Flows of vector fields with point singularities and the vortex-wave system

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Abstract

The vortex-wave system is a version of the vorticity equation governing the motion of 2D incompressible fluids in which vorticity is split into a finite sum of Diracs, evolved through an ODE, plus an L^p part, evolved through an active scalar transport equation. Existence of a weak solution for this system was recently proved by Lopes Filho, Miot and Nussenzveig Lopes, for $p > 2$, but their result left open the existence and basic properties of the underlying Lagrangian flow. In this article we study existence, uniqueness and the qualitative properties of the (Lagrangian flow for the) linear transport problem associated to the vortex-wave system. To this end, we study the flow associated to a two-dimensional vector field which is singular at a moving point. We first observe that existence and uniqueness of the regular Lagrangian flow are ensured by combining previous results by Ambrosio and by Lacave and Miot. In addition we prove that, generically, the Lagrangian trajectories do not collide with the point singularity. In the second part we present an approximation scheme for the flow, with explicit error estimates obtained by adapting results by Crippa and De Lellis for Sobolev vector fields.

1 Introduction

The purpose of this article is to study the flow associated to a particular class of vector fields that contain a point singularity, which arise as weak solutions of the vortex-wave system. For a smooth vector field $b : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, the flow of b is the unique map $X : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{cases} \dfrac{d}{dt}X(t,x) = b\,(t, X(t,x)) \quad t \in [0,T], \\ X(0,x) = x \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

It turns out that, in some cases, even if b is not smooth, it is still possible to define an extended notion of flow for b, nowadays called regular Lagrangian flow (see, e.g., Definition 1.1 below). In their pioneering work, DiPerna and Lions [8] proved the existence and uniqueness of the flow for vector fields belonging to $L^1(W^{1,1}_{\mathrm{loc}})$ with suitable decay at infinity and with bounded divergence (see assumptions (H_1) and (H_3) below). The Sobolev-type regularity assumptions on b were later relaxed by Ambrosio [1], allowing for BV vector fields (see assumption (H_2)). There is a wide literature devoted to this issue, see e.g. [2, 3, 4] and references therein for additional or related results. The problem we address here is that the vector field associated to the vortex-wave system is not BV.

We will focus on the case where b is given by

$$b(t,x) = v(t,x) + H(t,x), \tag{1.2}$$

where the field v enters the class of vector fields considered in the theory of DiPerna and Lions and Ambrosio, and where H is a special vector field which is singular along a curve in space time. More precisely, we assume that the first component v satisfies the same assumptions as in [1]:

$$\begin{aligned}
&(H_1) \quad \frac{v}{1+|x|} \in L^1\left([0,T], L^1(\mathbb{R}^2)\right) + L^1\left([0,T], L^\infty(\mathbb{R}^2)\right),\\
&(H_2) \quad v \in L^1\left([0,T], \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2)\right),\\
&(H_3) \quad \mathrm{div}\,(v) \in L^1\left([0,T], L^\infty(\mathbb{R}^2)\right).
\end{aligned}$$

The result of Ambrosio [1] ensures existence and uniqueness of the regular Lagrangian flow associated to such fields. In addition, in our context, we require the following assumption:

$$(H_4) \quad v \in L^\infty\left([0,T], L^q(\mathbb{R}^2)\right) \quad \text{for some } 2 < q \le +\infty.$$

Next, we define our singular part H as follows. We consider a given Lipschitz trajectory in $\mathbb{R}^2$:

$$z \in W^{1,\infty}([0,T], \mathbb{R}^2). \tag{1.3}$$

We introduce the map

$$K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad K(x) = \frac{x^\perp}{|x|^2} = \frac{(-x_2, x_1)}{|(x_1,x_2)|^2}$$

and we define

$$H(t,x) = K\left(x - z(t)\right). \tag{1.4}$$

Then H satisfies (H_1) and (H_3): actually, it is divergence free. It does not satisfy (H_2) therefore such a field is not covered by the result of Ambrosio [1]. However note that H is smooth away from the set $\{(t, z(t)),\ t \in [0, T]\}$.

The structure described by (1.2) includes that of solutions of the vortex-wave system in the special case of a single vortex together with compactly supported L^p vorticity, $p > 1$.

Next we recall, following DiPerna and Lions [8] and Ambrosio [1], the definition of regular Lagrangian flow. We denote by $\mathcal{L}^2$ the Lebesgue measure on $\mathbb{R}^2$.

Definition 1.1 (Regular Lagrangian flow)**.** We say that a map $X : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a regular Lagrangian flow for the vector field b if:

(i) There exists an $\mathcal{L}^2$-negligible set $S \subset \mathbb{R}^2$ such that for all $x \in \mathbb{R}^2 \setminus S$ the map $t \mapsto b(t, X(t, x))$ belongs to $L^1([0, T])$, and

$$X(t, x) = x + \int_0^t b(s, X(s, x))\, ds, \quad \forall t \in [0, T].$$

(ii) For all $R > 0$ there exists $L_R > 0$ such that[1]

$$X(t, \cdot)_{\#}(\mathcal{L}^2 \llcorner B_R) \leq L_R \mathcal{L}^2, \quad \forall t \in [0, T],$$

i.e. $\mathcal{L}^2 \left(X(t, \cdot)^{-1}(A) \cap B_R \right) \leq L_R \mathcal{L}^2(A)$ for every Borel set $A \subset \mathbb{R}^2$.

In Section 2 we combine the abstract theory by Ambrosio [2] (Theorem 2.1 below), see also [3], exploiting the link between the ODE and the continuity and transport equations (see (2.1)-(2.2)), with a version of the renormalization result established by Lacave and Miot [12] to show existence and uniqueness for the regular Lagrangian flow of b. Moreover, we prove the additional property that for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$ the trajectory starting from the point x does not collide with the singularity point. More precisely, we prove the following theorem:

Theorem 1.2. *Let b be as in* (1.2)*, where v satisfies* $(H_1) - (H_2) - (H_3) - (H_4)$ *and where H is given by* (1.4)*. Then there exists a unique regular Lagrangian flow. Moreover, for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$ we have*

$$X(t, x) \neq z(t), \quad \forall t \in [0, T].$$

[1] Here and in the following, given a Borel measure μ, we denote by $X(t, \cdot)_{\#}\mu$ the push-forward of the measure μ by the map $X(t, \cdot)$.

Observe that, by the very definition of regular Lagrangian flow, the absolute continuity of the measure $X(t,\cdot)_\# \mathcal{L}^2$ with respect to $\mathcal{L}^2$ implies, by Fubini's theorem, that for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$ we have $X(t,x) \neq z(t)$ for $\mathcal{L}^1$-a.e. $t \in [0,T]$. The main point of Theorem 1.2 is that collisions between the Lagrangian trajectories and the singularity point are avoided for *all* $t \in [0,T]$. Indeed Proposition 2.4 yields a quantitative control of the amount of Lagrangian trajectories getting closer than ε to the point singularity: the proof of this proposition uses the additional assumption (H_4). We mention that an analogous control on the trajectories was performed in the setting of the Vlasov-Poisson equation with singular fields by Caprino, Marchioro, Miot and Pulvirenti [6].

In the second part of this work we present an effective construction of the regular Lagrangian flow by an approximation argument. In contrast with the point of view adopted in the first part, this construction does not rely on the link between the ODE and the PDE. Moreover, we provide a quantitative rate of convergence, by extending to our setting the estimates performed by Crippa and De Lellis [7] for vector fields without singular part. We restrict ourselves to vector fields satisfying the stronger assumptions:

$$
\begin{aligned}
&(H_1') \quad v \in L^\infty([0,T]\times\mathbb{R}^2),\\
&(H_2') \quad \nabla v \in L^1\left([0,T], L^p(\mathbb{R}^2)\right) \quad \text{for some } 1 < p \leq +\infty,\\
&(H_3') \quad \operatorname{div}(v) \in L^1\left([0,T], L^\infty(\mathbb{R}^2)\right).
\end{aligned}
$$

In Section 3 we define a suitable smooth approximation $(b_n)_{n\in\mathbb{N}}$ of b, and we denote by X_n the unique corresponding classical flow. We prove the following theorem:

Theorem 1.3. *Let v satisfy $(H_1')-(H_2')-(H_3')$. Let $R>0$. There exists $\widetilde{R}$ and C, depending on R, T, $\|v\|_{L^\infty(L^\infty)}$, $\|\operatorname{div}(v)\|_{L^1(L^\infty)}$, $\|\nabla v\|_{L^1(L^p)}$, and $\|z\|_{W^{1,\infty}}$, such that, denoting by*

$$\delta(n,m) = \|b_n - b_m\|_{L^1([0,T]\times B_{\widetilde{R}})}$$

the following estimate holds:

$$\int_{B_R} \sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)|\,dx \leq \frac{C}{|\ln \delta(n,m)|^{1/3}}.$$

In particular, $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1_{\mathrm{loc}}\left(\mathbb{R}^2, L^\infty([0,T])\right)$, and it converges to X in $L^1_{\mathrm{loc}}\left(\mathbb{R}^2, L^\infty([0,T])\right)$, where X is the regular Lagrangian flow relative to b as in Theorem 1.2.

To conclude this introduction, we describe the vortex-wave system and the connection between the present work and this system. In two-dimensional incompressible fluids, we consider a flow with initial vorticity consisting of the superposition of a diffuse part $\omega_0 \in L^p$ for some $p \geq 1$ and a point vortex located at $z_0 \in \mathbb{R}^2$, with unit strength. The evolution of vorticity can be described by a system of equations called the vortex-wave system (with one single point vortex):

$$\begin{cases} \partial_t \omega + (v + H) \cdot \nabla \omega = 0 \\ v = \dfrac{1}{2\pi} K * \omega \\ H(t,x) = \dfrac{1}{2\pi} K(x - z(t)) \\ \dot{z}(t) = v(t, z(t)). \end{cases} \tag{1.5}$$

This system was introduced by Marchioro and Pulvirenti [14, 15]. It is an idealized model for two-dimensional flows where regions of sharply concentrated vorticity interact with a distributed vorticity background. Such flows arise naturally as simplified models in geophysical flows and in plasma dynamics, see [9, 16, 17, 18]. There are two natural notions of weak solution for this system, one is a solution in the sense of distributions, called *Eulerian solution*, while the other is a solution for which the diffuse part of the vorticity is constant along the trajectories of the flow, called *Lagrangian solution*, see [10, 12] for precise definitions. (By 'trajectories of the flow' we mean the flow associated to the vector field $b = v + H$ above.) In [14, 15], Marchioro and Pulvirenti established global existence of a Lagrangian solution with $\omega \in L^\infty(L^1 \cap L^\infty)$. In [10], Lopes Filho, Miot and Nussenzveig Lopes established global existence of an Eulerian solution with vorticity $\omega \in L^\infty(L^1 \cap L^p)$, with $p > 2$. For any $p > 2$ Lagrangian solutions to the vortex-wave system are Eulerian. The converse was left open in [10]. The issue of the Lagrangian formulation is the natural requirement that flow trajectories should not collide with the point vortex. When $p = +\infty$, almost-Lipschitz regularity for the velocity $(1/2\pi)K * \omega$ enables to define flow trajectories in the classical sense, which do not intersect with the point vortex, starting from *any* $x \neq z_0$ [14, 15]. For $p < +\infty$ this property is unclear. In Section 4, we use the results established in Sections 2 and 3 to show that any Eulerian solution with $\omega \in L^\infty(L^1 \cap L^p)$, $p > 2$, gives rise to a regular Lagrangian flow such that ω is constant along the flow trajectories, which do not, generically, collide with the point vortex. As it happens, when $p > 2$, the assumptions $(H_1') - (H_2') - (H_3')$ are all satisfied and, in particular, the point vortex trajectory $t \mapsto z(t)$ is Lipschitz.

2 Proof of Theorem 1.2

In the theory of DiPerna and Lions and Ambrosio [1, 2, 8], the existence, uniqueness and the stability properties of the flow associated to a field b are linked to the well-posedness of the corresponding continuity equation

$$\partial_t u + \operatorname{div}(bu) = 0 \quad \text{on } (0,T)\times\mathbb{R}^2, \quad u(0) = u_0 \tag{2.1}$$

and transport equation

$$\partial_t u + b\cdot\nabla u = 0 \quad \text{on } (0,T)\times\mathbb{R}^2, \quad u(0) = u_0. \tag{2.2}$$

Note that one passes formally from the ODE to the continuity and transport equations by noticing that if X solves (1.1) then $X(t,\cdot)_{\#}u_0$ solves (2.1), and $u_0 \circ X(t,\cdot)^{-1}$ solves (2.2). In the non-smooth case, we consider distributional solutions to (2.1) and (2.2). Such distributional formulations make sense as soon as bu and $u\operatorname{div}(b)$ belong to L^1_{loc}.

As a matter of fact, we have the following general abstract result due to Ambrosio [2], somewhat extending this connection to the non smooth context:

Theorem 2.1 (Ambrosio [2], Theorems 3.3 and 3.5)**.** *Let b be a given vector field in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^2)$. If existence and uniqueness for* (2.1) *hold in $L^\infty\left(L^1\cap L^\infty\right)$ then the regular Lagrangian flow of b exists and is unique.*

And, besides, existence and uniqueness for (2.1) hold for vector fields satisfying the assumptions $(H_1)-(H_2)-(H_3)$ or $(H_1')-(H_2')-(H_3')$ [1, 8].

Now, in the case where b is given by (1.2), the PDE well-posedness results cannot be applied directly because of the singular field H. However, the following holds:

Proposition 2.2. *Let b be given by* (1.2)*.*
(1) Let v satisfy the assumptions $(H_1)-(H_2)-(H_3)$. Let $u_0\in L^1\cap L^\infty$. Then (2.1) *has a unique solution $u\in L^\infty\left(L^1\cap L^\infty\right)$.*
(2) Let v satisfy the assumptions $(H_1')-(H_2')-(H_3')$. Let $u_0\in L^1\cap L^r$, with $r>2$. Then (2.1) *has a unique solution $u\in L^\infty\left(L^1\cap L^r\right)$.*

Remark 2.3. Since H belongs to L^a_{loc} if and only if $1\le a<2$, the condition $r>2$ in (2) is a natural requirement to give sense to the product $\operatorname{div}(Hu) = H\cdot\nabla u$ in the sense of distributions.

Proof. First, existence of a distributional solution follows in both cases from standard regularization arguments.

The argument for uniqueness is strictly analogous to the one of Lacave and Miot [12]. We give the main lines for the reader's convenience. First, using the by now standard methods introduced in [1, 8], it suffices to show that any solution u satisfies the renormalization property:

$$\partial_t |u| + \operatorname{div}\left((v+H)|u|\right) = 0. \tag{2.3}$$

This is proved in [12] for the case (1). Similar arguments, which we sketch below, lead to (2.3) in the case (2). First, we observe that (2.3) holds in the sense of distributions on the complement of the set $\{(t, z(t)),\, t \in [0,T]\}$. Indeed, $v+H$ is regular enough in that set so that the renormalization results of [1, 8] hold. We next establish (2.3) in $(0,T)\times\mathbb{R}^2$. Let $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^2)$ and let $\chi \in C^\infty(\mathbb{R}^2)$ be a radial function such that $0 \le \chi \le 1$, $\chi = 0$ on $B_{1/2}$ and $\chi = 1$ on B_1^c. For $\varepsilon > 0$ we set $\chi_\varepsilon(t,x) = \chi((x - z(t))/\varepsilon)$ and $\varphi_\varepsilon = \varphi\chi_\varepsilon$. Since φ_ε is compactly supported away from the set $\{(t, z(t)),\, t \in [0,T]\}$ we have

$$\iint |u|\big(\partial_t\varphi_\varepsilon + (v+H)\cdot\nabla\varphi_\varepsilon\big)\,dx\,dt = 0.$$

Expanding the previous expression yields

$$\begin{aligned}
&\iint |u|\chi_\varepsilon(\partial_t\varphi + (v+H)\cdot\nabla\varphi)\,dx\,dt\\
&+\iint |u|\varphi\left(-\frac{\dot z(t)}{\varepsilon}\cdot\nabla\chi\left(\frac{x-z(t)}{\varepsilon}\right) + (v+H)(t,x)\cdot\nabla\chi_\varepsilon(t,x)\right)\,dx\,dt = 0.
\end{aligned}$$

We remark that $H\cdot\nabla\chi_\varepsilon = 0$. Therefore by Hölder's inequality the second term is bounded by $C(\varphi)(\|\dot z\|_{L^\infty} + \|v\|_{L^\infty(L^\infty)})\|u\|_{L^\infty(L^r)}\|\varepsilon^{-1}\nabla\chi(\varepsilon^{-1}\cdot)\|_{L^{r'}}$ which vanishes when ε tend to 0 because $r' < 2$. Hence by applying Lebesgue's theorem to the first term we finally obtain

$$\iint |u|\big(\partial_t\varphi + (v+H)\cdot\nabla\varphi\big)\,dx\,dt = 0.$$

□

Combining Part (1) of Proposition 2.2 and Theorem 2.1 we obtain the existence and uniqueness of the regular Lagrangian flow X in Theorem 1.2. Therefore we only have to prove that for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$ no collision between $X(t,x)$ and $z(t)$ occurs on $[0,T]$. This is a direct consequence of the following

Proposition 2.4. *For $0 < \varepsilon < 1$ and $R > 0$, let*

$$P(\varepsilon, R) = \left\{ x \in B_R \setminus S \quad s.t. \quad \min_{t \in [0,T]} |X(t,x) - z(t)| < \varepsilon \right\},$$

where S is as in Definition 1.1. Then

$$\mathcal{L}^2(P(\varepsilon, R)) \leq C(T, L_R, \|v\|_{L^\infty(L^q)} + \|\dot{z}\|_{L^\infty}) \varepsilon^{1-\frac{2}{q}}.$$

Proof. We adapt the strategy introduced in [6] for the Vlasov-Poisson equation. Here, we set $\alpha = 1 - 2/q > 0$. We introduce

$$\Delta T = \lambda \varepsilon^\beta,$$

where $0 < \lambda < 1$ is a parameter to be determined later, and where

$$\beta = \frac{1 + \alpha/q}{1 - 1/q} \geq 1.$$

We set

$$N = \left[\frac{T}{\Delta T} \right] - 1$$

and we define

$$t_i = i \Delta T, \quad i = 0, \ldots, N, \quad t_{N+1} = T,$$

so that

$$[0,T] = \bigcup_{i=0}^{N} [t_i, t_{i+1}] \quad \text{with} \ \ |t_{i+1} - t_i| \leq \Delta T, \quad \forall i = 0, \ldots, N.$$

We first consider the case $2 < q < +\infty$. We set

$$A = \left\{ x \in B_R \setminus S : \quad \int_{t_i}^{t_{i+1}} |v(s, X(s,x))|^q \, ds \leq \varepsilon^{-\alpha}, \quad \forall i \in \{0, \ldots, N\} \right\}$$

and for $i \in \{0, \ldots, N\}$ we set

$$B_i = \left\{ x \in B_R \setminus S : \quad \int_{t_i}^{t_{i+1}} |v(s, X(s,x))|^q \, ds \geq \varepsilon^{-\alpha} \right\}.$$

By Chebyshev's inequality and Fubini's theorem we have

$$\begin{aligned} \mathcal{L}^2(B_i) &\leq \varepsilon^\alpha \int_{B_R} \int_{t_i}^{t_{i+1}} |v(s, X(s,x))|^q \, ds \, dx \\ &= \varepsilon^\alpha \int_{t_i}^{t_{i+1}} \int_{B_R} |v(s, X(s,x))|^q \, dx \, ds. \end{aligned}$$

Using Property (ii) in Definition 1.1 for $X(s,\cdot)$ and the assumption (H_4) for v we get

$$\mathcal{L}^2(B_i) \le L_R \varepsilon^{\alpha} \|v\|_{L^\infty(L^q)}(t_{i+1}-t_i).$$

Therefore

$$\mathcal{L}^2\left(\bigcup_{i=0}^{N} B_i\right) \le L_R T \|v\|_{L^\infty(L^q)}\varepsilon^{\alpha}. \tag{2.4}$$

Then, let $x \in P(\varepsilon,R)\cap A$ and let $s_0 \in [0,T]$ such that

$$|X(s_0,x)-z(s_0)| < \varepsilon.$$

By continuity, we may assume that $s_0 \in [0,T)$, hence we have $s_0 \in [t_i,t_{i+1})$ for some $i \in \{0,\dots,N\}$. Let $s_1 \le t_{i+1}$ maximal such that $|X(t,x)-z(t)| < 2\varepsilon$ on $[s_0,s_1)$. If $s_1 = t_{i+1}$ then $x \in X(t_{i+1},\cdot)^{-1}\big(B(z(t_{i+1}),2\varepsilon)\big)\cap B_R$. We assume then that $s_1 < t_{i+1}$. For $\mathcal{L}^1$-a.e. $t \in [s_0,s_1)$ we have $\dot{X}(t,x) = b(t,X(t,x))$. Now we observe that, even though b is not uniformly bounded, the map $t \mapsto |X(t,x)-z(t)|$ is Hölder continuous for each $x \in A$. Indeed, for $\mathcal{L}^1$-a.e. $t \in [s_0,s_1)$ such that $X(t,x) \neq z(t)$ we get, using that

$$K(y)\cdot y = 0, \quad \forall y \in \mathbb{R}^2\setminus\{0\},$$

$$\frac{d}{dt}|X(t,x)-z(t)| = \frac{X(t,x)-z(t)}{|X(t,x)-z(t)|}\cdot\big(v(t,X(t,x))-\dot{z}(t)\big),$$

hence

$$\left|\frac{d}{dt}|X(t,x)-z(t)|\right| \le |v(t,X(t,x))| + |\dot{z}(t)|. \tag{2.5}$$

Hence for all $t \in [s_0,s_1]$ we have by Hölder inequality

$$\begin{aligned}
|X(t,x)-z(t)| &= |X(s_0,x)-z(s_0)| + \int_{s_0}^{t}\frac{d}{ds}|X(s,x)-z(s)|\,ds\\
&< \varepsilon + \int_{s_0}^{t}|v(s,X(s,x))|\,ds + \int_{s_0}^{t}|\dot{z}(s)|\,ds\\
&\le \varepsilon + (t_{i+1}-t_i)^{1-\frac{1}{q}}\left(\int_{t_i}^{t_{i+1}}|v(s,X(s,x))|^q\,ds\right)^{\frac{1}{q}} + \|\dot{z}\|_{L^\infty}(t_{i+1}-t_i).
\end{aligned}$$

Finally, by definition of the set A and by definition of $\beta \geq 1$ we get

$$
\begin{aligned}
|X(t,x) - z(t)| &\leq \varepsilon + \lambda^{1-\frac{1}{q}} \varepsilon^{\beta(1-\frac{1}{q})-\frac{\alpha}{q}} + \|\dot{z}\|_{L^\infty} \lambda \varepsilon^{\beta} \\
&\leq \varepsilon + \lambda^{1-\frac{1}{q}} \varepsilon + \|\dot{z}\|_{L^\infty} \lambda \varepsilon.
\end{aligned}
$$

Now we choose λ so that

$$
\lambda^{1-\frac{1}{q}} + \|\dot{z}\|_{L^\infty} \lambda < 1.
$$

For this choice of λ we obtain $|X(s_1,x) - z(s_1)| < 2\varepsilon$, which contradicts the definition of s_1 and shows that we must have $s_1 = t_{i+1}$. It follows that

$$
A \cap P(\varepsilon, R) \subset \bigcup_{i=0}^{N} X(t_{i+1}, \cdot)^{-1} \big(B(z(t_{i+1}), 2\varepsilon) \big) \cap B_R.
$$

Therefore in view of (ii) in Definition 1.1,

$$
\begin{aligned}
\mathcal{L}^2(A \cap P(\varepsilon, R)) &\leq \sum_{i=0}^{N} \mathcal{L}^2 \Big(X(t_{i+1}, \cdot)^{-1} \big(B(z(t_{i+1}), 2\varepsilon) \big) \cap B_R \Big) \\
&\leq (N+1) L_R (4\pi\varepsilon^2)
\end{aligned}
$$

and finally

$$
\mathcal{L}^2(A \cap P(\varepsilon, R)) \leq 4\pi L_R T \lambda^{-1} \varepsilon^{2-\beta}. \tag{2.6}
$$

Combining (2.4), (2.6) and using the definition of λ we obtain

$$
\mathcal{L}^2(P(\varepsilon, R)) \leq C(T, L_R, \|v\|_{L^\infty(L^q)}, \|\dot{z}\|_{L^\infty})(\varepsilon^{\alpha} + \varepsilon^{2-\beta}).
$$

Since $2 - \beta = \alpha$, this yields the conclusion.

We now study the case where $q = \infty$, which is easier and does not require to introduce the sets B_i and A. Indeed, let $x \in P(\varepsilon, R)$. Coming back to (2.5) and proceeding similarly as before we obtain for $s \in [s_0, s_1)$

$$
|X(t,x) - z(t)| \leq \varepsilon + \lambda\varepsilon(\|v\|_{L^\infty(L^\infty)} + \|\dot{z}\|_{L^\infty}) < 2\varepsilon
$$

provided that

$$
\lambda(\|v\|_{L^\infty(L^\infty)} + \|\dot{z}\|_{L^\infty}) < 1.
$$

This shows that

$$
P(\varepsilon, R) \subset \bigcup_{i=0}^{N} X(t_{i+1}, \cdot)^{-1} \big(B(z(t_{i+1}), 2\varepsilon) \big) \cap B_R,
$$

and the conclusion then follows as before.

$\square$

3 Proof of Theorem 1.3

We start by defining the smooth approximation involved in Theorem 1.3. Let $(\rho_n)_{n\in\mathbb{N}}$ be the usual sequence of Friedrichs mollifiers, namely $\rho_n \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$, $\int_{\mathbb{R}^2} \rho_n = 1$ and $\mathrm{supp}(\rho_n) \subset B(0, 1/n)$ for all $n \in \mathbb{N}^*$. Let $v_n = \rho_n * v$ and let

$$K_n(x) = \frac{x^\perp}{|x|^2 + \frac{1}{n^2}}, \quad x \in \mathbb{R}^2,$$

which defines a globally bounded, divergence free and smooth vector field on $\mathbb{R}^2$. We finally set

$$b_n(t,x) = v_n(t,x) + K_n(x - z(t)).$$

We first remark that $(|X_n(t,x) - z(t)|)_{n\in\mathbb{N}}$ is uniformly Lipschitz in time even though b_n is not uniformly bounded in L^∞. Indeed, by the same computation leading to (2.5), using that $K_n(y)\cdot y = 0$ and (H_1'), we have

$$\left|\frac{d}{dt}|X_n(t,x) - z(t)|\right| \le |v_n(t, X_n(t,x))| + |\dot z(t)| \le \|v\|_{L^\infty(L^\infty)} + \|\dot z\|_{L^\infty}. \tag{3.1}$$

In particular, we have the local equiboundedness property

$$\|X_n\|_{L^\infty([0,T]\times B_R)} \le R + 2\|z\|_{L^\infty} + (\|v\|_{L^\infty(L^\infty)} + \|\dot z\|_{L^\infty})T. \tag{3.2}$$

On the other hand, since $\mathrm{div}\,(b_n) = \rho_n * \mathrm{div}\,(v)$ we infer from (H_3') that

$$\sup_{n\ge 0} \int_0^T \|\mathrm{div}\,(b_n)(s)\|_{L^\infty}\, ds \le L_0 < \infty. \tag{3.3}$$

In particular it follows from the standard theory on Jacobians that

$$X_n(t,\cdot)_\# \mathcal{L}^2 \le e^{L_0}\mathcal{L}^2, \quad \forall t \in [0,T]. \tag{3.4}$$

Part of our subsequent analysis is borrowed from [7]: we introduce

$$\widetilde{R} = R + 2\|z\|_{L^\infty} + (\|v\|_{L^\infty(L^\infty)} + \|\dot z\|_{L^\infty})T$$

and

$$\delta(n,m) = \|b_n - b_m\|_{L^1([0,T]\times B_{\widetilde{R}})}.$$

We consider the positive quantity

$$g_{n,m} = \int_{B_R} \sup_{t\in[0,T]} \ln\left(\frac{|X_n(t,x) - X_m(t,x)|}{\delta(n,m)} + 1\right) dx. \tag{3.5}$$

Lemma 3.1. *We have*

$$g_{m,n} \leq C|\ln \delta(n,m)|^{2/3}$$

where C depends only on R, T, L_0, $\|v\|_{L^\infty(L^\infty)}$, $\|\dot{z}\|_{L^\infty}$, and $\|\nabla v\|_{L^1(L^p)}$.

From now on C will denote a positive constant depending only on R, T, L_0, $\|v\|_{L^\infty(L^\infty)}$, $\|\dot{z}\|_{L^\infty}$, and $\|\nabla v\|_{L^1(L^p)}$.

Before proving Lemma 3.1 we show how it implies Theorem 1.3. In the following we will sometimes write δ instead of $\delta(m,n)$.

Proof of Theorem 1.3 with Lemma 3.1.

We fix $\eta > 0$ to be determined later. By Chebychev's inequality and Lemma 3.1 we can find a set $K \subset B_R$ such that $\mathcal{L}^2(B_R \setminus K) \leq \eta$ and

$$\sup_{t\in[0,T]} \ln\left(\frac{|X_n(t,x) - X_m(t,x)|}{\delta} + 1\right) \leq \frac{C|\ln\delta|^{2/3}}{\eta}, \quad \text{for } x \in K. \tag{3.6}$$

Using (3.2), it follows that

$$\begin{aligned}
&\int_{B_R} \sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)|\,dx \\
&\quad\leq \int_{B_R\setminus K} \sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)|\,dx + \int_K \sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)|\,dx \\
&\quad\leq C\mathcal{L}^2(B_R\setminus K) + C \sup_{x\in K}\sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)| \\
&\quad\leq C\Big(\eta + \delta \exp\big(C|\ln\delta|^{2/3}/\eta\big)\Big),
\end{aligned}$$

where we have used (3.6) in the last inequality. We finally optimize the choice of the parameter η as follows. We set

$$\eta \equiv \frac{2C}{|\ln\delta|^{1/3}},$$

so that $\exp(C|\ln\delta|^{2/3}/\eta) = \exp(|\ln\delta|/2) = \delta^{-1/2}$. This yields

$$\int_{B_R} \sup_{t\in[0,T]} |X_n(t,x) - X_m(t,x)|\,dx \leq \frac{C}{|\ln\delta(n,m)|^{1/3}}. \tag{3.7}$$

In particular, we infer that $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence converging to some $Y : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ in the space $L^1_{\text{loc}}\big(\mathbb{R}^2, L^\infty([0,T])\big)$. Finally, the fact that Y is the (unique) regular Lagrangian flow associated to b is standard, and we omit the proof. $\square$

Remark 3.2. Given the strong convergence of $(X_n)_{n\in\mathbb{N}}$ to X together with the uniform bound (3.4) we infer that under assumptions $(H_1')-(H_2')-(H_3')$ the constant L_R in Definition 1.1 actually does not depend on R.

We finally give the
Proof of Lemma 3.1.
Let $\varepsilon > 0$ be a small parameter to be chosen later. We consider the set

$$P(n,\varepsilon) = \left\{ x \in \mathbb{R}^2 \quad \text{s.t.} \quad \min_{t\in[0,T]} |X_n(t,x) - z(t)| < \varepsilon \right\}.$$

Using Proposition 2.4 applied to X_n, with $q=\infty$, and thanks to (3.4), which yields a uniform bound with respect to n, we obtain

$$\mathcal{L}^2(P(n,\varepsilon)) \leq C\varepsilon, \tag{3.8}$$

where C depends only on T, L_0, $\|v\|_{L^\infty(L^\infty)}$, $\|\dot z\|_{L^\infty}$, and $\|\nabla v\|_{L^1(L^p)}$. Next,

$$g_{n,m} = \mathcal{G}_{n,m} + \mathcal{B}_{n,m},$$

where

$$\mathcal{G}_{n,m} = \int_{B_R\setminus[P(n,\varepsilon)\cup P(m,\varepsilon)]} \sup_{t\in[0,T]} \ln\left(\frac{|X_n(t,x) - X_m(t,x)|}{\delta} + 1\right) dx,$$
$$\mathcal{B}_{n,m} = \int_{P(n,\varepsilon)\cup P(m,\varepsilon)} \sup_{t\in[0,T]} \ln\left(\frac{|X_n(t,x) - X_m(t,x)|}{\delta} + 1\right) dx.$$

By (3.2) and (3.8),

$$\mathcal{B}_{n,m} \leq C|\ln\delta|\varepsilon. \tag{3.9}$$

We next estimate the second part, for which we can adapt the proof of Theorem 2.9 in [7] for Sobolev vector fields since H is regular away from the set $\{(t,z(t)), t\in[0,T]\}$. We have

$$\begin{aligned}
&\sup_{t\in[0,T]} \ln\left(\frac{|X_n(t,x) - X_m(t,x)|}{\delta} + 1\right) \\
&\qquad \leq \int_0^T \left|\frac{d}{dt}X_n(\tau,x) - \frac{d}{dt}X_m(\tau,x)\right| \left(|X_n(\tau,x) - X_m(\tau,x)| + \delta\right)^{-1} d\tau \\
&\qquad \leq \int_0^T \frac{|b_n(\tau, X_n(\tau,x)) - b_m(\tau, X_m(\tau,x))|}{|X_n(\tau,x) - X_m(\tau,x)| + \delta} d\tau.
\end{aligned}$$

Writing

$$b_n(X_n) - b_m(X_m) = \big[b_n(X_n) - b_m(X_n)\big] + \big[b_m(X_n) - b_m(X_m)\big]$$

we further obtain $\mathcal{G}_{n,m} \le \mathcal{G}^1_{n,m} + \mathcal{G}^2_{n,m}$, where

$$\mathcal{G}^1_{n,m} = \frac{1}{\delta}\int_0^T \int_{B_R} |b_n(\tau, X_n(\tau,x)) - b_m(\tau, X_n(\tau,x))| \, dx \, d\tau$$

and

$$\mathcal{G}^2_{n,m} = \int_0^T \int_{B_R \setminus [P(n,\varepsilon)\cup P(m,\varepsilon)]} \frac{|b_m(\tau, X_n(\tau,x)) - b_m(\tau, X_m(\tau,x))|}{|X_n(\tau,x) - X_m(\tau,x)|} \, dx \, d\tau.$$

By definition of $\widetilde{R}$ and by (3.4) and (3.2) we obtain

$$\mathcal{G}^1_{m,n} \le \frac{e^{L_0}}{\delta}\int_0^T \int_{B_{\widetilde{R}}} |b_n - b_m|(\tau, y)\, dy\, d\tau = e^{L_0}. \tag{3.10}$$

We now estimate $\mathcal{G}^2_{m,n}$. Let $0 \le \chi_\varepsilon \le 1$ be a smooth function such that $\chi_\varepsilon = 0$ on $B(0,\varepsilon/2)$ and $\chi_\varepsilon = 1$ on $B(0,\varepsilon)^c$ and let

$$H_{m,\varepsilon}(t,x) = (K_m\chi_\varepsilon)(x - z(t)), \quad b_{m,\varepsilon} = v_m + H_{m,\varepsilon}.$$

For $x \in B_R \setminus [P(n,\varepsilon)\cup P(m,\varepsilon)]$ we have $b_m(\tau, X_n(\tau,x)) = b_{m,\varepsilon}(\tau, X_n(\tau,x))$ and $b_m(\tau, X_m(\tau,x)) = b_{m,\varepsilon}(\tau, X_m(\tau,x))$ for $\tau \in [0,T]$.

In the following Mf denotes the maximal function of f. Using the classical estimate of the difference quotient of a function in terms of the maximal function of the derivative (see e.g. Lemma A.3 in [7]) we find

$$\begin{aligned}
&\int_0^T \int_{B_R} \frac{|b_{m,\varepsilon}(\tau, X_n(\tau,x)) - b_{m,\varepsilon}(\tau, X_m(\tau,x))|}{|X_n(\tau,x) - X_m(\tau,x)|} \, dx\, d\tau \\
&\qquad \le C\int_0^T \int_{B_R} \big[M\nabla b_{m,\varepsilon}(\tau, X_m(\tau,x)) + M\nabla b_{m,\varepsilon}(\tau, X_n(\tau,x))\big]\, dx\, d\tau.
\end{aligned}$$

By using (3.2) and (3.4) we get

$$\begin{aligned}
\mathcal{G}^2_{m,n} &\le Ce^{L_0}\int_0^T \int_{B_{\widetilde{R}}} |M\nabla b_{m,\varepsilon}(\tau, y)| \, dy\, d\tau \\
&\le Ce^{L_0}\widetilde{R}^{1-1/p}\int_0^T \|M\nabla b_{m,\varepsilon}(\tau)\|_{L^p(B_{\widetilde{R}})}\, d\tau \\
&\le Ce^{L_0}\widetilde{R}^{1-1/p}\Big(\|\nabla v_m\|_{L^1(L^p)} + \|\nabla H_{m,\varepsilon}\|_{L^1(L^p)}\Big).
\end{aligned}$$

In view of (H_2') and of the expression of $H_{m,\varepsilon}$ we get

$$\mathcal{G}_{m,n}^2 \le \frac{C}{\varepsilon^{2-\frac{2}{p}}} \le \frac{C}{\varepsilon^2}. \tag{3.11}$$

We gather (3.9), (3.10) and (3.11), obtaining

$$g_{m,n} \le C\big(\varepsilon|\ln\delta| + \varepsilon^{-2}\big).$$

We now optimize our choice of ε, setting $\varepsilon = |\ln\delta|^{-1/3}$, so that

$$g_{m,n} \le C|\ln\delta|^{2/3}$$

and the conclusion of Lemma 3.1 follows. □

4 Lagrangian solutions to the vortex-wave system

We finally comment on the applications of the previous results to the vortex-wave system (1.5). Two notions of weak solution for the vortex-wave system have been introduced: Eulerian solutions and Lagrangian solutions, see [10, 12]. These notions coincide when the vorticity ω belongs to $L^\infty(L^1 \cap L^\infty)$ [14, 15, 12]. In [10] the authors establish global existence of an Eulerian solution with ω belonging to $L^\infty(L^1 \cap L^p)$ for $p > 2$. We claim that to this Eulerian solution corresponds a unique regular Lagrangian flow and that ω is constant along the flow trajectories. Indeed, the velocity field defined by $v = \frac{1}{2\pi} K * \omega$ is divergence free, therefore (H_3') is satisfied. Moreover, since $p > 2$ it is well-known that v satisfies (H_1'), see e.g. Lemma 1 in [11]. Finally, v satisfies also (H_2') by the Calderón-Zygmund inequality, see [19] (Chapter II, Theorem 3). In particular, $(H_1) - (H_2) - (H_3) - (H_4)$ are satisfied as well. Hence, in view of Theorem 1.2 of the present article, there exists a unique regular Lagrangian flow associated to the divergence free velocity field $b = v + H$. Moreover, it can be readily checked (adapting, e.g., the proof of Theorem 1.3 in [12]), that the function $\widetilde{\omega} = X(t,\cdot)_{\#}\omega_0$ is a distributional solution in $L^\infty(L^1 \cap L^p)$ of the PDE

$$\partial_t \widetilde{\omega} + (v + H) \cdot \nabla \widetilde{\omega} = 0, \quad \widetilde{\omega}(0) = \omega_0.$$

Now, invoking the uniqueness part of Proposition 2.2 we obtain $\omega = \widetilde{\omega}$, which establishes our claim.

Finally, we mention that Theorems 1.2 and 1.3 can be extended to vector fields H containing several point singularities

$$H(t,x) = \sum_{i=1}^{N} d_i K(x - z_i(t)), \quad d_i \in \mathbb{R},$$

under the condition

$$\min_{i \neq j} \min_{t \in [0,T]} |z_i(t) - z_j(t)| > 0,$$

which corresponds to the interaction of several point vortices in the setting of the point vortex system.

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