NUMERICAL SOLUTION OF THE POISSON EQUATION ON DOMAINS WITH A THIN LAYER OF RANDOM THICKNESS

M. DAMBRINE *, I. GREFF *, H. HARBRECHT †, AND B. PUIG *

Abstract. The present article is dedicated to the numerical solution of the Poisson equation on domains with a thin layer of different conductivity and of random thickness. By changing the boundary condition, the boundary value problem given on a random domain is transformed into a boundary value problem on a fixed domain. The randomness is then contained in the coefficients of the new boundary condition. This thin coating can be expressed by a random Robin boundary condition which yields a third order accurate solution in the scale parameter ε of the layer's thickness. With the help of the Karhunen-Loève expansion, we transform this random boundary value problem into a deterministic parametric one with a possibly high-dimensional parameter $\mathbf{y}$. Based on the decay of the random fluctuations of the layer's thickness, we prove rates of decay of the derivatives of the random solution with respect to this parameter $\mathbf{y}$ which are robust in the scale parameter ε. Numerical results validate our theoretical findings.

Key words. thin layer equation, random boundary value problems, random domains

1. Introduction.

1.1. Motivation and background. Many practical problems in engineering lead to boundary value problems for an unknown function. In this article, we consider uncertainties in the geometric definition of the domain motivated by tolerances in the manufacturing processes or in a damaged boundary during the life of a mechanical device. Manufactured or damaged devices are close to a nominal geometry but differ of course from its mathematical definition. Since we are motivated by tolerances, we can make the crucial assumption that the random perturbations are small. By identifying domains with their boundary, domains close to the nominal domain D can be seen as a perturbation in the normal direction of the nominal boundary ∂D.

The most common approach to study boundary value problems with stochastic inputs is the Monte-Carlo method, see e.g. [5, 13, 22] and the references therein. In many situations, this approach is easy to implement since it only requires a sufficiently large number of samples. However, for boundary value problems on random domains, each sample means a new domain and thus a new mesh, the building of new mass and stiffness matrices, etc. All these steps are mandatory to compute the quantity of interest. Therefore, the Monte-Carlo method is extremely costly and not so easy to implement in our context. This article is a contribution to the development of a method for solving boundary value problems in random domains that requires only a single, fixed mesh.

The treatment of boundary value problems on random domains is of high interest. There are several approaches for dealing with boundary value problems on random domains. Besides the fictitious domain approach considered in [7], one might essentially distinguish two approaches: the domain mapping method, cf. [8, 19, 16, 28, 27], and the perturbation method, cf. [14, 17], which is based on shape derivatives. They result from a description of the random domain either in Lagrangian coordinates or in Eulerian coordinates, see e.g. [24]. In this article, we consider Eulerian coordinates

*Département de Mathématiques, Université de Pau et des Pays de l'Adour, Avenue de l'université, 64000 Pau, France (marc.dambrine@univ-pau.fr, isabelle.greff@univ-pau.fr, benedicte.puig@univ-pau.fr)

†Departement für Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Schweiz (helmut.harbrecht@unibas.ch)

and propose a perturbation method based on approximate boundary conditions. This is especially motivated by the fact that shape derivatives are in general hard to compute.

1.2. Geometrical setting. Let us make precise the geometrical situation as illustrated in Figure 1.1: a given smooth domain D is surrounded by a thin coating layer L_ε. The precise regularity will specified later in Remark 2.1. The thickness h of the layer is a smooth, real-valued function which is defined on ∂D. We make the assumptions that the layer coats D everywhere and that its characteristic size is a small parameter $\varepsilon > 0$ so that the layer L_ε is described as:

$$L_\varepsilon = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}) : 0 \leq t < \varepsilon h(\mathbf{x}),\ \mathbf{x} \in \partial D\}.$$

Moreover, there exist nonnegative real numbers $h_{\min}$ and $h_{\max}$ such that

$$0 < h_{\min} \leq h(\mathbf{x}) \leq h_{\max} \text{ for all } \mathbf{x} \in \partial D.$$

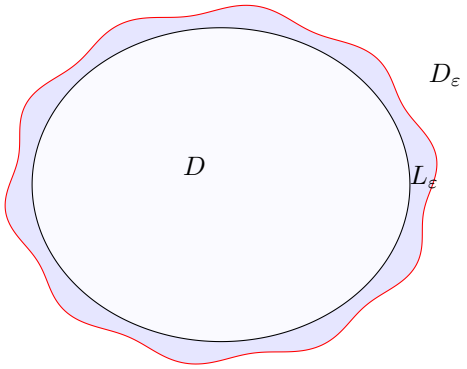

FIG. 1.1. *The geometrical setting – the domain D and the layer L_ε.*

Now, we are interested in the numerical solution of the following model boundary value problem posed in $D_\varepsilon = D \cup L_\varepsilon$: for a given function $f \in \mathrm{L}^2(D_\varepsilon)$, find the function u_ε such that

$$\left\{ \begin{array}{rcl} -\mathrm{div}(\sigma \nabla u_\varepsilon) & = & f \text{ in } D_\varepsilon, \\ u_\varepsilon & = & 0 \text{ on } \partial D_\varepsilon, \end{array} \right. \tag{1.1}$$

where the conductivity σ is piecewise constant, taking the value σ_0 in D and 1 in the layer L_ε. The presented theory extends to spatially varying conductivities σ in D but not to general variations in the layer. The change in the conductivity is motivated by the possibility to take into account surface treatments or surface damage. Notice that when the conductivity takes a constant value in the body and the layer, we recover Poisson's equation in a random domain with small fluctuations of the boundary.

1.3. Approximate boundary conditions. In order to efficiently compute a numerical approximation of the restriction to D of the solution u_ε to (1.1), a classical idea is to introduce impedance boundary conditions (see [4, 11] and derived works) to

avoid the meshing of the thin layer. The strategy is thus the following: work only in D and search for a boundary condition on ∂D so that the solution of the new boundary value problem defined in D (that is without the thin layer) is a good approximation of the restriction to D of the solution of the real boundary value problem set in D_ε.

A heuristic way to derive such a condition is as follows. Consider a point $\mathbf{x}$ on ∂D. The thickness εh being small, Taylor's formula provides

$$u_\varepsilon(\mathbf{x}+\varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})) = u_\varepsilon(\mathbf{x}) + \varepsilon h(\mathbf{x})\partial^+_{\mathbf{n}} u_\varepsilon(\mathbf{x}) + \mathcal{O}(\varepsilon^2),$$

where

$$\partial^+_{\mathbf{n}} u_\varepsilon(\mathbf{x}) = \lim_{t\to 0+} \frac{u_\varepsilon(\mathbf{x}+t\mathbf{n}(\mathbf{x})) - u_\varepsilon(\mathbf{x})}{t}.$$

Since the point $\mathbf{x}+\varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})$ lies on the boundary ∂D_ε, we get $u_\varepsilon(\mathbf{x}+\varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})) = 0$. Hence, a natural choice is to solve the Robin boundary value problem:

$$\left\{ \begin{array}{rcl} -\sigma_0 \Delta u^{[1]} &=& f \text{ in } D, \\ u^{[1]} + \varepsilon h \sigma_0 \partial_{\mathbf{n}} u^{[1]} &=& 0 \text{ on } \partial D. \end{array} \right. \tag{1.2}$$

Notice that $u^{[1]}$ is defined only in D where the diffusion coefficient is σ_0. In addition, σ_0 appears in the boundary condition as a consequence of the flux continuity at the interface ∂D. We will rigorously derive (1.2) in Section 2. It is well-known in the literature [1, 6, 21, 23] that indeed (1.2) is pertinent. There is a constant C, independent of ε, such that

$$\|u_\varepsilon - u^{[1]}\|_{\mathrm{H}^1(D)} \leq C\varepsilon^2. \tag{1.3}$$

A more precise but less intuitive approximate boundary value problem is

$$\left\{ \begin{array}{rcl} -\sigma_0 \Delta u^{[2]} &=& f \qquad \text{in } D, \\ \left(1 + \dfrac{\varepsilon\kappa h}{2}\right) u^{[2]} + \varepsilon\sigma_0 h \partial_{\mathbf{n}} u^{[2]} &=& \dfrac{\varepsilon^2 h^2}{2} f \text{ on } \partial D, \end{array} \right. \tag{1.4}$$

where κ stands for the curvature of D. Of course, this approximation obviously needs additional regularity of the right hand side f to be well defined. This will be specified later in Subsection 2.2. Then, for the solution of this boundary value problem, there is another constant C, independent of ε, such that

$$\|u_\varepsilon - u^{[2]}\|_{\mathrm{H}^1(D)} \leq C\varepsilon^3. \tag{1.5}$$

Proving the error estimates (1.3) and (1.5) requires a careful asymptotic analysis of problem (1.1) as already performed in [4, 11]. Note that the method has also been used for other boundary conditions and other differential operators, see [1, 11].

Since we will consider families of random layers, a crucial point for our work is the dependency of the constant C on the thickness h. In fact, we are interested in knowing if this constant is uniform in h or not since uniformity will later on be needed when randomness comes into play. This point has not been specified in [4, 11], so that we shall present it in Section 2. Our result is that the constants C appearing in (1.3) and (1.5) depend only on the $\mathcal{C}^1(\partial D)$-norm of h (see Remark 2.1). The proof of this crucial point is based on asymptotic expansions of the solution to (1.1) in D_ε.

1.4. Layers of random thickness. Once the uniform error estimates (1.3) and (1.5) are obtained, we shall consider the situation where the layer's thickness h is random. To that end, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space and assume that $h : \partial D \times \Omega \to \mathbb{R}$ is a stochastic process which satisfies the following assumptions:

(UB) Uniform boundedness: there exist two nonnegative real numbers $h_{\min} \leq h_{\max}$ and $q < 1$ such that the stochastic process

$$h(\mathbf{x}, \omega) = \overline{h}(\mathbf{x}) + \widetilde{h}(\mathbf{x}, \omega) \quad \text{with} \quad \overline{h}(\mathbf{x}) = \mathbb{E}\big(h(\mathbf{x}, \omega)\big)$$

satisfies

$$0 < h_{\min} \leq \overline{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\widetilde{h}(\mathbf{x}, \omega)| \leq q|\overline{h}(\mathbf{x})| \tag{1.6}$$

for all $\mathbf{x} \in \partial D$ and for $\mathbb{P}$-almost all $\omega \in \Omega$.

(UR) Uniform regularity: the function $\mathbf{x} \mapsto h(\mathbf{x}, \omega)$ is uniformly bounded in $\mathcal{C}^1$ for all ω in Ω, that is, the stochastic process h belongs to the Bochner space $\mathrm{L}^\infty_{\mathbb{P}}(\Omega, \mathcal{C}^1(\partial D))$.

Let us recall for the reader's convenience the definition of Bochner spaces. Consider a real number $p \geq 1$. Then, for a Banach space X, the Bochner space $\mathrm{L}^p_{\mathbb{P}}(\Omega, \mathrm{X})$ consists of all functions $v : \Omega \to \mathrm{X}$ whose norm

$$\|v\|_{\mathrm{L}^p_{\mathbb{P}}(\Omega,\mathrm{X})} := \begin{cases} \displaystyle\left(\int_\Omega \|v(\cdot,\omega)\|^p_{\mathrm{X}} \, \mathrm{d}\mathbb{P}(\omega)\right)^{1/p}, & p < \infty, \\ \displaystyle\operatorname*{ess\,sup}_{\omega \in \Omega} \|v(\cdot,\omega)\|_{\mathrm{X}}, & p = \infty, \end{cases}$$

is finite. If $p = 2$ and X is a Hilbert space, then the Bochner space is isomorphic to the tensor product space $\mathrm{L}^2_{\mathbb{P}}(\Omega) \otimes \mathrm{X}$.

In the sequel, we tacitly assume that the right hand side f is defined in a sufficiently large hold-all $\mathcal{D}$ such that $f \in \mathrm{L}^2(D_\varepsilon(\omega))$ holds for $\mathbb{P}$-almost all $\omega \in \Omega$. We then continue with the second order approximation of the solution to the thin layer equation. The reason for this is that it is more accurate (the committed error is of third order instead of second order) without being computationally much more demanding. Rewriting the deterministic problem (1.4) obtained for each realization ω, we shall thus finally consider the following elliptic partial differential equation with random Robin boundary condition:

$$\left.\begin{aligned} -\sigma_0 \Delta u^{[2]}(\omega) &= f && \text{in } D \\ \left(1 + \frac{\varepsilon\kappa h(\omega)}{2}\right) u^{[2]}(\omega) + \varepsilon\sigma_0 h(\omega) \partial_{\mathbf{n}} u^{[2]}(\omega) &= \frac{\varepsilon^2 h^2(\omega)}{2} f && \text{on } \partial D \end{aligned}\right\} \ \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{1.7}$$

1.5. Organization of the article. The rest of this article is organized as follows. As mentioned above, the asymptotic analysis for the thin layer is performed in Section 2. Then, in Section 3, we investigate the existence of solutions to the Robin boundary value problem (1.7) obtained to approximate the solution of the original boundary value problem in the case of a layer with random thickness and estimate the systematic error made in the expectation and the variance by this solution. In order to solve the random Robin problem, we assume in Section 4 that the random fluctuations are given in the form of a Karhunen-Loève expansion. Then, for a special model of the stochastic process h, we transform the problem into a deterministic parametric

problem. Note that the parameter $\mathbf{y}$ that takes the place of the event ω in (1.7) lives in a high-dimensional space. Our main result is then Theorem 4.2 which gives precise estimates on the derivatives of the random solution with respect to the variable $\mathbf{y}$. These estimates allow the use of quasi Monte-Carlo methods and anisotropic collocation schemes with convergence rates that are independent of the number of terms of Karhunen-Loève expansion provided that the individual terms satisfy a certain summability condition. Numerical experiments are performed in Section 5 to show the feasibility of our approach and to validate the theoretical findings. We present conclusions in Section 6.

2. Approximate boundary conditions for layers of variable thickness. In this section, we consider a deterministic layer. Let us introduce the notations and the objects needed to derive the approximate boundary conditions. For ease of notation, we deal with the bidimensional case and assume that D is simply connected so that its boundary has a parametrization by the arclength $s \mapsto \boldsymbol{\vartheta}(s)$ defined on the segment $[0, |\partial D|]$ where $|\partial D|$ is the perimeter of D. At the point $\boldsymbol{\vartheta}(s)$, the unit tangent vector $\mathbf{t}(s)$ is $\boldsymbol{\vartheta}'(s)$ and the curvature $\kappa(s)$ is defined by the equality $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$.

With the help of the above notation, the boundary of D_ε is parametrized by $s \mapsto \boldsymbol{\vartheta}_h(s) = \boldsymbol{\vartheta}(s) + \varepsilon h(s)\mathbf{n}(s)$. Of course, this parametrization is not by the arclength and the unit tangential and outward normal field are

$$\mathbf{t}_h(s) = \frac{(1+\varepsilon h\kappa)\mathbf{t}(s) + \varepsilon h'\mathbf{n}(s)}{\sqrt{(1+\varepsilon h\kappa)^2 + (\varepsilon h')^2}} \quad \text{and} \quad \mathbf{n}_h(s) = \frac{(1+\varepsilon h\kappa)\mathbf{n}(s) - \varepsilon h'\mathbf{t}(s)}{\sqrt{(1+\varepsilon h\kappa)^2 + (\varepsilon h')^2}}.$$

Here and in the following, we drop the dependency on s for κ and h for ease of notation.

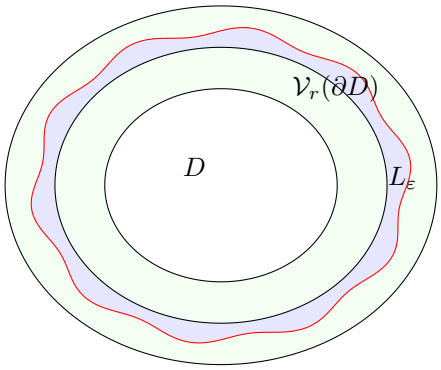

FIG. 2.1. *The geometrical setting – the domain D, the layer L_ε (in blue) and a tubular neighborhood $\mathcal{V}_r(\partial D)$ with $r < \rho$ (in green).*

Let χ be the cut locus of ∂D. This is the maximum length ρ such that the orthogonal projection on ∂D is uniquely defined in the tubular neighborhood $\mathcal{V}_\rho(\partial D)$ of radius ρ of ∂D. An illustration of the tubular neighborhood is found in Figure 2.1. In such a neighborhood, the curvilinear coordinates $(s,t) \in [0, |\partial D|] \times (-\chi, \chi)$ are uniquely defined for all $\mathbf{x}$ in $\mathcal{V}_\rho(\partial D)$ by $\mathbf{x} = \boldsymbol{\vartheta}\big(s\big) + t\,\mathbf{n}(s)$. In particular, each function $v : \mathcal{V}_\rho(\partial D) \to \mathbb{R}$ can also be expressed in these curvilinear coordinates as

$\tilde{v} : [0, |\partial D|] \times (-\chi, \chi) \to \mathbb{R}$ so that $v(\mathbf{x}) = \tilde{v}(s,t)$ for all $\mathbf{x} \in \mathcal{V}_\rho(\partial D)$. The gradient and the Laplace operator are expressed in the curvilinear coordinates by

$$\nabla = \frac{1}{1+t\kappa}\,\frac{\partial}{\partial s}\mathbf{t}(s) + \frac{\partial}{\partial t}\mathbf{n}(s) \tag{2.1}$$

and

$$\Delta = \frac{1}{1+t\kappa}\,\frac{\partial}{\partial s}\left(\frac{1}{1+t\kappa}\,\frac{\partial}{\partial s}\right) + \frac{\kappa}{1+t\kappa}\,\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}. \tag{2.2}$$

2.1. Asymptotic expansion in D and in the layer L_ε. The usual strategy relies on an asymptotic expansion of u with respect to the scaling factor ε with a double ansatz, one valid in D and the other in the layer L_ε. Namely, we postulate that there are real-valued functions $u_{\mathrm{int},k}$ defined on D and $u_{\mathrm{ext},k}$ defined for

$$(s,\tau) = \left(s, \frac{t}{\varepsilon h}\right) \in [0, |\partial D|] \times [0,1]$$

such that

$$\begin{cases} u_\varepsilon(\mathbf{x}) = u_{\mathrm{int}}(\mathbf{x}) = \displaystyle\sum_{k=0}^{\infty} \varepsilon^k u_{\mathrm{int},k}(\mathbf{x}) \text{ in } D, \\ u_\varepsilon(\mathbf{x}) = u_{\mathrm{ext}}(\mathbf{x}) = \displaystyle\sum_{k=0}^{\infty} \varepsilon^k u_{\mathrm{ext},k}(s,\tau) \text{ in } L_\varepsilon. \end{cases} \tag{2.3}$$

In contrast to the earlier work [6], the anisotropy in the second curvilinear coordinate takes the variation of the thickness into account. With these ansätze at hand, we can reformulate the boundary value problem (1.1) as a transmission problem:

$$\begin{cases} -\sigma_0 \Delta u_{\mathrm{int}} &= f & \text{in } D, \\ -\Delta u_{\mathrm{ext}} &= f & \text{in } L_\varepsilon, \\ u_{\mathrm{int}} &= u_{\mathrm{ext}} & \text{on } \partial D, \\ \sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int}} &= \partial_{\mathbf{n}} u_{\mathrm{ext}} & \text{on } \partial D, \\ u_{\mathrm{ext}} &= 0 & \text{on } \partial D_\varepsilon. \end{cases} \tag{2.4}$$

In accordance with [6], we write the equation in the layer L_ε in the new coordinate system by employing (2.2):

$$\mathcal{L} u_{\mathrm{ext}} = -(1+t\kappa)\, f \quad \text{with} \quad \mathcal{L} = \partial_s \left(\frac{1}{1+t\kappa}\,\partial_s\right) + \kappa \partial_t + (1+t\kappa)\partial_t^2. \tag{2.5}$$

In order to rewrite equation (2.5) with respect to the anisotropic, curvilinear coordinates $(s,\tau) = (s, t/(\varepsilon h))$, corresponding to the ansätze (2.3), we shall have the following relations in mind:

$$\kappa \partial_t = \frac{\kappa}{\varepsilon h}\partial_\tau, \quad (1+t\kappa)\partial_t^2 = \frac{1}{\varepsilon^2 h^2}\partial_\tau^2 + \frac{\kappa\tau}{\varepsilon h}\partial_\tau^2.$$

With the help of the decomposition of $(1+t\kappa)^{-1}$ into a power series and the product rule, we moreover arrive at

$$\partial_s \left(\frac{1}{1+t\kappa}\,\partial_s\right) = \sum_{n\geq 0} (-1)^n \varepsilon^n \tau^n \kappa^n h^n \partial_s^2 - \sum_{n\geq 1} (-1)^n \varepsilon^n \tau^n n(\kappa' \kappa^{n-1} h^n + \kappa^n h' h^{n-1}) \partial_s.$$

As a consequence of these computations, the operator $\mathcal{L}$ from (2.5) can be split in powers of the small parameter ε according to $\mathcal{L} = \sum_{n\geq -2} \varepsilon^n \mathcal{L}_n$ with

$$\mathcal{L}_{-2} = \frac{1}{h^2}\partial_\tau^2, \quad \mathcal{L}_{-1} = \frac{\kappa}{h}\left(\partial_\tau + \tau\partial_\tau^2\right), \quad \mathcal{L}_0 = \partial_s^2,$$

and, for $n \geq 1$, with

$$\mathcal{L}_n = (-1)^n \tau^n \left[\kappa^n h^n \partial_s^2 - n(\kappa'\kappa^{n-1}h^n + \kappa^n h' h^{n-1})\partial_s\right].$$

Hence, combining the ansatz for u_{ext} with this expansion of $\mathcal{L}$, we obtain

$$\begin{aligned}\mathcal{L}u_{\text{ext}} = \varepsilon^{-2}\frac{1}{h^2}\partial_\tau^2 u_{\text{ext},0} + \varepsilon^{-1}\left(\frac{1}{h^2}\partial_\tau^2 u_{\text{ext},1} + \frac{\kappa}{h}\left(\partial_\tau u_{\text{ext},0} + \tau\partial_\tau^2 u_{\text{ext},0}\right)\right)\\ + \sum_{n\geq 0}\varepsilon^n\left(\frac{1}{h^2}\partial_\tau^2 u_{\text{ext},n+2} + \frac{\kappa}{h}\left(\partial_\tau u_{\text{ext},n+1} + \tau\partial_\tau^2 u_{\text{ext},n+1}\right) + \sum_{k=0}^{n}\mathcal{L}_{n-k}u_{\text{ext},k}\right).\end{aligned} \tag{2.6}$$

We expand next the inhomogeneity f of the boundary value problem (2.5) by a Taylor series:

$$(1+t\kappa)f(\cdot,t) = (1+\varepsilon h\kappa\tau)f(\cdot,\varepsilon h\tau) = \sum_{n=0}^{N}\frac{\varepsilon^n}{n!}f_n(\cdot,\tau) + \mathcal{O}(\varepsilon^{N+1}). \tag{2.7}$$

This, of course, requires that f is smooth in the layer. Since $f_0(s,\tau) = f(s,0)$, we obtain from (2.5), (2.6) and (2.7) the following sequence of differential equations for the functions u_{ext}^k by matching the coefficients of terms with the same power of ε:

$$\begin{aligned}&\partial_\tau^2 u_{\text{ext},0} = 0,\\ &\partial_\tau^2 u_{\text{ext},1} = -\kappa h\left(\partial_\tau u_{\text{ext},0} + \tau\partial_\tau^2 u_{\text{ext},0}\right),\\ &\partial_\tau^2 u_{\text{ext},n+2} = -\left(\kappa h\left(\partial_\tau u_{\text{ext},n+1} + \tau\partial_\tau^2 u_{\text{ext},n+1}\right) + h^2\sum_{k=0}^{n}\mathcal{L}_{n-k}u_{\text{ext},k}\right) - h^2 f_n,\ n\geq 0.\end{aligned}$$

The boundary conditions on the outer boundary, that is for $\tau = 1$, are $u_{\text{ext},k} = 0$. Whereas, in view of the ansätze (2.3), the boundary conditions on the interface, that is for $\tau = 0$, are given by the transmission conditions on ∂D stated in (2.4):

$$\begin{gathered}\sigma_0\partial_{\mathbf{n}}u_{\text{int}} = \partial_t u_{\text{ext}} = \frac{1}{\varepsilon h}\partial_\tau u_{\text{ext}} \Rightarrow \sigma_0\partial_{\mathbf{n}}u_{\text{int},k-1} = \frac{1}{h}\partial_\tau u_{\text{ext},k}\\ u_{\text{int}} = u_{\text{ext}} \Rightarrow u_{\text{int},k} = u_{\text{ext},k}\end{gathered}$$

They are obtained by matching the coefficients of terms with the same power of ε. The resolution of (2.4) is then iterative with the following strategy. At step k, we first compute $u_{\text{ext},k}$: we consider the equation in the layer, we use the transmission condition for the flux at the interface and the Dirichlet boundary condition on the outer boundary. Then, we use the transmission condition $u_{\text{int},k} = u_{\text{ext},k}$ on ∂D to derive the boundary value problem solved by $u_{\text{int},k}$. In order to derive the equivalent boundary conditions (1.2) and (1.4), we need the first terms. Let us make these terms explicit.

Order $n = 0$. Since $\partial_\tau^2 u_{\text{ext},0} = 0$, the first function $u_{\text{ext},0}$ should be affine in the variable τ, taking the value 0 for $\tau = 1$ and with derivative $\partial_\tau u_{\text{ext},0} = 0$ at $\tau = 0$.

Hence, it follows that $u_{\mathrm{ext},0} = 0$ and thus, in particular, $u_{\mathrm{ext},0} = 0$ on ∂D. The transmission conditions on ∂D then provide that $u_{\mathrm{int},0}$ solves $-\Delta u = f$ in $\mathrm{H}^1_0(D)$.

Order $n = 1$. The equations for $u_{\mathrm{ext},1}$ are

$$\begin{aligned} \partial_\tau^2 u_{\mathrm{ext},1} &= -\kappa h \left(\partial_\tau u_{\mathrm{ext},0} + \tau \partial_\tau^2 u_{\mathrm{ext},0}\right) = 0, \\ \partial_\tau u_{\mathrm{ext},1}(s,0) &= \sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},0}(s,0), \\ u_{\mathrm{ext},1}(s,1) &= 0. \end{aligned}$$

Consequently, $u_{\mathrm{ext},1}$ is the affine (in τ) function $u_{\mathrm{ext},1}(s,\tau) = (\tau - 1)\sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},0}(s,0)$ so that

$$u_{\mathrm{ext},1} = -\sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},0} \text{ on } \partial D.$$

Thus, $u_{\mathrm{int},1}$ solves $\Delta u_{\mathrm{int},1} = 0$ with $u_{\mathrm{int},1} = u_{\mathrm{ext},1} = -\sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},0}$ on ∂D.

Order $n = 2$. Since $u_{\mathrm{ext},0} = 0$ and $u_{\mathrm{ext},1}$ is affine in τ, the equations for u^2_{ext} are

$$\begin{aligned} \partial_\tau^2 u_{\mathrm{ext},2} &= -\kappa h \partial_\tau u_{\mathrm{ext},1} - h^2 f_0 = -\kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int},0}(s,0) - h^2 f_0, \\ \partial_\tau u_{\mathrm{ext},2}(s,0) &= \sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},1}(s,0), \\ u_{\mathrm{ext},2}(s,1) &= 0. \end{aligned}$$

These equations for u^2_{ext} can still be solved analytically. Indeed, since $f_0(s,.) = f(s,0)$, it is independent of τ, the second order primitive integral of $-f_0(s,\tau)$ which vanishes at 1 and which has a derivative that vanishes at 0 is $f(s,0)(1-\tau^2)/2$. Namely, it holds

$$u_{\mathrm{ext},2}(s,\tau) = \frac{1-\tau^2}{2}\kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int},0}(s,0) + (\tau - 1)\sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},1}(s,0) + f(s,0)\frac{1-\tau^2}{2},$$

which, for $\tau = 0$, leads to

$$u_{\mathrm{ext},2} = \frac{1}{2}\kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int},0} - \sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},1} + \frac{f}{2} \text{ on } \partial D.$$

Thus, $u_{\mathrm{int},2}$ solves $\Delta u = 0$ in D with $u_{\mathrm{int},2} = u_{\mathrm{ext},2}$ on ∂D.

2.2. Derivation of the approximate boundary conditions. Let us now derive the first and second order approximate boundary conditions. To that end, we introduce the partial sums $u^{[1]}_{\mathrm{int}} = u_{\mathrm{int},0} + \varepsilon u_{\mathrm{int},1}$ and $u^{[2]}_{\mathrm{int}} = u_{\mathrm{int},0} + \varepsilon u_{\mathrm{int},1} + \varepsilon^2 u_{\mathrm{int},2}$. By construction, we check that $-\sigma_0 \Delta u^{[i]}_{\mathrm{int}} = f$ in D for $i = 1, 2$.

Order $n = 1$. On ∂D, one has $u^{[1]}_{\mathrm{int}} = -\varepsilon\sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},0} = -\varepsilon\sigma_0 h \partial_{\mathbf{n}}[u^{[1]}_{\mathrm{int}} - \varepsilon u_{\mathrm{int},1}]$ and it follows

$$u^{[1]}_{\mathrm{int}} + \varepsilon\sigma_0 h \partial_{\mathbf{n}} u^{[1]}_{\mathrm{int}} = \varepsilon^2 \sigma_0 h \partial_{\mathbf{n}} u_{\mathrm{int},1}.$$

Neglecting the second order term, we introduce $v^{[1]}_\varepsilon$ as the solution of the following Robin boundary value problem:

$$\begin{aligned} -\sigma_0 \Delta v^{[1]}_\varepsilon &= f \text{ in } D, \\ v^{[1]}_\varepsilon + \varepsilon\sigma_0 h \partial_{\mathbf{n}} v^{[1]}_\varepsilon &= 0 \text{ on } \partial D. \end{aligned}$$

Notice that $u_{\rm int}^{[1]}$ and $v_\varepsilon^{[1]}$ solve two distinct boundary value problems differing from a second order term (in ε) in the boundary condition.

Order $n = 2$. On the boundary ∂D, one has

$$\begin{aligned} u_{\rm int}^{[2]} &= -\varepsilon\sigma_0 h\partial_{\mathbf{n}} u_{{\rm int},0} + \varepsilon^2\left(\frac{1}{2}\kappa h^2\sigma_0\partial_{\mathbf{n}} u_{{\rm int},0} - \sigma_0 h\partial_{\mathbf{n}} u_{{\rm int},1} + \frac{h^2}{2}\, f\right) \\ &= -\varepsilon\sigma_0 h\left(1 - \frac{\varepsilon\kappa h}{2}\right)\partial_{\mathbf{n}} u_{{\rm int},0} - \varepsilon^2\sigma_0 h\left(1 - \frac{\varepsilon\kappa h}{2}\right)\partial_{\mathbf{n}} u_{{\rm int},1} \\ &\qquad + \frac{\varepsilon^2 h^2}{2}\, f - \frac{\varepsilon^3}{2}\kappa h^2\sigma_0\partial_{\mathbf{n}} u_{{\rm int},1}. \end{aligned}$$

Therefore, by using the Taylor expansion $(1+x)^{-1} = 1 - x + \mathcal{O}(x^2)$, we obtain

$$\begin{aligned} u_{\rm int}^{[2]} &= -\varepsilon\sigma_0 h\left(\frac{1}{1+\frac{\varepsilon\kappa h}{2}} + \mathcal{O}(\varepsilon^2)\right)\partial_{\mathbf{n}}(u_{{\rm int},0} + \varepsilon u_{{\rm int},1}) + \frac{\varepsilon^2 h^2}{2}\, f - \frac{\varepsilon^3}{2}\kappa h^2\sigma_0\partial_{\mathbf{n}} u_{{\rm int},1} \\ &= -\varepsilon\sigma_0 h\frac{1}{1+\frac{\varepsilon\kappa h}{2}}\partial_{\mathbf{n}}(u_{{\rm int},0} + \varepsilon u_{{\rm int},1}) + \frac{\varepsilon^2 h^2}{2}\, f + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Since $\partial_{\mathbf{n}} u_{\rm int}^{[2]} = \partial_{\mathbf{n}}(u_{{\rm int},0} + \varepsilon u_{{\rm int},1}) + \varepsilon^2\partial_{\mathbf{n}} u_{{\rm int},2}$, we get

$$\left(1 + \frac{\varepsilon\kappa h}{2}\right) u_{\rm int}^{[2]} + \varepsilon\sigma_0 h\partial_{\mathbf{n}} u_{\rm int}^{[2]} - \frac{\varepsilon^2 h^2}{2} f = \mathcal{O}(\varepsilon^3).$$

By neglecting the third order term, we introduce $v_\varepsilon^{[2]}$ as the solution of the Robin boundary value problem (1.4) that we recall for convenience:

$$\begin{aligned} -\sigma_0\Delta v_\varepsilon^{[2]} &= f \qquad \text{in } D, \\ \left(1 + \frac{\varepsilon\kappa h}{2}\right) v_\varepsilon^{[2]} + \varepsilon\sigma_0 h\partial_{\mathbf{n}} v_\varepsilon^{[2]} &= \frac{\varepsilon^2 h^2}{2} f \text{ on } \partial D. \end{aligned}$$

Notice that $u_{\rm int}^{[2]}$ and $v_\varepsilon^{[2]}$ solve two distinct boundary value problems differing from a third order term in the boundary condition.

2.3. Error estimates and dependency on the layer thickness ε. We proceed in two steps: first, we obtain an error estimate for the remainders in the truncated asymptotic expansions of the solution u_ε of (1.1). Then, in a second step, we obtain an asymptotic expansion of $u_{\rm int}^{[i]}$ for $i = 1, 2$. Both steps are adapted from the proofs of [6] in the case of layers with constant thickness, i.e. $h(\mathbf{x})$ takes a constant value. We therefore explain the main lines of the proof without entering into the details.

To estimate the truncation error for u_ε, the first step is to write a precise error estimate for the remainder r_ε^N in the asymptotic expansion of $u_{\rm ext}$ and $u_{\rm int}$. The reminder r_ε^N is piecewise defined as

$$r_\varepsilon^N = r_{\varepsilon,{\rm int}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{{\rm int},k} \text{ in } D \quad \text{and} \quad r_\varepsilon^N = r_{\varepsilon,{\rm ext}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{{\rm ext},k} \text{ in } L_\varepsilon.$$

These remainders satisfy the boundary value problems

$$\left\{\begin{array}{rll} \sigma_0 \Delta r^N_{\varepsilon,\mathrm{int}} &= 0 & \text{in } D, \\ \Delta r^N_{\varepsilon,\mathrm{ext}} &= f_N & \text{in } L_\varepsilon, \\ \sigma_0 \partial_{\mathbf{n}} r^N_{\varepsilon,\mathrm{int}} &= \partial_{\mathbf{n}} r^N_{\varepsilon,\mathrm{ext}} + g_N & \text{on } \partial D, \\ r^N_{\varepsilon,\mathrm{int}} &= r^N_{\varepsilon,\mathrm{ext}} & \text{on } \partial D, \\ r^N_{\varepsilon,\mathrm{ext}} &= 0 & \text{on } \partial D_\varepsilon, \end{array}\right. \tag{2.8}$$

with $f_N = \mathcal{O}(\varepsilon^{N-1})$ and $g_N = \mathcal{O}(\varepsilon^N)$ by construction. Precisely, the normal derivatives satisfy:

$$\sigma_0 \partial_{\mathbf{n}} r^N_{\varepsilon,\mathrm{int}} - \partial_{\mathbf{n}} r^N_{\varepsilon,\mathrm{ext}} = \sum_{n=0}^{N} \varepsilon^n \left[\sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int},n} - \frac{1}{\varepsilon h}\, \partial_t u_{\mathrm{ext},n} \right] = \sigma_0 \varepsilon^N \partial_{\mathbf{n}} u_{\mathrm{int},N}.$$

The proof is similar for the Laplacian in the layer.

In order to derive the variational formulation of (2.8), we define the bilinear form a_ε on $\mathrm{H}^1_0(D_\varepsilon)$ by

$$a_\varepsilon(u,v) = \sigma_0 \int_D \nabla u \nabla v \,\mathrm{d}\mathbf{x} + \int_{L_\varepsilon} \nabla u \nabla v \,\mathrm{d}\mathbf{x}$$

and the linear form F^N_ε on $\mathrm{H}^1_0(D_\varepsilon)$ by

$$\langle F^N_\varepsilon, v\rangle = \int_{\partial D} g_N v \,\mathrm{d}o - \int_{L_\varepsilon} f_N v \,\mathrm{d}\mathbf{x}.$$

Here, do stands for the surface measure. Thus, (2.8) has the variational formulation: $\forall v \in \mathrm{H}^1_0(D_\varepsilon),\ a_\varepsilon(r^N_\varepsilon, v) = F^N_\varepsilon(v)$.

For applying Lax Milgram's theorem, we check the uniform coercivity and continuity of the bilinear forms a_ε. Notice that from now on, in this section, the term "uniform" means uniform with respect to both ε and h. Since, for $\varepsilon \le \varepsilon_0$ and h such that $\|h\|_{\mathrm{L}^\infty(\partial D)} \le C_0$, the family D_ε is uniformly contained in a fixed domain, a uniform Poincaré inequality holds in $\mathrm{H}^1_0(D_\varepsilon)$ which provides the uniform coercivity of a_ε. The uniform continuity of a_ε is clear.

Concerning the linear form F^N_ε, it follows for all $v \in \mathrm{H}^1_0(D_\varepsilon)$ that

$$|\langle F^N_\varepsilon, v\rangle| \le \|g_N\|_{\mathrm{L}^2(\partial D)} \|v\|_{\mathrm{L}^2(\partial D)} + \|f_N\|_{\mathrm{L}^2(L_\varepsilon)} \|v\|_{\mathrm{L}^2(L_\varepsilon)}.$$

Moreover, by the trace inequality, we find $\|v\|_{\mathrm{L}^2(\partial D)} \le c\|v\|_{\mathrm{H}^1_0(L_\varepsilon)}$. Hence, there is a constant k such that $|\langle F^N_\varepsilon, v\rangle| \le k\|v\|_{\mathrm{H}^1_0(L_\varepsilon)}$. Consequently, Lax Milgram's theorem can be applied and combining the uniform coercivity of a_ε with respect to ε and the continuity of F^N_ε, we check that there is a constant C, independent of $\varepsilon \le \varepsilon_0$ and of h, such that

$$\|r^N_\varepsilon\|_{\mathrm{H}^1_0(D_\varepsilon)} \le C\varepsilon^{N-1} \tag{2.9}$$

since $f_N = \mathcal{O}(\varepsilon^{N-1})$ and $g_N = \mathcal{O}(\varepsilon^N)$.

Finally, we split the remainder of order N according to

$$r^N_\varepsilon = r^{N+2}_\varepsilon + \varepsilon^{N+1} u_{\varepsilon,N+1} + \varepsilon^{N+2} u_{\varepsilon,N+2},$$

where $u_{\varepsilon,k}$ is $u_{\mathrm{int},k}$ in D and $u_{\mathrm{ext},k}$ in the layer. The triangle inequality and (2.9) applied to the reminder r_ε^{N+2} of order $N+2$ give

$$\|r_\varepsilon^N\|_{\mathrm{H}_0^1(D_\varepsilon)} \leq \|r_\varepsilon^{N+2}\|_{\mathrm{H}_0^1(D_\varepsilon)} + \varepsilon^{N+1}\|u_{\varepsilon,N+1}\|_{\mathrm{H}_0^1(D_\varepsilon)} + \varepsilon^{N+2}\|u_{\varepsilon,N+2}\|_{\mathrm{H}_0^1(D_\varepsilon)} \leq C\varepsilon^{N+1}.$$

We thus immediately get

$$\|r_{\varepsilon,\mathrm{int}}^N\|_{\mathrm{H}_0^1(D)} \leq C\varepsilon^{N+1}. \tag{2.10}$$

Of course, this constant C depends of the truncation order N.

Let us now consider the Robin boundary problems for $u_{\mathrm{int}}^{[i]}$ for $i = 1, 2$. We compute asymptotic expansions of $u_{\mathrm{int}}^{[i]}$. Thanks to the ansatz

$$u_{\mathrm{int}}^{[i]} = \sum_{k=0}^{\infty} \varepsilon^k u_{\mathrm{int},k}^{[i]},$$

we obtain recursion formulae. For example, for $i = 1$, we get

$$\left\{ \begin{array}{rcll} -\sigma_0 \Delta u_{\mathrm{int},k}^{[1]} & = & \delta_{0,k} f & \text{in } D, \\ u_{\mathrm{int},k}^{[1]} & = & -\sigma_0 \partial_{\mathbf{n}} u_{\mathrm{int},k-1}^{[1]} & \text{on } \partial D. \end{array} \right. \tag{2.11}$$

We obtain the same i first order terms: $u_{\mathrm{int},k}^{[i]} = u_{\mathrm{int},k}$ for $k \leq i$. The previous error estimate allows to conclude that

$$\left\|u_\varepsilon - u_{\mathrm{int}}^{[i]}\right\|_{\mathrm{H}^1(D)} \leq C\varepsilon^{i+1} \quad \text{for } i = 1, 2. \tag{2.12}$$

Notice that these error estimates are optimal since the next terms in the expansion of $u_{\mathrm{int}}^{[i]}$ differ from those in the expansion of u_{int}.

REMARK 2.1 (On the regularity of ∂D and of h). *When one checks the regularity of ∂D needed to validate the previous computations, one sees that it depends on the order of the approximate boundary condition. The local coordinates (s, τ) exist if ∂D and h are of class $\mathcal{C}^1$. Moreover, the differential operators $\mathcal{L}_n$ involve the curvature κ of ∂D if $n \geq -1$ and its derivative if $n \geq 1$. Hence, the requirement for ∂D is that it is a curve of class $\mathcal{C}^2$ to derive approximate boundary conditions up to second order. The error estimate is provided by the uniform boundedness of the family D_ε. Hence, what we really need is that h is $\mathcal{C}^1$ and with a uniform (in h) L^∞ bound depending only on D. For sake of simplicity, we work with the stronger assumption that h is bounded in the $\mathcal{C}^1$-norm.*

3. Randomly varying thin layers and random Robin boundary condition. From now on, we assume that h is a stochastic process which satisfies the requirements from Subsection 1.4. Then, the elliptic partial differential equation (1.4) becomes the problem with random Robin boundary condition (1.7). To obtain the variational formulation, we multiply (1.7) with an arbitrary test function from $L^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$: seek $u^{[2]} \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$ such that

$$\begin{aligned} \int_\Omega \left\{ \sigma_0 \int_D \nabla u^{[2]}(\omega) \nabla v(\omega) \,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} \left[\frac{1}{h(\omega)} + \frac{\varepsilon}{2}\kappa \right] u^{[2]}(\omega) v(\omega) \,\mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega) \\ = \int_\Omega \left\{ \int_D f v(\omega) \,\mathrm{d}\mathbf{x} + \int_{\partial D} \frac{\varepsilon h(\omega)}{2\sigma_0} f v(\omega) \,\mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega) \end{aligned} \tag{3.1}$$

holds for all $v \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$.

THEOREM 3.1. *Under the conditions* (1.6)*, there exists a unique solution* $u^{[2]}$ *in* $\mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$ *to the variational formulation* (3.1) *provided that* ε *is small enough that*

$$|\varepsilon\kappa(\mathbf{x})| \leq \frac{1}{(1+q)h_{\max}} \text{ for all } \mathbf{x} \in \partial D. \tag{3.2}$$

In particular, introducing the spatial energy norm

$$|\!|\!| v |\!|\!| := \sqrt{\sigma_0 |v|^2_{\mathrm{H}^1(D)} + \frac{1}{\varepsilon\sigma_0} \|T(v)\|^2_{\mathrm{L}^2(\partial D)}}, \tag{3.3}$$

where $T : \mathrm{H}^1(D) \to \mathrm{L}^2(\partial D)$ *is the trace operator, we have the stability estimate*

$$\sqrt{\int_\Omega |\!|\!| u^{[2]}(\omega) |\!|\!|^2 \,\mathrm{d}\mathbb{P}(\omega)} \leq C\big\{ \|f\|_{\widetilde{\mathrm{H}}^{-1}(D)} + \|T(f)\|_{\mathrm{L}^2(\partial D)} \big\} \tag{3.4}$$

uniformly as ε *tends to* 0*, where* $\|\cdot\|_{\widetilde{\mathrm{H}}^{-1}(D)}$ *denotes as usual the dual norm to* $\|\cdot\|_{\mathrm{H}^1(D)}$.

Proof. By introducing the bilinear form

$$\begin{aligned} a &: \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D)) \times \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D)) \to \mathbb{R}, \\ &a(u,v) := \int_\Omega \left\{ \sigma_0 \int_D \nabla u \nabla v \,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} \left[\frac{1}{h} + \frac{\varepsilon}{2}\kappa \right] uv \,\mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega) \end{aligned}$$

and the linear form

$$\ell : \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D)) \to \mathbb{R}, \quad \ell(v) := \int_\Omega \left[\int_D fv \,\mathrm{d}\mathbf{x} + \int_{\partial D} \frac{\varepsilon h}{2\sigma_0} fv \,\mathrm{d}o \right] \mathrm{d}\mathbb{P}(\omega),$$

the variational formulation (3.1) is equivalent to the problem:

$$\begin{aligned} &\text{seek } u^{[2]} \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D)) \text{ such that} \\ &\qquad a(u^{[2]}, v) = \ell(v) \text{ for all } v \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D)). \end{aligned} \tag{3.5}$$

In view of (1.6), it holds that

$$0 < (1-q)h_{\min} \leq (1-q)\overline{h}(\mathbf{x}) \leq h(\mathbf{x},\omega) \leq (1+q)\overline{h}(\mathbf{x}) \leq (1+q)h_{\max}$$

and hence with (3.2) that

$$\frac{1}{2(1+q)h_{\max}} \leq \frac{1}{h(\mathbf{x},\omega)} + \frac{\varepsilon}{2}\kappa(\mathbf{x}) \leq \frac{3}{2(1-q)h_{\min}} \tag{3.6}$$

for all $\mathbf{x} \in D$ and for $\mathbb{P}$-almost all $\omega \in \Omega$. The bound (3.6) ensures the uniform ellipticity and boundedness of the bilinear form $a(\cdot,\cdot)$:

$$\begin{aligned} \min\left\{1, \frac{1}{2(1+q)h_{\max}}\right\} \int_\Omega |\!|\!| u(\omega) |\!|\!|^2 \,\mathrm{d}\mathbb{P}(\omega) &\leq a(u,u), \\ |a(u,v)| \leq \max\left\{1, \frac{3}{2(1-q)h_{\min}}\right\} &\sqrt{\int_\Omega |\!|\!| u(\omega) |\!|\!|^2 \,\mathrm{d}\mathbb{P}(\omega)} \sqrt{\int_\Omega |\!|\!| v(\omega) |\!|\!|^2 \,\mathrm{d}\mathbb{P}(\omega)}. \end{aligned}$$

In addition, the linear form $\ell(\cdot)$ satisfies

$$|\ell(v)| \leq \|f\|_{\widetilde{\mathrm{H}}^{-1}(D)} \sqrt{\int_\Omega \|v(\omega)\|^2_{\mathrm{H}^1(D)}\,\mathrm{d}\mathbb{P}(\omega)} + \frac{\varepsilon}{2\sigma_0}(1+q)h_{\max}\|T(f)\|_{\mathrm{L}^2(\partial D)} \sqrt{\int_\Omega \|T(v(\omega))\|^2_{\mathrm{L}^2(\partial D)}\,\mathrm{d}\mathbb{P}(\omega)}$$

provided that $f \in \mathrm{L}^2(D) \cap \mathrm{L}^2(\partial D)$. Due to the Sobolev norm equivalence theorem, there exist constants $0 < \underline{c} < \overline{c}$ such that

$$\underline{c}\|v\|_{\mathrm{H}^1(D)} \leq \sqrt{|v|^2_{\mathrm{H}^1(D)} + \|T(v)\|^2_{\mathrm{L}^2(\partial D)}} \leq \overline{c}\|v\|_{\mathrm{H}^1(D)}$$

for all $v \in \mathrm{H}^1(D)$, see e.g. [25]. This implies that the energy norm (3.3) is equivalent to the $\mathrm{H}^1(D)$-norm with

$$\underline{c}\min\left\{\sqrt{\sigma_0}, \frac{1}{\sqrt{\varepsilon\sigma_0}}\right\}\|v(\omega)\|_{\mathrm{H}^1(\Omega)} \leq |\!|\!|v(\omega)|\!|\!| \leq \overline{c}\max\left\{\sqrt{\sigma_0}, \frac{1}{\sqrt{\varepsilon\sigma_0}}\right\}\|v(\omega)\|_{\mathrm{H}^1(\Omega)}.$$

Hence, we can set

$$c_f := \frac{1}{\underline{c}}\max\left\{\frac{1}{\sqrt{\sigma_0}}, \sqrt{\varepsilon\sigma_0}\right\}\|f\|_{\widetilde{\mathrm{H}}^{-1}(D)} + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}}(1+q)h_{\max}\|T(f)\|_{\mathrm{L}^2(\partial D)} < \infty$$

to arrive at the continuity of the linear form $\ell(\cdot)$:

$$|\ell(v)| \leq c_f \sqrt{\int_\Omega |\!|\!|v(\omega)|\!|\!|^2\,\mathrm{d}\mathbb{P}(\omega)}.$$

Herein, the constant c_f does not depend on the layer's thickness ε any more provided that $\varepsilon \leq \varepsilon_0$. According to the Lax Milgram's theorem, we conclude thus the desired result. □

This theorem implies the well-posedness of the thin layer equation (3.1) with random thickness. In particular, as an immediate consequence of our analysis in Subsection 2.3, we conclude from (2.12) that

$$\|u(\omega) - u^{[2]}(\omega)\|_{\mathrm{H}^1(D)} \leq C\varepsilon^3 \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{3.7}$$

where the constant $C > 0$ is independent of ω. Hence, the random solution $u^{[2]}$ of (3.1) is in the Bochner space $L^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$, satisfying the error estimate

$$\|u - u^{[2]}\|_{\mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))} \leq C\varepsilon^3. \tag{3.8}$$

Therefore, we derive the following proposition.

PROPOSITION 3.2. *The random solution $u^{[2]} \in L^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$ satisfies the error estimates*

$$\|\mathbb{E}(u) - \mathbb{E}(u^{[2]})\|_{\mathrm{H}^1(D)} \leq C\varepsilon^3, \quad \|\mathbb{V}(u) - \mathbb{V}(u^{[2]})\|_{\mathrm{W}^{1,1}(D)} \leq C\varepsilon^4.$$

Proof. The first assertion follows immediately from (3.8). For the second assertion, one has to consider the difference of the covariances

$$\begin{aligned}\operatorname{Cov}(u)(\mathbf{x},\mathbf{x}') &= \mathbb{E}\Big(\big[u(\mathbf{x},\omega)-\mathbb{E}(u(\mathbf{x},\omega))\big]\big[u(\mathbf{x}',\omega)-\mathbb{E}(u(\mathbf{x}',\omega))\big]\Big),\\ &= \mathbb{E}\Big(\big[u(\omega)-\mathbb{E}(u(\omega))\big]\otimes\big[u(\omega)-\mathbb{E}(u(\omega))\big]\Big)\Big|_{(\mathbf{x},\mathbf{x}')}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Cov}(u^{[2]})(\mathbf{x},\mathbf{x}') &= \mathbb{E}\Big(\big[u^{[2]}(\mathbf{x},\omega)-\mathbb{E}(u^{[2]}(\mathbf{x},\omega))\big]\big[u^{[2]}(\mathbf{x}',\omega)-\mathbb{E}(u^{[2]}(\mathbf{x}',\omega))\big]\Big)\\ &= \mathbb{E}\Big(\big[u^{[2]}(\omega)-\mathbb{E}(u^{[2]}(\omega))\big]\otimes\big[u^{[2]}(\omega)-\mathbb{E}(u^{[2]}(\omega))\big]\Big)\Big|_{(\mathbf{x},\mathbf{x}')}\end{aligned}$$

in the space $\mathrm{H}^1_{\mathrm{mix}}(D\times D) := \mathrm{H}^1(D)\otimes\mathrm{H}^1(D)$. Due to

$$u^{[2]}(\omega)-\mathbb{E}(u^{[2]}(\omega)) = \big[u^{[2]}(\omega)-u(\omega)\big]+\big[u(\omega)-\mathbb{E}(u(\omega))\big]+\mathbb{E}\big(u(\omega)-u^{[2]}(\omega)\big),$$

using linearity and symmetry, we find

$$\begin{aligned}&\|\operatorname{Cov}(u)-\operatorname{Cov}(u^{[2]})\|_{\mathrm{H}^1_{\mathrm{mix}}(D\times D)}\\ &\quad\le \Big\|\mathbb{E}\Big(\big[u^{[2]}(\omega)-u(\omega)\big]\otimes\big[u^{[2]}(\omega)-u(\omega)\big]\Big)\Big\|_{\mathrm{H}^1_{\mathrm{mix}}(D\times D)}\\ &\quad+2\Big\|\mathbb{E}\Big(\big[u^{[2]}(\omega)-u(\omega)\big]\otimes\big[u(\omega)-\mathbb{E}(u(\omega))\big]\Big)\Big\|_{\mathrm{H}^1_{\mathrm{mix}}(D\times D)}\\ &\quad+3\Big\|\mathbb{E}\big(u^{[2]}(\omega)-u(\omega)\big)\otimes\mathbb{E}\big(u^{[2]}(\omega)-u(\omega)\big)\Big\|_{\mathrm{H}^1_{\mathrm{mix}}(D\times D)}.\end{aligned}$$

Here, in view of (3.7), the first term and the last term on the right hand side of this estimate are of order $\mathcal{O}(\varepsilon^6)$. The second term is of order $\mathcal{O}(\varepsilon^4)$ since we only know that $\|u(\omega)-\mathbb{E}(u(\omega))\|_{\mathrm{H}^1(D)} = \mathcal{O}(\varepsilon)$ which follows by a linearization in terms of the local shape derivative, see [17] for the details. Hence, we arrive at

$$\|\operatorname{Cov}(u)-\operatorname{Cov}(u^{[2]})\|_{\mathrm{H}^1_{\mathrm{mix}}(D\times D)} \le C\varepsilon^4.$$

Taking the trace $\mathbf{x}=\mathbf{x}'$ gives finally the desired result. ∎

4. Regularity of the random solution. For ease of notation, we will drop the suffix of the solution $u^{[2]}$, i.e., we will denote the solution to (3.1) only by u. Moreover, we shall assume that the random fluctuations $(\mathbf{x},\omega)\mapsto\widetilde{h}(\mathbf{x},\omega)$ are given by a possibly infinite Karhunen-Loève expansion, that is

$$\widetilde{h}(\mathbf{x},\omega) = \sum_{k=1}^{m} h_k(\mathbf{x})Y_k(\omega) \tag{4.1}$$

where the coefficient functions $\{h_k(\mathbf{x})\}$ are pairwise orthonormal in $\mathrm{L}^2(D)$ and the random variables $\{Y_k(\omega)\}$ are assumed to be independently and uniformly distributed in $[-1/2,1/2]$. Although a finite Karhunen-Loève expansion is assumed here, we shall derive estimates which are independent of the number of terms m. This means, the situation of $m\to\infty$ shall be covered by the following theory. To that end, we have to assume that

$$\gamma_k := \|h_k\|_{\mathrm{L}^\infty(D)} < \infty \quad \text{for all } k\in\{1,2,\dots,m\} \tag{4.2}$$

and that the sequence $\{\gamma_k\}$ is always summable as $m \to \infty$, i.e. $\sum_{k=1}^{m} \gamma_k \leq c_\gamma$ independently of m.

The assumption that the random variables $\{Y_k(\omega)\}_k$ are stochastically independent implies that the pushforward measure $\mathbb{P}_{\mathbf{Y}} := \mathbb{P} \circ \mathbf{Y}^{-1}$ with respect to the measurable mapping

$$\mathbf{Y} : \Omega \to \Gamma := [-1/2, 1/2]^m, \quad \omega \mapsto \mathbf{Y}(\omega) := \big(Y_1(\omega), \dots, Y_m(\omega)\big)$$

is given by the joint density function 1. With this representation at hand, we can reformulate the stochastic problem (1.4) as a parametric, deterministic problem where, for ease of notation, we take the same function names as before. To that end, we replace the space $\mathrm{L}^2_{\mathbb{P}}(\Omega)$ by $\mathrm{L}^2(\Gamma)$ and substitute the random variables Y_k by the coordinates $y_k \in [-1/2, 1/2]$. Then, we have to seek $u \in \mathrm{L}^2(\Gamma, \mathrm{H}^1(D))$ such that

$$\begin{aligned} \int_\Gamma \Big\{ \sigma_0 \int_D \nabla u(\mathbf{x},\mathbf{y}) \nabla v(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \\ + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} \left[\frac{1}{h(\mathbf{x},\mathbf{y})} + \frac{\varepsilon}{2}\kappa(\mathbf{x}) \right] u(\mathbf{x},\mathbf{y}) v(\mathbf{x},\mathbf{y}) \,\mathrm{d}o_{\mathbf{x}} \Big\} \,\mathrm{d}\mathbf{y} \\ = \int_\Gamma \Big\{ \int_D f(\mathbf{x}) v(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} + \int_{\partial D} \frac{\varepsilon h(\mathbf{x},\mathbf{y})}{2\sigma_0} f(\mathbf{x}) v(\mathbf{x},\mathbf{y}) \,\mathrm{d}o_{\mathbf{x}} \Big\} \,\mathrm{d}\mathbf{y} \end{aligned} \tag{4.3}$$

holds for all $v \in \mathrm{L}^2(\Gamma, \mathrm{H}^1(D))$. Herein, the function $h(\mathbf{x},\mathbf{y})$ is affine in the stochastic parameter $\mathbf{y}$:

$$h(\mathbf{x},\mathbf{y}) = \overline{h}(\mathbf{x}) + \widetilde{h}(\mathbf{x},\mathbf{y}) = \overline{h}(\mathbf{x}) + \sum_{k=1}^{m} h_k(\mathbf{x}) y_k. \tag{4.4}$$

In particular, the solvability condition (1.6) is equivalent to

$$0 < h_{\min} \leq \overline{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\widetilde{h}(\mathbf{x},\mathbf{y})| \leq q\overline{h}(\mathbf{x}) \text{ for some } 0 \leq q < 1 \tag{4.5}$$

for all $\mathbf{x} \in \partial D$ and $\mathbf{y} \in \Gamma$.

The next lemma is concerned with the decay of the derivatives of the function

$$g(\mathbf{x},\mathbf{y}) := \frac{1}{h(\mathbf{x},\mathbf{y})} + \frac{\varepsilon}{2}\kappa(\mathbf{x}) \tag{4.6}$$

with respect to the stochastic parameter $\mathbf{y}$ and stress that $g(\mathbf{x},\mathbf{y})$ is well defined for $\mathbf{x} \in \partial D$ and $\mathbf{y} \in \Gamma$ provided that (3.2) holds.

LEMMA 4.1. *Let* (4.2), (4.4), *and* (4.5) *hold. Then, the derivatives of* $g(\mathbf{x},\mathbf{y})$ *satisfy*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} g(\mathbf{x},\mathbf{y})\|_{\mathrm{L}^\infty(\partial D)} \leq \frac{|\boldsymbol{\alpha}|!}{((1-q)h_{\min})^{|\boldsymbol{\alpha}|+1}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}$$

for all $\boldsymbol{\alpha} \in \mathbb{N}^m$ *and for all* $\mathbf{y} \in \Gamma$. *Here,* $\boldsymbol{\gamma}^{\boldsymbol{\alpha}}$ *has to be understood as the product* $\prod_{k=1}^{m} \gamma_k^{\alpha_k}$.

Proof. Due to (4.4), we conclude

$$\partial_{y_k}^m h(\mathbf{x},\mathbf{y}) = \begin{cases} h_k(\mathbf{x}), & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases} \tag{4.7}$$

It thus follows by the chain rule that

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} g(\mathbf{x},\mathbf{y}) = (-1)^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}|! \frac{\prod_{k=1}^{m}(h_k(\mathbf{x}))^{\alpha_k}}{h(\mathbf{x},\mathbf{y})^{|\boldsymbol{\alpha}|+1}} \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{N}^m.$$

Hence, because of

$$h(\mathbf{x},\mathbf{y}) \geq (1-q)\overline{h}(\mathbf{x}) \geq (1-q)h_{\min} > 0$$

for all $\mathbf{x} \in \partial D$ and for all $\mathbf{y} \in \Gamma$, cf. (4.5), we derive the assertion (4.2) for all $\boldsymbol{\alpha} \in \mathbb{N}^m$:

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} g(\mathbf{x},\mathbf{y})\|_{\mathrm{L}^\infty(\partial D)} \leq |\boldsymbol{\alpha}|! \frac{\prod_{k=1}^{m}\|h_k\|_{\mathrm{L}^\infty(\partial D)}^{\alpha_k}}{((1-q)h_{\min})^{|\boldsymbol{\alpha}|+1}} = \frac{|\boldsymbol{\alpha}|!}{((1-q)h_{\min})^{|\boldsymbol{\alpha}|+1}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}.$$

□

With the help of this lemma, we can prove the main theorem in this section which provides estimates on the derivatives of the random solution with respect to the stochastic parameter. These estimates are robust in the scale parameter of the layer's thickness and show the analytic dependence of the random solution on the stochastic parameter $\mathbf{y}$. For the sake of readability, we drop the dependency in $\mathbf{x}$ in next theorem.

THEOREM 4.2. *Under the assumptions of Lemma 4.1, the derivatives of the solution $u \in \mathrm{L}^2(\Gamma, \mathrm{H}^1(D))$ to (4.3) satisfy the pointwise estimate*

$$\|\!|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})|\!\| \leq c_f |\boldsymbol{\alpha}|! \left(\frac{c_u}{(1-q)h_{\min}}\right)^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \tag{4.8}$$

for all $\mathbf{y} \in \Gamma$ and $\boldsymbol{\alpha} \in \mathbb{N}^m$. Herein, the constant c_f depends only on $\|f\|_{\widetilde{\mathrm{H}}^{-1}(D)}$, $\|T(f)\|_{\mathrm{L}^2(\partial D)}$ and σ_0, but not on the layer thickness ε, while the constant c_u is given by

$$c_u = 2\max\left\{\frac{1}{(1-q)h_{\min}}, \frac{2(1+q)h_{\max}}{(1-q)h_{\min}}\right\} \geq 2.$$

Proof. For $|\boldsymbol{\alpha}| = 0$, the assertion follows by straightforward modification of the proof to Theorem 3.1. For $|\boldsymbol{\alpha}| > 0$, we shall have a look at the parametrized problem (4.3) which, for given $\mathbf{y} \in \Gamma$, implies the identity

$$\sigma_0 \int_D \nabla u(\mathbf{y}) \nabla v \,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} g(\mathbf{y}) u(\mathbf{y}) v \,\mathrm{d}o = \int_D f v \,\mathrm{d}\mathbf{x} + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h(\mathbf{y}) f v \,\mathrm{d}o$$

for all $v \in \mathrm{H}^1(D)$, where the function g is given as in (4.6). Thus, in view of the Leibniz rule, differentiation with respect to $\mathbf{y}$ on both sides of this equality leads to

$$\begin{aligned} \sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) \nabla v \,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} \sum_{\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) v \,\mathrm{d}o \\ = \frac{\varepsilon}{2\sigma_0} \int_{\partial D} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} h(\mathbf{y}) f v \,\mathrm{d}o. \end{aligned} \tag{4.9}$$

Due to (4.7), the term on the right hand side of (4.9) vanishes if $|\boldsymbol{\alpha}| > 1$. Hence, we shall first consider the case $|\boldsymbol{\alpha}| = 1$ where we obtain

$$\begin{aligned} \sigma_0 \int_D \nabla \partial_{y_k} u(\mathbf{y}) \nabla v \,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} g(\mathbf{y}) \partial_{y_k} u(\mathbf{y}) v \,\mathrm{d}o \\ = -\frac{1}{\varepsilon\sigma_0} \int_{\partial D} \partial_{y_k} g(\mathbf{y}) u(\mathbf{y}) v \,\mathrm{d}o + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h_k f v \,\mathrm{d}o. \end{aligned}$$

In view of (3.3) and (3.6), the special choice $v = \partial_{y_k} u(\mathbf{y})$ yields

$$\begin{aligned}
&\min\left\{1, \frac{1}{2(1+q)h_{\max}}\right\} |\!|\!| \partial_{y_k} u(\mathbf{y}) |\!|\!|^2 \\
&\quad \le -\frac{1}{\varepsilon\sigma_0} \int_{\partial D} \partial_{y_k} g(\mathbf{y}) u(\mathbf{y}) \partial_{y_k} u(\mathbf{y}) \, \mathrm{d}o + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h_k f \partial_{y_k} u(\mathbf{y}) \, \mathrm{d}o \\
&\quad \le \|\partial_{y_k} g(\mathbf{y})\|_{\mathrm{L}^\infty(\partial D)} |\!|\!| u(\mathbf{y}) |\!|\!| |\!|\!| \partial_{y_k} u(\mathbf{y}) |\!|\!| + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}} \|h_k\|_{\mathrm{L}^\infty(\partial D)} \|T(f)\|_{\mathrm{L}^2(\partial D)} |\!|\!| \partial_{y_k} u(\mathbf{y}) |\!|\!|.
\end{aligned}$$

By possibly increasing c_f, this leads to the assertion in the case $|\boldsymbol{\alpha}| = 1$:

$$\begin{aligned}
|\!|\!| \partial_{y_k} u(\mathbf{y}) |\!|\!| &\le \frac{c_u}{2(1-q)h_{\min}} \gamma_k |\!|\!| u(\mathbf{y}) |\!|\!| + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}} \max\{1, 2(1+q)h_{\max}\} \gamma_k \|T(f)\|_{\mathrm{L}^2(\partial D)} \\
&\le c_f \frac{c_u}{(1-q)h_{\min}} \gamma_k.
\end{aligned}$$

Next, we consider the case of arbitrary multiindices $|\boldsymbol{\alpha}| > 1$, where we rewrite (4.9) in accordance with

$$\begin{aligned}
\sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) \nabla v \, \mathrm{d}\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} g(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) v \, \mathrm{d}o \\
= -\frac{1}{\varepsilon\sigma_0} \int_{\partial D} \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) v \, \mathrm{d}o.
\end{aligned}$$

We use again (3.3), (3.6) and the special choice $v = \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})$ to conclude

$$\begin{aligned}
&\min\left\{1, \frac{1}{2(1+q)h_{\max}}\right\} |\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) |\!|\!|^2 \\
&\quad \le -\frac{1}{\varepsilon\sigma_0} \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \int_{\partial D} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) \, \mathrm{d}o \\
&\quad \le \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} g(\mathbf{y})\|_{\mathrm{L}^\infty(\partial D)} |\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) |\!|\!| |\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) |\!|\!|,
\end{aligned}$$

which means that

$$|\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) |\!|\!| \le \max\{1, 2(1+q)h_{\max}\} \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} g(\mathbf{y})\|_{\mathrm{L}^\infty(\partial D)} |\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) |\!|\!|.$$

Inserting the result of Lemma 4.1 on the decay of the derivatives of g, we arrive at

$$|\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) |\!|\!| \le \frac{c_u}{2} \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \frac{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|!}{((1-q)h_{\min})^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} |\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} u(\mathbf{y}) |\!|\!|.$$

By induction, we may further estimate this expression according to

$$|\!|\!| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) |\!|\!| \le c_f \frac{c_u}{2} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \left(\frac{1}{(1-q)h_{\min}}\right)^{|\boldsymbol{\alpha}|} \sum_{\substack{\boldsymbol{\alpha}' \le \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' \ne \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} |\boldsymbol{\alpha} - \boldsymbol{\alpha}'|! |\boldsymbol{\alpha}'|! c_u^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}. \qquad (4.10)$$

We find that

$$\sum_{\substack{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}\\ \boldsymbol{\alpha}'\neq\boldsymbol{\alpha}}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}|\boldsymbol{\alpha}'|!|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|!c_u^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}=\sum_{j=0}^{|\boldsymbol{\alpha}|-1}j!(|\boldsymbol{\alpha}|-j)!c_u^{|\boldsymbol{\alpha}|-j}\sum_{\substack{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}\\ |\boldsymbol{\alpha}'|=j}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}.$$

By employing the combinatorial identity

$$\sum_{\substack{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}\\ |\boldsymbol{\alpha}'|=j}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}=\binom{|\boldsymbol{\alpha}|}{j},$$

we thus obtain

$$\begin{aligned}\sum_{\substack{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}\\ \boldsymbol{\alpha}'\neq\boldsymbol{\alpha}}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}|\boldsymbol{\alpha}'|!|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|!c_u^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}&=\sum_{j=0}^{|\boldsymbol{\alpha}|-1}j!(|\boldsymbol{\alpha}|-j)!\binom{|\boldsymbol{\alpha}|}{j}c_u^{|\boldsymbol{\alpha}|-j}\\ &=|\boldsymbol{\alpha}|!\sum_{j=0}^{|\boldsymbol{\alpha}|-1}c_u^{|\boldsymbol{\alpha}|-j}=|\boldsymbol{\alpha}|!\frac{c_u^{|\boldsymbol{\alpha}|}-1}{c_u-1}.\end{aligned}$$

Since $c_u\ge 2$, it holds that

$$\frac{c_u}{2}\frac{c_u^{|\boldsymbol{\alpha}|}-1}{c_u-1}\le c_u^{|\boldsymbol{\alpha}|}.$$

Inserting the latter two estimates into (4.10) implies finally the desired assertion for arbitrary $|\boldsymbol{\alpha}|\ge 1$. □

The decay estimate (4.8) coincides with the one which is obtained in the case of a diffusion problem with uniformly elliptic random coefficient, see e.g. [2, 9]. It is sufficient to conclude that the solution u admits an analytic extension into the complex plane with respect to each particular direction y_k (see [2]). In fact, there exists even an analytic extension with respect to the variable $\mathbf{y}$ to an appropriately chosen Bernstein ellipse (see [3, 10]). As a consequence, several approaches are available to deal with the possibly very high-dimensional stochastic parameter. For example, the anisotropic stochastic collocation method [2, 20] can be applied. Moreover, besides the Monte-Carlo method, also the quasi Monte-Carlo method produces convergence rates which are essentially independent of the stochastic dimension m provided that it holds $\gamma_k\lesssim k^{-2-\delta}$ for some $\delta>0$, see [18, 26]. In our numerical examples, we will employ a quasi Monte-Carlo method since it is easy to implement.

5. Numerical results.

5.1. The Poisson equation on a random domain. In our first example, we consider the Poisson equation

$$-\Delta u(\omega)=4 \text{ in } D_\varepsilon(\omega),\quad u(\omega)=0 \text{ on } \partial D_\varepsilon(\omega), \tag{5.1}$$

on the randomly perturbed unit disc. To treat this problem within our framework, we choose $\sigma_0=1$ and split the domain $D_\varepsilon(\omega)$ according to

$$D_\varepsilon(\omega)=D\cup L_\varepsilon(\omega)\quad\text{where}\quad L_\varepsilon(\omega)=\{\mathbf{x}+t\mathbf{n}(\mathbf{x}):0\le t<\varepsilon h(\mathbf{x},\omega),\ \mathbf{x}\in\partial D\}. \tag{5.2}$$

We shall approximately solve problem (5.1), (5.2) by employing the parametrized problem (4.3) derived in this article. To that end, we will compute the random solution's expectation

$$\mathbb{E}\big(u^{[2]}\big)(\mathbf{x}) = \int_\Gamma u^{[2]}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in D, \tag{5.3}$$

and variance

$$\mathbb{V}\big(u^{[2]}\big)(\mathbf{x}) = \int_\Gamma \big[u^{[2]}(\mathbf{x},\mathbf{y}) - \mathbb{E}\big(u^{[2]}\big)(\mathbf{x})\big]^2\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in D, \tag{5.4}$$

by the quasi Monte-Carlo method using 10000 Halton points, see [12].

We consider the specific choices $\varepsilon = 0.02, 0.04, \ldots, 0.2$ of the perturbation parameter ε to examine the asymptotic estimates given in Proposition 3.2. To that end, we choose the reference domain D as the disc of radius $1-\varepsilon$ which leads to a layer of constant thickness ε in the mean. This means that $\overline{h}(\mathbf{x}) = \mathbb{E}\big(h(\mathbf{x},\omega)\big) \equiv 1$. The random fluctuations are defined by the finite Karhunen-Loève expansion

$$h(\varphi,\omega) = 1 + \frac{1}{8}\sum_{k=0}^{5}\left\{a_k(\omega)\cos(k\varphi) + b_k(\omega)\sin(k\varphi)\right\},$$

where $0\le\varphi<2\pi$ is the polar angle of a given point $\mathbf{x}\in\partial D$ and $a_k, b_k \in [-1/2, 1/2]$ are independent and uniformly distributed random variables. Notice that we have $0.5 \le h(\varphi,\omega) \le 1.5$ for all $0\le\varphi<2\pi$ and $\omega\in\Omega$.

In order to quantify the modelling error, we compare the mean (5.3) and the variance (5.4) with the related quantities $\mathbb{E}(u)$ and $\mathbb{V}(u)$ for the original problem (5.1), (5.2). This reference solution is also determined by a quasi Monte-Carlo method based on 10000 Halton points, where each sample corresponds to a new domain and thus to a new mesh. All samples are restricted onto the *fixed* disc K of radius 0.8 (K coincides with the domain D if $\varepsilon = 0.2$) which is contained in the reference domain D for all values of ε. This is done by re-interpolating the sample onto a mesh on K consisting of about 2000 triangles. Likewise, the approximations for (5.3) and (5.4) are re-interpolated onto the mesh on the disc K.

Finally, for comparison reasons, we also apply the shape derivative approach from [17] which employs a linearization of (5.1), (5.2) by means of a shape Taylor expansion. The shape derivative approach approximates on the disc K the random solution's expectation with the order $\mathcal{O}(\varepsilon^2)$ and its variance with the order $\mathcal{O}(\varepsilon^3)$, see [17] for the details.

For all approaches under consideration, the spatial discretization consists of about 33000 continuous, piecewise linear, triangular finite elements. The numerical results are found in Figure 5.1, where we plotted the deviation (the ℓ^2-errors in the nodal values of the mesh on K) of the approximate expectation and variance from the reference solution. Proposition 3.2 predicts a modelling error of order $\mathcal{O}(\varepsilon^3)$ for the expectation (5.3) and a modelling error of order $\mathcal{O}(\varepsilon^4)$ for the variance (5.4). This asymptotic behaviour is indeed observed in Figure 5.1 (the respective graphs are labeled as "second order thin layer equation"). Although these approximation orders are higher than for the shape derivative approach (the respective graphs are labeled as "shape derivative approach"), the approximation of the expectation is somewhat less accurate if $\varepsilon > 0.08$. In case of the variance, (5.4) seems to yield better result

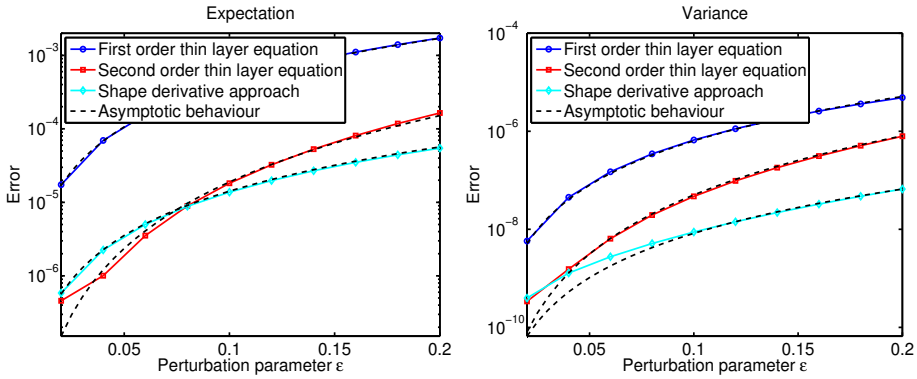

FIG. 5.1. L^2*-errors of the approximate expectation (left) and the approximate variance (right) of the random solution versus the perturbation parameter* ε.

than the shape derivative approach only for very small values of ε. Nonetheless, for $\varepsilon = 0.2$, the relative error of the expectation and of the variance is only about 0.5% and 5%, respectively. Notice finally that also the modelling errors with respect to the approximation $u^{[1]}$ from equation (1.2) are presented in these plots (the respective graphs are labeled as "first order thin layer equation"). It is clearly seen that it produces much less accurate results which are only second order accurate in ε in the case of the expectation and only third order accurate in ε in the case of the variance.

5.2. The Poisson equation with thin random layer. Our second example is dedicated to the illustration of the thin layer equation for the unit disc if the conductivity in D and L_ε is different. The layer is assumed to have the constant mean $\varepsilon = 0.01$. The conductivity is 1 in the layer and $\sigma_0 = 10$ in the unit disc. The inhomogeneity is chosen like before as $f \equiv 4$. In order to approximate the solution's mean (5.3) and variance (5.4), we use about 4000 continuous, piecewise linear, triangular finite elements for the spatial discretization and a quasi Monte-Carlo method with 10 000 Halton points for the quadrature in the stochastic parameter.

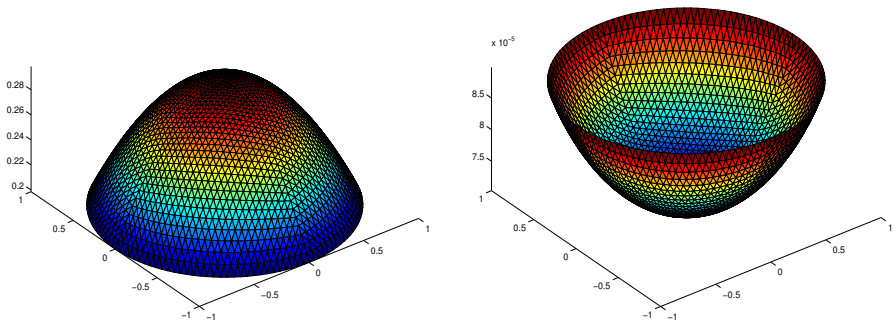

FIG. 5.2. *Expectation (left) and variance (right) of the random solution in the case of a unit disc surrounded by a random layer of average thickness* $\varepsilon = 0.01$ *and random perturbations with exponential covariance.*

The random boundary fluctuations $\widetilde{h}(\mathbf{x},\omega)$ under consideration are given according to (4.1) where we prescribe the exponential covariance kernel $\exp(-\|\mathbf{x}-\mathbf{x}'\|)/16$. The Karhunen-Loève expansion of this covariance kernel is computed with the help of the pivoted Cholesky decomposition as proposed in [15]. We obtain $m = 126$ terms if we demand the truncation error 10^{-3}. It turns out that a point on the random boundary varies at most ± 0.0065 around the nominal interface. The results of our computations are shown in Figure 5.2.

6. Conclusion. In the present article, we considered the Dirichlet boundary value problem for the Poisson equation where the domain is surrounded by a randomly varying thin layer of size $\mathcal{O}(\varepsilon)$ which has a different conductivity. We replaced this Dirichlet boundary value problem which is stated on a random domain by a random Robin boundary value problem which is stated on a deterministic domain. Decay estimates for the derivates of its random solution with respect to the stochastic parameter are provided which are required to develop appropriate solution schemes. Numerical experiments were given to validate the approach and to quantify the modelling errors.

REFERENCES

[1] B. Aslanyurek, H. Haddar, and H. Sahinturk. Generalized impedance boundary conditions for thin dielectric coatings with variable thickness. *Wave Motion*, 48(7):680–699, 2011.

[2] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.*, 45(3):1005–1034, 2007.

[3] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Mod. Methods Appl. Sci.*, 22(9):1250023, 2012.

[4] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.

[5] R.E. Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numer.*, 7:1–49, 1998.

[6] G. Caloz, M. Costabel, M. Dauge, and G. Vial. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptot. Anal.*, 50(1-2):121–173, 2006.

[7] C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. *Numer. Math.*, 107(2):257–293, 2007.

[8] J.E. Castrillon-Candas, F. Nobile, and R. Tempone. Analytic regularity and collocation approximation for PDEs with random domain deformations. *Comput. Math. Appl.*, 71:1173–1197, 2016.

[9] A. Cohen, R. DeVore and C. Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. *J. Found. Comp. Math.*, 10(6):615–646, 2010.

[10] A. Cohen, R. DeVore and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs. *Anal. Appl.*, 9(1):11–47, 2010.

[11] B. Engquist and J.-C. Nédélec. Effective boundary conditions for acoustic and electro-magnetic scattering in thin layers. Technical Report RI 278, CMAP – Ecole Polytechnique, 1993.

[12] J.H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, 2:84–90, 1960.

[13] J.M. Hammersley and D.C. Handscomb. *Monte Carlo Methods*. Chapman and Hall, London & New York, 1964.

[14] H. Harbrecht and J. Li. First order second moment analysis for stochastic interface problems based on low-rank approximation. *ESAIM Math. Model. Numer. Anal.*, 47:1533–1552, 2013.

[15] H. Harbrecht, M. Peters, and R. Schneider. On the low-rank approximation by the pivoted Cholesky decomposition. *Appl. Numer. Math.*, 62(4):428–440, 2012.

[16] H. Harbrecht, M. Peters, and M. Siebenmorgen. Analysis of the domain mapping method for elliptic diffusion problems on random domains. *Numer. Math.*, to appear.

[17] H. Harbrecht, R. Schneider, and C. Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numer. Math.*, 109(3):385–414, 2008.

[18] F.Y. Kuo and C. Schwab and I.H. Sloan. Quasi-Monte Carlo methods for high dimensional integration. The standard (weighted Hilbert space) setting and beyond. *ANZIAM J.*, 53(1):1–37, 2011.

[19] P.S. Mohan, P.B. Nair, and A.J. Keane. Stochastic projection schemes for deterministic linear elliptic partial differential equations on random domains. *Int. J. Numer. Meth. Eng.*, 85(7):874–895, 2011.

[20] F. Nobile, R. Tempone, and C.G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.* 46(5):2309–2345, 2008.

[21] C. Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Math. Methods Appl. Sci.*, 32(4):435–453, 2009.

[22] P.E. Protter. Stochastic Integration and Differential Equations. 2nd edition, Springer, Berlin, 2004.

[23] K. Schmidt and S. Tordeux. Asymptotic modelling of conductive thin sheets. *Z. Angew. Math. Phys.*, 61(4):603–626, 2010.

[24] J. Sokolowski and J.P. Zolésio. *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Springer series in computational mathematics. Springer, Berlin-Heidelberg, 1992.

[25] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements*. Springer, New York, 2008.

[26] X. Wang. A constructive approach to strong tractability using quasi-Monte Carlo algorithms. *J. Complex.*, 18:683–701, 2002.

[27] D.M. Tartakovsky and D. Xiu. Stochastic analysis of transport in tubes with rough walls. *J. Comput. Phys.*, 217(1):248–259, 2006.

[28] D. Xiu and D.M. Tartakovsky. Numerical methods for differential equations in random domains. *SIAM J. Sci. Comput.*, 28(3):1167–1185, 2006.