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Moduli Spaces of Quadratic Rational Maps with a Marked Periodic Point of Small Order

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The surface corresponding to the moduli space of quadratic endomorphisms of $\mathbb{P}^1$ with a marked periodic point of order n is studied. It is shown that the surface is rational over $\mathbb{Q}$ when $n \leq 5$ and is of general type for $n = 6$. An explicit description of the $n = 6$ surface lets us find several infinite families of quadratic endomorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{Q}$ with a rational periodic point of order 6. In one of these families, f also has a rational fixed point, for a total of at least 7 periodic and 7 preperiodic points. This is in contrast with the polynomial case, where it is conjectured that no polynomial endomorphism defined over $\mathbb{Q}$ admits rational periodic points of order $n > 3$.

1 Introduction

A classical question in arithmetic dynamics concerns periodic, and more generally preperiodic, points of a rational map (endomorphism) $f\colon \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$. A point p is said to be *periodic of order* n if $f^n(p) = p$ and if $f^i(p) \neq p$ for $0 < i < n$. A point p is said to be *preperiodic* if there exists a non-negative integer m such that the point $f^m(p)$ is periodic.

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In [10, Conjecture 2], it was conjectured that if f is a polynomial of degree 2 defined over $\mathbb{Q}$, then f admits no rational periodic point of order $n > 3$. This conjecture, also called Poonen's conjecture (because of the refinement made in [20], see [11]), was proved in the cases $n = 4$ [18, Theorem 4] and $n = 5$ [10, Theorem 1]. Some evidence for $n = 6$ is given in [10, Section 10; 11, 24]. The bound $n > 3$ is needed because the polynomial maps f of degree 2 constitute an open set in $\mathbb{A}^3$, and for any p_i the condition $f(p_i) = p_{i+1}$ cuts out a hyperplane in this $\mathbb{A}^3$.

In this article, we study the case where f is not necessarily a polynomial but a rational map of degree 2. Here the space of such maps is an open set in $\mathbb{P}^5$, and again each condition $f(p_i) = p_{i+1}$ cuts out a hyperplane. Hence the analog of Poonen's conjecture would be that there is no map defined over $\mathbb{Q}$ with a rational periodic point of order $n > 5$. However, we show that in fact there are infinitely many pairs (f, p), even up to automorphism of $\mathbb{P}^1$, such that $f \colon \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$ is a rational map of degree 2 defined over $\mathbb{Q}$ and p is a rational periodic point of order 6. We do this by studying the structure of the algebraic variety parameterizing such pairs, called as usual the moduli space.

The study of the moduli spaces considered in this article is also motivated by some more general facts. For example, Morton and Silverman [19] stated the so-called Uniform Boundedness Conjecture. The conjecture asserts that for every number field K the number of preperiodic points in $\mathbb{P}^N(K)$ of a morphism $\Phi \colon \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$ defined over K is bounded, by a number depending only on the integers d, N and on the degree $D = [K : \mathbb{Q}]$ of the extension $K/\mathbb{Q}$. It seems very hard to settle this conjecture, even in the case $(N, D, d) = (1, 1, 2)$. As usual, the way to solve a problem in number theory starts with the study of the geometrical aspects linked to the problem. The moduli spaces studied in this article are some geometrical objects naturally related with the Uniform Boundedness Conjecture.

We next give a more precise structure to our moduli spaces. All our varieties will be algebraic varieties defined over the field $\mathbb{Q}$ of rational numbers, and thus also over any field of characteristic zero.

We denote by Rat_d the algebraic variety parameterizing all endomorphisms (rational functions) of degree d of $\mathbb{P}^1$; it is an affine algebraic variety of dimension $2d + 1$. The algebraic group $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ of automorphisms of $\mathbb{P}^1$ acts by conjugation on Rat_d. Milnor [16] proved that the moduli space $\mathrm{M}_2(\mathbb{C}) = \mathrm{Rat}_2(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$ is analytically isomorphic to $\mathbb{C}^2$. Silverman [21] generalized this result: for each positive integer d, the quotient space $\mathrm{M}_d = \mathrm{Rat}_d/\mathrm{PGL}_2$ exists as a geometric quotient scheme over $\mathbb{Z}$ in the sense of Mumford's geometric invariant theory, and $\mathrm{Rat}_2/\mathrm{PGL}_2$ is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$.

More recently, Levy [13] proved that the quotient space M_d is a rational variety for all positive integers d.

Let $n \geq 1$ be an integer, and let $\mathrm{Rat}_d(n)$ be the subvariety of $\mathrm{Rat}_d \times (\mathbb{P}^1)^n$ given by the points $(f, p_1, \ldots, p_n)$ such that $f(p_i) = p_{i+1}$ for $i = 1, \ldots, n-1$, $f(p_n) = p_1$ and all points p_i are distinct (note that here (f, p_1) carries the same information as $(f, p_1, \ldots, p_n)$). The variety $\mathrm{Rat}_d(n)$ has dimension $2d+1$. For $n \geq 2$, $\mathrm{Rat}_d(n)$ is moreover affine, since Rat_d is affine and the subset of $(\mathbb{P}^1)^n$ corresponding to n-uples of pairwise distinct points of $\mathbb{P}^1$ is also affine.

The group PGL_2 naturally acts on $\mathrm{Rat}_d(n)$, and Manes [14] proved that the quotient $\mathrm{Rat}_d(n)/\mathrm{PGL}_2$ exists as a geometric quotient scheme. (In [14], points of "formal period" n are considered, so our varieties $\mathrm{Rat}_d(n)$ are PGL_2-invariant open subsets of the varieties called by the same name in [14].) We will denote by $\mathcal{M}_d(n)$ the quotient surface $\mathrm{Rat}_d(n)/\mathrm{PGL}_2$, which is an affine variety for $n > 1$ and $d = 2$.

In [14, Theorem 4.5], it is shown that the surfaces $\mathcal{M}_2(n)$ are geometrically irreducible for every $n > 1$ (a fact which is also true for $n = 1$), but not much else is known about these surfaces.

The closed curve $\mathcal{C}_2(n) \subset \mathcal{M}_2(n)$ corresponding to periodic points of polynomial maps is better known; it is rational for $n \leq 3$, of genus 2 for $n = 4$, of genus 14 for $n = 5$, and its genus rapidly increases with n. Bousch studied these curves from an analytic point of view in his thesis [4] and Morton from an algebraic point of view in [17]. See [22, Chapter 4] for a compendium of the known results on $\mathcal{C}_2(n)$.

Note that $\mathcal{M}_d(n)$ has an action of the automorphism σ_n of order n which sends the class of $(f, p_1, p_2, \ldots, p_n)$ to the class of $(f, p_2, \ldots, p_n, p_1)$. For $n \geq 5$, the quotient surface $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ parameterizes the set of orbits of size n of endomorphisms of $\mathbb{P}^1$ of degree 2 (see Lemma 2.1). One approach to Poonen's conjecture, carried out in [10, 18, 24] and elsewhere, is to study the quotient curve $\mathcal{C}_2(n)/\langle\sigma_n\rangle$, which has a lower genus than $\mathcal{C}_2(n)$.

The aim of this article is to understand the geometry of $\mathcal{M}_2(n)$ and $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ for small n, that is, to describe the birational type of the surfaces and to determine whether they contain rational points. Our main result is the following.

Theorem 1.

(1) For $1 \leq n \leq 5$, the surfaces $\mathcal{M}_2(n)$ and $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ are rational over $\mathbb{Q}$.

(2) The surface $\mathcal{M}_2(6)$ is an affine smooth surface, birational to a projective surface of general type, whereas $\mathcal{M}_2(6)/\langle\sigma_6\rangle$ is rational over $\mathbb{Q}$.

(3) The set $\mathcal{M}_2(6)(\mathbb{Q})$ of $\mathbb{Q}$-rational points of $\mathcal{M}_2(6)$ is infinite. □

For $n=6$, we also show that $\mathcal{M}_2(6)/\langle\sigma_6^2\rangle$ is of general type, whereas $\mathcal{M}_2(6)/\langle\sigma_6^3\rangle$ is rational.

The proof of Theorem 1 is obtained by considering some suitable models for the surfaces $\mathcal{M}_2(n)$, in particular for $n=6$ we used the one contained in the following statement.

Proposition 1.1.

(i) The surface $\mathcal{M}_2(6)$ is isomorphic to an open subset of the quintic irreducible hypersurface $S_6 \subset \mathbb{P}^3$ given by

$$W^2 F_3(X, Y, Z) = F_5(X, Y, Z),$$

where

$$F_3(X, Y, Z) = (X+Y+Z)^3 + (X^2Z + XY^2 + YZ^2) + 2XYZ,$$

$$F_5(X, Y, Z) = (Z^3X^2 + X^3Y^2 + Y^3Z^2) - XYZ(YZ + XY + XZ),$$

and the action of σ_6 corresponds to the restriction of the automorphism

$$[W:X:Y:Z] \mapsto [-W:Z:X:Y].$$

(ii) The complement of $\mathcal{M}_2(6)$ in S_6 is the union of 9 lines and 14 conics, and is also the trace of an ample divisor of $\mathbb{P}^3$: the points of $\mathcal{M}_2(6)$ are points $[W:X:Y:Z] \in S_6$ satisfying that $W^2 \neq XY + YZ + XZ$ and that W^2, X^2, Y^2, Z^2 are pairwise distinct. □

The proof of the above result, that is essentially Lemma 4.1, is contained in Section 4.

Remark 1.2. The blow-up of the singular points of S_6 (the triple point $[1:0:0:0]$ and some double points, see the proof of Corollary 4.4) gives a birational morphism $\hat{S}_6 \to S_6$, and the linear system $|K_{\hat{S}_6}|$ corresponds thus to the system of hyperplanes through the triple point, that is, to the projection $[w:x:y:z] \dashrightarrow [x:y:z]$. This induces a double covering $\hat{S}_6 \to \mathbb{P}^2$. The branch locus in $\mathbb{P}^2$ contains the quintic $F_5 = 0$, as one sees from the equation, but also the cubic $F_3 = 0$, whose corresponding ramification locus lies on the exceptional divisor of the blow-up (as can be checked by computing the blow-up locally). One can also see that the surface $\hat{S}_6$ is a Horikawa surface since $c_2 = 46 = 5c_1^2 + 36$. □

From the explicit description, it directly follows that S_6 is of general type (Corollary 4.4). The set of rational points should thus not be Zariski dense, according to the Bombieri–Lang conjecture. We have, however, infinitely many rational points in $\mathcal{M}_2(6)$, which are contained in the preimage in S_6 of the rational cubic curve $X^3+Y^3+Z^3=X^2Y+Y^2Z+Z^2X$, which is again rational, and the preimages in S_6 of the lines $X=0$, $Y=0$, $Z=0$, which are elliptic curves of rank 1 over $\mathbb{Q}$. But, for any number field K and for any finite fixed set S of places of K containing all the archimedean ones, the set of S-integral points of $\mathcal{M}_2(6)$ is finite.

2 Preliminaries

2.1 The variety Rat_2

Associating to $(a_0:\dots:a_5)\in\mathbb{P}^5$ the rational map (endomorphism) of $\mathbb{P}^1$

$$[u:v]\mapsto\left[a_0u^2+a_1uv+a_2v^2:a_3u^2+a_4uv+a_5v^2\right],$$

the variety Rat_2 can be viewed as the open subset of $\mathbb{P}^5$, where $a_0u^2+a_1uv+a_2v^2$ and $a_3u^2+a_4uv+a_5v^2$ have no common roots; explicitly, it is equal to the open subset of $\mathbb{P}^5$ which is the complement of the quartic hypersurface defined by the polynomial

$$\mathrm{Res}(a_0,\dots,a_5)=a_2^2a_3^2+a_0^2a_5^2-2a_3a_2a_0a_5-a_1a_2a_3a_4-a_4a_1a_0a_5+a_0a_4^2a_2+a_1^2a_3a_5,$$

where the polynomial Res is the homogeneous resultant of the two polynomials $a_0u^2+a_1uv+a_2v^2$ and $a_3u^2+a_4uv+a_5v^2$.

2.2 Embedding $\mathcal{M}_2(n)$ into $\mathbb{P}^5\times\mathbb{A}^{n-3}$

When $n\geq 3$, any element of $\mathrm{Rat}_d(n)$ is in the orbit under $\mathrm{Aut}(\mathbb{P}^1)=\mathrm{PGL}_2$ of exactly one element of the form

$$(f,[0:1],[1:0],[1:1],[x_1:1],\dots,[x_{n-3}:1]),$$

where $f\in\mathrm{Rat}_2$ and $(x_1,\dots,x_{n-3})\in\mathbb{A}^{n-3}$.

In particular, the surface $\mathcal{M}_2(n)$ is isomorphic to a locally closed subset (and hence a subvariety) of $\mathrm{Rat}_2\times\mathbb{A}^{n-3}\subset\mathbb{P}^5\times\mathbb{A}^{n-3}$.

Lemma 2.1. Viewing $\mathcal{M}_2(n)$ as a subvariety of $\mathbb{P}^5\times\mathbb{A}^k$, where $k=n-3$ as before, and assuming that $n\geq 5$, the projection $\mathbb{P}^5\times\mathbb{A}^k\to\mathbb{A}^k$ restricts to an isomorphism from $\mathcal{M}_2(n)$ with its image, which is locally closed in $\mathbb{A}^k$, and is an affine surface.

The inverse map sends $(x_1, \dots, x_k)$ to $(a_0 : \cdots : a_5)$, where

$$\begin{aligned}
a_0 &= x_1(x_2x_k + x_1 - x_2 - x_k),\\
a_1 &= x_1(x_k^2 - x_k^2x_2 - x_2x_1 + x_2 + x_2x_1^2 - x_1^2),\\
a_2 &= x_1x_k(x_kx_2 - x_1x_k + x_2x_1 - x_2 - x_2x_1^2 + x_1^2),\\
a_3 &= x_1(x_2x_k + x_1 - x_2 - x_k),\\
a_4 &= -x_1x_k^2 + x_k^2 + x_kx_1^2 - x_k + x_1x_k + x_2x_1^2 - x_1^2x_kx_2 + x_1 - 2x_1^2,\\
a_5 &= 0.
\end{aligned}$$

□

Proof. Let $(f, x_1, \dots, x_k)$ be an element of $\mathcal{M}_2(n) \subset \mathrm{Rat}_2 \times \mathbb{A}^k$. Recall that f corresponds to the endomorphism

$$f\colon [u : v] \mapsto \left[a_0u^2 + a_1uv + a_2v^2 : a_3u^2 + a_4uv + a_5v^2\right].$$

The equalities $f([0:1]) = [1:0]$ and $f([1:0]) = [1:1]$ correspond, respectively, to saying that $a_5 = 0$ and $a_0 = a_3$. Adding the conditions $f([x_k:1]) = [0:1]$, $f([1:1]) = [x_1:1]$, and $f([x_1:1]) = [x_2:1]$ yields

$$\begin{pmatrix} x_k & 1 & x_k^2 & 0 \\ 1 & 1 & 1-x_1 & -x_1 \\ x_1 & 1 & x_1^2(1-x_2) & -x_1x_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

We now prove that the 3×4 matrix above has rank 3, if $(f, x_1, \dots, x_k) \in \mathcal{M}_2(n) \subset \mathrm{Rat}_2 \times \mathbb{A}^k$.

The third minor (determinant of the matrix obtained by removing the third column) is equal to $-x_1(x_1 - x_k + x_2x_k - x_2)$. Since $x_1 \neq 0$, we only have to consider the case where $x_1 = x_k - x_2x_k + x_2$. Replacing this in the fourth minor, we get $-x_2(x_k-1)^2(x_2-1)(x_2 - x_2x_k - 1 + 2x_k)$. Since $x_2, x_k \notin \{0, 1\}$, the only case is to study is when $x_2 - x_2x_k - 1 + 2x_k = 0$. Writing $x_k = t$, this yields $(x_1, x_2, x_k) = \left(1-t, \frac{1-2t}{1-t}, t\right)$. The solutions of the linear system (1) are in this case given by $a_1 = -a_3t + a_4$ and $a_2 = -a_4t$, and yields a map f which is not an endomorphism of degree 2, since $a_3u + a_4v$ is a factor of both coordinates.

The fact that the matrix has rank 3 implies that the projection yields an injective morphism $\pi\colon \mathcal{M}_2(n) \to \mathbb{A}^k$. It also implies that we can find the coordinates $(a_0 : \cdots : a_5)$ of f as polynomials in x_1, x_2, x_k. A direct calculation yields the formula given in the statement. It remains to see that the image $\pi(\mathcal{M}_2(n))$ is locally closed in $\mathbb{A}^k$, and that it is an affine surface.

To do this, we describe open and closed conditions that define $\pi(\mathcal{M}_2(n))$. First, the coordinates x_i have to be pairwise distinct and different from 0 and 1. Secondly, we replace x_1, x_2, x_k in the formulas that give $a_0, \dots, a_5$, compute the resultant $\mathrm{Res}(a_0, \dots, a_5)$ (see Section 2.1) and ask that this resultant is not zero. These open conditions give the existence of a unique map $f \in \mathrm{Rat}_2$ associated to any given x_1, x_2, x_k. We then ask that $f([x_i : 1]) = [x_{i+1} : 1]$ for $i = 2, \dots, k-1$, which are closed conditions. This shows that $\mathcal{M}_2(n)$ is locally closed in $\mathbb{A}^k$, and moreover that it is an affine surface since all open conditions are given by the nonvanishing of a finite set of equations. ■

Remark 2.2. Lemma 2.1 is quite technical. The fact that $\mathcal{M}_2(n)$ maps isomorphically to its image in $\mathbb{A}^{n-3}$, when $n \geq 5$, follows from the elementary assertion that two rational maps of degree d, that have the same value in $2d+1$ points, are identical. Indeed, for any different $f, g \in \mathrm{Rat}_d$ it is enough to consider the equation $f - g = 0$ that admits at most $2d$ roots in $\mathbb{P}^1$. The proof of the above lemma makes a similar argument, but is useful because it constructs an explicit inverse that we shall use later. □

Corollary 2.3. The surface $\mathcal{M}_2(5)$ is isomorphic to an open affine subset of $\mathbb{A}^2$, and is thus smooth, and rational over $\mathbb{Q}$. □

Proof. This result follows directly from Lemma 2.1. ■

3 The Surfaces $\mathcal{M}_2(n)$ for $n \leq 5$

The proof of the rationality of the surfaces $\mathcal{M}_2(n)$ and $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ for $n \leq 5$ is done by case-by-case analysis.

Note that the rationality of $\mathcal{M}_2(n)$ implies that $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ is unirational, and thus geometrically rational (rational over the algebraic closure of $\mathbb{Q}$) by Castelnuovo's rationality criterion for algebraic surfaces. However, over $\mathbb{Q}$ there exist rational surfaces with nonrational quotients, for example some double coverings of smooth cubic surfaces with only one line defined over $\mathbb{Q}$. The contraction of the line in the cubic gives a nonrational minimal del Pezzo surface of degree 4, which admits a rational double covering [15, IV, Theorem 29.4].

Hence, the rationality of $\mathcal{M}_2(n)/\langle\sigma_n\rangle$ is not a direct consequence of the rationality of $\mathcal{M}_2(n)$.

Lemma 3.1. The surfaces $\mathcal{M}_2(3)$ and $\mathcal{M}_2(4)$ are isomorphic to affine open subsets of $\mathbb{A}^2$ and are thus smooth, and rational over $\mathbb{Q}$. The surfaces $\mathcal{M}_2(3)/\langle\sigma_3\rangle$ and $\mathcal{M}_2(4)/\langle\sigma_4\rangle$ are also rational over $\mathbb{Q}$. □

Proof. (i) We embed $\mathcal{M}_2(4)$ in $\mathrm{Rat}_2 \times \mathbb{A}^1 \subset \mathbb{P}^5 \times \mathbb{A}^1$ as in Section 2.2. Recall that an element (f, x) of $\mathcal{M}_2(4)$ corresponds to an endomorphism

$$f\colon [u:v] \mapsto \left[a_0u^2 + a_1uv + a_2v^2 : a_3u^2 + a_4uv + a_5v^2\right]$$

and a point $x \in \mathbb{A}^1$ which satisfy

$$f([0:1]) = [1:0],\quad f([1:0]) = [1:1],\quad f([1:1]) = [1:x],\quad f([1:x]) = [0:1].$$

This implies that $[a_0 : \cdots : a_5]$ is equal to

$$[-a_1x - a_2x^2 : a_1 : a_2 : -a_1x - a_2x^2 : -a_1x^2 - a_2x^3 + 2a_1x + xa_2 + a_2x^2 : 0],$$

so $\mathcal{M}_2(4)$ is isomorphic to an open subset of $\mathbb{P}^1 \times \mathbb{A}^1$, with coordinates $([a_1 : a_2], x)$. Because the resultant of the map equals

$$-a_2x^2(x-1)(x-2)(a_1 + a_2 + xa_2)(a_1 + xa_2)(a_1x - a_1 + a_2x^2),$$

we obtain an open affine subset of $\mathbb{A}^2$.

(ii) The map σ_4 sends $(f, [0:1], [1:0], [1:1], [1:x])$ to $(f, [1:0], [1:1], [1:x], [0:1])$, which is in the orbit of

$$(gfg^{-1}, g([1:0]), g([1:1]), g([1:x]), g([0:1])) = (gfg^{-1}, [0:1], [1:0], [1:1], [x-1:x]),$$

where $g\colon [u:v] \mapsto [(1-x)v : x(u-v)]$. The endomorphism gfg^{-1} corresponds to $[u:v] \mapsto [b_0u^2 + b_1uv + b_2v^2 : b_3u^2 + b_4uv + b_5v^2]$, where $[b_0 : \cdots : b_5]$ is equal to $[(x-1)(xa_2 + a_1 + a_2)x^2 : (x-1)(a_2 - a_2x^2 - a_1x - xa_2)x : (x-1)^2(a_1 + xa_2) : (x-1)(xa_2 + a_1 + a_2)x^2 : -(a_2x^3 + a_1x^2 - a_2x^2 - 2a_1x - xa_2 + a_1)x : 0]$. Hence, σ_4 corresponds to

$$([a_1 : a_2], x) \mapsto \left([(a_2 - a_2x^2 - a_1x - xa_2)x : (x-1)(a_1 + xa_2)], \frac{x}{x-1}\right).$$

The birational map $\kappa : \mathbb{P}^1 \times \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$ taking $([a_1 : a_2], x)$ to

$$\left(\frac{a_2(a_1 + (x+1)a_2)x}{(x^3 + x^2 - x)a_2^2 + (x-1)a_1^2 + 2(x^2-1)a_1a_2}, \frac{(xa_1 + a_2x^2 - a_1)(a_1 + xa_2)}{(x^3 + x^2 - x)a_2^2 + (x-1)a_1^2 + 2(x^2-1)a_1a_2}\right),$$

whose inverse is

$$\kappa^{-1}\colon (x,y) \dashrightarrow \left(\left[\frac{2(x^2-4yx+2x-y^2+1)y}{(1+x-y)(1+x+y)(1-x-y)}:1\right], \frac{-4xy}{(1+x+y)(1-x-y)}\right),$$

conjugates σ_4 to the automorphism $\tau\colon (x,y)\mapsto(-y,x)$ of $\mathbb{A}^2$. This automorphism τ has eigenvalues $\pm\mathbf{i}$ over $\mathbb{C}$. If v_1, v_2 are eigenvectors, then the invariant subring is generated, over $\mathbb{C}$, by v_1v_2, $(v_1)^4$, $(v_2)^4$. This implies that

$$\mathbb{Q}[x,y]^\tau = \mathbb{Q}\left[x^2+y^2, x^4+y^4, xy(x^2-y^2)\right] = \mathbb{Q}[x_1,x_2,x_3]/(x_1^4+2x_3^2+2x_2^2-3x_1^2x_2).$$

The surface $\mathcal{M}_2(4)/\langle\sigma_4\rangle$ is thus birational, over $\mathbb{Q}$, to the hypersurface of $\mathbb{A}^3$ given by $x_1^4 + 2x_3^2+2x_2^2-3x_1^2x_2=0$. This hypersurface is a conic bundle over the x_1 line, with a section $(x_2,x_3)=(0,0)$, and is therefore rational. An explicit birational map to $\mathbb{A}^2$ is $(x_1,x_2,x_3)\dashrightarrow \left(x_1, \frac{x_1x_3}{x_1^2-x_2}\right)$, whose inverse is $(x,y)\dashrightarrow (x, \frac{(x^2+2y^2)x^2}{2(x^2+y^2)}, \frac{x^3y}{2(x^2+y^2)})$.

(iii) We embed $\mathcal{M}_2(3)$ in Rat_2 as in Section 2.2. It is given by maps f which satisfy $f([0:1])=[1:0]$, $f([1:0])=[1:1]$, $f([1:1])=[1:0]$, and is thus parameterized by an open subset of $\mathbb{P}^2$. A point $[a_1:a_3:a_4]$ corresponds to an endomorphism

$$[u:v]\mapsto\left[a_3u^2+a_1uv+(-a_1-a_3)v^2, a_3u^2+a_4uv\right].$$

Because the corresponding resultant is $a_3(a_3+a_4)(a_3+a_1)(a_1+a_3-a_4)$, the surface $\mathcal{M}_2(3)$ is the complement of three lines in $\mathbb{A}^2$.

(iv) The map σ_3 sends $(f,[0:1],[1:0],[1:1])$ to $(f,[1:0],[1:1],[0:1])$, which is in the orbit of

$$(gfg^{-1}, g([1:0]), g([1:1]), g([0:1])) = (gfg^{-1},[0:1],[1:0],[1:1]),$$

where $g\colon [u:v]\mapsto[u-v:u]$. Because the endomorphism gfg^{-1} equals

$$[u:v]\mapsto\left[(a_1+a_3)u^2-(a_1+2a_3+a_4)uv+(a_3+a_4)v^2, (a_1+a_3)u^2-(a_1+2a_3)uv\right],$$

the automorphism σ_3 corresponds to the automorphism

$$[a_1:a_3:a_4]\mapsto[-a_1-2a_3-a_4:a_1+a_3:-a_1-2a_3]$$

of $\mathbb{P}^2$. The affine plane where $a_4-a_1-a_3\neq 0$ is invariant, and the action, in coordinates $x_1=\frac{a_1}{a_4-a_1-a_3}$, $x_2=\frac{-a_1-2a_3-a_4}{a_4-a_1-a_3}$, corresponds to $(x_1,x_2)\mapsto(x_2,-x_1-x_2)$. The quotient of $\mathbb{A}^2$ by this action is rational over $\mathbb{Q}$ (see the proof of Lemma 4.7, where the quotient of the same action on $\mathbb{A}^2$ is computed), so $\mathcal{M}_2(3)/\langle\sigma_3\rangle$ is rational over $\mathbb{Q}$. ■

Lemma 3.2. The varieties $\mathcal{M}_2(1)$, $\mathcal{M}_2(2)$, and $\mathcal{M}_2(2)/\langle\sigma_2\rangle$ are surfaces that are rational over $\mathbb{Q}$. □

Proof. (i) Let us denote by $U \subset \mathrm{Rat}_2(1)$ the open subset of pairs (f, p), where p is not a critical point, which means here that $f^{-1}(p)$ consists of two distinct points, namely p and another one. This open set is dense (its complement has codimension 1) and is invariant by PGL_2. We consider the morphism

$$\begin{aligned}\tau\colon \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\} &\to U\\ (a, b, c) &\mapsto ([u:v] \mapsto \left[uv : au^2 + buv + cv^2\right], [0:1]),\end{aligned}$$

and observe that τ is a closed embedding. Moreover, the multiplicative group $\mathbb{G}_m$ acts on $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$ via $(t, (a, b, c)) \mapsto (t^2 a, tb, c)$, and the orbits of this action correspond to the restriction of the orbits of the action of PGL_2 on U.

In consequence, U/PGL_2 is isomorphic to $(\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m$, which is isomorphic to $\mathrm{Spec}\,\mathbb{Q}\left[\frac{b^2}{a}, c, \frac{1}{c}\right] = \mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$. Hence, U/PGL_2 is rational, and thus $\mathcal{M}_2(1)$ is rational as well.

(ii) We take coordinates $[a:b:c:d]$ on $\mathbb{P}^3$ and consider the open subset $W \subset \mathbb{P}^3$, where $ad \neq bc$, $b \neq 0$, and $c \neq 0$. The morphism

$$\begin{aligned}W &\to \mathrm{Rat}_2(2),\\ [a:b:c:d] &\mapsto ([u:v] \mapsto [v(au+bv) : u(cu+dv)], [0:1], [1:0])\end{aligned}$$

is a closed embedding. Moreover, the multiplicative group $\mathbb{G}_m$ acts on W via $(t, [a:b:c:d]) \mapsto [a\mu^2 : b\mu^3 : c : \mu d]$, and the orbits of this action correspond to the restriction of the orbits of the action of PGL_2 on $\mathrm{Rat}_2(2)$.

In consequence, $\mathrm{Rat}_2(2)/\mathrm{PGL}_2$ is isomorphic to $W/\mathbb{G}_m$. Denote by $\hat{W} \subset \mathbb{P}^3$ the open subset where $bc \neq 0$, which is equal to $\mathrm{Spec}(\mathbb{Q}[\frac{a}{b}, \frac{a}{c}, \frac{b}{c}, \frac{c}{b}, \frac{d}{b}, \frac{d}{c}])$. Writing $t_1 = \frac{d}{c}$, $t_2 = \frac{a}{c}$, $t_3 = \frac{b}{c}$, we also have $\hat{W} = \mathrm{Spec}(\mathbb{Q}[t_1, t_2, t_3, \frac{1}{t_3}])$, and the action of $\mathbb{G}_m$ corresponds to $t_i \mapsto \mu^i t_i$. This implies that $\hat{W}/\mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[\frac{(t_1)^3}{t_3}, \frac{t_1 t_2}{t_3}, \frac{(t_2)^3}{(t_3)^2}])$, and is thus isomorphic to the singular rational affine hypersurface $\Gamma \subset \mathbb{A}^3$ given by $xz = y^3$. The variety $W/\mathbb{G}_m \cong \mathrm{Rat}_2(2)/\mathrm{PGL}_2$ corresponds to the open subset of Γ where $y \neq 1$.

(iii) The map σ_2 corresponds to the automorphism $[a:b:c:d] \mapsto [d:c:b:a]$ of $\mathbb{P}^3$, to the automorphism $(t_1, t_2, t_3) \mapsto \left(\frac{t_2}{t_3}, \frac{t_1}{t_3}, \frac{1}{t_3}\right)$ of $\hat{W}$, and to the automorphism $(x, y, z) \mapsto (z, y, x)$ of Γ. The invariant subalgebra of $\mathbb{Q}[x, y, z]/(xz - y^2)$ is thus generated by $\{x + z, xz, y\}$. Hence the quotient of Γ by the involution is the rational variety $\mathrm{Spec}(\mathbb{Q}[x + z, y]) = \mathbb{A}^2$. This shows that $\mathcal{M}_2(2)/\langle\sigma_2\rangle$ is rational over $\mathbb{Q}$. ■

Remark 3.3. The proof of Lemma 3.2 shows that $\mathcal{M}_2(2)$ is an affine singular surface but that $\mathcal{M}_2(2)/\langle\sigma_2\rangle$ is smooth. One can also see that the surface $\mathcal{M}_2(1)$ is singular, and that it is not affine. □

Lemma 3.4. The surface $\mathcal{M}_2(5)/\langle\sigma_5\rangle$ is rational over $\mathbb{Q}$. □

Proof. The surface $\mathcal{M}_2(5)$ is isomorphic to an open subset of $\mathbb{A}^2$, and an element $(x, y) \in \mathcal{M}_2(5) \subset \mathbb{A}^2$ corresponds to a map $(f, ([0:1],[1:0],[1:1],[x:1],[y:1])) \in \mathrm{Rat}_2(5)$. The element $(f, ([1:0],[1:1],[x:1],[y:1],[0:1]))$ is in the orbit under $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ of $(gfg^{-1}, [0:1],[1:0],[1:1],[x-1:y-1],[1-x:1])$, where $g\colon [u:v] \mapsto [(x-1)v : u-v]$. Therefore, the automorphism σ_5 of $\mathcal{M}_2(5)$ is the restriction of the birational map $(x, y) \dashrightarrow \left(\frac{x-1}{y-1}, 1-x\right)$ of $\mathbb{A}^2$, which is the restriction of the following birational map of $\mathbb{P}^2$ (viewing $\mathbb{A}^2$ as an open subset of $\mathbb{P}^2$ via $(x, y) \mapsto [x:y:1]$):

$$\tau\colon [x:y:z] \dashrightarrow [(x-z)z : (z-x)(y-z) : (y-z)z]\,.$$

The map τ has order 5 and the set of base-points of the powers of τ are the four points $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$, $p_4 = [1:1:1]$. Denoting by $\pi\colon S \to \mathbb{P}^2$ the blow-up of these four points, the map $\hat{\tau} = \pi^{-1}\tau\pi$ is an automorphism of the surface S. Because p_1, p_2, p_3, p_4 are in general position (no 3 are collinear), the surface S is a del Pezzo surface of degree 5, and thus the anti-canonical morphism gives an embedding $S \to \mathbb{P}^5$ as a surface of degree 5. The map $\pi^{-1}\colon \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ corresponds to the system of cubics through the four points, so we can assume, up to automorphism of $\mathbb{P}^5$, that it is equal to

$$\begin{aligned}\pi^{-1}([x:y:z]) = \big[&-xz(y-z) : y(x-z)(x-y) : z(x^2-yz) :\\ &(2yz-y^2-xz)z : (y-z)(yz+xy-xz) : x(z-y)(y-z+x)\big]\,,\end{aligned}$$

and the choice made here implies that $\hat{\tau} \in \mathrm{Aut}(S)$ is given by

$$[X_0 : \cdots : X_5] \mapsto [X_0 : X_1 : X_3 : X_4 : X_5 : -X_2 - X_3 - X_4 - X_5]\,.$$

The affine open subset of $\mathbb{P}^5$ where $X_0 \neq 0$ has the coordinates $x_1 = \frac{X_1}{X_0}, x_2 = \frac{X_2}{X_0}, \ldots, x_5 = \frac{X_5}{X_0}$, and is invariant. The action of $\hat{\tau}$ on these coordinates is linear, with eigenvalues $1, \zeta, \ldots, \zeta^4$ where ζ is a 5th of unity. We diagonalize the action over $\mathbb{Q}[\zeta]$, obtaining eigenvectors $\mu_0, \mu_1, \ldots, \mu_4$. Then the field of invariant functions is generated by

$$\mu_0,\ \mu_1\mu_4,\ \mu_2\mu_3,\ (\mu_1)^2\mu_3,\ \mu_1(\mu_2)^2,\ (\mu_3)^2\mu_4,\ \mu_2(\mu_4)^2.$$

In consequence, the field $\mathbb{Q}(S)^{\hat{\tau}}$ is generated by x_1 and by all invariant homogenous polynomials of degree 2 and 3 in $x_2, \dots, x_5$. The invariant homogeneous polynomials of degree 2 in $x_2, \dots, x_5$ are linear combinations of v_1 and v_2, where

$$v_1 = x_5^2 + x_3x_5 - x_3x_4 + 2x_2x_5 + x_2x_4 + x_2^2,$$

$$v_2 = x_4x_5 + x_4^2 + 2x_3x_4 + x_3^2 - x_2x_5 + x_2x_3.$$

By replacing the x_i by the composition with τ^{-1} given above, we observe that $-1 - 11x_1 + x_1^2 - v_1 - 4v_2$ is equal to zero on S, so we can eliminate v_1.

The space of invariant homogeneous polynomials of degree 3 in $x_2, \dots, x_5$ has dimension 4, but by again replacing the x_i by their composition with τ^{-1} we can compute that the following invariant suffices:

$$v_3 = x_4x_5^2 + x_4^2x_5 + x_2x_5^2 + 2x_2x_4x_5 + 2x_2x_3x_5 + x_2x_3^2 + x_2^2x_5 + x_2^2x_3.$$

This shows that the field of invariants $\mathbb{Q}(S)^{\hat{\tau}}$ is generated by x_1, v_2, v_3. In consequence, the map $S \dashrightarrow \mathbb{A}^3$ given by (x_1, v_2, v_3) factors through a birational map from $S/\hat{\tau}$ to an hypersurface $S' \subset \mathbb{A}^3$, and it suffices to prove that this latter is rational. To get the equation of the hypersurface, we observe that

$$3 + 10v_2 + 11v_2^2 + 4v_2^3 + 70x_1 + 445x_1^2 + 410x_1^3 - 85x_1^4 + 4x_1^5$$
$$+ 66x_1v_2^2 - x_1^2v_2^2 - 30x_1^3v_2 + 320x_1^2v_2 + 140x_1v_2 + v_3^2$$

is equal to zero on S. Because the above polynomial is irreducible, it is the equation of the surface S' in $\mathbb{A}^3$. It is not clear from the equation that surface is rational, so we will change coordinates. Choosing $\mu = \frac{x_1^2 - 11x_1 - 1}{v_2}$ and $\nu = \frac{v_3}{v_2}$, we have $\mathbb{Q}(S)^{\hat{\tau}} = \mathbb{Q}(x_1, \mu, \nu)$, and replacing $v_2 = \frac{x_1^2 - 11x_1 - 1}{\mu}$ and $v_3 = \nu \cdot \frac{x_1^2 - 11x_1 - 1}{\mu}$ in the equation above, we find a simpler equation, which is

$$4x_1^2 - \mu x_1^2 + 66\mu x_1 + 4x_1\mu^3 - 44x_1 - 30\mu^2x_1 - 4 + 3\mu^3 + 11\mu + \nu^2\mu - 10\mu^2 = 0.$$

We do another change of coordinates, which is $\kappa = \frac{nu}{x_1 + 7 - 5\mu}$, $\rho = \frac{\mu^2 - 5\mu + 5}{x_1 + 7 - 5\mu}$, and replace $\nu = \frac{\kappa(\mu^2 - 5\mu + 5)}{\rho}$, $x_1 = \frac{-7\rho + 5\rho\mu + \mu^2 - 5\mu + 5}{\rho}$ in the equation, to obtain

$$4\rho\mu - \mu + \kappa^2\mu + 4 + 20\rho^2 - 20\rho = 0,$$

which is obviously the equation of a rational surface. Moreover, this procedure gives us two generators of $\mathbb{Q}(S)^{\hat{\tau}}$, which are ρ and κ. ∎

Remark 3.5. The proof of Lemma 3.4 could also be seen more geometrically, using more sophisticated arguments. The quotient of $(\mathbb{P}^1)^5$ by PGL_2 is the del Pezzo surface $S\subset\mathbb{P}^5$ of degree 5 constructed in the proof. The action of σ_5 on S has two fixed points over $\mathbb{C}$, which are conjugate over $\mathbb{Q}$. The quotient S/σ_5 has thus two singular points of type A_4, and is then a weak del Pezzo surface of degree 1. Its equation in a weighted projective space $\mathbb{P}(1,1,2,3)$ is in fact the homogenization of the one given in the proof of the lemma. The fact that it is rational can be computed explicitly, as done in the proof, but can also be viewed by the fact that the elliptic fibration given by the anti-canonical divisor has five rational sections of self-intersection -1, and the contraction of these yields another del Pezzo surface of degree 5. Moreover, any unirational del Pezzo surface of degree 5 contains a rational point, and is then rational [12, p. 642]. □

4 The Surface $\mathcal{M}_2(6)$

4.1 Explicit embedding of the surfaces $\mathcal{M}_2(6)$ into S_6

Using Lemma 2.1, one can see $\mathcal{M}_2(6)$ as a locally closed surface in $\mathbb{A}^3$. However, this surface has an equation which is not very nice, and its closure in $\mathbb{P}^3$ has bad singularities (in particular, a whole line is singular). Moreover, the action of σ_6 on $\mathcal{M}_2(6)$ is not linear. We thus take another model of $\mathcal{M}_2(6)$.

The variety $\mathcal{M}_2(6)$ embeds into the moduli space P_1^6 of 6 ordered points of $\mathbb{P}^1$, modulo $\mathrm{Aut}(\mathbb{P}^1)=\mathrm{PGL}_2$. This variety P_1^6 can be viewed in $\mathbb{P}^5$ as the rational cubic threefold defined by the equations

$$\begin{aligned} x_0+x_1+x_2+x_3+x_4+x_5&=0,\\ x_0^3+x_1^3+x_2^3+x_3^3+x_4^3+x_5^3&=0 \end{aligned} \tag{2}$$

(see [8, Example 2, p. 14]). We obtain $\mathcal{M}_2(6)$ as an open subset of the projective surface in P_1^6 given by

$$x_0^3+x_1^2x_2+x_2^2x_3+x_3^2x_1+x_4^2x_5+x_4x_5^2=0,$$

with σ_6 acting by $[x_0:\cdots:x_5]\mapsto[x_0:x_2:x_3:x_1:x_5:x_4]$. The quotient $\mathcal{M}_2(6)/\langle\sigma_6^3\rangle$ can thus be explicitly computed; using tools of birational geometry we obtain that it is rational, so $\mathcal{M}_2(6)$ is birational to a double cover of $\mathbb{P}^2$. The ramification curve obtained is the union of a smooth cubic with a quintic having four double points. Choosing coordinates on $\mathbb{P}^2$ so that the action corresponding to σ_6 is an automorphism of order 3, and contracting some curves (see Remark 1.2), we obtain an explicit description of the surface

$\mathcal{M}_2(6)$ as an open subset of the projective hypersurface S_6 of $\mathbb{P}^3$ given by

$$W^2F_3(X,Y,Z)=F_5(X,Y,Z),$$

where

$$F_3(X,Y,Z)=(X+Y+Z)^3+(X^2Z+XY^2+YZ^2)+2XYZ,$$

$$F_5(X,Y,Z)=(Z^3X^2+X^3Y^2+Y^3Z^2)-XYZ(YZ+XY+XZ).$$

The following result shows that it is a projectivization of $\mathcal{M}_2(6)$ that has better properties, and directly shows Proposition 1.1.

Lemma 4.1. Let $\varphi\colon \mathbb{A}^3 \dashrightarrow \mathbb{P}^3$ be the birational map given by

$$\varphi((x,y,z))=[-y+z-yz+xy:-y-z+yz+2x-xy:y-z-yz+xy:y+z-xy-yz],$$

$$\varphi^{-1}([W:X:Y:Z])=\left(\frac{(X+Z)(W+Y)}{W^2+XY+YZ+XZ},\frac{W+Y}{X+Y},\frac{(W+Z)(W+Y)}{W^2+XY+YZ+XZ}\right).$$

Then, the following hold:

(i) The map φ restricts to an isomorphism from $\mathcal{M}_2(6)\subset\mathbb{A}^3$ to the open subset of S_6 which is the complement of the union of the nine lines

$$L_1: W=Z=-Y,\quad L_2: W=Y=-Z,\quad L_3: W=Y=-X,$$

$$L_4: W=X=-Y,\quad L_5: W=X=-Z,\quad L_6: W=Z=-X,$$

$$L_7: -X=Y=Z,\quad L_8: X=-Y=Z,\quad L_9: X=Y=-Z,$$

and of the 14 conics

$$\mathcal{C}_1:\begin{cases} W=X+Y+Z,\\ X^2+Y^2+Z^2+3(XY+XZ+YZ)=0,\end{cases}$$

$$\mathcal{C}_2:\begin{cases} W=-(X+Y+Z),\\ X^2+Y^2+Z^2+3(XY+XZ+YZ)=0,\end{cases}$$

$$\mathcal{C}_3:\begin{cases} W=X,\\ X^2+XY+3XZ-YZ=0,\end{cases}\qquad \mathcal{C}_4:\begin{cases} W=-X,\\ X^2+XY+3XZ-YZ=0,\end{cases}$$

$$\mathcal{C}_5 : \begin{cases} W = Y, \\ Y^2 + YZ + 3YX - ZX = 0, \end{cases} \qquad \mathcal{C}_6 : \begin{cases} W = -Y, \\ Y^2 + YZ + 3YX - ZX = 0, \end{cases}$$

$$\mathcal{C}_7 : \begin{cases} W = Z, \\ Z^2 + ZX + 3ZY - XY = 0, \end{cases} \qquad \mathcal{C}_8 : \begin{cases} W = -Z, \\ Z^2 + ZX + 3ZY - XY = 0, \end{cases}$$

$$\mathcal{C}_9 : \begin{cases} Z = X, \\ W(Y + 3X) + X(X - Y) = 0, \end{cases} \qquad \mathcal{C}_{10} : \begin{cases} Z = X, \\ W(Y + 3X) - X(X - Y) = 0, \end{cases}$$

$$\mathcal{C}_{11} : \begin{cases} Y = Z, \\ W(X + 3Z) + Z(Z - X) = 0, \end{cases} \qquad \mathcal{C}_{12} : \begin{cases} Y = Z, \\ W(X + 3Z) - Z(Z - X) = 0, \end{cases}$$

$$\mathcal{C}_{13} : \begin{cases} X = Y, \\ W(Z + 3Y) + Y(Y - Z) = 0, \end{cases} \qquad \mathcal{C}_{14} : \begin{cases} X = Y. \\ W(Z + 3Y) - Y(Y - Z) = 0. \end{cases}$$

Moreover, the union of the 23 curves is the support of the zero locus on S_6 of

$$(W^2 + XY + YZ + XZ)(W^2 - X^2)(W^2 - Y^2)(W^2 - Z^2)(X^2 - Y^2)(Y^2 - Z^2)(Y^2 - Z^2),$$

which corresponds to the set of points where two coordinates are equal up to sign, or where $W^2 + XY + YZ + XZ = 0$.

(ii) The automorphism σ_6 of $\mathcal{M}_2(6)$ is the restriction of the automorphism

$$[W : X : Y : Z] \mapsto [-W : Y : Z : X]$$

of $\mathbb{P}^3$.

(iii) To any point $[W : X : Y : Z] \in \mathcal{M}_2(6) \subset \mathbb{P}^3$ corresponds the element $[a_0 : \cdots : a_5] \in \mathrm{Rat}_2 \subset \mathbb{P}^5$ given by

$$a_0 = 1,$$

$$a_1 = \frac{(W - X)(W(X + Y + 2Z) + Z(Y - X))}{(W^2 + XY + XZ + YZ)(X - Z)} - 1,$$

$$a_2 = \frac{(W + Y)(W + Z)(X - W)(W(X + Y + 2Z) + XY - Z^2)}{(W^2 + XY + XZ + YZ)^2(X - Z)},$$

$$a_3 = 1,$$

$$a_4 = \frac{(Y+Z)(W-Z)(X-W)(W(2X+Y+Z)+YZ-X^2)}{((W^2+XY+XZ+YZ)(X+Z)(Y+W)(X-Z))} - 1,$$

$$a_5 = 0,$$

and its orbit of six points is given by

$$[0:1],\ [1:0],\ [1:1],$$

$$\left[(X+Z)(W+Y) : W^2+XY+YZ+XZ\right],\ [W+Y : X+Y],$$

$$\left[(W+Z)(W+Y) : W^2+XY+YZ+XZ\right]. \qquad \square$$

Remark 4.2. Applying the automorphism of $\mathbb{P}^1$ given by

$$[u:v] \mapsto [(W+X)u - (W+Y)v : (W-X)u],$$

we can send the six points of the orbits to

$$[1:0],\ [W+Y : W-X],\ [0:1],\ [Z-W : X+Z],\ [1:1],\ [Y+Z, W+Z],$$

respectively. This also changes the coefficients of the endomorphism of $\mathbb{P}^1$. $\square$

Proof. The explicit description of φ and φ^{-1} implies that φ restricts from an isomorphism $U \to V$, where $U \subset \mathbb{A}^3$ is the open set where $y(y-1)(x-z) \neq 0$ and $V \subset \mathbb{P}^3$ is the open set where $(W^2+XY+YZ+XZ)(X+Y)(W+Y)(W-X) \neq 0$ (just compute $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$).

Since $\mathcal{M}_2(6) \subset \mathbb{A}^3$ is contained in U, the map φ restricts to an isomorphism from $\mathcal{M}_2(6)$ to its image, contained in V. Substituting $x_1 = \frac{(X+Z)(W+Y)}{W^2+XY+YZ+XZ}$, $x_2 = \frac{W+Y}{X+Y}$, $x_3 = \frac{(W+Z)(W+Y)}{W^2+XY+YZ+XZ}$ into the formula of Lemma 2.1 yields assertion (iii). The fact that the map $f \in \mathrm{Rat}_2$ constructed by this process sends $[x_1:1]$ to $[x_2:1]$ corresponds to the equation of the surface S_6. This shows that $\mathcal{M}_2(6)$ can be viewed, via φ, as an open subset of S_6.

The automorphism σ_6 sends a point $(x, y, z) \in \mathcal{M}_2(6) \subset \mathbb{A}^3$, corresponding to an element $(f, [0:1], [1:0], [1:1], [x:1], [y:1], [z:1])$ (see Section 2.2), to the point corresponding to $\alpha = (f, [1:0], [1:1], [x:1], [y:1], [z:1], [0:1])$. The automorphism of $\mathbb{P}^1$ given by $\nu : [u:v] \mapsto [v(x-1) : u-v]$ sends α to

$$(\nu f \nu^{-1}, [0:1], [1:0], [1:1], [x-1 : y-1], [x-1 : z-1], [1-x : 1]),$$

so the action of σ_6 on $\mathcal{M}_2(6) \subset \mathbb{A}^3$ is the restriction of the birational map of order 6 given by

$$\tau\colon (x, y, z) \dashrightarrow \left(\frac{x-1}{y-1}, \frac{x-1}{z-1}, 1-x\right).$$

Assertion (ii) is then proved by observing that

$$\varphi^{-1}\tau\varphi([W:X:Y:Z]) = [-W:Y:Z:X].$$

In order to prove (i), we need to show that the complement of $\mathcal{M}_2(6) \subset S_6$ is the union of $L_1, \ldots, L_9$ and $\mathcal{C}_1, \ldots, \mathcal{C}_{14}$, and that it is the set of points given by $W^2 + XY + YZ + XZ = 0$ or where two coordinates are equal up to sign. Note that this complement is invariant under $[W:X:Y:Z] \mapsto [-W:Y:Z:X]$, since this automorphism corresponds to $\sigma_6 \in \mathrm{Aut}(\mathcal{M}_2(6))$. This simplifies the calculations.

Let us show that the union of $L_1, \ldots, L_8, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_9, \mathcal{C}_{10}$ is the zero locus of the polynomial $(W^2 + XY + YZ + XZ)(W - X)(W + Y)(X + Y)(X - Z)$:

(1) The zero locus of $W^2 + XY + YZ + XZ$ on the quintic gives the degree-10 curve

$$\begin{cases} W^2 + XY + YZ + XZ = 0, \\ (X+Z)(X+Y)(Y+Z)(X^2 + Y^2 + Z^2 + 3XY + 3XZ + 3YZ) = 0, \end{cases}$$

which is the union of $L_1, L_2, \ldots, L_6$ and $\mathcal{C}_1$ and $\mathcal{C}_2$ because the linear system of quadrics given by

$$\lambda(W^2 + XY + YZ + XZ) + \mu(X^2 + Y^2 + Z^2 + 3(XY + XZ + YZ)) = 0$$

with $(\lambda : \mu) \in \mathbb{P}^1$ corresponds to

$$\lambda(W^2 - (X + Y + Z)^2) + (\lambda + \mu)(X^2 + Y^2 + Z^2 + 3(XY + XZ + YZ)) = 0,$$

and thus its base-locus is the union of $\mathcal{C}_1$ and $\mathcal{C}_2$.

(2) Substituting $W = X$ in the equation of S_6 yields $(X+Z)(Y+X)^2(X^2 + XY + 3XZ - YZ) = 0$, so the locus of $W = X$ on S_6 is the union of L_4, L_5, and $\mathcal{C}_3$.

(3) Substituting $W = -Y$ in the equation yields $(X+Y)(Y+Z)^2(3XY + Y^2 - XZ + YZ) = 0$ which corresponds to L_1, L_4, and $\mathcal{C}_6$.

(4) Substituting $X = -Y$ yields $(-W+Y)(W+Y)(Y-Z)(Y+Z)^2 = 0$ which corresponds to L_3, L_4, L_7, and L_8.

(5) Substituting $Z = X$ yields $(X+Y)(W(Y+3X) + X(X-Y))(W(Y+3X) - X(X-Y))$, which corresponds to $L_8, \mathcal{C}_9, \mathcal{C}_{10}$.

Applying the automorphism $[W:X:Y:Z]\mapsto[-W:Y:Z:X]$, we obtain that the union of $L_1,\dots,L_9,\mathcal{C}_1,\dots,\mathcal{C}_{14}$ is given by the zero locus of $W^2+XY+YZ+XZ$ and all hyperplanes of the form $x_i\pm x_j$ where $x_i,x_j\in\{W,X,Y,Z\}$ are distinct. This shows that (i) implies (ii).

The steps (1)–(4) imply that the complement of $S_6\cap V$ in S_6 is the union of the lines $L_1,\dots,L_8$ and the conics $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_6$.

On the affine surface $S_6\cap V$, the map φ^{-1} is an isomorphism. To any point $[W:X:Y:Z]\in S_6\cap V$ we associate an element of $\mathbb{P}^5$, via the formula described in (iii), which should correspond to an endomorphism of degree 2 if the point belongs to $\mathcal{M}_2(6)$. Computing the resultant with the formula of Section 2.1, we get a polynomial R with many factors:

$$
\begin{aligned}
R=&(W^2+XY+XZ+XY)^4(Z-X)(X+Z)^2(Y+Z)\\
&(W-Z)(W+Z)(Y+W)^3(W-X)^3\\
&(W(X+Y+2Z)+XY-Z^2)(W(X+Y+2Z)+Z(Y-X))\\
&(W(2X+Y+Z)+YZ-X^2)(W(2X+Y+Z)+X(Y-Z)).
\end{aligned}
$$

The surface $\mathcal{M}_2(6)$ is thus the complement in S_6 of the curves $L_1,\dots,L_8$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_6$, and the curves given by the polynomial R. The components $(W-X)(W+Y)$ $(W^2+XY+XZ+YZ)(Z-X)$ were treated before. In particular, this shows (using again the action given by σ_6) that each of the curves $L_1,\dots,L_9,\mathcal{C}_1,\dots,\mathcal{C}_{14}$ is contained in $S_6\setminus\mathcal{M}_2(6)$. It remains to see that the trace of any irreducible divisor of R on S_6 is contained in this union. The case of $(W^2+XY+XZ+XY)$ and all factors of degree 1 were done before, so it remains to study the last four factors. Writing $\Gamma=W^2F_3-F_5$, which is the polynomial defining S_6, we obtain

$$
\begin{aligned}
&(W(X+Y+2Z)+XY-Z^2)(-W(X+Y+2Z)+XY-Z^2)(X+Y)+\Gamma\\
&\quad=(Y-Z)(Y+Z)(X-Z)(W^2+XY+YZ+XZ),\\
&(W(X+Y+2Z)+Z(Y-X))(-W(X+Y+2Z)+Z(Y-X))(X+Y)+\Gamma\\
&\quad=(Y-Z)(Y+Z)(X-Z)(W-X)(W+X).
\end{aligned}
$$

The last two factors being in the image of these two factors by $[W:X:Y:Z]\mapsto[W:Y:Z:X]$, we have shown that $S_6\setminus\mathcal{M}_2(6)$ is the union of $L_1,\dots,L_9,\mathcal{C}_1,\dots,\mathcal{C}_{14}$. ■

Remark 4.3.

(1) In the above proof, we show just that the birational map φ satisfies the property described in the statement of the Lemma 4.1. The proof is thus significantly shorter than the original derivation of the map φ, explained at the beginning of this section.

(2) Note that our technique uses the fact that $n=6$ is even, and that the quotient $\mathcal{M}_2(n)/\langle\sigma_n^{n/2}\rangle$ is rational in this case, a fact that is not clear to be true for $n>6$. Another peculiarity of the case $n=6$ is the description of P_1^6 as the rational cubic three-fold defined by the simple equations in $\mathbb{P}^5$ (see Equation (3)). □

Corollary 4.4. The variety $\mathcal{M}_2(6)$ is an affine smooth surface, which is birational to S_6, a projective surface of general type. □

Proof. Computing the partial derivatives of the equation of S_6, one directly sees that it has exactly 11 singular points, of which two are fixed by

$$[W:X:Y:Z]\mapsto[-W:Y:Z:X]$$

and the nine others form a set that consists of an orbit of size 3 and an orbit of size 6:

$$[1:0:0:0],\quad [0:1:1:1],$$

$$[0:1:0:0],\quad [0:0:1:0],\quad [0:0:0:1],$$

$$[-1:-1:1:1],\quad [-1:1:-1:1],\quad [-1:1:1:-1],$$

$$[1:-1:1:1],\quad [1:1:-1:1],\quad [1:1:1:-1].$$

Since none of the points belongs to $\mathcal{M}_2(6)$, viewed in S_6 using Lemma 4.1, the surface $\mathcal{M}_2(6)$ is smooth. Because the complement of $\mathcal{M}_2(6)$ in S_6 is the zero locus of a homogeneous polynomial (by Lemma 4.1(ii)), the surface $\mathcal{M}_2(6)$ is affine. It remains to see that S_6 is of general type.

The point $[1:0:0:0]$ is a triple point, and all others are double points. Denoting by $\pi:\hat{\mathbb{P}}^3\to\mathbb{P}^3$ the blow-up of the 11 points, the strict transform $\hat{S}_6$ of S_6 is a smooth surface.

We denote by $E_1,\ldots,E_{11}\in\mathrm{Pic}(\hat{\mathbb{P}}^3)$ the exceptional divisors obtained (according to the order above), and by H the pull-back of a general hyperplane of $\mathbb{P}^3$. The ramification formula gives the canonical divisor $K_{\hat{\mathbb{P}}^3}=-4H+2\sum_{i=1}^{11}E_i$. The divisor of $\hat{S}_6$ is then equivalent to $5H - 3E_1 - 2\sum_{i=2}^{11} E_i$. Applying the adjunction formula, we find that $K_{\hat{S}_6} = (K_{\hat{\mathbb{P}}^3} + \hat{S}_6)|_{\hat{S}_6} = (H - E_1)|_{\hat{S}_6}$.

The linear system $H - E_1$ corresponds to the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by $[W : X : Y : Z] \mapsto [X : Y : Z]$. The map $K_{\hat{S}_6} \xrightarrow{|K_{\hat{S}_6}|} \mathbb{P}^2$ is thus surjective, which implies that $\hat{S}_6$, and thus S_6, is of general type. ■

Remark 4.5. One can see that the divisor of $\hat{S}_6$ given by $D = \sum_{1\le i\le 9} \hat{L}_i + \sum_{1\le i\le 14} \hat{\mathcal{C}}_i$ is normal crossing, where the $\hat{L}_i$'s and the $\hat{\mathcal{C}}_i$'s are the lines and the conics in $\hat{S}_6$ associated to the L_i's and $\mathcal{C}_i$'s in S_6. This follows from the study of the intersections of the L_i's and $\mathcal{C}_i$'s in S_6. They are not normal only in the following situations:

(i) $\mathcal{C}_3 \cap \mathcal{C}_9 = \{[0:0:1:0]\}$ and the curves have a common tangent line, which is $W = X = Z$. By applying the automorphism σ_6, we find that the intersection is not normal also in $\mathcal{C}_6 \cap \mathcal{C}_{14}, \mathcal{C}_7 \cap \mathcal{C}_{11}, \mathcal{C}_4 \cap \mathcal{C}_{10}, \mathcal{C}_5 \cap \mathcal{C}_{13}$, and $\mathcal{C}_8 \cap \mathcal{C}_{12}$. We note that in each of the previous cases the intersection point is a singular point of S_6 and the multiplicity of intersection is 2 because the intersection point is between two noncoplanar conics. Therefore, in $\hat{S}_6$ the intersections become normal.

(ii) $\mathcal{C}_9 \cap \mathcal{C}_{10} = \{[0:0:1:0], [0:1:1:1], [1:0:0:0]\}$. In this case, the two conics are coplanar and the tangent point is $[1:0:0:0]$ with tangent line $Z = X = -Y/3$. Note that all three intersection points, in particular $[1:0:0:0]$, are singular points of S_6. Hence $\hat{\mathcal{C}}_9$ and $\hat{\mathcal{C}}_{10}$ have normal crossings in $\hat{S}_6$. By applying the automorphism σ_6, we find the similar situation with $\mathcal{C}_{11} \cap \mathcal{C}_{12}$ and $\mathcal{C}_{13} \cap \mathcal{C}_{14}$.

Therefore, the divisor D of $\hat{S}_6$ is normal crossing. □

Remark 4.6. The condition for the divisor at infinity to be normal crossing is sometimes connected to the notion of integral points of surfaces (see Section 4.3 for the definition of S-integral point). For an example in this direction, see Vojta's conjecture [25]. In our situation it will be easy to prove the finiteness of integral points of $M_2(6)$ and we shall do this in Section 4.3. But in general the study of the integral points on surfaces could be a very difficult problem. See, for example, [6] for some results in this topic. □

4.2 Quotients of $\mathcal{M}_2(6)$

It follows from the description of $\mathcal{M}_2(6) \subset S_6$ given in Lemma 4.1 that the quotient $(\mathcal{M}_2(6))/\langle\sigma_6^3\rangle$ is rational over $\mathbb{Q}$: the quotient map corresponds to the projection $\mathcal{M}_2(6) \to \mathbb{P}^2$ given by $[W : X : Y : Z] \mapsto [X : Y : Z]$, whose image is an affine open subset $\mathcal{U}$ of $\mathbb{P}^2$, isomorphic to $(\mathcal{M}_2(6))/\langle\sigma_6^3\rangle$. The ramification of $\mathcal{M}_2(6) \to (\mathcal{M}_2(6))/\langle\sigma_6^3\rangle$ is the zero locus of F_5 on the open subset $\mathcal{U}$. We now describe the other quotients:

Lemma 4.7. The quotient $(\mathcal{M}_2(6))/\langle\sigma_6^2\rangle$ is birational to a projective surface of general type, but the quotient $(\mathcal{M}_2(6))/\langle\sigma_6\rangle$ is rational. □

Proof. (i) Recall that $S_6 \subset \mathbb{P}^3$ has equation $W^2F_3(X,Y,Z) = F_5(X,Y,Z)$ and σ_6 corresponds to $[W:X:Y:Z] \mapsto [-W:Y:Z:X]$ (Lemma 4.1). In particular, the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by $[W:X:Y:Z] \mapstochar\dashrightarrow [X:Y:Z]$ corresponds to the quotient map $U \to U/\langle\sigma_6^3\rangle$, where $U \subset S_6$ is the open subset where $F_3 \neq 0$. This implies that the surfaces $S_6/\langle\sigma_6^3\rangle$ and $\mathcal{M}_2(6)/\langle\sigma_6^3\rangle$ are rational over $\mathbb{Q}$, and that $\mathcal{M}_2(6)/\langle\sigma_6\rangle$ is birational to the quotient of $\mathbb{P}^2$ by the cyclic group of order 3 generated by $\mu\colon [X:Y:Z] \mapsto [Y:Z:X]$. We next prove that this quotient of $\mathbb{P}^2$ is rational.

The open subset of $\mathbb{P}^2$ where $X+Y+Z \neq 0$ is an affine plane $\mathbb{A}^2$ invariant under μ. We choose coordinates $x_1 = \frac{X-Y}{X+Y+Z}$ and $x_2 = \frac{Y-Z}{X+Y+Z}$ on this plane, and compute that the action of μ on $\mathbb{A}^2$ corresponds to $(x_1, x_2) \mapsto (x_2, -x_1 - x_2)$. This action is linear, with eigenvalues ω, ω^2, where ω is a third root of unity. We diagonalize the action over $\mathbb{Q}[\omega]$, obtaining eigenvectors w_1, w_2. the invariant ring is generated by w_1w_2, w_1^3, and w_2^3. In consequence, the ring $\mathbb{Q}[x_1, x_2]^\mu$ is generated by the invariant homogeneous polynomials of degree 2 and 3. An easy computation gives the following generators of the vector spaces of invariant polynomials of degree 2 and 3:

$$v_1 = x_2^2 + x_1x_2 + x_1^2,$$

$$v_2 = x_1x_2^2 + x_1^2x_2,$$

$$v_3 = x_1^3 - x_2^3 - 3x_1x_2^2.$$

Hence, $\mathbb{Q}[x_1, x_2]^\mu = \mathbb{Q}[v_1, v_2, v_3]$. Since $v_1^3 - 9v_2^2 - 3v_2v_3 - v_3^2 = 0$, the quotient $\mathbb{A}^2/\langle\mu\rangle$ is equal to the affine singular cubic hypersurface of $\mathbb{A}^3$ defined by the corresponding equation. The projection from the origin gives a birational map from the cubic surface to $\mathbb{P}^2$. Hence $\mathbb{A}^2/\langle\mu\rangle$, and thus $\mathcal{M}_2(6)/\langle\sigma_6\rangle$, is rational over $\mathbb{Q}$.

(ii) We compute the quotient $\mathcal{M}_2(6)/\langle\sigma_6^2\rangle$. The open subset of $\mathbb{P}^3$ where $W \neq 0$ is an affine space; we choose coordinates $x_0 = \frac{X+Y+Z}{W}$, $x_1 = \frac{X-Y}{W}$, and $x_2 = \frac{Y-Z}{W}$, and the action of σ_6^4 (which is the inverse of σ_6^2) corresponds to $(x_0, x_1, x_2) \mapsto (x_0, x_2, -x_1 - x_2)$. In these coordinates, $\mathbb{Q}[x_0, x_1, x_2]^{\langle\sigma_6^2\rangle} = \mathbb{Q}[x_0, v_1, v_2, v_3]$, where v_1, v_2, v_3 are polynomials of degree $2, 3, 3$ in x_1, x_2, which are the same as above.

This implies that the quotient of $\mathbb{A}^3$ by σ_6^2 is the rational singular three-fold $V \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{Q}[x_0, v_1, v_2, v_3])$ with the equation $v_1^3 - 9v_2^2 - 3v_2v_3 - v_3^2 = 0$. The three-fold V is birational to $\mathbb{P}^3$ via the map $(x_0, v_1, v_2, v_3) \mapstochar\dashrightarrow [v_1 : v_1x_0 : v_2 : v_3]$, whose inverse is $[W : X : Y : Z] \mapstochar\dashrightarrow \left(\frac{X}{W}, \frac{9Y^2+3YZ+Z^2}{W^2} : \frac{Y(9Y^2+3YZ+Z^2)}{W^3}, \frac{Z(9Y^2+3YZ+Z^2)}{W^3}\right)$. We write the equation for S_6 in

$\mathbb{A}^3$ in terms of our invariants, and find that its image in V is given by the zero locus of

$$x_0^3(32-2v_1)+3v_3x_0^2-6v_1x_0-12v_2+v_3-v_1(v_3-3v_2),$$

which is this birational with $\mathcal{M}_2(6)/\langle\sigma_6^2\rangle$. That zero locus is in turn birational, via the map $V \dashrightarrow \mathbb{P}^3$ defined above, to the quintic hypersurface of $\mathbb{P}^3$ given by

$$W^2\tilde{F}_3(X,Y,Z)=\tilde{F}_5(X,Y,Z),$$

where

$$\tilde{F}_3(X,Y,Z)=(Z+2X)(16X^2-8XZ+Z^2-27Y^2-9YZ)-108Y^3,$$

$$\tilde{F}_5(X,Y,Z)=X^2(9Y^2+3YZ+Z^2)(2X-3Z)-(3Y-Z)(9Y^2+3YZ+Z^2)^2.$$

We can see that $[1:0:0:0]$ is the only triple point of this surface, and that all other singularities are ordinary double points. As in the proof of Corollary 4.4, this implies that the surface is of general type. ■

4.3 Rational points of $\mathcal{M}_2(6)$

Because $\mathcal{M}_2(6)$ is of general type, the Bombieri–Lang conjecture asserts that the rational points of $\mathcal{M}_2(6)$ should not be Zariski dense. Since $\mathcal{M}_2(6)$ has dimension 2, this means that there should be a finite number of curves that contain all but finitely many of the rational points; moreover, by Faltings' theorem (proof of Mordell's conjecture, which is the 1D case of Bombieri–Lang), each curve that contains infinitely many rational points has genus 0 or 1.

We already found 23 rational curves on S_6, namely the components of $S_6 \setminus \mathcal{M}_2(6)$. We searched in several ways for other curves that would yield infinite families of rational points on $\mathcal{M}_2(6)$. One approach was to intersect the quintic surface $W^2F_3=F_5$ with planes and other low-degree surfaces that contain some of the known rational curves, hoping that the residual curve would have new components of low genus. We found no new rational curves this way but did discover several curves of genus 1, some of which have infinitely many rational points. Another direction was to pull back curves of low degree on the $[X:Y:Z]$ plane on which F_3F_5 has several double zeros (where the curve is either tangent to $F_3F_5=0$ or passes through a singular point). This way we found a few of the previous elliptic curves, and later also a 24th rational curve. We next describe these new curves of genus 0 and 1 on $\mathcal{M}_2(6)$.

Lemma 4.8. The equation

$$X^3 + Y^3 + Z^3 = X^2Y + Y^2Z + Z^2X$$

defines a rational cubic in $\mathbb{P}^2$, birational with $\mathbb{P}^1$ via the map $c: \mathbb{P}^1 \to \mathbb{P}^2$ taking $[m:1]$ to

$$\left[-m^3 + 2m^2 - 3m + 1 : m^3 - m + 1 : m^3 - 2m^2 + m - 1\right].$$

It is smooth except for the node at $[X:Y:Z] = [1:1:1]$. Its preimage under the $2:1$ map $\mathcal{M}_2(6) \to \mathbb{P}^2$ taking $[W:X:Y:Z]$ to $[X:Y:Z]$ is a rational curve C that is mapped to itself by σ_6. Every point of C parameterizes a quadratic endomorphism $f: \mathbb{P}^1 \to \mathbb{P}^1$ that has a rational fixed point in addition to its rational 6-cycle. □

Proof. We check that the coordinates of c are relatively prime cubic polynomials, whence its image is a rational cubic curve, and that these coordinates satisfy the cubic equation $X^3 + Y^3 + Z^3 = X^2Y + Y^2Z + Z^2X$. This proves that c is birational to its image, and thus that our cubic is a rational curve. A rational cubic curve in $\mathbb{P}^2$ has only one singularity, and since ours has a node at $[1:1:1]$ there can be no other singularities. (The parameterization c was obtained in the usual way by projecting from this node.) We may identify the function field of the cubic with $\mathbb{Q}(m)$. Substituting the coordinates of c into F_3 and F_5 yields $-4(m^2 - m)^3$ and $-4(m^2 - m)^3(m^2 - m + 1)^3$, respectively. Thus, adjoining a square root of F_5/F_3 yields the quadratic extension of $\mathbb{Q}(m)$ generated by a square root of $m^2 - m + 1$. The conic $j^2 = m^2 - m + 1$ is rational (e.g., because it has the rational point $(m, j) = (0, 1)$), so this function field is rational as well. The cubic $X^3 + Y^3 + Z^3 - (X^2Y + Y^2Z + Z^2X)$ is invariant under cyclic permutations of X, Y, Z (which act on the projective m-line by the projective linear transformations taking m to $1/(1-m)$ and $(m-1)/m$); hence σ_6^2 acts on C. Also σ_6^3 acts because it fixes X, Y, Z and takes W to $-W$, which fixes m and takes j to $-j$ in $j^2 = m^2 - m + 1$. This proves that σ_6 takes C to itself.

Choosing a parameterization of the conic given by

$$(j, m) = \left(\frac{p^2 + p + 1}{1 - p^2}, \frac{2p + 1}{1 - p^2}\right),$$

and putting the 1st, 3rd, and 5th points of the cycle at ∞, 0, and 1, respectively (see Remark 4.2), we find that the generic quadratic endomorphism f parameterized by C

has the 6-cycle

$$\begin{aligned} x_1 &= [1:0], \quad x_2 = \left[p^3+5p^2+2p+1 : (2p+1)(p^3+p^2+1)\right], \\ x_3 &= [0:1], \quad x_4 = \left[(p+2)(p^3-p^2-2p-1) : 2(p-1)(p+1)^2(2p+1)\right], \\ x_5 &= [1:1], \quad x_6 = \left[2(p+2)(2p+1) : (p+1)(p^3+p^2+4p+3)\right]. \end{aligned}$$

The quadratic map that realizes the above cycle $x_1, \ldots, x_6$ is

$$[u:v] \mapsto \left[(p^3+5p^2+2p+1)(u-\lambda_1 v)(u-\lambda_2 v) : (2p+1)(p^3+p^2+1)(u-\lambda_3 v)(u-\lambda_4 v)\right],$$

where

$$\lambda_1 = \frac{p^3+5p^2+2p+1}{(1+2p)(p^3+p^2+1)}, \quad \lambda_2 = \frac{(p+2)^2(p^3-p^2-2p-1)^2}{(p^3+5p^2+2p+1)(p^3+p^2+4p+3)(p+1)^2},$$
$$\lambda_3 = \frac{2(p+2)(1+2p)}{(p+1)(p^3+p^2+4p+3)}, \quad \lambda_4 = \frac{(p^2-1)(p^3+5p^2+2p+1)(p^3-p^2-2p-1)}{(2p+1)^2(p^3+p^2+1)^2}.$$

The point $x_0 = [1:p+1]$ is fixed by f. [We refrain from exhibiting the coefficients of f itself, which are polynomials of degree 11, 12, and 13 in p. However, we include a machine-readable formula for f in a comment line of the LaTeX source following the displayed formula for the λ_i, so that the reader may copy the formula for f from the arXiv preprint and check the calculation or build on it.] ■

Lemma 4.9. The zero locus on $\mathcal{M}_2(6)$ of XYZ is the union of three isomorphic elliptic curves, which contain infinitely many rational points. □

Proof. The 3-cycle $[W:X:Y:Z] \mapsto [W:Y:Z:X]$ permutes the factors X, Y, Z of XYZ, so it is enough to consider only curve defined by $Z=0$. Substituting $Z=0$ in the equation of S_6, we obtain

$$W^2(X^3+3X^2Y+4XY^2+Y^3) - Y^2X^3 = 0.$$

This is a singular plane curve of degree 5, birational to the smooth elliptic curve $E \subset \mathbb{A}^2$ given by

$$y^2 = x^3+4x^2+3x+1 = (x+1)^3+x^2 \tag{3}$$

via the map $[W:X:Y] \dashrightarrow \left(\frac{Y}{X}, \frac{Y}{W}\right)$ whose inverse is $(x,y) \mapsto [\frac{1}{y} : \frac{1}{x} : 1]$.

We show that E has positive Mordell–Weil rank by showing that the points $(0, \pm 1) \in E$ have infinite order. Using to the duplication formula in [23, p. 31], we calculate that the x coordinate of $2(x, y)$ is

$$\frac{x^4 - 6x^2 - 8x - 7}{4x^3 + 16x^2 + 12x + 4}.$$

Note that the x coordinate of $2(0, 1)$ is $-\frac{7}{4}$. Hence $(-\frac{7}{4}, \frac{13}{8}) \in E$. The Nagell–Lutz Theorem (e.g., see [23, p. 56]) implies that $(-\frac{7}{4}, \frac{13}{8})$ is not a torsion point. According to Lemma 4.1, the curve E meets the boundary $S_6 \setminus \mathcal{M}_2(6)$ in finitely many points; therefore, infinitely many rational points of E belong to $\mathcal{M}_2(6)$. ∎

Remark 4.10. In fact, the curve E has a simple enough equation that we readily determine the structure of its Mordell–Weil group $E(\mathbb{Q})$ using 2-descent as implemented in Cremona's program MWRANK: $E(\mathbb{Q})$ is the direct sum of the 3-element torsion group generated by $(-1, 1)$ with the infinite cyclic group generated by $(0, 1)$. The curve E has conductor 124, small enough that it already appeared in Tingley's "Antwerp" tables [2] of curves of conductor at most 200, where E is named 124B (see p. 97); Cremona's label for the curve is 124A1 [7, p. 102]. (Both sources give the standard minimal equation $y^2 = x^3 + x^2 - 2x + 1$ for E, obtained from (3) by translating x by 1.) □

Lemma 4.9 yields the existence of infinitely many classes of endomorphisms of $\mathbb{P}^1$ defined over $\mathbb{Q}$ that admit a rational periodic point of primitive period 6. We next give an example by using the previous arguments.

Example 4.11. The point $\left(-\frac{7}{4}, \frac{13}{8}\right)$ is a point of the elliptic curve E defined in (3). Using the birational map in the proof of Lemma 4.9, we see that the point $(x, y) = (-\frac{7}{4}, \frac{13}{8})$ is sent to the point $[W : X : Y : Z] = \left[\frac{8}{13} : -\frac{4}{7} : 1 : 0\right]$. Applying the map ϕ^{-1} of Lemma 4.1, we obtain the point $\left(\frac{91}{19}, \frac{49}{13}, -\frac{98}{19}\right) \in \mathbb{A}^3$. Finally we apply Lemma 2.1, finding the endomorphism

$$f[u : v] = [(19u + 98v)(133u - 441v) : 19u(133u - 529v)],$$

which admits the 6-cycle

$$[0 : 1] \mapsto [1 : 0] \mapsto [1 : 1] \mapsto [91 : 19] \mapsto [49 : 13] \mapsto [-98 : 19] \mapsto [0 : 1].$$ □

Apart from the elliptic curves in S_6 corresponding to $XYZ = 0$, there are other elliptic curves which can be found using the special form of the equation of S_6. The following lemma shows that none of these curves provides a rational point of $\mathcal{M}_2(6)$.

Lemma 4.12. The intersection with S_6 of the hyperplanes $W=\pm(X+Y+Z)$ is the union of two conics, contained in $S_6\setminus\mathcal{M}_2(6)$, and two isomorphic elliptic curves, which contain only finitely many rational points, all contained in $S_6\setminus\mathcal{M}_2(6)$ too. □

Proof. Thanks to the automorphism $[W:X:Y:Z]\mapsto[-W:X:Y:Z]$, it is sufficient to study the curve defined by $W=X+Y+Z$. Replacing $W=X+Y+Z$ in the equation of S_6 yields the following reducible polynomial of degree 5:

$$(X^2+Y^2+Z^2+3(XY+XZ+YZ))((X+Y+Z)^3-X^2Y-Y^2Z-XZ^2)).$$

The first factor corresponds to the conic $\mathcal{C}_1\subset S_6\setminus\mathcal{M}_2(6)$ (Lemma 4.1) and the second yields a smooth plane cubic $E\subset S_6$ birational to the elliptic curve $\mathcal{E}\subset\mathbb{P}^2$ given in Weierstrass form by

$$y^2z=4x^3+z^3$$

via the birational transformation $\psi\in\mathrm{Bir}(\mathbb{P}^2)$ given by

$$\psi:[X:Y:Z]\dashrightarrow\left[(-X-Y)(Y+Z):-X^2-2YZ-(X+Y+Z)^2:(Y+Z)^2\right],$$

whose inverse is given by

$$[x:y:z]\dashrightarrow\left[-2x^2-2xz+yz+z^2:-2x^2+2xz-yz-z^2:2x^2+2xz+yz+z^2\right].$$

Note that the image of $\psi(E\setminus\mathcal{M}_2(6))$ is contained in open set where $xz\neq 0$. The result will then follow from the fact that $\mathcal{E}(\mathbb{Q})=\{[0:1:0],[0:1:1],[0,-1,1]\}$, which we prove next. We could again do this using MWRANK, or by finding the curve in tables (the conductor is 27), but here it turns out that the result is much older, reducing to the case $n=3$ of Fermat's Last Theorem.

It is clear that the three points are in $\mathcal{E}$. Conversely, let $[x:y:z]\in\mathcal{E}$ be a rational point. Write the equation of $\mathcal{E}$ as $4x^3=z(y-z)(y+z)$, and make the linear change of coordinates $(y,z)=(r+s,r-s)$. This yields $z(y-z)(y+z)=4rs(s-r)$ and $x^3=rs(s-r)$, and we are to show that $x=0$. We may assume that x,y,z are integers with no common factor. Then $\gcd(r,s)=1$ (since any prime factor of both r and s would divide x^3 and would thus also be a factor of x). Therefore, $r,s-r,s$ are pairwise coprime, and if their product is a nonzero cube then each of them is a cube individually. But then the cube roots, call them α,β,γ, satisfy $\alpha^3+\beta^3=\gamma^3$. Hence by the $n=3$ case of Fermat $\alpha\beta\gamma=0$, and we are done. ■

There may be yet further rational or elliptic curves to be found: we searched for rational points on $\mathcal{M}_2(6)$ using the p-adic variation of the technique of [9], finding more than 100 orbits under the action of $\langle\sigma_6\rangle$ that are not accounted for by the known rational and elliptic curves, such as $[W:X:Y:Z]=[-46572:20403:35913:16685]$ and $[-75523:54607:72443:62257]$.

We conclude this section with a curiosity involving the curves $F_3(X,Y,Z)=0$ and $F_5(X,Y,Z)=0$, which lift to points of S_6 fixed by σ_6^3. These curves have genus 1 and 2, respectively, and do not yield any rational points on $\mathcal{M}_2(6)$. But they are birational with the modular curves $X_1(14)$ and $X_1(18)$ which parameterize elliptic curves with a 14- or 18-torsion point, respectively. Could one use the modular structure to explain these curves' appearance on S_6?

As already written in the introduction of the present paper, Morton proved that does not exist any rational number c for which the quadratic polynomial x^2+c admits a rational periodic point with exact period 4 (see [18]). Also in that problem one has the presence of modular curves. The variety that parameterized the pairs (c,x), where x is a periodic point with exact period 4 for the map x^2+c, is a curve birational to the modular curve $X_1(16)$.

4.4 S-integral points of $M_2(6)$

In this section, we consider the S-integral points of $M_2(6)$ viewed as $S_6\setminus D$, where D is the effective ample divisor $D=\sum_{1\le i\le 9}L_i+\sum_{1\le i\le 14}\mathcal{C}_i$ and the lines L_i and the conics $\mathcal{C}_i$ are the one defined in Lemma 4.1. We shall apply the so-called S-unit Equation Theorem, but before to state it we have to set some notation.

Let K be a number field and S a finite set of places of K containing all the archimedean ones. The set of S-integers is the following one:

$$\mathcal{O}_S := \{x\in K \mid |x|_v\le 1 \text{ for any } v\notin S\}$$

and we denote by $\mathcal{O}_S^*$ its group of units

$$\mathcal{O}_S^* := \{x\in K \mid |x|_v = 1 \text{ for any } v\notin S\},$$

which elements are called S-units. (See, for example, [3] for more information about these objects.)

We shall use the following classical result:

Theorem 2. Let K, S, and $\mathcal{O}_S^*$ be as above. Let a and b be nonzero fixed elements of K. Then the equation

$$ax + by = 1$$

has only finitely many solutions $(x, y) \in (\mathcal{O}_S^*)^2$. □

This result, due to Mahler, was proved in some less general form also by Siegel. Theorem 2 can be viewed as a particular case of the result proved by Beukers and Schlickewei [1] that gives also a bound for the number of solutions.

We recall briefly the notion of S-integral points. Let $X \subset \mathbb{A}^n$ be an affine variety defined over a number field K with $\mathcal{O}$ its ring of algebraic integers. Let $K[X]$ be the ring of regular functions on X. Recall that $K[X]$ is a quotient of the polynomial ring $K[x_1, x_2, \ldots, x_n]$. Denote by $\mathcal{O}_S[X]$ the image in this quotient of the ring $\mathcal{O}_S[x_1, x_2, \ldots, x_n]$. If $P = (p_1, p_2, \ldots, p_n)$ is a point of X whose coordinates are all S-integers, then P defines a morphism of specialization $\phi_P : \mathcal{O}_S[X] \to \mathcal{O}_S$. It is clear that also the converse holds: to any such morphism corresponds a point $P \in X$ with S-integral coordinates.

Let $\tilde{X}$ be a projective variety, D an effective ample divisor on $\tilde{X}$, and $X = \tilde{X} \setminus D$. Also, by considering an embedding $\tilde{X} \to \mathbb{P}^n$ associated to a suitable multiple of D, we can view D as the intersection of $\tilde{X}$ with the hyperplane H at infinity. By choosing an affine coordinate system for $\mathbb{P}^n \setminus H$, we can consider the ring of regular functions $\mathcal{O}[X]$. Note that the choice of the ring $\mathcal{O}[X]$ gives an integer model for X. We can define the set $X(\mathcal{O}_S)$ of the S-integral points of X as the set of morphisms of algebras $\mathcal{O}[X] \to \mathcal{O}_S$. There is a bijection between this set and the points of X whose reduction modulo p are not in D.

For example, we can see that Theorem 2 implies that there are only finitely many S-integral points in $\mathbb{P}^1 \setminus \{0, \infty, 1\}$. Thus, consider the ring of regular functions

$$\mathcal{O}\left[\mathbb{P}^1 \setminus \{0, \infty, 1\}\right] = \mathcal{O}\left[T, T^{-1}, (T-1)^{-1}\right],$$

to deduce that S-integral points of $\mathbb{P}^1 \setminus \{0, \infty, 1\}$ correspond to morphisms from $\mathcal{O}_S[T, T^{-1}, (T-1)^{-1}]$ to $\mathcal{O}_S$. But such a morphism is the specialization of T to an S-unit u such that $1 - u$ is an S-unit too. Therefore, if we write $v = 1 - u$, we obtain the equation $u + v = 1$; by Theorem 2, it follows that there are only finitely many possible values for the S-unit u.

As a direct application of the previous arguments we prove the following result:

Proposition 4.13. Let K and S be as above. Let D be the effective divisor sum of the lines and the conics defined in Lemma 4.1. Then the set of S-integral points of $M_2(6) = S_6 \setminus D$ is finite. □

Proof. We can consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the compactification of $\mathbb{A}^3$ and consider the restriction to $M_2(6)$ of the rational map $\Phi \colon \mathbb{P}^3 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ obtained in the canonical way from the map φ^{-1} defined in Lemma 4.1. The map Φ is an isomorphism from $M_2(6)$ to its image, which is locally closed in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 4.1, we see that each S-integral point is sent via the map Φ into a point $(x, y, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where x, y, z are S-integral points in $\mathbb{P}^1 \setminus \{0, \infty, 1\}$, because the preimage of the locus in $(\mathbb{P}^1)^3$ where at least one coordinate is $[0:1]$, $[1:0]$, or $[1:1]$ is contained in the zero locus of

$$(W^2 + XY + YZ + XZ)(W^2 - X^2)(W^2 - Y^2)(W^2 - Z^2)(X^2 - Y^2)(Y^2 - Z^2)(Y^2 - Z^2).$$

By the argument described before the present proposition, there are finitely many such S-integral points. Now the proposition follows from the fact that Φ is a one-to-one map. ■

Remark 4.14. Proposition 4.13 also follows from [5, Theorem 1.2]. An S-integral point of $M_2(6) = S_6 \setminus D$ corresponds to an $(n+1)$-tuple

$$(f, 0, \infty, 1, x, y, z) = (f, [0:1], [1:0], [1:1], [x:1], [y:1], [z:1]),$$

where x, y, z are S-units and f is a quadratic map defined over K with good reduction outside S. See [22] or [5] for the definition of good reduction, but roughly speaking it means that the homogeneous resultant of the two p-coprime polynomials defining f is a p-unit for any $p \notin S$. In particular, to an S-integral point of $M_2(6)$ corresponds a rational map f defined over K, with good reduction outside S, which admits a K-rational periodic point of minimal period 6; and this set is finite by [5, Theorem 1.2]. Now Proposition 4.13 follows from the previous argument because for any point $[W : X : Y : Z] \in M_2(6)$ there exists a unique f that admits the cycle $([0:1], [1:0], [1:1], [x:1], [y:1], [z:1])$, where $(x, y, z) = \phi^{-1}([W : X : Y : Z])$ and the map ϕ^{-1} is the one defined in Lemma 4.1. □

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