducible components B_i, $i = 0, \ldots, r$ of B are ordered in such a way that

$$B_i \cdot B_j = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1, \end{cases}$$

and we write $B = B_0 \triangleright B_1 \triangleright \cdots \triangleright B_r$ for such an ordered zigzag. The sequence of integers $((B_0)^2, \ldots, (B_r)^2)$ is then called the *type* of B.

Definition 2.1. A *standard pair* (These were called 1-standard in [1].) is a pair (X, B) consisting of a normal rational projective surface X and an ordered zigzag B that can be written as $B = F \triangleright C \triangleright E$, where F and C are smooth irreducible rational curves with self-intersections $F^2 = 0$ and $C^2 = -1$, and where $E = E_1 \triangleright \cdots \triangleright E_r$ is a (possibly empty) chain of irreducible rational curves with self-intersections $(E_i)^2 \leq -2$ for every $i = 1, \ldots, r$. The *type* of the pair (X, B) is the type $(0, -1, -a_1, \ldots, -a_r)$ of its ordered zigzag B. □

2.1.1

The underlying projective surface of a standard pair $(X, B = F \triangleright C \triangleright E)$ comes equipped with a rational fibration $\bar{\pi} = \bar{\pi}_{|F|} : X \to \mathbb{P}^1$ defined by the complete linear system $|F|$. The latter restricts on the quasi-projective surface $S = X \setminus B$ to a faithfully flat morphism $\pi : S \to \mathbb{A}^1$ with generic fiber isomorphic to the affine line over the function field of $\mathbb{A}^1$, called an $\mathbb{A}^1$-*fibration*. (It follows in particular from this description that these quasi-projective surfaces have negative logarithmic Kodaira dimension.) We use the notations $(X, B, \bar{\pi})$ and $(X \setminus B, \bar{\pi}\mid_{X\setminus B})$ (or simply, $(X \setminus B, \pi)$ when we consider the corresponding surfaces as equipped with these respective fibrations).

When B is the support of an ample divisor, S is an affine surface and $\pi : S \to \mathbb{A}^1$ has a unique degenerate fiber $\pi^{-1}(\bar{\pi}(E))$ consisting of a nonempty disjoint union of affine lines, possibly defined over finite algebraic extensions of k, when equipped with its reduced scheme structure (see, e.g., [11, 1.4]). Furthermore, if any, the singularities of S are all supported on the degenerate fiber of π and admit of a minimal resolution whose exceptional set consists of a chain of rational curves possibly defined over a finite algebraic extension of k. In particular, if k is algebraically closed of characteristic 0, then S has at worst Hirzebruch–Jung cyclic quotient singularities. Furthermore, according to [1, Lemma 1.0.7], the minimal resolution of singularities $\mu : (Y, B) \to (X, B)$ of the pair (X, B) can be obtained from the first Hirzebruch surface $\rho \colon \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ by a uniquely determined sequence of blow-ups $\eta \colon Y \to \mathbb{F}^1$ restricting to isomorphisms outside the degenerate fibers of $\bar{\pi} \circ \mu$ in such a way that we have a commutative diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\mu}{\swarrow} & \downarrow{\scriptstyle \bar{\pi}\circ\mu} & \overset{\eta}{\searrow} & \\ X & & & & \mathbb{F}_1 \\ & \underset{\bar{\pi}}{\searrow} & & \underset{\rho}{\swarrow} & \\ & & \mathbb{P}^1. & & \end{array}$$

2.2 Birational maps of standard pairs

A birational map $\phi : (X, B) \dashrightarrow (X', B')$ between standard pairs is a birational map $X \dashrightarrow X'$ which restricts to an isomorphism $X \setminus B \stackrel{\sim}{\to} X' \setminus B'$. It is an isomorphism of pairs if it is moreover an isomorphism from X to X'. The birational maps between standard pairs play a central role in the study of the automorphism groups of $\mathbb{A}^1$-fibered affine surfaces as in Section 2.1.1. The main result of [1] asserts the existence of a decomposition of every such birational map into a finite sequence of "basic" birational maps of standard pairs called *fibered modifications* and *reversions* which can be defined, respectively, as follows:

Definition 2.2 ([1, Definition 2.2.1 and Lemma 2.2.3]). A *fibered modification* is a strictly birational map (birational and not biregular) of standard pairs

$$\phi : (X, B = F \triangleright C \triangleright E) \dashrightarrow (X', B' = F' \triangleright C' \triangleright E'),$$

which induces an isomorphism of $\mathbb{A}^1$-fibered quasi-projective surfaces

$$\begin{array}{ccc} S = X \setminus B & \xrightarrow[\phi]{\sim} & S' = X' \setminus B' \\ {\scriptstyle \bar{\pi}|_S} \downarrow & & \downarrow {\scriptstyle \bar{\pi}'|_{S'}} \\ \mathbb{A}^1 & \xrightarrow{\sim} & \mathbb{A}^1 \end{array}$$

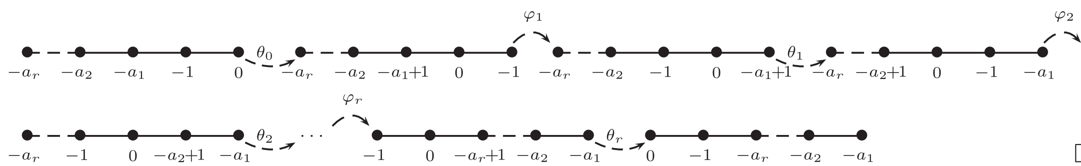
where $\bar{\pi}\mid_S$ and $\bar{\pi}'\mid_S'$ denote the restrictions of the rational pencils defined by the complete linear systems $|F|$ and $|F'|$ on X and X', respectively. Equivalently, with the notation of Section 2.1.1, the birational map $(\mu')^{-1} \circ \phi \circ \mu : Y \dashrightarrow Y'$ induced by ϕ is the lift via η and η' of a nonaffine (i.e., of degree ≥ 1) isomorphism of $\mathbb{A}^1$-fibered affine surfaces

$$\begin{array}{ccc} \mathbb{A}^2 = \mathbb{F}_1 \setminus (\eta(F) \cup \eta(C)) & \xrightarrow[\sim]{} & \mathbb{A}^2 = \mathbb{F}_1 \setminus (\eta'(F') \cup \eta'(C')) \\ {\scriptstyle \rho|_{\mathbb{A}^2}} \downarrow & & \downarrow {\scriptstyle \rho|_{\mathbb{A}^2}} \\ \mathbb{A}^1 & \xrightarrow[\sim]{} & \mathbb{A}^1 \end{array}$$

which maps the base-points of η^{-1} onto those of η'^{-1}. □

Definition 2.3 ([1, Section 2.3]). A *reversion* is a special kind of a birational map of standard pairs uniquely determined, up to isomorphisms of pairs, by the choice of a k-rational point $p \in F \setminus C$ and obtained by the following construction.

Starting from a pair $(X, B = F \triangleright C \triangleright E)$ of type $(0, -1, -a_1, \ldots, -a_r)$, the contraction of the (-1)-curve C followed by the blow-up of $p \in F \setminus C$ yields a birational map $\theta_0 : (X, B) \dashrightarrow (X_0, B_0)$ to a pair with a zigzag of type $(-1, 0, -a_1 + 1, \ldots, -a_r)$. Preserving the fibration given by the (0)-curve, one can then construct a unique birational map $\varphi_1 : (X_0, B_0) \dashrightarrow (X_1', B_1')$, where B_1' is a zigzag of type $(-a_1 + 1, 0, -1, -a_2, \ldots, -a_r)$. The blow-down of the (-1)-curve followed by the blow-up of the point of intersection of the (0)-curve with the curve immediately after it yields a birational map $\theta_1 : (X_1', B_1') \dashrightarrow (X_1, B_1)$, where B_1 is a zigzag of type $(-a_1, -1, 0, -a_2+1, \dots, -a_r)$. Repeating this procedure eventually yields birational maps $\theta_0, \varphi_1, \theta_1, \dots, \varphi_r, \theta_r$ described by the following figure.



□

Remark 2.4. In [1, Definition 2.3.1], reversions were defined in terms of their minimal resolution as birational maps and then given in [1, Section 2.3.5] in terms of elementary links as above, inspired by the construction of [5].

Note that working with pairs of the form $(0, 0, -a_1, \dots, -a_r)$ as in [5] has the advantage of giving reversions having one step less as here (see [5, Section 2.11]), but has the disadvantage that it is not possible to distinguish reversions and fibered modifications by looking at their proper base-points. See [1, Section 2.3.4, Section 2.3.5] for a complete comparative description. □

The *reversion of* (X, B) *with center at* p is then the strictly birational map of standard pairs

$$\phi = \theta_r \varphi_r \dots \theta_1 \varphi_0 \theta_0 : (X, B) \dashrightarrow (X_r, B_r) = (X', B').$$

Note that the above construction is symmetric so that the inverse $\phi^{-1} : (X', B') \dashrightarrow (X, B)$ of ϕ is again a reversion, with center at its unique proper base point (here proper means not infinitely near) $p' = \phi(E) \in F' \setminus C'$.

With these definitions, the decomposition results established in [1, Theorem 3.0.2 and Lemma 3.2.4] can be summarized as follows:

Proposition 2.5. For a birational map of standard pairs $\phi : (X, B) \dashrightarrow (X', B')$, the following hold

1. The map ϕ is either an isomorphism of pairs or it can be decomposed into a finite sequence

$$\phi = \phi_n \circ \cdots \circ \phi_1 : (X, B) = (X_0, B_0) \overset{\phi_1}{\dashrightarrow} (X_1, B_1) \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} (X_n, B_n) = (X', B'),$$

where each ϕ_i is either a fibered modification or a reversion. The integer n is called *length* of the decomposition.

2. If ϕ is not an isomorphism, then a decomposition as above of minimal length is unique up to isomorphisms of the intermediate pairs occurring

in the decomposition. Furthermore, a decomposition of minimal length is characterized by the property that it is *reduced*, which means that for every $i = 1, \ldots, n-1$ the induced birational map $\phi_{i+1}\phi_i : (X_{i-1}, B_{i-1}) \dashrightarrow (X_{i+1}, B_{i+1})$ is neither a reversion, nor a fibered modification, nor an automorphism.

3. A composition $\phi_{i+1}\phi_i : (X_{i-1}, B_{i-1}) \dashrightarrow (X_{i+1}, B_{i+1})$ as above is not reduced if and only if one of the following holds:

   (a) ϕ_i and ϕ_{i+1} are both fibered modifications;
   (b) ϕ_i and ϕ_{i+1} are both reversions, and ϕ_{i+1} and $(\phi_i)^{-1}$ have the same proper base-point;
   (c) ϕ_i and ϕ_{i+1} are both reversions, ϕ_{i+1} and $(\phi_i)^{-1}$ do not have the same proper base-point but each irreducible component of B_{i-1} (equivalently B_{i+1}) has self-intersection ≥ -2;

   In case (a), $\phi_{i+1}\phi_i$ is either a fibered modification (length 1) or an automorphism of pairs (length 0). In case (b), $\phi_{i+1}\phi_i$ is an automorphism of pairs (length 0), and in case (c), it is a reversion (length 1). □

Proposition 2.5 allows one to define the *length* of a birational map $(X, B) \dashrightarrow (X', B')$, which is the number of fibered modifications and reversions occurring in a minimal decomposition of the map (by convention an isomorphism of pairs has length 0).

Note that starting from a pair (X, B) of type $(0, -1, -a_1, \ldots, -a_r)$, the pairs that appear in the sequence are either of the same type or of type $(0, -1, -a_r, \ldots, -a_1)$. In particular, the property of having an irreducible curve of self-intersection ≤ -3 in the boundary depends only on the surface $X \backslash B$.

2.3 Graphs of $\mathbb{A}^1$-fibrations and associated graphs of groups

The existence of the above decomposition of birational maps between standard pairs into sequences of fibered modifications and reversions enables to associate to every normal affine surface S completable by a standard pair an oriented graph that encodes equivalence classes of $\mathbb{A}^1$-fibrations on S and links between these. This graph $\mathcal{F}_S$ is defined as follows (see [1, Definition 4.0.5]).

Definition 2.6. Given a normal affine surface S completable by a standard pair, we let $\mathcal{F}_S$ be the oriented graph with the following vertices and edges:

(a) A vertex of $\mathcal{F}_S$ is an equivalence class of standard pairs (X, B) such that $X \setminus B \cong S$, where two standard pairs $(X_1, B_1, \overline{\pi_1})$, $(X_2, B_2, \overline{\pi_2})$ define the same vertex if and only if the $\mathbb{A}^1$-fibered surfaces $(X_1 \setminus B_1, \pi_1)$ and $(X_2 \setminus B_2, \pi_2)$ are isomorphic.

(b) An arrow of $\mathcal{F}_S$ is an equivalence class of reversions. If $\phi : (X, B) \dashrightarrow (X', B')$ is a reversion, then the class $[\phi]$ of ϕ is an arrow starting from the class $[(X, B)]$ of (X, B) and ending at the class $[(X', B')]$ of (X', B'). Two reversions $\phi_1 : (X_1, B_1) \dashrightarrow (X_1', B_1')$ and $\phi_2 : (X_2, B_2) \dashrightarrow (X_2', B_2')$ define the same arrow if and only if there exist isomorphisms $\theta : (X_1, B_1) \to (X_2, B_2)$ and $\theta' : (X_1', B_1') \to (X_2', B_2')$, such that $\phi_2 \circ \theta = \theta' \circ \phi_1$. □

2.3.1

By definition, a vertex of $\mathcal{F}_S$ represents an equivalence class of $\mathbb{A}^1$-fibrations on S, where two $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$ and $\pi' : S \to \mathbb{A}^1$ are said to be equivalent if there exist automorphisms Ψ and ψ of S and $\mathbb{A}^1$, respectively, such that $\pi' \circ \Psi = \psi \circ \pi$. By virtue of [1, Proposition 4.0.7], the graph $\mathcal{F}_S$ is connected. Furthermore, if there exists a standard pair $(X, B = F \triangleright C \triangleright E)$ completing S for which B has an irreducible component of self-intersection ≤ -3, then there exists a natural exact sequence

$$0 \to H \to \mathrm{Aut}(S) \to \Pi_1(\mathcal{F}_S) \to 0,$$

where H is the normal subgroup of the automorphism group $\mathrm{Aut}(S)$ generated by all automorphisms of $\mathbb{A}^1$-fibrations on S and where $\Pi_1(\mathcal{F}_S)$ is the fundamental group of $\mathcal{F}_S$. This implies in particular that $\mathrm{Aut}(S)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations if and only if $\mathcal{F}_S$ is a tree. In contrast, in the case where all irreducible components of B have self-intersection ≥ 2, the information carried by the graph $\mathcal{F}_S$ is not longer really relevant, due to the fact that the composition of two reversions can be again a reversion. Alternative methods to handle this specific case exist, see, for example, [4].

In both cases (if B contains curves of self-intersection ≤ -3 or not) we will show (Proposition 2.7) that for any algebraic subgroup G of $\mathrm{Aut}(S)$, the image in $\Pi_1(\mathcal{F}_S)$ is finite, which implies in particular that if $\Pi_1(\mathcal{F}_S)$ is not countable, then $\mathrm{Aut}(S)$ cannot be generated by a countable set of algebraic subgroups.

2.3.2

Recall that a graph of groups in the sense of [13, 4.4] consists of an oriented graph $\mathcal{G}$ together with a choice consisting of a group G_v for every vertex v of $\mathcal{G}$, a group G_a for every arrow a in $\mathcal{G}$ and an injective homomorphism $\rho_a : G_a \to G_{t(a)}$, where $t(a)$ denotes the target of the oriented arrow a, and for every arrow a admitting of an inverse arrow $\bar{a}$ an anti-isomorphism of groups $\bar{}: G_a \to G_{\bar{a}}$ such that $\bar{\bar{x}} = x$ for any $x \in G_a$.

It was observed in [1] that under mild hypothesis on a normal affine surface S completable by a standard pair (X, B) (for instance, whenever the type $(0, -1, -a_1, \ldots, -a_r)$ of B does not satisfy $(a_1, \ldots, a_r) = (a_r, \ldots, a_1)$, but also in most of the cases where this is satisfied, see Section 4.0.10 in [1] for a complete discussion), the following choices determine a structure of a graph of groups on $\mathcal{F}_S$:

(a) for any vertex v of $\mathcal{F}_S$, we let $G_v = \mathrm{Aut}(X_v \setminus B_v, \pi_v)$ for a fixed standard pair $(X_v, B_v, \overline{\pi_v})$ in the class v;

(b) for any arrow a of $\mathcal{F}_S$, we let $G_a = \{(\phi, \phi') \in \mathrm{Aut}(X_a, B_a) \times \mathrm{Aut}(X'_a, B'_a) \mid r_a \circ \phi = \phi' \circ r_a\}$ for a fixed reversion $r_a : (X_a, B_a, \overline{\pi_a}) \overset{r_a}{\dashrightarrow} (X'_a, B'_a, \overline{\pi'_a})$ in the class of a, and we let $\rho_a : G_a \to G_{t(a)}$, $(\phi, \phi') \mapsto \mu_a \circ \phi' \circ (\mu_a)^{-1}$ for a fixed isomorphism μ between $(X'_a, B'_a, \overline{\pi'_a})$ and the fixed standard pair on the target vertex $t(a)$ of a.

We further require that the chosen reversion $r_{\bar{a}}$ for the inverse $\bar{a}$ of the arrow a is equal to $(r_a)^{-1}$ and that the structural anti-isomorphism $\bar{}: G_a \to G_{\bar{a}}$ is the map $(\phi, \phi') \mapsto (\phi', \phi)$.

When such a structure exists on $\mathcal{F}_S$, we established in [1, Theorem 4.0.11] that the automorphism group $\mathrm{Aut}(S)$ of S is isomorphic to the fundamental group of $\mathcal{F}_S$ as a graph of groups. This means that after choosing a base vertex v in $\mathcal{F}_S$, $\mathrm{Aut}(S)$ can be identified with the set of paths $g_n a_n g_{n-1} \cdots a_2 g_2 a_1 g_1$, where a_i is an arrow from v_i to v_{i+1}, $g_i \in G_{v_i}$ and $v_1 = v_n = v$ modulo the relations $\rho_a(h) \cdot a = a \cdot \rho_{\bar{a}}(\bar{h})$ and $a\bar{a} = 1$ for any arrow a and any $h \in G_a$.

Note that a graph of groups was already defined and used in the work of Danilov and Gizatullin [2]. Our graph, which can be thought of as a "contraction" of their graph, has a simpler structure (for instance, their graphs are usually infinite, even in the cases where ours are finite). See [1] for a more detailed comparison between these different graphs.

2.4 Actions of algebraic groups on standard pairs

Here we derive from the decomposition results some additional informations about algebraic subgroups of automorphism groups of affine surfaces S completable by a standard pair.

2.4.1

Let us first observe that the automorphism group $\mathrm{Aut}(S)$ of an affine surface $S = X \setminus B$ completable by a standard pair (X, B) contains many algebraic subgroups. For instance, it follows from Definition 2.2 (see also [1, Lemma 5.2.1 or Lemma 2.2.3]) that automorphisms of S preserving the $\mathbb{A}^1$-fibration $\pi = \bar{\pi}\,|_S \colon S = X \setminus B \to \mathbb{A}^1$ come as lifts of suitable triangular automorphisms ψ of $\mathbb{A}^2$ of the form $(x, y) \mapsto (ax + R(y), cy)$, where $a, c \in k^*$ and $R(y) \in k[y]$. It follows in particular that the subgroup $\mathrm{Aut}(X \setminus B, \pi)$ of $\mathrm{Aut}(X \setminus B)$ consisting of automorphisms preserving the $\mathbb{A}^1$-fibration π is a (countable) increasing union of algebraic subgroups. Furthermore, the group $\mathrm{Aut}(X, B)$ of automorphisms of the pair (X, B) is itself algebraic as B is the support of an ample divisor and coincides with the subgroup of $\mathrm{Aut}(X \setminus B, \pi)$ consisting of lifts of affine automorphisms ψ as above.

The following proposition describes more generally the structure of all possible algebraic subgroups of $\mathrm{Aut}(S)$.

Proposition 2.7. Let S be an affine normal surface completable by a standard pair, and let $G \subset \mathrm{Aut}(S)$ be an algebraic subgroup. Then there exists a standard pair (X, B) and an isomorphism $\psi : S \xrightarrow{\sim} X \setminus B$ such that for the conjugate $G^\psi = \psi G \psi^{-1} \subset \mathrm{Aut}(X \setminus B)$ of G the following alternative holds:

1. G^ψ is a subgroup of $\mathrm{Aut}(X \setminus B, \pi)$;
2. G^ψ contains a reversion $(X, B) \dashrightarrow (X, B)$ and every other element of G^ψ is either a reversion from (X, B) to itself or an element of $\mathrm{Aut}(X, B)$. More precisely, one of the following holds:

    (a) There exists a k-rational point $p \in B$ such that every reversion in G^ψ is centered at p, and every element in $G_0^\psi = G^\psi \cap \mathrm{Aut}(X, B)$ fixes the point p. We then have an exact sequence $1 \to G_0^\psi \to G^\psi \to \mathbb{Z}/2\mathbb{Z} \to 0$, and G_0^ψ is an algebraic group of dimension ≤ 2.

    (b) Every irreducible component of B has self-intersection ≥ -2 and the contraction $(X, B) \to (Y, D)$ of all irreducible components of negative self-intersection in B conjugates G^ψ whence G to a subgroup of $\mathrm{Aut}(Y, D)$, where Y is a projective rational surface and $D \cong \mathbb{P}^1$ is the support of an ample divisor. □

Proof. Up to fixing a standard pair such that $X \setminus B$ is isomorphic to S, we may assume from the very beginning that $S = X \setminus B$. Let $G \subset \mathrm{Aut}(S)$ be an algebraic group. According to [14], there exists a projective surface Y which contains S as an open dense subset, and such that G extends to a group of automorphisms of Y. The induced birational map $\theta \colon X \dashrightarrow Y$ has a finite number r of base-points (including infinitely near ones), and its inverse has r' base-points. It follows that every element $g \in G$ considered as a birational self-map of X has at most $r + r'$ base-points (again including infinitely near ones); indeed, each element is a composition of θ, an automorphism of Y, and θ^{-1}. Recall that the number n of fibered modifications and reversions occurring in a minimal decomposition $g = \phi_n \dots \phi_1$ (see Proposition 2.5) of an element $g \in G$ is called *the length of g*. The number of base-points of g is bigger or equal to its length (each map has base-points and there is no simplification between the base-points), so length of elements of G is bounded. Let $m = m(G)$ be the maximal length.

(a) If $m \leq 1$ and no element of G is a reversion, then $G \subset \mathrm{Aut}(X \setminus B, \pi)$ and we get (1).

(b) Otherwise, if $m \leq 1$ and G contains a reversion, then $m = 1$ and all elements of G are either reversions or automorphisms of (X, B) because the composition of a fibered modification and a reversion always has length 2. If there exists a k-rational point $p \in B$ such that every reversion in G is centered at p and every element in $G_0 = G \cap \mathrm{Aut}(X, B)$ fixes the point p, then the product of any two reversions in G is an automorphism (Proposition 2.5 3(c)). It follows that $G/G_0 \cong \mathbb{Z}/2\mathbb{Z}$, which gives 2*a*). Otherwise, if G contains two reversions ϕ, ϕ' with distinct proper base-points, then since $\phi'\phi^{-1}$ has length at most 1 by hypothesis, it cannot be reduced. By Proposition 2.5 3(c), it follows that all components of B have self-intersection ≥ -2. But in this case, reversions simply correspond to contracting all components of $B = F \triangleright C \triangleright E$ of negative self-intersection to a point p on the proper transform D of F and then blowing-up a new chain of the same type starting from a point $p' \in D$ distinct from p. The conjugation by the corresponding contraction $(X, B) \to (Y, D)$ identifies G with a subgroup of $\mathrm{Aut}(Y, D)$, which gives 2(*b*).

To complete the proof, it remains to show that we can always reduce by an appropriate conjugation to the case $m \leq 1$. If $m \geq 2$, then we consider an element $g \in G$ of length m and we fix a reduced decomposition $g = \phi_m \cdots \phi_1$. Writing $\phi_1 : (X, B) \dashrightarrow (X_1, B_1)$ and arguing by induction on m, it is enough to show that the length of every element in $\phi_1 G \phi_1^{-1}$ considered as a group of birational self-maps $(X_1, B_1) \dashrightarrow (X_1, B_1)$ is at most $m - 1$. Given an element $h \in G$ of length $n \geq 0$, we have the following possibilities:

(1) If $n = 0$, then h is an automorphism of (X, B). If it fixes the proper base-point of ϕ_1, then $(\phi_1 h)\phi_1^{-1}$ is not reduced whence has length at most $1 \leq m - 1$. Otherwise, if the proper base-point of ϕ_1 is not a fixed point of h, then since by hypothesis $ghg^{-1} = \phi_m \cdots \phi_1 h \phi_1^{-1} \cdots \phi_m^{-1}$ has length $\leq m$, it cannot be reduced.

It follows necessarily that $(\phi_1 h)\phi_1^{-1}$ is not reduced, which implies in turn that ϕ_1 is a reversion and that every irreducible curve in B has self-intersection ≥ -2 (using again case 3(c) of Proposition 2.5). In this case, $(\phi_1 h)\phi_1^{-1}$ is a nonreduced composition of two reversions, and has thus length at most $1 \leq m-1$.

(2) If h has length $n \geq 1$, then we consider a reduced decomposition $h = \psi_n \cdots \psi_1$ of h into fibered modifications and reversions. Since $gh^{-1} = \phi_m \cdots \phi_1 \psi_1^{-1} \cdots \psi_n^{-1}$ is not reduced, then so is $\nu = \phi_1 \psi_1^{-1}$, which has thus length ≤ 1. This implies that ϕ_1 and ψ_1 are both fibered modifications or both reversions.

(a) If $n = 1$ and ϕ_1 and $h = \psi_1$ are both fibered modifications, then $\phi_1 h \phi_1^{-1}$ is a fibered modification or an automorphism of the pair (X_1, B_1), and hence has length $\leq 1 \leq m-1$. Otherwise, if $n = 1$ and ϕ_1 and $h = \psi_1$ are both reversions, then either ν is an isomorphism of pairs, in which case $\phi_1 h \phi_1^{-1}$ has length 1, or it is a reversion and then all irreducibles curves of B and B' have self-intersection ≥ -2. This implies that $\phi_1 h \phi_1^{-1}$ is again a reversion or an automorphism, and hence has length ≤ 1.

(b) Finally, if $n \geq 2$, then the composition $\phi_2 \nu \psi_2^{-1}$ is not reduced. Let us observe that this implies that ν is an isomorphism of pairs. If ν is a fibered modification, so are ϕ_1, ψ_1, and hence ϕ_2, ψ_2 are reversions because $\phi_2 \phi_1$ and $\psi_2 \psi_1$ is reduced. This contradicts the fact that $\phi_2 \nu \psi_2^{-1}$ is not reduced. If ν is a reversion, then so are ϕ_1, ψ_1, and all irreducibles curves of B and B' have self-intersection ≥ -2. This implies, as before, that ϕ_2, ψ_2 are fibered modification and contradicts the fact that $\phi_2 \nu \psi_2^{-1}$ is not reduced.

Replacing h with h^{-1}, we conclude that $\phi_1 \psi_n$ is an isomorphism of pairs. This implies that $\phi_1 h \phi_1^{-1}$ has length at most $n-2 \leq m-2$ and completes the proof. ∎

Remark 2.8. Writing $S = X \setminus B$, Proposition 2.7 implies that the image of an algebraic subgroup of $\mathrm{Aut}(S)$ under the morphism $\mathrm{Aut}(S) \to \Pi_1(\mathcal{F}_S)$ described in Section 2.3.1 is very special: there is at most one nontrivial element in the image, which consists, if it exists, of one path of the form $\varphi^{-1}\sigma\varphi$, where φ is a path from $[(X, B)]$ to another vertex $[(X', B')]$ and σ a loop of length 1 based at the vertex $[(X', B')]$ representing a reversion $(X', B') \dashrightarrow (X'', B'')$ between two isomorphic pairs (see Definition 2.3.1). This implies in turn that in most cases, in particular whenever the type $(0,-1,-a_1,\dots,-a_r)$ of B does not satisfy $(a_1,\dots,a_r)=(a_r,\dots,a_1)$, the image of an algebraic subgroup of $\mathrm{Aut}(S)$ in $\Pi_1(\mathcal{F}_S)$ is trivial. In other words, if these conditions are satisfied, every algebraic subgroup of $\mathrm{Aut}(S)$ is contained in the subgroup H generated by automorphisms preserving an $\mathbb{A}^1$-fibration (see Section 2.3.1). □

3 Affine Surfaces Completable by a Standard Pair of Type $(0,-1,-a,-b)$

In this section, we classify all models of standard pairs (X,B) of type $(0,-1,-a,-b)$, $a,b\geq 2$ for which $X\setminus B$ is a normal affine surface.

3.1 Construction of standard pairs

Here we construct standard pairs of type $(0,-1,-a,-b)$, $a,b\geq 2$, in terms of the base points of the birational morphism $\eta\colon Y\to\mathbb{F}_1$ from their minimal resolution of singularities as in Section 2.1.1.

3.1.1

In what follows, we consider $\mathbb{F}_1$ embedded into $\mathbb{P}^2\times\mathbb{P}^1$ as

$$\mathbb{F}_1=\{((x:y:z),(s:t))\subset\mathbb{P}^2\times\mathbb{P}^1\mid yt=zs\};$$

the projection on the first factor yields the birational morphism $\tau:\mathbb{F}_1\to\mathbb{P}^2$, which is the blow-up of $(1:0:0)\in\mathbb{P}^2$ and the projection on the second factor yields the $\mathbb{P}^1$-bundle $\rho:\mathbb{F}_1\to\mathbb{P}^1$, corresponding to the projection of $\mathbb{P}^2$ from $(1:0:0)$. We denote by $F,L\subset\mathbb{P}^2$ the lines with equations $z=0$ and $y=0$, respectively. We also call $F,L\subset\mathbb{F}_1$ their proper transforms on $\mathbb{F}_1$, and denote by $C\subset\mathbb{F}_1$ the exceptional curve $\tau^{-1}((1:0:0))=(1:0:0)\times\mathbb{P}^1$. The affine line $L\setminus C\subset\mathbb{F}_1$ and its image $L\setminus(1:0:0)\subset\mathbb{P}^2$ will be called L_0. The morphism $\mathbb{A}^2=\mathrm{Spec}(k[x,y])\to\mathbb{P}^2\times\mathbb{P}^1$, $(x,y)\to((x:y:1),(y:1))$ induces an open embedding of $\mathbb{A}^2$ into $\mathbb{F}_1$ as the complement of $F\cup C$ for which L_0 coincides with the line $y=0$. With this notation each of the points blown-up by η belongs, as proper or infinitely near point, to the affine line L_0 and is defined over $\bar{k}$ but not necessarily over k; however, the set of all points blown-up by η is defined over k.

3.1.2

Given polynomials $P,Q\in k[w]$ of degrees $a-1$ and $b-1$, respectively, we define a birational morphism $\eta_{P,Q}:Y\to\mathbb{F}_1$ that blows-up $a+b-1$ points that belong, as proper or infinitely near points, to $L \setminus C$. The morphism $\eta_{P,Q}$ is equal to $\eta_{wP} \circ \epsilon_{P,Q}$, where η_{wP} and $\epsilon_{P,Q}$ are birational morphisms that blow-up, respectively, a and $b-1$ points, defined as follows:

1. The map $\eta_{wP} : W \to \mathbb{F}_1$ is a birational morphism associated to $wP(w)$ that blows-up a points as follows. Let $\alpha_0 = 0, \alpha_1, \ldots, \alpha_l$ be the distinct roots of $wP(w)$ with respective multiplicities $r_0 + 1 > 0$ and $r_i > 0$, $i = 1, \ldots, l$. Then η_{wP} is obtained by first blowing each of the points $(\alpha_i, 0) \in (L \setminus C)(\bar{k})$ and then $r_i - 1$ other points in their respective infinitesimal neighborhoods, each one belonging to the proper transform of L. We denote by A_i the last exceptional divisor produced by this sequence of blow-ups over each point $(\alpha_i, 0)$, $i = 0, \ldots, l$ and by $\mathcal{A}_i$ the union of the $r_i - 1$ other components in the inverse image of $(\alpha_i, 0)$ of self-intersection (-2). The r_0 blow-ups above $(0, 0)$ are described locally by $(u, v) \mapsto (x, y) = (u, u^{r_0} v)$; the curve $\{v = 0\}$ corresponds to the proper transform of L. The last step consists of the blow-up of $(u, v) = (0, 0)$ with exceptional divisor E_2.
2. The map $\epsilon_{P,Q} : Y \to W$ is the blow-up of $b-1$ points as follows. If $P(0) \neq 0$, then we let $\beta_1, \ldots, \beta_m$ be the distinct roots of Q in $\bar{k}$. Otherwise, if $P(0) = 0$, then we let $\beta_0 = 0$ and we denote by $\beta_1, \ldots, \beta_m$ the nonzero distinct roots of Q in $\bar{k}$. In each case, we denote by s_j the multiplicity of β_j as a root of Q. Then $\epsilon_{P,Q}$ consists for every $j = 0, 1, \ldots, m$ of the blow-up of the point in $E_2 \setminus L(\bar{k})$ corresponding to the direction $u + \beta_j v = 0$ followed by the blow-up of $s_j - 1$ other points in its infinitesimal neighborhood, each one belonging to the proper transform of E_2. For every $j = 0, \ldots, m$, we denote, respectively, by B_j and $\mathcal{B}_j$ the last exceptional divisor and the union of the $s_j - 1$ other components of self-intersection -2 of the exceptional locus of $\epsilon_{P,Q}$ over the corresponding point of E_2.

We denote by E_1 the proper transform of L on the smooth projective surface Y obtained by the above procedure and we let $E = E_1 \triangleright E_2$. Then the contraction of every exceptional divisor of $\eta_{P,Q}$ not intersecting E yields a birational morphism $\mu_{P,Q} : Y \to X$ to a normal projective surface X for which the zigzag $B = F \triangleright C \triangleright E_1 \triangleright E_2$ has type $(0, -1, -a, -b)$. The definition of η_{wP} and $\epsilon_{P,Q}$ implies that when $P(0) = 0$ the direction of the line $u = 0$ is a special point in $E_2 \subset W$ corresponding to the curve contracted by η_{wP} which intersects E_2. Furthermore, this point is blown-up by $\epsilon_{P,Q}$ if and only if $Q(0) = 0$.

This leads to the following three possible cases below:

3.1.3 *Case* I $P(0) \neq 0$

The unique degenerate fiber F_0 of the rational pencil $\bar{\pi} : X \to \mathbb{P}^1$ defined by the proper transform of F consists of the total transform $(\mu_{P,Q})_*(\eta_{P,Q})^* L$ of L. The multiplicities of the roots of P and Q in $\bar{k}$ coincide with that of the corresponding irreducible components of F_0. Furthermore, each multiple root of P (respectively, of Q) yields a cyclic quotient rational double point of X of order r_i (respectively, s_j) supported on the corresponding irreducible component of F_0.

3.1.4 *Case* II

$P(0) = 0$ and $Q(0) \neq 0$. In the unique degenerate fiber $F_0 = (\mu_{P,Q})_*(\eta_{P,Q})^* L$ of the induced rational pencil $\bar{\pi} : X \to \mathbb{P}^1$ the multiplicities of the roots of P in $\bar{k}$ coincide with that of the corresponding irreducible components A_i, $i = 0, \ldots, l$, whereas each irreducible component B_j, $j = 1, \ldots, m$ corresponding to a root β_j of Q has multiplicity $(r_0 + 1)s_j$ in F_0. Similarly, as in case I, each multiple root of P (respectively, of Q) yields a cyclic quotient rational double point of X supported on the corresponding irreducible component of F_0.

3.1.5 *Case* III*:* $P(0) = Q(0) = 0$

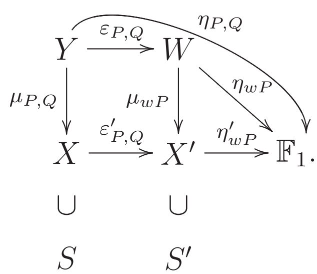
In this model again, the multiplicity of the nonzero roots of P coincide with that of the corresponding irreducible components of the degenerate fiber $F_0 = (\mu_{P,Q})_*(\eta_{P,Q})^* L$ of $\bar{\pi} : X \to \mathbb{P}^1$, each supporting a cyclic quotient rational double point of order r_i. A component B_j of F_0 corresponding to a nonzero root of Q has multiplicity $(r_0 + 1)s_j$ and supports a cyclic quotient rational double point of order s_j. Finally, the irreducible component B_0 of F_0 corresponding to the common root 0 of P and Q has multiplicity $(s_0 + 1)(r_0 + 1) - 1$. Furthermore, it supports a singular point of X whose minimal resolution is a zigzag $\mathcal{B}_0 \triangleright E_3 \triangleright \mathcal{A}_0$, where E_3 is rational curve with self-intersection -3 and where $\mathcal{B}_0$ and $\mathcal{A}_0$ are chains of $s_0 - 1$ and $r_0 - 1$ (-2)-curves, respectively.

Remark 3.1. Case III always leads to a singular surface $X \setminus B$ while in case I and II, the resulting affine surface $X \setminus B$ is smooth if and only if the polynomials $wP(w)$ and $Q(w)$ both have simple roots in $\bar{k}$. The induced $\mathbb{A}^1$-fibration $\pi = \bar{\pi}\mid_{X \setminus B} : X \setminus B \to \mathbb{A}^1$ has a unique degenerate fiber $\pi^{-1}(0)$ consisting of $m + l + 1$ components (recall that $m + 1$ and l are the number of distinct roots of the polynomials P and Q, respectively). If $X \setminus B$ is smooth, then the latter is reduced in case I whereas in case II each root of Q gives rise to an irreducible component of $\pi^{-1}(0)$ of multiplicity two. □

3.1.6

In each of the above three cases, it follows from the construction that the quasi-projective surface $S = X \setminus B$ does not contain any complete curve. Furthermore, one checks for instance that the divisor $D = 4abF + 3abC + 2bE_1 + E_2$ has a positive intersection with its irreducible components hence positive self-intersection. Hence, B is the support of an ample divisor by virtue of the Nakai–Moishezon criterion and so S is a normal affine surface.

The contraction in the intermediate projective surface W of every exceptional divisor of η_{wP} not intersecting E_1 yields a birational morphism $\mu_{wP} : W \to X'$ to a normal projective surface X' for which the zigzag $B' = F \triangleright C \triangleright E_1$ has type $(0, -1, -a)$. The morphisms $\eta_{wP} : W \to \mathbb{F}_1$ and $\varepsilon_{P,Q} : Y \to X$ descend, respectively, to birational morphisms $\eta'_{wP} : X' \to \mathbb{F}_1$ and $\varepsilon'_{P,Q} : X \to X'$ for which the following diagram is commutative:



With the choice of coordinates made in Section 3.1.1, the affine surface $S' = X' \setminus B'$ embeds into $\mathbb{A}^3 = \mathrm{Spec}(k[x, y, u])$ as the subvariety defined by the equation $yu = xP(x)$ in such a way that the restriction of η'_{wP} to it coincides with the projection $\mathrm{pr}_{x,y}|_{S'} : S' \to \mathbb{A}^2 \subset \mathbb{F}_1$ (see, e.g., [1, Lemma 5.4.4]). One checks further that $S = X \setminus B$ can be embedded into $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ as the subvariety S given by the following system of equations:

$$yu = xP(x),$$

$$xv = uQ(u),$$

$$yv = P(x)Q(u),$$

so that $\varepsilon'_{P,Q}: X \to X'$ restricts on S to the projection $\mathrm{pr}_{x,y,u}|_S: S \to S' \subset \mathbb{A}^3$. In this description, the intersection with S of the irreducible components A_i and B_j of F_0 coincide, respectively, with the irreducible components $\{y = x - \alpha_i = 0\}$, $i = 1, \ldots, l$, and $\{y = x = u - \beta_j\}$, $j = 1, \ldots, m$, of the degenerate fiber of the induced $\mathbb{A}^1$-fibration $\bar{\pi}|_S = \mathrm{pr}_y: S \to \mathbb{A}^1$.

3.2 Isomorphism classes

Here we show that the construction of the previous subsection describes all possible isomorphism types of normal affine $\mathbb{A}^1$-fibered surfaces admitting of a completion by a standard pair of type $(0, -1, -a, -b)$. We characterize their isomorphism classes in terms of the corresponding polynomials P and Q.

Proposition 3.2. Let $(X, B = F \triangleright C \triangleright E_1 \triangleright E_2, \bar{\pi})$ be a standard pair of type $(0, -1, -a, -b)$, $a, b \geq 2$, with a minimal resolution of singularities $\mu: (Y, B, \bar{\pi} \circ \mu) \to (X, B, \bar{\pi})$ and let $\eta: Y \to \mathbb{F}_1$ be the birational morphism as in Section 3.1.1. If $X \setminus B$ is affine, then the morphisms η, μ are equal to that $\eta_{P,Q}$, $\mu_{P,Q}$ defined in Section 3.1.2, for some polynomials $P, Q \in k[w]$ of degree $a - 1$ and $b - 1$, respectively.

In particular, every normal affine surface completable by a zigzag of type $(0, -1, -a, -b)$ $(a, b \geq 2)$ is isomorphic to one in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by a system of equations of the form

$$yu = xP(x),$$

$$vx = uQ(u),$$

$$yv = P(x)Q(u). \qquad \square$$

Proof. Since η maps C to the (-1)-curve of $\mathbb{F}_1$, it follows that E_1 is the strict transform of $L \subset \mathbb{F}_1$. We may factor η as $\eta_1 \circ \eta_2$, where $\eta_1: Y' \to \mathbb{F}_1$ is the minimal blow-up which extracts E_2 and $\eta_2: Y \to Y'$ is another birational morphism. By definition η_1 is the blow-up of a sequence of points $p_1, \ldots, p_n$ such that for every $i = 2, \ldots, n$, p_i is in the first neighborhood of p_{i-1} and such that E_2 is the exceptional divisor of the blow-up of p_n. The fact that E_2 and E_1 intersect each other implies that p_n and hence each p_i belong to the strict transform of $L_0 = L \setminus C$. Since $S = X \setminus B$ is affine, it follows from [11, Lemma 1.4.2 p. 195] that every (-1)-curve in the degenerate fiber of $\bar{\pi} \circ \mu$ intersects either E_1 or E_2. This implies that there exist points $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k \in Y'$ where each α_i belongs to $E_1 \setminus (E_2 \cup C) \cong \mathbb{A}^{1*}_{\bar{k}}$, each β_i belongs to $E_2 \setminus E_1 \cong \mathbb{A}^1{}_{\bar{k}}$, and some multiplicities associated to them, so that η_2 is the blow-up of the points α_i and β_i and of infinitely near points belonging only to E_1 and E_2, respectively. Taking an appropriate parameterization for the α_i's in $E_1 \setminus (C \cup E_2) \cong \mathbb{A}^{1*}_{\bar{k}}$ and the β_i's in $E_2 \setminus E_1 \cong \mathbb{A}^1{}_{\bar{k}}$ yields the polynomials P and Q, respectively. ■

Notation 3.3. In the sequel, we will say that two polynomials $P, Q \in k[w]$ are equivalent, and write $P \sim Q$, if there exist $\alpha, \beta \in k^*$ such that $Q(w) = \alpha P(\beta w)$. This yields an equivalence relation on the set of polynomials in one variable. □

Proposition 3.4. Let $(X, B, \bar{\pi})$ and $(X', B', \bar{\pi}')$ be two standard pairs of type $(0, -1, -a, -b)$ obtained from pairs of polynomials (P, Q) and (P', Q') via the construction of Section 3.1.2.

1. The pairs (X, B) and (X', B') are isomorphic if and only if the $\mathbb{A}^1$-fibered surfaces $(X \setminus B, \bar{\pi}|_{X\setminus B})$ and $(X' \setminus B', \bar{\pi}'|_{X'\setminus B'})$ are isomorphic.
2. The pairs (X, B) and (X', B') are isomorphic if and only if one of the following holds:
   (a) $P(0)P'(0) \neq 0$ and $P' \sim P$ (in the sense of Notation 3.3), $Q'(w) \sim Q(w+t)$ for some $t \in k$.
   (b) $P(0) = P'(0) = 0$ and $P' \sim P$, $Q' \sim Q$.
3. Letting r_0 be the multiplicity of 0 in P, the automorphism group $\mathrm{Aut}(X, B)$ of the pair (X, B) consists of lifts of automorphisms of $\mathbb{A}^2 \subset \mathbb{F}_1$ of the form

$$\{(x, y) \mapsto (ax + by, cy) | P(aw)/P(w) \in k^*,\ Q((aw - b)/c)/Q(w) \in k^*\} \quad \text{if } r_0 = 0,$$

$$\{(x, y) \mapsto (ax + by, cy) | P(aw)/P(w) \in k^*,\ Q(a^{r_0+1}/c \cdot w)/Q(w) \in k^*\} \quad \text{if } r_0 \geq 1. \quad □$$

Proof. Let $\mu_{P,Q} : Y \to X$ and $\eta_{P,Q} : Y \to \mathbb{F}_1$ be the morphisms defined in Section 3.1.2, and the same with primes. By virtue of [1, Lemma 5.2.1 or Lemma 2.2.3], $(X \setminus B, \bar{\pi}|_{X\setminus B})$ and $(X' \setminus B', \overline{\pi'}|_{X'\setminus B'})$ are isomorphic if and only if there exists an automorphism ψ of $\mathbb{A}^2 \subset \mathbb{F}_1$ preserving the $\mathbb{A}^1$-fibration pr_y and sending the base locus Z of $\eta_{P,Q}^{-1}$ isomorphically onto that Z' of $\eta_{P',Q'}^{-1}$ while $(X, B, \bar{\pi})$ and $(X', B', \bar{\pi}')$ are isomorphic if and only if there exists an affine automorphism ψ of this type. An automorphism f preserving the fibration pr_y and mapping Z isomorphically onto Z' must preserve the fiber $L_0 = \mathrm{pr}_y^{-1}(0)$ and fix the point $(0, 0)$. Thus, f has the form f: $(x, y) \mapsto (ax + yR(y), cy)$, with $a, c \in k^*$ and $R \in k[y]$. Such an automorphism acts on L_0 by $x \mapsto ax$. We identify points of $E_2 \setminus L$ with directions $u + \beta v = 0$ in the blow-up $(u, v) \mapsto (u, u^{r_0} v)$ as in Section 3.1.2, where $r_0 \geq 0$ denotes the multiplicity of 0 as a root of P. We claim that f acts on E_2 in the following way:

$$\beta \mapsto a\beta - R(0)/c \quad \text{if } r_0 = 0,$$

$$\beta \mapsto a^{r_0+1}/c \cdot \beta \quad \text{if } r_0 \geq 1.$$

Indeed, if $r_0 = 0$, then the action of f^{-1} on the tangent directions is given by $u + \beta v \mapsto au + vR(0) + \beta cv = a(u + v(R(0) + \beta c)/a)$ and so f maps β to $(a\beta - R(0))/c$. Otherwise, if $r_0 > 0$, then the lift of f by $(u, v) \dashrightarrow (u, u^{r_0}v)$ takes the form

$$(u, v) \mapsto (au + u^{r_0}vR(u^{r_0}v), cv/(a + u^{r_0-1}vR(u^{r_0}v))^{r_0}).$$

In the local chart $(\hat{u}, v) \mapsto (\hat{u}v, v) = (u, v)$ of the blow-up of the origin $(0, 0)$, the latter lifts further to the map

$$(\hat{u}, v) \mapsto \left(\frac{(a\hat{u} + (\hat{u}v)^{r_0}R((\hat{u}v)^{r_0}v))(a + (\hat{u}v)^{r_0-1}vR((\hat{u}v)^{r_0}v))^{r_0}}{c}, \frac{cv}{(a + (\hat{u}v)^{r_0-1}vR((\hat{u}v)^{r_0}v))^{r_0}} \right).$$

By construction the tangent direction $u + \beta v = 0$ corresponds to the point $(\hat{u}, v) = (-\beta, 0)$ which is thus mapped to $(-\beta \cdot a^{r_0+1}/c, 0)$ as claimed.

It follows from the above description that the affine automorphism $\psi: (x, y) \mapsto (ax + yR(0), cy)$ also maps Z isomorphically onto Z' and so, we obtain the equivalence between isomorphism classes of standard pairs and isomorphism classes between induced $\mathbb{A}^1$-fibered surfaces. The second assertion then follows immediately from the description of the action of ψ on L_0 and E_2.

Finally, as explained earlier, the group $\mathrm{Aut}(X, B)$ consists of lifts of automorphisms of $\mathbb{F}_1$ which preserve the set Z. Since they fix the origin $(0, 0)$, these automorphism can be written in the form $(x, y) \mapsto (ax + by, cy)$, where $a, c \in k^*$, $b \in k$. By virtue of the above description, the induced action on the line $L_0 = \mathrm{pr}_Y^{-1}(0)$ which supports the points of Z corresponding to roots of P is given by $x \mapsto ax$, whereas the action on the line $E_2 \setminus L$ which supports the points of Z corresponding to roots of Q is either $\beta \mapsto (a\beta - b)/c$, or $\beta \mapsto a^{r_0+1}/c \cdot \beta$, depending on whether r_0 is 0 or positive. This yields the last assertion. ■

Corollary 3.5. Let $(X, B, \bar{\pi})$ and $(X', B', \bar{\pi}')$ be two standard pairs of type $(0, -1, -a, -b)$ obtained from pairs of polynomials (P, Q) and (P', Q') via the construction of

Section 3.1.2. If the pairs (X, B) and (X', B') are isomorphic, then one of the following holds:

1. both pairs are of type I, that is, $P(0)P'(0) \neq 0$;
2. both pairs are of type II, that is, $P(0) = P'(0) = 0$ and $Q(0)Q'(0) \neq 0$;
3. both pairs are of type III, that is, $P(0) = P'(0) = Q(0) = Q'(0) = 0$. □

Proof. Follows directly from Proposition 3.4. ■

Remark 3.6. In general, in contrast with assertion (1) in Proposition 3.4, nonisomorphic standard pairs $(X, B, \bar{\pi})$ and $(X', B', \bar{\pi}')$ can give rise to isomorphic $\mathbb{A}^1$-fibered quasi-projective surfaces $(X \setminus B, \pi|_{X\setminus B})$ and $(X' \setminus B', \pi'|_{X'\setminus B'})$. For instance, let $(X_a, B_a, \bar{\pi}_a), a \in k$ be the family of smooth standard pairs obtained from $\mathbb{F}_1$ by the following sequences of blow-ups $\eta_a : X_a \to \mathbb{F}_1$: we first blow-up the points $(1, 0)$ and $(0, 0)$ in $\mathbb{A}^2 \subset \mathbb{F}_1$ with respective exceptional divisors A_1 and E_2. Then we blow-up the point on $E_2 \setminus L$ corresponding to the direction $x = 0$ with exceptional divisor E_3. The last step is the blow-up of a k-rational point $a \in E_3 \setminus E_2 \simeq \mathbb{A}^1_k$, with exceptional divisor A_2. We denote the corresponding surface by X_a and we let $B_a = F \triangleright C \triangleright E_1 \triangleright E_2 \triangleright E_3$, where E_1 denotes the proper transform of L.

We claim that the standard pairs (X_a, B_a) are pairwise nonisomorphic, while the corresponding $\mathbb{A}^1$-fibered surfaces $(X_a \setminus B_a, \bar{\pi}_a|_{X_a\setminus B_a})$, $a \in k$ are all isomorphic. Indeed, recall that by virtue of [1, Lemma 5.2.1 or Lemma 2.2.3], the $\mathbb{A}^1$-fibered surfaces $(X_a \setminus B_a, \bar{\pi}_a|_{X_a\setminus B_a})$ and $(X_{a'} \setminus B_{a'}, \bar{\pi}_{a'}|_{X_{a'}\setminus B_{a'}})$ are isomorphic if and only if the points blown-up by η_a and $\eta_{a'}$ belong to a same orbit of the action of the group $\mathbf{Jon} \subset \mathrm{Aut}(\mathbb{A}^2)$ of automorphisms of the form $\{(x, y) \mapsto (ax + Q(y), cy) \mid Q \in k[y]\}$, while the standard pairs (X_a, B_a) and $(X_{a'}, B_{a'})$ are isomorphic if and only if the corresponding points belong to a same orbit of the action of the subgroup $\mathbf{Aff} \cap \mathbf{Jon}$ consisting of maps where the polynomial Q has degree ≤ 1. Now on the one hand, the same computation as in the proof of Proposition 3.4 shows that an element of $\mathbf{Jon}$ mapping the points blown-up by η_a onto those blown-up by $\eta_{a'}$ must have the form $(x, y) \mapsto (x + y^2R(y), cy)$ with $c \in k^*$ and $R(y) \in k[y]$. This implies in particular that the standard pairs (X_a, B_a) are pairwise nonisomorphic. On the other hand, since automorphisms of the form $(x, y) \mapsto (x + by^2, cy)$, $c \in k^*$ and $b \in k$, act transitively on the directions corresponding to points in $E_3 \setminus E_2$, it follows that the $\mathbb{A}^1$-fibered surfaces $(X_a \setminus B_a, \bar{\pi}_a|_{X_a\setminus B_a})$, $a \in k$ are all isomorphic, as desired.

Of course, the same procedure but involving the blow-up of more points leads to similar families of isomorphic $\mathbb{A}^1$-fibered surfaces arising from nonisomorphic standard pairs. □

4 Reversions Between Standard Pairs of Type $(0, -1, -a, -b)$

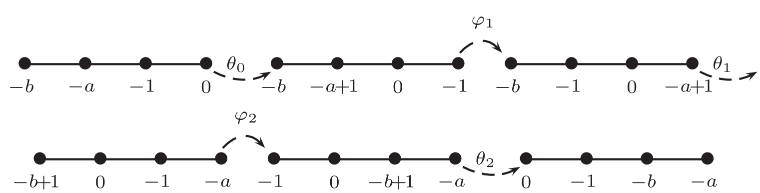
To classify the existing $\mathbb{A}^1$-fibrations on the affine surfaces constructed in the previous section, the next step consists of studying birational maps between the corresponding standard pairs (X, B). In view of the description recalled in Section 2.2, this amounts to describing all possible fibered modifications and reversions between these pairs. Since Proposition 3.4 guarantees that there cannot exists fibered modifications between two nonisomorphic pairs, it remains to characterize possible reversions between these.

4.1 Preliminaries

Here we set up notations that will be used in the sequel to describe the geometry of the different pairs that can obtained by reversing a given standard pair $(X, B = F \triangleright C \triangleright E)$ of type $(0, -1, -a, -b)$.

4.1.1

For such pairs, the general description of reversions given in Definition 2.3 specializes to the following simpler form: Given a k-rational point $p \in F \setminus C$, the contraction of C followed by the blow-up of p yields a birational map $\theta_0 : (X, B) \dashrightarrow (X_0, B_0)$ to a pair with a zigzag of type $(-b, -a+1, 0, -1)$. Then we produce a birational map $\varphi_1 : (X_0, B_0) \dashrightarrow (X_1', B_1')$, where B_1' is of type $(-b, -1, 0, -a+1)$. The blow-down of the (-1)-curve in B_1' followed by the blow-up of the point of intersection of its (0)-curve with the curve immediately after it yields a birational map $\theta_1 : (X_1', B_1') \dashrightarrow (X_1, B_1)$, where B_1 is a zigzag of type $(-b+1, 0, -1, -a)$. Repeating this process yields birational maps $\theta_0, \varphi_1, \theta_1, \varphi_2, \theta_2$ described by the following figure.



The reversion of (X, B) with center at $p \in F \setminus C$ is then the composition

$$\phi = \theta_2 \varphi_2 \theta_1 \varphi_1 \theta_0 : (X, B) \dashrightarrow (X', B') = (X_2, B_2).$$

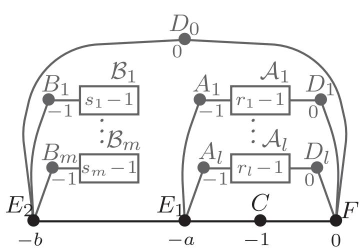
4.1.2

By virtue of Proposition 3.2, we may assume that the initial pair (X, B) is obtained from a pair of polynomial $P, Q \in k[w]$ of respective degrees $a-1, b-1 \geq 1$ by means of the construction described in Section 3.1.2. We let

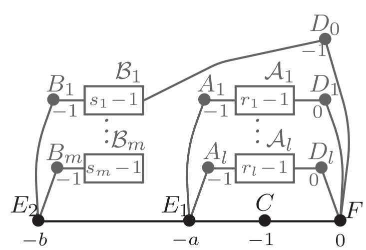
$$(X, B = F \triangleright C \triangleright E_1 \triangleright E_2, \bar{\pi}) \xleftarrow{\mu_{P,Q}} (Y, B, \bar{\pi}\mu_{P,Q}) \xrightarrow{\eta_{P,Q}} (\mathbb{F}_1, F \triangleright C \triangleright L)$$

be the corresponding birational morphisms, where $\eta_{P,Q} = \eta_{wP} \circ \varepsilon_{P,Q}$. The k-rational point $p \in F \setminus C$ corresponds via $\tau : \mathbb{F}_1 \to \mathbb{P}^2$ to a point $(\lambda : 1 : 0) \in \mathbb{P}^2$ for some $\lambda \in k$. For every root α_i of P in $\bar{k}$ we denote by $D_i \subset Y$ the proper transform by $(\tau \circ \eta_{P,Q})^{-1}$ of the line $x - \lambda y - \alpha_i z = 0$, passing through $(\lambda : 1 : 0)$ and $\tau(\alpha_i) = (\alpha_i : 0 : 1)$. Recall that if $P(0) \neq 0$, then $\beta_1, \ldots, \beta_m$ denote the distinct roots of Q in $\bar{k}$. With this notation, we have the following description.

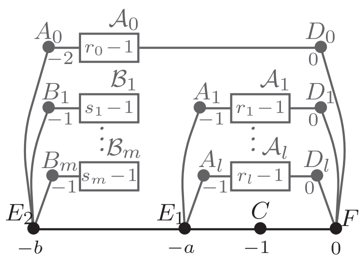
Lemma 4.1. The possible dual graphs for the divisor $\eta_{P,Q}^{-1}(F \triangleright C \triangleright L) \cup \bigcup_{i=0}^{m} D_i$ are the following:



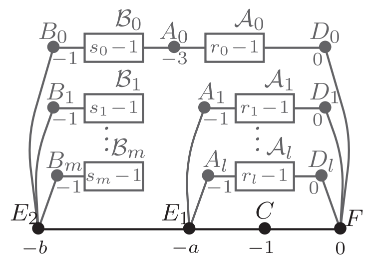
Ia: $P(0) \neq 0$, and $\lambda \notin \{\beta_1, \ldots, \beta_m\}$



Ib: $P(0) \neq 0$, and $\lambda \in \{\beta_1, \ldots, \beta_m\}$ (for the picture, $\lambda = \beta_1$)



II: $P(0) = 0$, and $Q(0) \neq 0$



III: $P(0) = Q(0) = 0$ □

Proof. The structure of the dual graph of $\eta_{P,Q}^{-1}(F \triangleright C \triangleright L)$ has already been discussed in Section 3.1.2 (see Figures 1–3). It remains to consider the properties of the additional

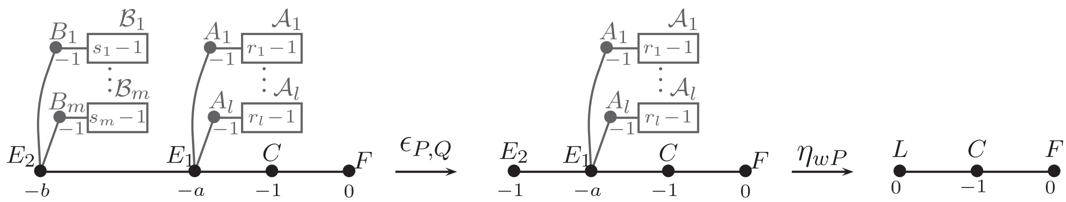


Fig. 1. The morphisms $(Y, B) \stackrel{\epsilon_{P,Q}}{\to} (W, B) \stackrel{\eta_{wP}}{\to} (\mathbb{F}_1, F \triangleright C \triangleright L)$ when $P(0) \neq 0$. A block with label t consists of a zigzag of t (-2)-curves.

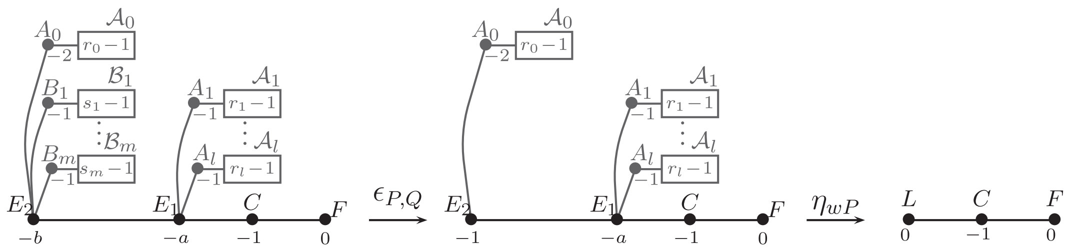


Fig. 2. The morphisms $(Y, B) \stackrel{\epsilon_{P,Q}}{\to} (W, B) \stackrel{\eta_{wP}}{\to} (\mathbb{F}_1, F \triangleright C \triangleright L)$ when $P(0) = 0$ and $Q(0) \neq 0$.

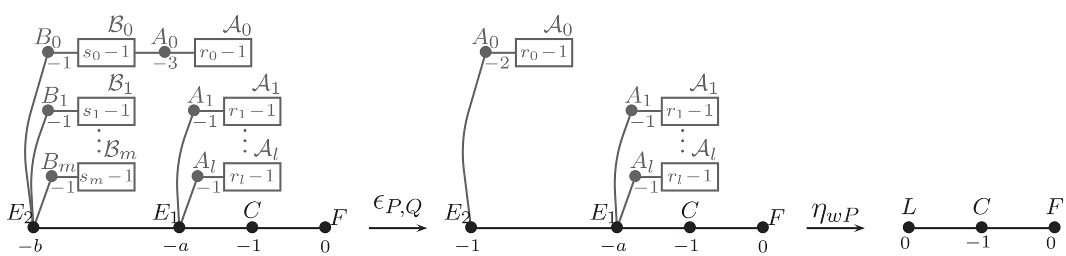


Fig. 3. The morphisms $(Y, B) \stackrel{\epsilon_{P,Q}}{\to} (W, B) \stackrel{\eta_{wP}}{\to} (\mathbb{F}_1, F \triangleright C \triangleright L)$ when $P(0) = Q(0) = 0$.

curves D_i, $i = 0, \ldots, l$. By definition, D_i is the proper transform of the line $L_i \subset \mathbb{P}^2$ of equation $x = \lambda y + \alpha_i z$. Since the latter does not pass through the point $(1:0:0)$ blown-up by τ, its proper transform in $\mathbb{F}_1$ still has self-intersection 1. Moreover, since D_i passes through α_i (corresponding to $(\alpha_i : 0 : 1)$) and not through any other α_j, with $j \neq i$, its proper transform by $\eta_{wP} : W \to \mathbb{F}_1$ has self-intersection 0 in W and it intersects $\eta_{wP}^{-1}(F \triangleright C \triangleright L)$ transversally at a point of A_i if α_i is a simple root of $wP(w)$ and at a point on the last component of $\mathcal{A}_i$ otherwise. The only case where a point belonging to the proper transform of L_i in W is blown-up by $\varepsilon_{P,Q}$ is when $i = 0$, $P(0) \neq 0$, and L_0 corresponds to a tangent direction $x = \beta_j y$ for a certain root of Q in $\bar{k}$. In this case, D_0 has self-intersection -1 in Y and it intersects $\eta_{P,Q}^{-1}(F \triangleright C \triangleright L)$ transversally at a point of B_j if

β_j is a simple root of Q and at a point on the last component of $\mathcal{B}_j$ otherwise. This gives all diagrams pictured above (recall that $A_0 = E_2$ if $P(0) \neq 0$). ■

4.2 Classification of reversions

Recall that two standard pairs (X, B) and (X', B') of respective types $(0, -1, -a, -b)$ and $(0, -1, -a', -b')$ can be linked by a reversion only if $a' = b$ and $b' = a$ (see, e.g., Section 4.1.1). To decide which types of reversions can occur between the different models of standard pairs, we may thus consider the situation when (X, B) and (X', B') are obtained by means of the construction of Section 3.1.2 for pairs of polynomials (P, Q) and (P', Q') of degrees $(a-1, b-1)$ and $(b-1, a-1)$, respectively. We let

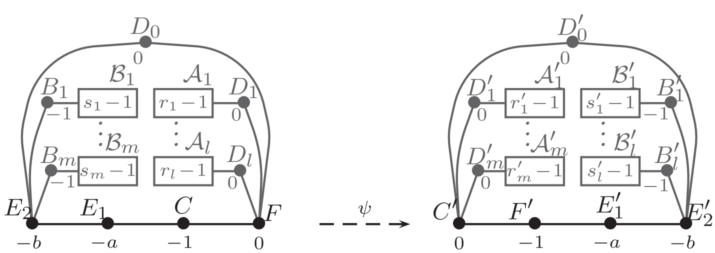
$$(X, B = F \triangleright C \triangleright E_1 \triangleright E_2, \bar{\pi}) \stackrel{\mu_{P,Q}}{\leftarrow} (Y, B, \bar{\pi}\mu_{P,Q}) \stackrel{\eta_{P,Q}}{\rightarrow} (\mathbb{F}_1, F \triangleright C \triangleright L),$$

$$(X', B' = F' \triangleright C' \triangleright E_1' \triangleright E_2', \bar{\pi}') \stackrel{\mu_{P',Q'}}{\leftarrow} (Y', B', \bar{\pi}'\mu_{P',Q'}) \stackrel{\eta_{P',Q'}}{\rightarrow} (\mathbb{F}_1, F \triangleright C \triangleright L)$$

be as in Section 3.1.2. Given a reversion $\phi : (X, B) \dashrightarrow (X', B')$ centered at $p \in F \setminus C$, with ϕ^{-1} centered at $p' \in F' \setminus C'$, which correspond, respectively, to $(\lambda : 1 : 0), (\lambda' : 1 : 0) \in \mathbb{P}^2$, we use the notation of Section 4.1.2 for $\alpha_1, \ldots, \alpha_l \in \bar{k}$, $A_i, B_i, \mathcal{A}_i, \mathcal{B}_i, D_i \subset Y$ and the same notation with primes on Y'.

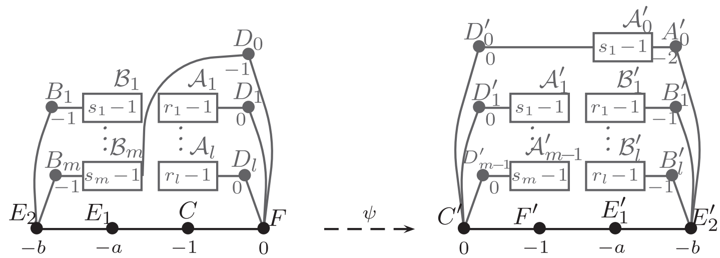
Lemma 4.2. With the notation above, let $\psi = (\mu_{P',Q'})^{-1}\phi\mu_{P,Q} : Y \dashrightarrow Y'$ be the lift of ϕ. Then one of the following three situations occurs:

(1) We have case **I***a* on both Y and Y': $P(0)P'(0)Q(\lambda)Q'(\lambda') \neq 0$. We have $l = m'$, $m = l'$ and up to renumbering, ψ sends $B_i, \mathcal{B}_i, \mathcal{A}_i$, and D_i to $D_i', \mathcal{A}_i', \mathcal{B}_i'$, and B_i, respectively. Moreover, $\psi(D_0) = D_0'$ and the situation is described by the following diagram:



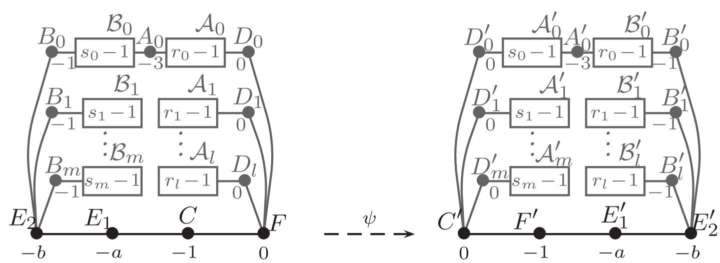
Furthermore, $P'(w) = cQ(dw + \lambda)$, and $P(w) = eQ'(fw + \lambda')$ for some $c, d, e, f \in k^*$.

(2) Up to an exchange of Y and Y', we have case Ib on Y and case **II** on Y': $P(0)Q(0)Q'(0) \neq 0$, $P'(0) = Q(\lambda) = 0$. Up to renumbering, $\lambda = \beta_m$, ψ sends $B_m, \mathcal{B}_m$, and D_0 onto D'_0, $\mathcal{A}'_0$, and A'_0, respectively, and sends the other B_i, $\mathcal{B}_i$, $\mathcal{A}_i$ and D_i onto D'_i, $\mathcal{A}'_i$, $\mathcal{B}'_i$, and B_i, respectively. The situation is described by the following diagram:



Furthermore, $P'(x) = cQ(dx + \lambda)$ for some $c, d \in k^*$, and $P \sim Q'$.

(3) We have case **III** on Y and Y': $P(0) = P'(0) = Q(0) = Q'(0) = 0$. Up to renumbering ψ sends B_i, $\mathcal{B}_i$, $\mathcal{A}_i$, and D_i onto D'_i, $\mathcal{A}'_i$, $\mathcal{B}'_i$, and B_i, respectively. Moreover, ψ sends A_0 onto A'_0. The situation is described by the following diagram:



Furthermore, $P' \sim Q$, and $P \sim Q'$. □

Proof. We decompose ϕ into $\phi = \theta_2\varphi_2\theta_1\varphi_1\theta_0$ as in Section 4.1.1, and use this decomposition to see that E_2 and E'_2 correspond, respectively, to the curves $\mathcal{E}'_p$ and $\mathcal{E}_p$ obtained by blowing-up p' and p.

According to Lemma 4.1, there are four possibilities for the situation on Y, depending on P, Q, λ. We study the image of the curves $D_0, \ldots, D_l$, which intersect F at the point p.

Let $i \in \{0, \ldots, r\}$ and assume that $D_i \subset Y$ does not intersect the boundary $E_2 \cup E_1 \cup C \cup F$ at another point (which occurs in all cases, except for $i = 0$ in case Ia). In the decomposition of ϕ, the curve D_i is affected by the blow-up of $p \in D_i$ and then is not affected by all other maps. In consequence, the image $\phi(D_i)$ of D_i on Y' is a curve that intersects the boundary only at one point, being on E_2', and which has self-intersection $\phi(D_i)^2 = (D_i)^2 - 1$. The curve $\phi(D_i)$ is thus contained in the special fiber and corresponds, therefore, to one of the B_i' if $(D_i)^2 = 0$ and to A_0' in case **II** if $(D_i)^2 = -1$. This shows that we obtain case **II** if and only if we start from **I**b. We can only go to **III** if we start from **III**, because of the special singularity, and then we see that **I**a goes to **I**a.

The diagrams above follow from the discussion made on the image of the D_i. It remains to see the correspondence between P, Q, P', Q', λ, λ'. The map ϕ induces an isomorphism between the blow-up $\mathcal{E}_p$ of p and the line $E_2' \subset Y'$. This isomorphism sends the tangent direction of D_i, which has equation $x - \lambda y = \alpha_i z$, onto $\phi(D_i) \cap E_2$. It also sends the direction of F, which is $z = 0$, onto $E_2' \cap E_1'$. We obtain, therefore, an isomorphism $\mathbb{P}^1 \to E_2'$ that sends $(0:1)$ onto $E_2' \cap E_1'$ and $(1:\alpha_i)$ onto $E_2' \cap \phi(D_i)$ for each i. Studying each of the three diagrams gives P' and Q' in terms of Q and P.

In the first diagram (case **I**a on both sides), E_2' corresponds to the blow-up of $(0:0:1)$, and the intersection of B_i' with E_2' corresponds to the tangent direction of $x = \beta_i' y$ (Section 3.1.2). The curve D_0' is the tangent direction of $x = \lambda' y$. We obtain an affine automorphism of k which sends α_i to β_i' for $i = 1, \dots, l$ and which sends 0 onto λ'. This means that $P(x) = eQ'(fx + \lambda')$ for some $e, f \in k^*$. Doing the same in the other direction, we obtain $P'(x) = cQ(dx + \lambda)$ for $c, d \in k^*$.

In the second diagram (case **I**b on Y and **II** on Y'), the curve E_2' is the blow-up of the point $(u, v) = (0, 0)$ obtained after blowing-up $(0:0:1)$ via $(u, v) \mapsto (u, u^{r_0'} v)$, where $r_0' > 0$ is the multiplicity of 0 in $P'(x)$ (see Section 3.1.2). The intersection of B_i' with E_2' corresponds to the direction of $u = \beta_i' v$, the point $E_1' \cap E_2'$ corresponds to $v = 0$, and $A_0' \cap E_2'$ to $u = 0$. We obtain an automorphism of k^* that sends α_i to β_i' for $i = 1, \dots, l$, so $P(x) = eQ'(fx)$ for some $e, f \in k^*$. To obtain $P'(x)$ from $Q(x)$, we do the same computation in the other direction: we have an isomorphism $\mathbb{P}^1 \to E_2$ that sends $(0:1)$ onto $E_2 \cap E_1$ and $(1:\alpha_i')$ onto $E_2 \cap \phi^{-1}(D_i')$ for each i. Recall that $\phi^{-1}(D_0') = B_m$ and $\phi^{-1}(D_i') = B_i$ for $i = 1, \dots, m$. The curve E_2 corresponds to the blow-up of $(0:0:1)$, the intersection of B_i with E_2 corresponds to the direction $x = \beta_i y$, and $B_m = \phi^{-1}(D_0')$ corresponds to $x = \lambda y$, which is $x = \beta_m y$. We get an affine automorphism of k that sends α_i' onto β_i for $i = 1, \dots, m-1$ and $0 = \alpha_0'$ onto $\beta_m = \lambda$. This implies that $P'(x) = cQ(dx + \lambda)$ for some $c, d \in k^*$.

The last diagram is when Y, Y' are in case **III**, and is symmetrical. As in the second diagram, the curve E_2' is the blow-up of the point $(u, v) = (0, 0)$ obtained after blowing-up $(0:0:1)$ via $(u, v) \mapsto (u, u^{r_0'} v)$, where $r_0' > 0$ is the multiplicity of 0 in $P'(x)$. The intersection of B_i' with E_2' corresponds to the direction of $u = \beta_i' v$, the point $E_1' \cap E_2'$

corresponds to $v=0$, and $A_0' \cap E_2'$ to $u=0$. We obtain an automorphism of k^* that sends α_i onto β_i' for $i=1,\ldots,l$, so $P \sim Q'$. And in the other direction, we get $P' \sim Q$. ■

To conclude this section, we give a complete characterization of when two reversions are equivalent in the sense of Definition 2.6, which will be needed in the next section to describe the graphs associated to the surfaces $X \setminus B$.

Proposition 4.3. Let $(X, B = F \triangleright C \triangleright E_1 \triangleright E_2)$ be a pair constructed from polynomials $P, Q \in k[w]$. For two reversions $\phi_i\colon (X, B) \dashrightarrow (X_i, B_i)$, $i=1,2$, the following are equivalent:

1. The pairs (X_1, B_1) and (X_2, B_2) are isomorphic.
2. The reversions ϕ_1, ϕ_2 are equivalent, that is, there exists $\theta \in \mathrm{Aut}(X, B)$ and an isomorphism $\theta' : (X_1, B_1) \to (X_2, B_2)$, such that $\phi_2 \circ \theta = \theta' \circ \phi_1$.

Moreover, these equivalent properties are always satisfied if $P(0)=0$. □

Proof. The implication $(2) \Rightarrow (1)$ is obvious. Conversely, we may suppose that ϕ_1, ϕ_2 are, respectively, centered at points $p_1, p_2 \in F \setminus C$ which we identify in turn with the points $(\lambda_1 : 1 : 0), (\lambda_2 : 1 : 0) \in \mathbb{P}^2$ (see Section 3.1.2). We denote by (P_1, Q_1) and (P_2, Q_2) the polynomials associated to the pairs (X_1, B_1) and (X_2, B_2). Since the reversions are uniquely determined by the choice of their proper base-point, assertion (2) is equivalent to the existence of an automorphism $\theta \in \mathrm{Aut}(X, B)$ which sends p_1 to p_2. If $P(0)=0$, then the automorphism $(x, y) \mapsto (x + (\lambda_2 - \lambda_1)y, y)$ of $\mathbb{A}^2$ lifts to an automorphism $\theta \in \mathrm{Aut}(X, B)$ (Proposition 3.4) such that $\theta(p_1) = p_2$. So it remains to consider the case when $P(0) \neq 0$ (case **I**). By Lemma 4.2 (assertions (1) and (2)), we have $P_i(w) = c_i Q(d_i w + \lambda_i)$, for some $c_i, d_i \in k^*$. Since (X_1, B_1) and (X_2, B_2) are isomorphic, we also have $P_1(w) = \alpha P_2(\beta w)$, for some $\alpha, \beta \in k^*$ (Proposition 3.4). This yields

$$c_1 Q(d_1 w + \lambda_1) = P_1(w) = \alpha P_2(\beta w) = \alpha c_2 Q(d_2 \beta w + \lambda_2),$$

which implies (replacing w by $(w - \lambda_1)/d_1$) that $Q(d_2\beta(w-\lambda_1)/d_1 + \lambda_2)/Q(w) = \alpha c_2/c_1 \in k^*$. Letting $c = \frac{d_1}{d_2\beta}$ and $b = \lambda_1 - \lambda_2 c$, we obtain that $Q(\frac{w-b}{c})/Q(w) \in k^*$ and $c\lambda_2 = \lambda_1 + b$. The first condition guarantees that the automorphism $\nu\colon (x, y) \mapsto (x + by, cy)$ of $\mathbb{A}^2$ lifts to an automorphism of (X, B) (see Proposition 3.4), while the second equality says precisely that the extension of ν to $\mathbb{P}^2$ maps $p_1 = [\lambda_1 : 1 : 0]$ onto $p_2 = [\lambda_2 : 1 : 0]$. ■

Remark 4.4. Proposition 4.3 implies in particular that if (X, B) is of type $(0, -1, -a, -b)$, in the graph $\mathcal{F}_S$ associated to $S = X \setminus B$ as in Definition 2.6, two arrows corresponding to reversions are equal if and only if they have the same source and target. Consequently, the graph $\mathcal{F}_S$ does not contain any reduced cycle of length 2 and each of its vertices is the base vertex of at most one cycle of length 1. □

Example 4.5. As explained in Section 3.1.6, given polynomials $P, Q \in k[w]$ of degrees ≥ 1, the surface S in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the system of equations

$$yu = xP(x),$$
$$vx = uQ(u),$$
$$yv = P(x)Q(u)$$

comes equipped with the $\mathbb{A}^1$-fibration $\pi = \mathrm{pr}_y|_S$ induced by the restriction of the rational pencil $\bar{\pi}$ on the standard pair $(X, B = F \triangleright C \triangleright E, \bar{\pi})$ associated with the pair (P, Q) via the construction of Section 3.1.2.

The automorphism $(x, y, u, v) \mapsto (u, v, x, y)$ of $\mathbb{A}^4$ induces an isomorphism σ of S with the surface $S' \subset \mathbb{A}^4$ defined by the system of equations

$$yu = xQ(x),$$
$$vx = uP(u),$$
$$yv = Q(x)P(u),$$

which comes equipped with the $\mathbb{A}^1$-fibration $\pi' = \mathrm{pr}_y|_{S'}$ induced by the restriction of the rational pencil $\bar{\pi}'$ on the standard pair $(X', B' = F' \triangleright C' \triangleright E', \bar{\pi}')$ associated with the pair $(P', Q') = (Q, P)$ via the construction of Section 3.1.2.

With our choice of coordinates, the closures in X of general fibers of $\pi' \circ \sigma = \mathrm{pr}_v|_S$ all intersect B at the point $p \in F$ with image $\tau(p) = [0:1:0] \in \mathbb{P}^2$, and one checks that the birational map of standard pairs $(X, B) \dashrightarrow (X', B')$ corresponding to σ is a reversion centered at p.

Note that if $P(0) \neq 0$ and $Q(0) = 0$, then S equipped with π is of type **I**, while it is of type **II** when equipped with $\sigma\pi' = \mathrm{pr}_v|_S$. □

5 Graphs of $\mathbb{A}^1$-Fibrations and Associated Graphs of Groups

Here we apply the results of the previous section to characterize equivalence classes of $\mathbb{A}^1$-fibrations on normal affine surfaces admitting of a completion by a standard pair

(X, B) of type $(0, -1, -a, -b)$. We also give explicit description of automorphism groups of some of these surfaces.

5.1 Notation

Given polynomials $P, Q \in k[w]$, we denote by $[P, Q]$ the isomorphism class of the standard pair $(X, B, \bar{\pi})$ obtained by means of the construction of Section 3.1.2. By virtue of Proposition 3.4, $[P, Q] = [P', Q']$ if and only if the corresponding $\mathbb{A}^1$-fibered surfaces $(X \setminus B, \bar{\pi}\mid_{X\setminus B})$ and $(X' \setminus B', \bar{\pi}'\mid_{X'\setminus B'})$ are isomorphic. Recall that by virtue of Proposition 3.4, this holds if and only if $P'(w) = \alpha P(\beta w)$, $Q'(w) = \gamma Q(\delta w + t)$, where $\alpha, \beta, \gamma, \delta \in k^*$ and $t \in k$ being 0 if $P(0) = 0$.

We say that $[P, Q]$ is *equivalent* to $[P', Q']$ if $X \setminus B$ and $X' \setminus B'$ are isomorphic as abstract affine surfaces. With this convention, the vertices of the graph of $\mathbb{A}^1$-fibrations $\mathcal{F}_{X\setminus B}$ of $X \setminus B$ as defined in Section 2.6 are in one-to-one correspondence with pairs $[P', Q']$ equivalent to $[P, Q]$. In what follows, we denote this graph simply by $\mathcal{F}_{[P,Q]}$.

Note that arrows of the graph $\mathcal{F}_{[P,Q]}$ correspond to equivalence classes of reversions between pairs equivalent to $[P, Q]$. Given one such pair $[P', Q']$, represented by a pair $(X', B' = F' \triangleright C' \triangleright E_1' \triangleright E_2', \bar{\pi}')$, the possible reversions starting from it are parameterized by the k-rational points of the line $F' \setminus C'$. Moreover, if σ_1, σ_2 are two reversions centered at points $p_1, p_2 \in F' \setminus C'$, they are equivalent, or give the same arrow (see Definition 2.6) if and only if there exists an automorphism of (X', B') that sends p_1 onto p_2. By Proposition 3.4, this always holds when $P(0) = 0$.

5.2 Affine surfaces of type III

As noted above normal affine surfaces corresponding to case **III** in the construction of Section 3.1.2 are always singular and form a distinguished class stable under taking reversions. Such a surface S in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ is defined by a system of equations

$$yu = xP(x),$$

$$vx = uQ(u),$$

$$yv = P(x)Q(u)$$

corresponding to a pair $[P, Q]$ with $P(0) = Q(0) = 0$. The structure of the graph $\mathcal{F}_{[P,Q]}$ is particularly simple: Indeed, Lemma 4.2 implies that $[Q, P]$ is the only pair equivalent to $[P, Q]$ and that they can be obtained from each other by performing a reversion. Since

$[P, Q] = [Q, P]$ if and only if $Q \sim P$, the corresponding $\mathcal{F}_{[P,Q]}$ is thus

$$\circlearrowleft \quad [P, Q] \text{ if } Q \sim P \quad \text{and} \quad [P, Q] \longleftrightarrow [Q, P] \quad \text{if } Q \not\sim P.$$

Denoting by $J_y = \mathrm{Aut}(S, \mathrm{pr}_y)$ and $J_v = \mathrm{Aut}(S, \mathrm{pr}_v)$ the groups of automorphisms of S which preserve the $\mathbb{A}^1$-fibrations $\mathrm{pr}_y\colon S \to \mathbb{A}^1$ and $\mathrm{pr}_v\colon S \to \mathbb{A}^1$, respectively, and by $\mathrm{Diag}(S) \subset \mathrm{Aut}(S)$ the subgroup consisting of restrictions to S of diagonal automorphisms of $\mathbb{A}^4$ preserving S, we obtain the following description of automorphism groups of affine surfaces of type **III**:

Proposition 5.1. For an affine surface S in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the equations

$$yu = xP(x),$$

$$vx = uQ(u),$$

$$yv = P(x)Q(u),$$

where P, Q are nonconstant polynomials with $P(0) = Q(0) = 0$, the following holds:

1. Every $\mathbb{A}^1$-fibration on S is equivalent either to $\mathrm{pr}_y\colon S \to \mathbb{A}^1$ or to $\mathrm{pr}_v\colon S \to \mathbb{A}^1$ and these two fibrations are equivalent to each other if and only if $Q \sim P$.
2. If $Q \sim P$, then $[P, Q] = [P, P]$ and, assuming further that $Q = P$, the group $\mathrm{Aut}(S)$ is the amalgamated product $A \star_{\mathrm{Diag}(S)} J_y$ of $J_y = \mathrm{Aut}(S, \mathrm{pr}_y)$ and the subgroup A of $\mathrm{Aut}(S)$ generated by $\mathrm{Diag}(S)$ and the involution $\sigma\colon (x, y, u, v) \to (u, v, x, y)$.
3. If $Q \not\sim P$, then $J_y \cap J_v = \mathrm{Diag}(S)$ and $\mathrm{Aut}(S)$ is the amalgamated product $J_y \star_{\mathrm{Diag}(S)} J_v$ of $J_y = \mathrm{Aut}(S, \mathrm{pr}_y)$ and $J_v = \mathrm{Aut}(S, \mathrm{pr}_v)$. □

Proof. The first assertion is an immediate consequence of the description of $\mathcal{F}_S = \mathcal{F}_{[P,Q]}$. As in Example 4.5, we consider S as $X \setminus B$ where $(X, B = F \triangleright C \triangleright E, \bar{\pi})$ is associated with the pair (P, Q) via the construction of Section 3.1.2 in such a way that $\mathrm{pr}_y|_S$ coincides with $\bar{\pi}|_S$. We denote by $\sigma\colon (X, B) \dashrightarrow (X', B')$ the reversion corresponding to the morphism $(x, y, u, v) \mapsto (u, v, x, y)$, where (X', B') is the standard pair associated with the pair of polynomials (Q, P). By virtue of Proposition 3.4, elements of $\mathrm{Aut}(X, B)$ are lifts of automorphisms of $\mathbb{A}^2$ of the form $(x, y) \mapsto (ax + by, cy)$ satisfying $P(aw)/P(w) \in k^*$, $Q(\frac{a^{r_0+1}}{c} \cdot w)/Q(w) \in k^*$, where $r_0 \geq 1$ is the multiplicity of 0 as a root of P. The extension of such an automorphisms to $\mathbb{P}^2$ fixes the center $[1:0:0]$ of σ if and only if $b=0$. If so, we write $\lambda = P(aw)/P(w) = a^{r_0}$ and $\mu = Q(\frac{a^{r_0+1}}{c} \cdot w)/Q(w) = Q(\frac{a\lambda}{c} w)/Q(w)$, and check that the lift to S of the corresponding automorphism coincides with the restriction of the diagonal automorphism $(x, y, u, v) \mapsto (ax, cy, \frac{a\lambda}{c} u, \frac{\lambda\mu}{c} v)$ of $\mathbb{A}^4$. Furthermore, every diagonal automorphism of $\mathbb{A}^4$ which preserves S is necessarily of this form. This implies in particular that the group $\mathrm{Diag}(S)$ coincides precisely with the subgroup of $\mathrm{Aut}(X, B)$ consisting of lifts of automorphisms whose extensions to $\mathbb{P}^2$ fix the point $[1:0:0]$. By Proposition 2.5, every birational map $f\colon (X, B) \dashrightarrow (X, B)$ is either an element of $\mathrm{Aut}(X, B)$ (and in this case belongs to J_y) or decomposes into a finite sequence of fibered modifications and reversions. Since all reversions are equivalent to σ or σ^{-1}, we can assume that f is a product of σ, σ^{-1}, and fibered modifications and automorphisms of the pairs (X, B) and (X', B').

(2) If $Q \sim P$, we can assume further that $Q = P$ so that σ becomes in fact an automorphism of S. This implies that $\mathrm{Aut}(S)$ is generated by J_y and σ, and hence by J_y and $A = \langle \mathrm{Diag}(S), \sigma \rangle$. It remains to see that every element $h = j_m a_m \cdots j_2 a_2 j_1 a_1$ with $a_l \in A \setminus J_y$, $j_l \in J_y \setminus A$ is nontrivial. By definition every $j_l \in J_y \setminus A$ is either a fibered modification $(X, B) \dashrightarrow (X, B)$ or an automorphism that does not fix the center of the reversion σ. On the other hand, every $a_l \in A \setminus J_y$ is a reversion $(X, B) \dashrightarrow (X, B)$ which has the same center as σ. It follows that h is either an element of $A \setminus J_y$ or admits of a reduced decomposition containing at least a reversion. So h is never trivial, as desired.

(3) If $Q \not\sim P$, then (X', B') is not isomorphic to (X, B). In particular, every element $f\colon (X, B) \dashrightarrow (X', B')$ decomposes into

$$f = \sigma^{-1} a_n' \sigma \cdots \sigma^{-1} a_2' \sigma a_2 \sigma^{-1} a_1' \sigma a_1,$$

where the a_i and a_j' are either fibered modifications or automorphisms of (X, B) and (X', B'), respectively. In consequence, every a_i is an element of J_y and every $\sigma^{-1} a_i' \sigma$ is an element of J_v, which shows that $\mathrm{Aut}(S)$ is generated by J_y and J_v. Furthermore, $a \in J_y$ is an element of J_v if and only if it is equal to $\sigma^{-1} b \sigma$ for a certain automorphism or a fibered modification b of (X', B'). The equality $a = \sigma^{-1} b \sigma$ implies that a and b are automorphisms of (X, B) and (X', B') respectively, and that a (respectively, b) fixes the center of σ (respectively, of σ^{-1}). In particular, $J_y \cap J_v = \mathrm{Diag}(S)$.

It remains to show that every element $h = b_m a_m \cdots b_2 a_2 b_1 a_1$ with $a_l \in J_y \setminus J_v$, $b_l = \sigma^{-1} b_l' \sigma \in J_v \setminus J_y$ is nontrivial. By virtue of the above description, each a_i is either an automorphism of (X, B) not fixing the center of σ or a fibered modification while each b_j' is either an automorphism of (X', B') not fixing the center of σ^{-1} or a fibered modification of (X', B'). This implies that h is either an element of $\mathrm{Aut}(X, B)$ not fixing the center of σ or admits of a reduced decomposition containing at least a reversion. In any case, h is not trivial, which achieves the proof. ■

Example 5.2. Let S be the surface in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the equations

$$yu = x^2(x-1),$$

$$vx = u^2(u-1),$$

$$yv = x(x-1)u(u-1)$$

corresponding to the polynomials $P(w) = Q(w) = w(w-1)$. Since $P(aw)/P(w)$ and $Q(\frac{a^2}{c} \cdot w)/Q(w)$ belong to k^* if and only if $a = c = 1$, it follows from the proof of the above proposition that $\mathrm{Diag}(S) = \{\mathrm{id}_S\}$. Furthermore, one checks that the group $J_y = \mathrm{Aut}(S, \mathrm{pr}_y)$ consist of lifts to S via the projection $\mathrm{pr}_{x,y} : S \to \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ of automorphisms of $\mathbb{A}^2$ of the form $(x, y) \mapsto (x + y^2 R(y), y)$, where $R \in k[y]$ is an arbitrary polynomial. So J_y is isomorphic as a group to $(k[y], +) \simeq \mathbb{G}_{a,k}^{\infty}$ and we conclude that $\mathrm{Aut}(S)$ is isomorphic to the free product $\mathbb{Z}_2 \star \mathbb{G}_{a,k}^{\infty}$. □

Example 5.3. Let S be the surface in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the equations

$$yu = x^2(x-1),$$

$$vx = u(u-1),$$

$$yv = x(x-1)(u-1)$$

corresponding to the polynomials $P(w) = w(w-1)$ and $Q(w) = w-1$. Again, the choice of P and Q guarantees that $\mathrm{Diag}(S) = \{\mathrm{id}_S\}$. As in the previous example, the groups $J_y = \mathrm{Aut}(S, \mathrm{pr}_y)$ and $J_v = \mathrm{Aut}(S, \mathrm{pr}_v)$ consist of lifts to S via the projections $\mathrm{pr}_{x,y}$ and $\mathrm{pr}_{u,v}$, respectively, of automorphisms of $\mathbb{A}^2$ of the form $(x, y) \mapsto (x + y^2 R_1(y), y)$, where $R_1 \in k[y]$, and $(u, v) \mapsto (u + vR_2(v), v)$, where $R_2 \in k[v]$. It follows that $\mathrm{Aut}(S)$ is isomorphic to the free product $\mathbb{G}_{a,k}^{\infty} \star \mathbb{G}_{a,k}^{\infty}$. □

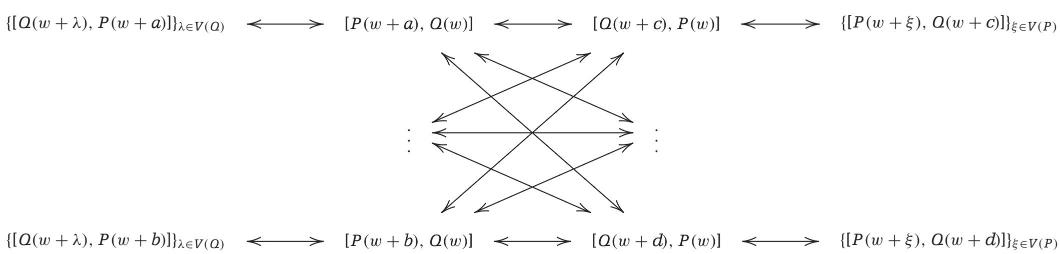
5.3 Affine surfaces of types I and II

In contrast to surfaces of type **III**, Example 4.5 shows that in general affine surfaces corresponding to case **I** and **II** in the construction of Section 3.1.2 can be obtained from each other by performing reversions. Recall that these models correspond to pairs $[P, Q]$ such that either $P(0) \neq 0$ or $P(0) = 0$ but $Q(0) \neq 0$. The associated graph $\mathcal{F}_{[P,Q]}$ is quite complicated, in particular infinite as soon as the field k is, as shown by the following result:

Proposition 5.4. Let P, Q be two polynomials of degree ≥ 1, and assume that $P(0) \neq 0$. The set of pairs equivalent to $[P, Q]$ is

$$\{[P(w+a), Q(w+b)], [Q(w+b), P(w+a)] | a, b \in k, (P(a), Q(b)) \neq (0, 0)\}.$$

The graph $\mathcal{F}_{[P,Q]}$ associated to $[P, Q]$ has the following structure:



where $P(a)P(b)Q(c)Q(d) \neq 0$ and $V(Q) = \{\lambda \in k \mid Q(\lambda) = 0\}$, $V(P) = \{\xi \in k \mid P(\xi) = 0\}$. (The arrows in the middle indicate that every $[P(w+a), Q(w)]$ is linked with any $[Q(w+c), P(w)]$, for $a, c \in k$, $P(a)Q(c) \neq 0$.) □

Proof. Follows from Lemma 4.2 and Propositions 3.4 and 4.3:

1. Starting from $[P(w+a), Q(w)]$ with $P(a) \neq 0$, we perform a reversion at a point λ (using the notation of Lemma 4.2).

   If $Q(\lambda) \neq 0$, it follows from Lemma 4.2(1) that the pair obtained is $[Q(w+\lambda), P(w+a)]$ and that in this case the first polynomial does not vanish at 0. It is one of the multiple arrows in the middle of our diagram.

   If $Q(\lambda) = 0$, it follows from Lemma 4.2(2) that the pair obtained is $[Q(w+\lambda), P(w+a)]$ and that in this case the first polynomial vanishes at $w = 0$. It is one of the multiple arrows in the middle of our diagram.
2. Starting from $[P(w+a), Q(w)]$ with $P(a) = 0$ and $Q(0) \neq 0$ and performing any reversion, we only obtain $[Q(w), P(w+a)]$, which is equivalent to $[Q(w), P(w+c)]$ for any $c \in k$ (this follows from Lemma 4.2, we can only go from **II** to **I**, and this link is the inverse of the one described in Lemma 4.2(2)).

This yields the result. ■

Proposition 5.5. Let k be an uncountable field and let $P, Q \in k[w]$ be polynomials having at least 2 distinct roots in the algebraic closure $\overline{k}$ of k and such that $P(0) \neq 0$. Then for the affine surface S in $\mathbb{A}^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the system of equations

$$yu = xP(x),$$

$$vx = uQ(u),$$

$$yv = P(x)Q(u),$$

the following holds:

1. S admits of uncountably many equivalence classes of $\mathbb{A}^1$-fibrations.
2. The subgroup $H \subset \mathrm{Aut}(S)$ generated by all automorphisms of $\mathbb{A}^1$-fibrations is not generated by a countable union of algebraic groups.
3. The subgroup $\mathrm{Aut}(S)_{\mathrm{alg}} \subset \mathrm{Aut}(S)$ generated by all algebraic subgroups of $\mathrm{Aut}(S)$ is not generated by a countable union of algebraic groups.
4. The quotient of $\mathrm{Aut}(S)$ by its normal subgroup $\mathrm{Aut}(S)_{\mathrm{alg}}$ contains a free group over an uncountable set of generators. Furthermore, the same holds for $\mathrm{Aut}(S)/H \simeq \Pi_1(\mathcal{F}_S)$ since $H \subset \mathrm{Aut}(S)_{\mathrm{alg}}$. □

Proof. Because P has at least 2 roots over $\bar{k}$, $P(w+t) \sim P(w) \in k^*$ for only finitely many $t \in k$, and the same holds for Q. Furthermore, the fact that the degree of P is at least 2 implies that for every standard pair (X, B) with $X \setminus B \simeq S$, the boundary B contains at least an irreducible component with self-intersection ≤ -3.

Suppose that $P \sim Q$. Choosing $t \in k$ general enough, we then have $P(w+t) \not\sim P(w)$. We replace in this case $Q(w)$ with $P(w+t)$ (this does not change the isomorphism class of the surface S, by Propositions 3.4 and 5.4), and obtain that $Q \not\sim P$, which corresponds to $[P(w), Q(w)] \neq [Q(w), P(w)]$. Thus, we may, and shall, further assume $P \not\sim Q$.

We can now choose an uncountable set $A \subseteq k$, containing 0, such that for every distinct $a_1, a_2 \in A$, we have

$$P(w+a_1) \not\sim P(w+a_2),\ Q(w+a_1) \not\sim Q(w+a2) \text{ and } P(w+a_1) \not\sim Q(w+a_2).$$

By the choice of A and by Proposition 3.4(a) the isomorphism classes of the corresponding standard pairs are distinct.

For every $a \in A$, we denote by (X_a, B_a) the standard pair obtained from the pair of polynomials $(P(w+a), Q(w))$ and by (X'_a, B'_a) the one obtained from the pair of polynomials $(Q(w+a), P(w))$. By Proposition 5.4, each $[(X_a, B_a)]$ and each $[(X'_a, B'_a)]$ is a vertex in the graph $\mathcal{F}_S = \mathcal{F}_{[P,Q]}$. The definition of A implies that the four pairs $[(X_a, B_a)]$, $[(X_b, B_b)]$, $[(X'_a, B'_a)]$, and $[(X'_b, B'_b)]$ are pairwise distinct for distinct $a, b \in A$. In particular, we obtain uncountably many vertices, which is equivalent to (1) by Proposition 3.4.

Let $(G_i)_{i\in\mathbb{N}}$ be a countable set of algebraic subgroups $G_i \subset \mathrm{Aut}(S)$. For any $i \in \mathbb{N}$, Proposition 2.7, gives a standard pair (X, B) (depending on i), and an isomorphism $\psi : S \xrightarrow{\sim} X \setminus B$ such that the conjugation of G_i by ψ consists of birational maps $(X, B) \dashrightarrow (X, B)$ being fibered modifications, automorphisms, or reversions. Viewing any element of G_i as a birational transformation of (X_0, B_0), we can factorize it into automorphisms of pairs, fibered modifications and reversions (Proposition 2.5), and the existence of ψ implies that the number of such factors is bounded; indeed, each element of G_i is equal to $\psi^{-1}\sigma\psi$ for some σ of length ≤ 1. In consequence, there exists a countable set $\mathcal{S}$ of equivalence classes of pairs (X, B) with $X \setminus B = S$, such that each element of each G_i can be decomposed into a sequence of automorphisms of pairs, fibered modifications, and reversions involving only pairs in $\mathcal{S}$. There exists, thus, $a \in A$ such that $[(X'_a, B'_a)] \notin \mathcal{S}$. We choose a reversion $\mu\colon (X_0, B_0) \dashrightarrow (X'_a, B'_a)$, and a nontrivial algebraic group $\hat{G}$ of fibered modifications $(X'_a, B'_a) \dashrightarrow (X'_a, B'_a)$. The group $\mu^{-1}\hat{G}\mu$ yields an algebraic subgroup of $\mathrm{Aut}(S)$, which preserves an $\mathbb{A}^1$-fibration and which is not contained in the group generated by the G_i. This yields (2) and (3).

By Proposition 5.4, for every $a \in A$, there exist reversions $\tau\colon (X_0, B_0) \dashrightarrow (X'_0, B'_0)$, $\sigma_a\colon (X'_0, B'_0) \dashrightarrow (X_a, B_a)$, $\tau_a\colon (X_a, B_a) \dashrightarrow (X'_a, B'_a)$ and $\sigma'_a\colon (X'_a, B'_a) \dashrightarrow (X_0, B_0)$, representing the cycle

$$\begin{array}{ccc}
[P(w), Q(w)] & \longrightarrow & [Q(w), P(w)] \\
\uparrow & & \downarrow \\
[Q(w+a), P(w)] & \longleftarrow & [P(w+a), Q(w)]
\end{array}$$

in $\mathcal{F}_S = \mathcal{F}_{[P,Q]}$. For every $a \in A\backslash\{0\}$, the map $\sigma'_a\tau_a\sigma_a\tau\colon (X_0, B_0) \dashrightarrow (X_0, B_0)$ restricts to an automorphism ζ_a of $S = X_0 \setminus B_0$. The decomposition $\sigma'_a\tau_a\sigma_a\tau$ is reduced, because $[P(w), Q(w)] \neq [P(w+a), Q(w)]$ and $[Q(w), P(w)] \neq [Q(w+a), P(w)]$ (recall that the composition of two reversions is of length 2 or is an isomorphism of pairs). We denote by $F \subset \mathrm{Aut}(S)$ the group generated by the ζ_a, $a \in A\backslash\{0\}$. In order to get (4), we will show that F is a free group over the ζ_a and intersects $\mathrm{Aut}(S)_{\mathrm{alg}}$ trivially.

1. First, we observe that for every $a, b \in A\backslash\{0\}$, $a \neq b$, the decomposition

$$\zeta_a\zeta_b = \sigma_a'\tau_a\sigma_a\tau\sigma_b'\tau_b\sigma_b\tau$$

is reduced. Indeed, $\tau\sigma_b'$ is reduced because otherwise it would be an isomorphism between (X_b', B_b') and (X_0', B_0'), which is not possible because $b \neq 0$.

2. Similarly, the following decomposition

$$\zeta_a(\zeta_b)^{-1} = \sigma_a'\tau_a\sigma_a(\sigma_b)^{-1}(\tau_b)^{-1}(\sigma_b')^{-1}$$

is reduced, because otherwise $\sigma_a(\sigma_b)^{-1}$ would be an isomorphism between (X_b, B_b) and (X_a, B_a).

3. Similarly, the following decomposition $(\zeta_a)^{-1}\zeta_b = \tau^{-1}(\sigma_a)^{-1}(\tau_a)^{-1}(\sigma_a')^{-1}\sigma_b'\tau_b\sigma_b\tau$, is again reduced, for otherwise $(\sigma_a')^{-1}\sigma_b'$ would be an isomorphism between (X_b', B_b') and (X_a', B_a').

These three observations imply that every element $h = (\zeta_{a_r})^{\delta_r}\cdots(\zeta_{a_1})^{\delta_1} \in F \setminus \{\mathrm{id}_S\}$, where $a_1, \ldots, a_r \in A$ and $\delta_1, \ldots, \delta_r \in \mathbb{Z}\backslash\{0\}$ and $a_i \neq a_{i+1}$ for $i = 1, \ldots, r-1$, is nontrivial because it admits of a reduced decomposition of positive length. This shows the freeness of F. By construction, the image of h in $\Pi_1(\mathcal{F}_S)$ consists of a product of loops based at $[(X_0, B_0)]$ of length ≥ 4. Since in contrast, the image in $\Pi_1(\mathcal{F}_S)$ of every element in $\mathrm{Aut}(S)_{\mathrm{alg}}$ can only contain loops of length 1 (see Remark 2.8), it follows that $F \cap \mathrm{Aut}(S)_{\mathrm{alg}}$ is trivial, which completes the proof. ∎

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