ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 83, No. 3 (May, 2015), 835–896

INVESTMENT AND COMPETITIVE MATCHING

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INVESTMENT AND COMPETITIVE MATCHING

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We study markets in which agents first make investments and are then matched into potentially productive partnerships. Equilibrium investments and the equilibrium matching will be efficient if agents can simultaneously negotiate investments and matches, but we focus on markets in which agents must first sink their investments before matching. Additional equilibria may arise in this sunk-investment setting, even though our matching market is competitive. These equilibria exhibit inefficiencies that we can interpret as coordination failures. All allocations satisfying a constrained efficiency property are equilibria, and the converse holds if preferences satisfy a separability condition. We identify sufficient conditions (most notably, quasiconcave utilities) for the investments of matched agents to satisfy an exchange efficiency property as well as sufficient conditions (most notably, a single crossing property) for agents to be matched positive assortatively, with these conditions then forming the core of sufficient conditions for the efficiency of equilibrium allocations.

KEYWORDS: Matching, competitive matching, investment, positive assortment.

1. INTRODUCTION

THERE ARE MANY MARKETS whose participants make investments before entering. Employers create firms before hiring employees, scientists develop inventions before taking them to market, developers construct commercial buildings and homes before finding buyers, people acquire human capital before embarking on careers, and so on. The agents in these markets are typically heterogeneous, in both their underlying characteristics and their investments, and hence the market must solve a matching problem rather than simply setting a market-clearing price. Perhaps the most obvious example is the market for skilled labor, requiring years of investment on the part of workers and the marshalling of significant physical and institutional capital on the part of firms, all before it is known who will match with whom.

The outcomes agents receive in the matching market will depend on their investments and hence will affect their investment incentives. A large literature has considered the question of how imperfections in the matching market will interact with the noncooperative nature of investment choices to yield inefficient investments. For example, Acemoglu and Shimer (1999), Cole, Mailath, and Postlewaite (2001a), de Meza and Lockwood (2010), and Felli and Roberts (2012), studied the hold-up problems (Grossman and Hart (1986) and Williamson (1985)) that can arise as a consequence of bargaining power at the matching stage. Bidner (2010), Cole, Mailath, and Postlewaite (1995),

[1]We thank Christian Hellwig, Werner Hildenbrand, George Mailath, and Karl Schlag for helpful comments and discussions. Larry Samuelson thanks the National Science Foundation (SES-1153893) for financial support. The present paper formed the basis of the Fisher–Schultz Lecture given by Larry Samuelson at the 2013 European Meeting of the Econometric Society in Gothenburg, Sweden.

 DOI: 10.3982/ECTA12349

Hopkins (2012), Hoppe, Moldovanu, and Sela (2009), and Rege (2008) studied the consequences of imperfect information at the matching stage. Burdett and Coles (2001) and Mailath, Samuelson, and Shaked (2000) studied models in which it is costly to search for a partner after one has invested.

While it is obviously important to understand how imperfections in the matching market affect incentives for investment, we examine a more basic question—even in the absence of such imperfections, can we expect investments to be efficient? We accordingly work throughout this paper with an economy whose matching market is competitive, in the sense that agents treat as fixed the utilities that must be provided to potential matching partners. In particular, we study equilibria in economies in which agents first make investments and then enter the matching market, where they form pairs whose productivity depends on their underlying characteristics as well as the investments they bring to the market. The structure of the underlying production process for a matched pair may give rise to imperfectly transferable utilities (as argued by Legros and Newman (2007b) and as in Iyigun and Walsh (2007)), and so we allow utility to be imperfectly transferable within a pair. Perfectly transferable utility is a special case. We identify when such economies will yield efficient outcomes and characterize the nature and causes of inefficiencies.

We first formulate a benchmark "ex ante" equilibrium concept in which agents can simultaneously choose investments and matching partners. Markets are complete in this economy, and forces analogous to those lying behind the familiar welfare theorems lead to the expected result that an allocation is an ex ante equilibrium if and only if it satisfies an appropriate (pairwise) efficiency condition. We then formulate an "ex post" equilibrium concept to capture the case in which investments must be sunk before matches are formed. We show that ex ante equilibria are also ex post equilibria, implying that efficient ex post equilibria exist whenever efficient allocations exist. The reasoning here is straightforward—the ex post setting affords agents fewer opportunities to deviate from a putative equilibrium allocation. Agents have no profitable deviations from an ex ante equilibrium allocation, and so must continue to have no profitable deviations from such an allocation in the ex post setting. Hence, sunk investments per se do not *preclude* efficiency in competitive markets.

Alas, not all ex post equilibria are efficient. The difficulty is that markets are incomplete—agents cannot simultaneously determine both investments and matches. There is no necessary link between competition and efficiency in the absence of complete markets. Which markets are available at the matching stage is determined endogenously by the agents' investment decisions. This gives rise to a coordination problem, with coordination failures leading to inefficient ex post equilibria. We formulate a "constrained efficiency" notion reflecting the more limited opportunities available to agents in the ex post setting, and show that all constrained efficient allocations are ex post equilibria, and that if the agents' preferences satisfy a separability condition, then all ex post equilibria are constrained efficient.

Ex post equilibria can be inefficient for any of three reasons. Matched agents may fail to coordinate on efficient investments, agents may have inadequate incentives to participate in the market, and agents may match with the "wrong" partners. We identify (independent) assumptions on the economy that suffice to eliminate each of these problems. First, we show that a quasiconcavity assumption on agents' utility functions suffices to ensure that the investments of matched agents are efficient. Second, an assumption that the optimal investments of unmatched agents allow them to be matched productively suffices to rule out inefficiencies stemming from too little participation in the market. Third, we examine mismatch in a model featuring unidimensional types and investments and satisfying our separability assumption. The first step is to show that if the utility frontiers satisfy a single crossing condition, then agents in an ex ante equilibrium must be positive assortatively matched. We then use separability and the constrained efficiency of ex post equilibria to show that the latter can be viewed as ex ante equilibria in an economy with restricted sets of possible investments, and hence must also be positive assortatively matched. We thus have conditions under which there can be no mismatch: every ex post equilibrium matches the agents just as does a pairwise efficient allocation. Finally, combining the assumptions that rule out each of the three sources of inefficiency gives us sufficient conditions for the Pareto efficiency of ex post equilibria.

The existing literature has considered the issues analyzed in this paper in a number of specific contexts. Cole, Mailath, and Postlewaite (2001b) were among the first to study the investment incentives generated by a competitive matching market, using an equilibrium concept akin to our ex post equilibrium. They considered a model with perfectly transferable utility, satisfying our single crossing and separability conditions, obtaining a counterpart of our constrained efficiency result. They identified cases in which constrained efficiency in itself eliminates the possibility that agents coordinate on inefficient investments (which is the only source of inefficiency that may arise in their setting). Dizdar (2012) noted that the efficiency result of Cole, Mailath, and Postlewaite (2001b) can fail in the absence of a counterpart to our quasiconcavity condition, and also presented examples showing that mismatch may arise in the absence of a single crossing property. Iyigun and Walsh (2007) considered a model in which consumption sharing within a match may give rise to imperfectly transferable utility, and argued that (their counterpart to) ex post equilibria are efficient. We explain in Section 2.1.3 how these and other examples fit into our framework. Our analysis unifies and extends these existing studies of investment in competitive matching markets, characterizing the nature and causes of inefficiency and identifying conditions under which equilibrium outcomes will be efficient.

Peters and Siow (2002) assumed that it is *impossible* to transfer utility ex post. In Nöldeke and Samuelson (2014), we explained how our analysis can be extended to the nontransferable case. Most of our results carry over, with one notable exception. In the absence of transfers, there is no counterpart to our result that agent with quasiconcave utility functions will necessarily coordinate on efficient investments. Perfect transferability is thus not critical to the primary results in the literature, but it is important that the agents have at least *some* ability to make ex post utility transfers.

2. THE MODEL

2.1. *The Economy*

2.1.1. *Agents and Preferences*

There are two distinct sets of agents that may be interpreted as buyers and sellers, firms and workers, men and women, and so on. We find it convenient to work with a consistent set of terms throughout and refer to the agents as buyers and sellers. Buyers (he) are indexed by their names i and sellers (she) by their names j. Names for both sets of agents are identically distributed on a compact set N of a Euclidean space.[2] The set N may be finite, giving us a model with finite, identical numbers of buyers and sellers that we refer to as the *finite case*, or may be infinite, in which case we assume names are distributed according to Lebesgue measure. The functions $\boldsymbol{\beta}: N \to \mathfrak{B}$ and $\boldsymbol{\sigma}: N \to \mathfrak{S}$ map each buyer i into his type $\boldsymbol{\beta}(i) \in \mathfrak{B}$ and each seller j into her type $\boldsymbol{\sigma}(j) \in \mathfrak{S}$, where $\mathfrak{B}$ and $\mathfrak{S}$ are compact subsets of a Euclidean space. The case in which names and types are unidimensional, that is, N, $\mathfrak{B}$, and $\mathfrak{S}$ are subsets of $\mathbb{R}$, is an important special case.

Each buyer chooses an investment $b \in B$ and each seller chooses an investment $s \in S$, with B and S again being (not necessarily unidimensional) compact subsets of a Euclidean space. Agents' types together with their investments determine their utility possibilities both when they stay unmatched and when they are part of a match, where a match pairs a single buyer with a single seller. A buyer of type β who chooses investment $b \in B$ and remains unmatched receives utility $\underline{U}(b, \beta)$. A seller of type σ who chooses investment $s \in S$ and remains unmatched receives utility $\underline{V}(s, \sigma)$. When a buyer of type β who chooses investment b matches with a seller of type σ who chooses investment s and the two agents agree on a transfer $t \in \mathbb{R}$, the resulting utility for the buyer is denoted by $U(b, s, \beta, \sigma, t)$ and the resulting utility for the seller by $V(s, b, \sigma, \beta, t)$.[3] It is natural to interpret the transfer t as a payment (either monetary or in terms of some consumption good) from the buyer to the seller,

[2]The assumption that the sets of names for buyers and sellers are identical and have the same measure is a convenient simplification, maintained in most of the related literature. Remark 4 in Section 2.2.3 comments on the most important implication of allowing unequal measures of buyers and sellers.

[3]These utility functions incorporate a practice that we follow whenever possible, of reversing the order of arguments in pairs of functions that have comparable roles, one from the perspective of the buyer and one from the perspective of the seller.

but we might also think of t as describing the allocation of effort in a joint production process, the allocation of consumption in a marriage, or the division of joint output.

The following assumption on the agents' utility functions is maintained throughout the paper.

ASSUMPTION 1:
(i) *The functions* $U : B \times S \times \mathfrak{B} \times \mathfrak{S} \times \mathbb{R} \to \mathbb{R}$, $V : S \times B \times \mathfrak{S} \times \mathfrak{B} \times \mathbb{R} \to \mathbb{R}$, $\underline{U} : B \times \mathfrak{B} \to \mathbb{R}$, *and* $\underline{V} : S \times \mathfrak{S} \to \mathbb{R}$ *are continuous*.
(ii) *The function* U *is strictly decreasing in* t *and for each* (b, s, β, σ) *has* $\mathbb{R}$ *as its image*.
(iii) *The function* V *is strictly increasing in* t *and for each* (s, b, σ, β) *has* $\mathbb{R}$ *as its image*.

In conjunction with our compactness assumptions, Assumption 1(i) ensures that solutions exist to the maximization problems (appearing in Sections 2.2.1 and 2.2.2) defining the utility possibilities available to pairs of matched agents. The requirement that U and V have $\mathbb{R}$ as their range in Assumptions 1(ii)–(iii) eliminates some special cases that we would otherwise have to explicitly address. The strict monotonicity properties in Assumptions 1(ii)–(iii) are consistent with the interpretation of the transfer t as a payment from the buyer to the seller.

Assumption 1, in particular the requirement that U and V are continuous and strictly monotonic in t, implies the following: for any (i) pair of types (β, σ), (ii) investments and transfer (b, s, t), and (iii) utility levels (u, v) satisfying

$$U(b, s, \beta, \sigma, t) \geq u, \quad V(s, b, \sigma, \beta, t) \geq v, \tag{1}$$

with at least one strict inequality, there exists t' such that

$$U(b, s, \beta, \sigma, t') > u, \quad V(s, b, \sigma, \beta, t') > v. \tag{2}$$

We refer to this property as the *strict Pareto property*.

The key role of transfers in our arguments is to ensure the strict Pareto property. As long as this property holds, we could just as well have allowed transfers to be multidimensional, accommodating interpretations that involve the allocation of effort in a joint production process or the allocation of multiple consumption goods.

2.1.2. *Allocations*

An allocation specifies for each buyer i a triple $(J(i), \mathbf{b}(i), \mathbf{u}(i))$ identifying the seller $J(i)$ (if any) with whom buyer i is matched and otherwise specifying that buyer i is unmatched ($J(i) = \emptyset$), the investment $\mathbf{b}(i)$ chosen by buyer i, and the level of utility $\mathbf{u}(i)$ received by buyer i. An allocation also specifies an analogous triple $(I(j), \mathbf{s}(j), \mathbf{v}(j))$ for each seller j.

DEFINITION 1: An allocation is a sextuple $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ of functions

$$\begin{aligned} &J : N \to N \cup \emptyset, \\ &I : N \to N \cup \emptyset, \\ &\mathbf{b} : N \to B, \\ &\mathbf{s} : N \to S, \\ &\mathbf{u} : N \to \mathbb{R}, \\ &\mathbf{v} : N \to \mathbb{R}. \end{aligned}$$

An allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is feasible if

$$I\big(J(i)\big) = i \quad \forall i \in N \quad \text{s.t.} \quad J(i) \neq \emptyset, \tag{3}$$
$$J\big(I(j)\big) = j \quad \forall j \in N \quad \text{s.t.} \quad I(j) \neq \emptyset,$$

(4) I and J are measure-preserving on $\{i \in N : J(i) \neq \emptyset\}$ and $\{j \in N : I(j) \neq \emptyset\}$,

and, for all (i, j) with $J(i) = j \in N$ (or, equivalently given (3), $I(j) = i \in N$), there exists $t \in \mathbb{R}$ such that

$$\mathbf{u}(i) = U\big(\mathbf{b}(i), \mathbf{s}(j), \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big), \tag{5}$$

$$\mathbf{v}(j) = V\big(\mathbf{s}(j), \mathbf{b}(i), \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big), \tag{6}$$

and, for all $i \in N$ with $J(i) = \emptyset$ and for all $j \in N$ with $I(j) = \emptyset$,

$$\mathbf{u}(i) = \underline{U}\big(\mathbf{b}(i), \boldsymbol{\beta}(i)\big), \tag{7}$$

$$\mathbf{v}(j) = \underline{V}\big(\mathbf{s}(j), \boldsymbol{\sigma}(j)\big), \tag{8}$$

and there exist measure-preserving bijections $\hat{J} : N \to N$ and $\hat{I} : N \to N$ that are inverses and for which $\hat{J}(i) = J(i)$ whenever $J(i) \neq \emptyset$ and $\hat{I}(j) = I(j)$ whenever $I(j) \neq \emptyset$.

Conditions (3)–(4) are the market balance conditions that matches are reciprocal and that any measurable set of buyers is matched with an equal-measure set of sellers. Conditions (5)–(6) ensure that the utility levels of matched agents are feasible given the investments and utility functions. Conditions (7)–(8) ensure that the utilities of unmatched agents are feasible. The final requirement, that there exist measure-preserving bijections $\hat{J}$ and $\hat{I}$ coinciding with J and I for matched agents, is a technical condition excluding counterintuitive constructions that arise out of the quirks of the continuum.[4] Intuitively, this final condition requires that a buyer can be unmatched only if some seller is also unmatched. We could interpret this by thinking of a first stage described by $\hat{J}$ and $\hat{I}$ in which the buyers and sellers are completely sorted into potential pairs, with some such pairs then electing to remain unmatched while the remaining matches are described by J and I. In the finite case, this requirement and condition (4) are satisfied by any pair of functions J and I satisfying (3).

REMARK 1: We have formulated our feasibility condition as a collection of pointwise requirements—feasibility places restrictions on the match of *every* agent and on the utility of *every* agent—rather than defining feasibility in terms of conditions that are required to hold only for almost all agents. These two approaches to feasibility coincide in the finite case, and we view it appropriate to use whichever most conveniently addresses the questions of interest. Our formulation allows us to avoid measure-theoretic technicalities. In particular, having excluded some perverse cases (cf. footnote 4) by building $\hat{J}$ and $\hat{I}$ into the definition of a feasible allocation, we find that our formulation simplifies many of the arguments.

REMARK 2: An alternative approach to defining a feasible allocation is to dispense with the functions I and J, specifying who is matched with whom, and instead to characterize the matching in terms of a measure on the product space of buyers and sellers. Dizdar (2012) followed this approach and Cole, Mailath, and Postlewaite (2001b, Appendix B) also considered this possibility. When utility is perfectly transferable, this approach makes powerful techniques from the optimal transport literature (cf. Villani (2009)) available, but it is less obviously useful when utility is imperfectly transferable.

Given a feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, we let $M \subset N \times N$ identify the collection of matched pairs, so that $(i, j) \in M$ whenever $j = J(i)$ or (equivalently) $i = I(j)$ holds. For every matched pair (i, j), we can identify from (5)–(6) the transfer t made by this pair. We refer to the corresponding (b, s, t) as the *exchange* made by the pair (i, j). An alternative formulation would be to express an allocation in terms of the matching and exchanges, which would in turn imply utilities.

Let $\underline{u}(i) = \max_{b \in B} \underline{U}(b, \boldsymbol{\beta}(i))$ and $\underline{v}(j) = \max_{s \in S} \underline{V}(s, \boldsymbol{\sigma}(j))$ denote the *outside options* of buyers i and sellers j. Assumption 1(i) ensures that outside options are well defined. We refer to any $b \in B$ satisfying $\underline{u}(i) = \underline{U}(b, \boldsymbol{\beta}(i))$ as an

[4] For example, the requirement that $\hat{J}$ and $\hat{I}$ are bijections excludes the possibility that $N = [0, 1]$ and the matching is described by the identity function except for agents $\{1, 1/2, 1/3, 1/4, \ldots\}$, with buyer 1 being matched with seller 1/2, buyer 1/2 with seller 1/3, and so on. This arrangement leaves every agent matched except seller 1. We use the existence of $\hat{I}$ and $\hat{J}$ in establishing Proposition 8.

autarchy investment of buyer i and to any $s \in S$ satisfying $\underline{v}(j) = \underline{V}(s, \boldsymbol{\sigma}(j))$ as an autarchy investment of seller j.

A feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is *individually rational* if it satisfies the *individual rationality* conditions

$$\text{(9)} \qquad \mathbf{u}(i) \geq \underline{u}(i) \quad \text{for all } i \in N \quad \text{and} \quad \mathbf{v}(j) \geq \underline{v}(j) \quad \text{for all } j \in N.$$

Throughout the following, we will focus on individually rational allocations, reflecting the idea that all agents are free to remain unmatched and choose an autarchy investment. The feasible allocation that results if all agents choose to exercise this option and then receive their outside options is the *autarchy allocation*.

A feasible allocation is *fully matched* if there are no unmatched agents, that is, J (and hence also I) maps onto N. In this case, $\hat{J}$ and J coincide, as do $\hat{I}$ and I. Feasible allocations $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ and $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ are *payoff equivalent* if $\mathbf{u}'(i) = \mathbf{u}(i)$ and $\mathbf{v}'(j) = \mathbf{v}(j)$ hold for all $i, j \in N$. When names are unidimensional, an allocation is *positive assortative* (or satisfies *positive assortative matching*) if J (or, equivalently I) is the identity map, that is, higher buyers are matched with higher sellers. Positive assortative allocations are fully matched.

2.1.3. *Special Cases*

Utility is *perfectly transferable* if there exist functions $\tilde{U} : B \times S \times \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}$ and $\tilde{V} : S \times B \times \mathfrak{S} \times \mathfrak{B} \to \mathbb{R}$ such that, for all (b, s, β, σ, t), we have

$$U(b, s, \beta, \sigma, t) = \tilde{U}(b, s, \beta, \sigma) - t,$$
$$V(s, b, \sigma, \beta, t) = \tilde{V}(s, b, \sigma, \beta) + t.$$

The requirement in Assumptions 1(ii)–(iii) that U and V have all of $\mathbb{R}$ as their range is automatic when utility is perfectly transferable.

When utility is perfectly transferable, then there exists a transfer such that (5)–(6) hold if and only if $\mathbf{u}(i) + \mathbf{v}(j) = Z(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j))$, where the *value function* $Z : B \times S \times \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}$ is defined by

$$Z(b, s, \beta, \sigma) = \tilde{U}(b, s, \beta, \sigma) + \tilde{V}(s, b, \sigma, \beta).$$

Consequently, the utility possibilities available to a pair of matched agents are completely described by the value function.

Preferences are *additively separable*[5] if there are continuous functions $\hat{f}$, $\hat{g}$, $\underline{f}$, $\underline{g}$, $\mathfrak{f}$, and $\mathfrak{g}$ such that

$$U(b,s,\beta,\sigma,t)=\hat{f}(b,s,t)-\mathfrak{f}(b,\beta), \tag{10}$$

$$V(s,b,\sigma,\beta,t)=\hat{g}(s,b,t)-\mathfrak{g}(s,\sigma), \tag{11}$$

$$\underline{U}(b,\beta)=\underline{f}(b)-\mathfrak{f}(b,\beta), \tag{12}$$

$$\underline{V}(s,\sigma)=\underline{g}(s)-\mathfrak{g}(s,\sigma). \tag{13}$$

We can interpret $\hat{f}$, $\hat{g}$, $\underline{f}$, and $\underline{g}$ as return functions and $\mathfrak{f}$ and $\mathfrak{g}$ as cost-of-investment functions, so that the payoffs of agents are additively separable in returns and costs, with the former not depending on agents' types. When considering additively separable preferences, we typically assume that the unmatched return functions $\underline{f}(b)$ and $\underline{g}(s)$ appearing in (12)–(13) are identically equal to zero.[6]

When utility is perfectly transferable and preferences are additively separable, there exist functions $\tilde{f}$ and $\tilde{g}$ such that (10)–(11) can be written as

$$U(b,s,\beta,\sigma,t)=\tilde{f}(b,s)-\mathfrak{f}(b,\beta)-t, \tag{14}$$

$$V(s,b,\sigma,\beta,t)=\tilde{g}(s,b)-\mathfrak{g}(s,\sigma)+t. \tag{15}$$

We can then define the *surplus function* $z(b,s)=\tilde{f}(b,s)+\tilde{g}(b,s)$ and write the value function as

$$Z(b,s,\beta,\sigma)=z(b,s)-\mathfrak{f}(b,\beta)-\mathfrak{g}(s,\sigma). \tag{16}$$

Referring to z as the surplus function is particularly apt when we invoke the normalization $\underline{f}(b)=\underline{g}(s)=0$ for all b and s. Then $z(b,s)$ identifies the surplus created by entering a match with investments (b,s), relative to choosing the same investments but remaining unmatched. With slight abuse of terminology, we refer to any model with perfectly transferable utility which satisfies (12)–(13) and (16) as having additively separable preferences.[7]

Much of the matching literature has followed the lead of Becker (1973) in focusing on conditions under which equilibrium matchings will be positive assortative. Building on insights from Legros and Newman (2007b), Section 4.3

[5]Section 3.4.2 offers a more general definition of separability, which does not impose the additive structure appearing in (10)–(13).

[6]Doing so is without loss of generality as we may redefine the cost functions $\mathfrak{f}(b,\beta)$ and $\mathfrak{g}(s,\sigma)$ to coincide with $\underline{U}(b,s)$ and $\underline{V}(s,\sigma)$ and then redefine the return functions for matched agents by deducting $\underline{f}(b)$ from $\hat{f}(b,s,t)$ and $\underline{g}(s)$ from $\hat{g}(b,s,t)$ to obtain an equivalent model.

[7]As the value function Z contains all relevant information about the utility possibilities of matched agents, any model satisfying the stated conditions is equivalent to one in which (14)–(15) also hold. To obtain such a model, it suffices to set $\tilde{f}(b,s)=\tilde{g}(s,b)=z(b,s)/2$.

identifies an appropriate single crossing property that ensures positive assortative matching in ex post equilibria. When utility is perfectly transferable, the assumption that the value function $Z(b, s, \beta, \sigma)$ is supermodular plays an important role in ensuring this single crossing property. If we also have additive separability, then (because the sum of supermodular functions is supermodular) the value function Z will be supermodular if the functions z, $-\mathfrak{f}$, and $-\mathfrak{g}$ appearing in (16) are supermodular.

We can now indicate how several existing models, all of which focus on fully matched allocations, fit into our framework:

1. Cole, Mailath, and Postlewaite (2001b) worked with unidimensional types and investments, perfectly transferable utility, additively separable preferences, and a supermodular value function. Very similar assumptions on preferences are maintained in Cole, Mailath, and Postlewaite (2001a) and Felli and Roberts (2012), who studied the finite case (without assuming matching to be competitive).

Cole, Mailath, and Postlewaite (2001b) introduced the concepts of ex ante contracting and ex post contracting equilibria, differing in technical details but analogous to our ex ante and ex post equilibria (Sections 2.2.1 and 2.2.2 below). They showed that in their setting, ex ante contracting equilibria exist, that ex ante contracting equilibria are efficient and are also ex post contracting equilibria, and that inefficient "coordination failure" ex post contracting equilibria also exist. They obtained a counterpart to our constrained efficiency result and identified cases in which constrained efficiency in itself eliminates the possibility that agents coordinate on inefficient investments.

2. Dizdar (2012) worked with multidimensional types and investments, while otherwise maintaining the framework from Cole, Mailath, and Postlewaite (2001b). Dizdar (2012) noted that the efficiency result of Cole, Mailath, and Postlewaite (2001b) can fail in the absence of a counterpart to our quasiconcavity condition and offered a sufficient condition for matched agents to avoid coordination failures in investments, which we discuss in Section 4.1. He showed that when investments and types are multidimensional, there exist ex post equilibria featuring a different matching than the one obtained in ex ante equilibrium, an impossibility in Cole, Mailath, and Postlewaite (2001b).

3. Acemoglu (1996) worked with unidimensional types and investments, perfectly transferable utility, additively separable preferences, and a supermodular value function. Buyers, corresponding to firms in his model, are ex ante identical, which in our setting corresponds to the assumption that the function $\boldsymbol{\beta}$ is constant. Acemoglu (1996) defined the concept of a Walrasian equilibrium and showed that there is a unique Walrasian equilibrium in his model. His Walrasian equilibrium is the counterpart of a collection of prices supporting a fully matched ex ante equilibrium (cf. Section 3.3 below), and is efficient. Acemoglu (1996, footnote 7) mentioned the issue which is at the center of our paper, namely the incompleteness of markets when investments are

chosen before markets operate, but his analysis concentrates on the implications of search and bargaining frictions that do not arise in our analysis (or the other papers cited here).

4. Iyigun and Walsh (2007) worked with unidimensional types and investments. Utility functions in their model can be written as

$$U(b, s, \beta, \sigma, t) = u_1(\beta - b) + u_2\big(f(b, s) + k - t\big),$$
$$V(s, b, \sigma, \beta, t) = v_1(\sigma - s) + v_2\big(g(s, b) + k + t\big),$$
$$\underline{U}(b, \beta) = u_1(\beta - b) + u_2\big(f(b, 0)\big),$$
$$\underline{V}(s, \sigma) = v_2(\sigma - s) + v_2\big(g(s, 0)\big),$$

where $k \geq 0$ is a constant. The interpretation is that agents' types correspond to their initial wealth, which they split between consumption in the first period and an investment into a technology. This technology produces second period consumption $f(b, 0)$ for an unmatched buyer and $g(s, 0)$ for an unmatched seller (with our buyers and sellers corresponding to men and women in Iyigun and Walsh (2007)). If a buyer and a seller match, the technology yields the amount $f(b, s) + g(s, b) + 2k$ of the second period consumption good, which the matched agents can share in any way they want.[8]

Preferences in this model are additively separable with $\mathfrak{f}(b, \beta) = -u_1(\beta - b)$, $\hat{f}(b, s, t) = u_2(f(b, s) + k - t)$, and $\underline{f}(b) = u_2(f(b, 0))$ for the buyers and an analogous specification for the sellers. If the functions u_2 and v_2 were linear, this would be a model with perfectly transferable utility, but instead the functions u_1, u_2, v_1, v_2 are all assumed to be strictly concave, resulting in a model with imperfectly transferable utility. Section 4 discusses conditions under which equilibria in such a model feature positive assortative matching and are efficient, extending corresponding results in Iyigun and Walsh (2007).

2.2. *Equilibrium*

In this section, we define two equilibrium notions, ex ante equilibrium and ex post equilibrium. The technology is the same in either case, requiring that investments be chosen before matches become productive. The ex ante equilibrium concept is appropriate for situations in which bilateral contracts, specifying matching partners and utilities, can be determined before investment decisions are made, while the ex post equilibrium concept is appropriate when

[8]The model in Iyigun and Walsh (2007) does not satisfy our Assumption 1 because they assumed that first and second period consumption must be positive for all agents, implying that investments are restricted to $b \in [0, \beta]$ and $s \in [0, \sigma]$ and, similarly, that transfers are restricted to the interval $[-g(s, b), f(b, s)]$. We could write a more general (though more tedious) version of Assumption 1 that would accommodate this setting.

investments must be made before matches are determined. We are primarily interested in the latter, with the simpler concept of an ex ante equilibrium serving as a useful benchmark.

We view both equilibria as being the functional equivalent of a competitive equilibrium, with complete markets in the case of ex ante equilibria and incomplete markets in the case of ex post equilibria. The standard notion of a competitive equilibrium combines three features: (i) prices, identifying the terms under which agents can trade the goods in the economy, with each agent viewing these prices as exogenously fixed, (ii) optimization, in the form of a requirement that each agent maximize utility, given the constraints imposed by the prices, and (iii) market-clearing, requiring that the excess demands emerging from the various agents' optimization problems balance, thus ensuring feasibility.

The counterpart of prices in the equilibrium definitions we give below is a pair of utility schedules $\mathbf{u}$ and $\mathbf{v}$, with $\mathbf{u}(i)$ identifying the "utility price" at which a seller can match with buyer type i and $\mathbf{v}(j)$ identifying the utility price at which a buyer can match with seller type j. The optimization requirement is that each buyer chooses a utility maximizing exchange and partner, given the utility possibilities presented by the schedule $\mathbf{v}$ (with sellers behaving similarly). Market-clearing is captured by the requirement that agents' choices yield a feasible allocation.

The key difference between ex ante equilibrium and ex post equilibrium is that in ex post equilibrium agents take not only the utility, but also the investments, of their potential partners as given when choosing a utility maximizing exchange and partner. The latter constraint is not present in ex ante equilibrium.

2.2.1. *Ex ante Equilibrium*

To define ex ante equilibrium, we find it convenient to formalize the maximization problem faced by the agents in two stages. We describe agents as first determining their optimal exchange conditional on matching with a particular partner and providing that partner with a particular utility level, and then, given the schedule of induced utilities from matching with various partners, deciding on the optimal partner (or choosing to stay unmatched).

To make this precise, let

$$\phi(i,j,v) = \max_{(b,s,t)\in B\times S\times\mathbb{R}} U\big(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t\big) \quad \text{s.t.} \quad V\big(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t\big) \ge v, \tag{17}$$

$$\psi(j,i,u) = \max_{(s,b,t)\in S\times B\times\mathbb{R}} V\big(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t\big) \quad \text{s.t.} \quad U\big(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t\big) \ge u. \tag{18}$$

Hence, $\phi : N \times N \times \mathbb{R} \to \mathbb{R}$ identifies the maximum utility a buyer of type i can achieve when matched with a seller of type j to whom he must provide utility v. The function $\psi : N \times N \times \mathbb{R} \to \mathbb{R}$ has an analogous interpretation. Assumption 1 ensures that these functions are well defined and have the properties asserted in the following lemma. The straightforward proof is in Appendix A.

LEMMA 1: *Let Assumption 1 hold. Then for every* $(i, j) \in N^2$,
(i) *ϕ is strictly decreasing in v and ψ is strictly decreasing in u*,
(ii) *ϕ and ψ are inverse: $u = \phi(i, j, \psi(j, i, u))$ for all $u \in \mathbb{R}$ and $v = \psi(j, i, \phi(i, j, v))$ for all $v \in \mathbb{R}$, and*
(iii) *ϕ is continuous in v and ψ is continuous in u.*

The interpretation of Lemma 1(ii) is that, for a given pair of types (i, j), the functions ϕ and ψ provide two equivalent ways of describing the Pareto frontier of the set of utilities available to this pair when forming a match. The Pareto frontier is strictly decreasing (Lemma 1(i)) and continuous (Lemma 1(iii)).

To conserve on notation, if the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is such that $\sup_{j \in N} \phi(i, j, \mathbf{v}(j)) \le \underline{u}(i)$, we say that $\emptyset$ maximizes $\phi(i, j, \mathbf{v}(j))$ over the set $N \cup \{\emptyset\}$ and that $\max_{j \in N \cup \{\emptyset\}} \phi(i, j, \mathbf{v}(i)) = \underline{u}(i)$. We adopt a similar convention for ψ. This gives us a convenient way of describing the maximization problem in which buyer i maximizes his utility by either choosing a seller with whom to match or choosing to remain unmatched.

DEFINITION 2: An ex ante equilibrium is a feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ satisfying, for all $i \in N$ and $j \in N$,

$$J(i) \in \arg\max_{j \in N \cup \{\emptyset\}} \phi\big(i, j, \mathbf{v}(j)\big) \quad \text{and} \quad \mathbf{u}(i) = \max_{j \in N \cup \{\emptyset\}} \phi\big(i, j, \mathbf{v}(j)\big), \tag{19}$$

$$I(j) \in \arg\max_{i \in N \cup \{\emptyset\}} \psi\big(j, i, \mathbf{u}(i)\big) \quad \text{and} \quad \mathbf{v}(j) = \max_{i \in N \cup \{\emptyset\}} \psi\big(j, i, \mathbf{u}(i)\big). \tag{20}$$

Notice that one of the requirements for equilibrium is that the maxima in (19)–(20) exist.

The (utility)-price-taking feature of competitive equilibrium appears in the incentive constraints (19)–(20), where each buyer i (for example) views the function $\mathbf{v}$ as a constraint requiring that the match (i, j) can form only if seller j receives at least utility $\mathbf{v}(j)$ from the match.[9]

The incentive conditions (19)–(20) incorporate the individual rationality conditions (9). A fully matched allocation is an ex ante equilibrium if and only

[9]Analogous competition assumptions are maintained in Cole, Mailath, and Postlewaite (2001b), Dizdar (2012), Mailath, Postlewaite, and Samuelson (2013a, 2013b), and Peters and Siow (2002). See Section 6.2 for further discussion.

if the individual rationality conditions hold and, for all $i \in N$ and $j \in N$,

$$J(i) \in \arg\max_{j \in N} \phi\big(i, j, \mathbf{v}(j)\big) \quad \text{and} \quad \mathbf{u}(i) = \max_{j \in N} \phi\big(i, j, \mathbf{v}(j)\big), \tag{21}$$

$$I(j) \in \arg\max_{i \in N} \psi\big(j, i, \mathbf{u}(i)\big) \quad \text{and} \quad \mathbf{v}(j) = \max_{i \in N} \psi\big(j, i, \mathbf{u}(i)\big). \tag{22}$$

Conditions (19)–(20) imply that an ex ante equilibrium $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ satisfies

$$\mathbf{u}(i) = \phi\big(i, j, \mathbf{v}(j)\big) \quad \text{and} \quad \mathbf{v}(j) = \psi\big(j, i, \mathbf{u}(i)\big) \quad \forall (i, j) \in M, \tag{23}$$

so that for every matched pair (i, j), there exists a transfer t such that the equilibrium utilities $\mathbf{u}(i) = U(\mathbf{b}(i), \mathbf{s}(j), \boldsymbol{\beta}(i), \sigma(j), t)$ and $\mathbf{v}(j) = V(\mathbf{s}(j), \mathbf{b}(i), \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t)$ lie on the utility frontier defined in (17)–(18). We say that an allocation satisfying (23) is *exchange efficient* and note that exchange efficiency is a necessary condition for a feasible allocation to be an ex ante equilibrium.

More generally, we find it useful to say that (b, s, t) is *exchange efficient for the pair* (i, j) if the exchange (b, s, t) solves both of the maximization problems appearing in (17)–(18) given the utility levels induced by (b, s, t), that is,

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) = \phi\big(i, j, V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big)\big), \tag{24}$$

$$V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) = \psi\big(j, i, U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big)\big). \tag{25}$$

By Lemma 1(ii), an exchange (b, s, t) is efficient for the pair (i, j) if and only if *one* of the two conditions appearing in (24)–(25) holds. Consequently, one of these two conditions is redundant. A corresponding observation applies to the two conditions for the exchange efficiency of an allocation given in (23), and also applies to the incentive constraints (21)–(22) for a fully matched equilibrium.

When utility is perfectly transferable, the two conditions (24)–(25) for an exchange (b, s, t) to be exchange efficient for a pair (i, j) reduces to the requirement that the pair of investments (b, s) maximize the value available to these two agents, or

$$(b, s) \in \arg\max_{b' \in B, s' \in S} Z\big(b', s', \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)\big). \tag{26}$$

2.2.2. *Ex post Equilibrium*

When markets open after investments have been chosen—so that the ex post equilibrium notion is applicable—buyer i (for example) faces sellers who are characterized not only by a schedule $\mathbf{v}$ of utility levels, but also by a schedule $\mathbf{s}$ of investments. The equilibrium incentive constraint for buyer i is that i's equilibrium payoff be at least the payoff i could obtain by matching with any seller j, given any exchange $(b, \mathbf{s}(j), t)$ that gives seller j at least her equilibrium utility. Unlike the case with an ex ante equilibrium, it is irrelevant whether player i could better his equilibrium payoff by matching with seller j with an exchange (b, s, t) that preserves player j's equilibrium payoff but for which $s \neq \mathbf{s}(j)$. As a first step to define ex post equilibrium, we let

$$\breve{\phi}(i, j, s, v) = \max_{(b,t)\in B\times\mathbb{R}} U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big)$$

$$\text{s.t.} \quad V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) \geq v,$$

$$\breve{\psi}(j, i, b, u) = \max_{(s,t)\in S\times\mathbb{R}} V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big)$$

$$\text{s.t.} \quad U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) \geq u.$$

Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be a feasible allocation. Analogously to our convention for ex ante equilibrium, if $\sup_{j\in N} \breve{\phi}(i, j, \mathbf{s}(j), \mathbf{v}(j)) \leq \underline{u}(i)$, we say that ∅ maximizes $\breve{\phi}(i, j, \mathbf{s}(j), \mathbf{v}(j))$ and that the maximum in that case is $\underline{u}(i)$, with a similar convention for sellers.

DEFINITION 3: An ex post equilibrium is a feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ satisfying, for all $i \in N$ and $j \in N$,

$$J(i) \in \arg\max_{j\in N\cup\{\emptyset\}} \breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big) \quad \text{and} \tag{27}$$

$$\mathbf{u}(i) = \max_{j\in N\cup\{\emptyset\}} \breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big),$$

$$I(j) \in \arg\max_{i\in N\cup\{\emptyset\}} \breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big) \quad \text{and} \tag{28}$$

$$\mathbf{v}(j) = \max_{i\in N\cup\{\emptyset\}} \breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big).$$

Again, one of the requirements for equilibrium is that the maxima in (27)–(28) exist.

The incentive conditions (27)–(28) imply the individual rationality conditions, which are again given by (9). As we have noted, every ex ante equilibrium satisfies the exchange efficiency condition (23). Conditions (27)–(28) imply less, namely that every matched pair $(i, j) \in M$ satisfies

$$\mathbf{u}(i) = \breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big) \quad \text{and} \quad \mathbf{v}(j) = \breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big). \tag{29}$$

We refer to a feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ satisfying (29) for all matched pairs as being *conditionally exchange efficient*. An exchange (b, s, t) is conditionally exchange efficient for a pair of agents $(i, j) \in M$ if it satisfies

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) = \breve{\phi}\big(i, j, s, V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big)\big), \tag{30}$$

$$V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) = \breve{\psi}\big(j, i, b, U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big)\big). \tag{31}$$

Condition (30) indicates that, conditional on a match between i and j, there is no possibility of increasing the buyer's utility without either changing the seller's investment or reducing her utility level. The interpretation of (31) is analogous.

In contrast to the equalities defining (unconditional) exchange efficiency, it is not the case that one of the conditions appearing in (29) is redundant. The utility possibilities created by buyer i contemplating a match with seller j are different from those created by seller j contemplating a match with buyer i, as it is j's investment $\mathbf{s}(j)$ that is held constant in the former instance and i's investment $\mathbf{b}(i)$ that is held constant in the latter. For similar reasons, it is not the case that one of the conditions (30)–(31) is redundant. This is most evident when utility is perfectly transferable, in which case the conditional exchange efficiency conditions (30)–(31) reduce to the requirement that both agents choose their own investment to maximize the value function Z while taking the investment of the other agent as given, that is, (b, s) satisfies

$$(32) \qquad b \in \arg\max Z\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)\big),$$

$$(33) \qquad s \in \arg\max Z\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)\big),$$

while t is arbitrary. Dizdar (2012) has noted that these conditions can be interpreted as the requirement that (b, s) is a Nash equilibrium in the *full appropriation game* in which both i and j have the value function as the payoff function. When utility is imperfectly transferable, there is no analogous simplification. In particular, we can no longer evaluate conditional exchange efficiency of investments (b, s) independently of the transfer t.

REMARK 3: Problems in which agents must invest before trading are notorious for giving rise to hold-up problems (Grossman and Hart (1986), Williamson (1985)). Felli and Roberts (2012) have studied the hold-up problem in a matching model in which agents first invest and thereafter engage in a bargaining process that prevents agents from capturing the full incremental return from a change in their investment. In contrast, the maximization problems appearing in (30) and (31) indicate that both agents in a partnership capture the incremental return from a change in their own investment. This precludes the existence of a hold-up problem in ex post equilibrium.

2.2.3. *Example*

The following example, adapted from Cole, Mailath, and Postlewaite (2001b, p. 338), illustrates the ex ante and ex post equilibrium concepts.

EXAMPLE 1: Names, types, and investments are unidimensional, with $N = [0, 1]$; with $\mathfrak{B} = \mathfrak{S} = [\gamma, \gamma + \alpha]$, where $\gamma > 0$ and $\alpha > 0$; and with $B \times S =$

$[0, \overline{b}] \times [0, \overline{s}]$, where $\overline{b}$ and $\overline{s}$ are assumed sufficiently large as not to pose constraints for the solutions of the maximization problems we consider below. Types are specified by $\boldsymbol{\beta}(i) = \gamma + \alpha i$ and $\boldsymbol{\sigma}(j) = \gamma + \alpha j$.

Utility is perfectly transferable and preferences are additively separable with the cost functions appearing in (10)–(13) given by

$$(34) \qquad \mathfrak{f}(b, \beta) = \frac{b^5}{5\beta} \quad \text{and} \quad \mathfrak{g}(s, \sigma) = \frac{s^5}{5\sigma}.$$

The return functions for unmatched agents satisfy $\underline{f}(b) = \underline{g}(s) = 0$ for all b and s, indicating that investments have no value outside a match. Autarchy investments are then zero for all agents, with resulting outside options $\underline{u}(i) = \underline{v}(j) = 0$, for all $i, j \in N$. The return functions for matched agents are given by $\hat{f}(b, s, t) = bs - t$ and $\hat{g}(s, b, t) = t - k$, where $k > 0$. The corresponding surplus and value functions are

$$(35) \qquad z(b, s) = bs - k \quad \text{and} \quad Z(b, s, \beta, \sigma) = bs - \frac{b^5}{5\beta} - \frac{s^5}{5\sigma} - k.$$

As the surplus function z is supermodular and the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ are submodular, the value function Z is supermodular.

We might interpret bs as the value of a product that is purchased by the buyer, with the buyer and seller each bearing the costs of their value-enhancing investment, given by $\mathfrak{f}(b, \beta)$ and $\mathfrak{g}(s, \sigma)$. The buyer purchases the product by making a transfer t to the seller, who bears the additional cost k whenever trade occurs. With $k = 0$, this model is a special case of the model examined by Cole, Mailath, and Postlewaite (2001b, p. 338), featuring functional forms that serve as a key example in their paper.

The assumption $k > 0$ implies that the autarchy allocation is an ex post equilibrium: From (35), the highest payoff an agent can obtain from matching with an agent on the other side of the market who refrains from investing and must be provided with his or her outside option is $-k$, so that choosing autarchy is optimal. On the other hand, even though the investments $(b, s) = (0, 0)$ satisfy the conditional exchange efficiency conditions (32)–(33) for every pair (i, j), there can be no ex post equilibrium in which the agents in a matched pair choose these investments, because any such allocation violates the individual rationality constraints. The only other solution of (32)–(33) for the pair (i, j) coincides with the solution to the exchange efficiency condition (26) and is given by

$$(36) \qquad b = \big(\boldsymbol{\beta}(i)\big)^{4/15}\big(\boldsymbol{\sigma}(j)\big)^{1/15} \quad \text{and} \quad s = \big(\boldsymbol{\beta}(i)\big)^{1/15}\big(\boldsymbol{\sigma}(j)\big)^{4/15}.$$

If the pair of agents (i, j) is matched in an (ex ante or ex post) equilibrium, it must choose these investments. In particular, every ex post equilibrium is exchange efficient.

Let us consider fully matched allocations. The submodularity of the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ implies that higher types of agents will choose larger equilibrium investments, while the supermodularity of the surplus function z ensures that higher investments will be matched with higher investments. This allows us to conclude that any fully matched (ex ante or ex post) equilibrium will be positive assortative, that is, each buyer i is matched with seller $j = i$. (Proposition 7 in Section 4.3.1 offers a general version of this result.) From (36), we thus obtain that investments in any fully matched (ex ante or ex post) equilibrium are given by

$$\mathbf{b}(i) = \big(\boldsymbol{\beta}(i)\big)^{1/3} \quad \text{and} \quad \mathbf{s}(j) = \big(\boldsymbol{\sigma}(j)\big)^{1/3}. \tag{37}$$

Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be a positive assortative and exchange efficient allocation. Such an allocation will be an ex ante equilibrium if and only if it satisfies the individual rationality conditions (9), which here reduce to $\mathbf{u}(i) \geq 0$ and $\mathbf{v}(j) \geq 0$, and the incentive conditions (21)–(22). As the two incentive conditions are equivalent, we can focus on (21). Because $J(i) = i$, this can be rewritten as

$$\mathbf{u}(i) = \phi\big(i, i, \mathbf{v}(i)\big) = \max_{0\leq j\leq 1} \phi\big(i, j, \mathbf{v}(j)\big). \tag{38}$$

Using (36), we can determine

$$\phi(i, j, v) = \frac{3}{5}\big(\boldsymbol{\beta}(i)\big)^{1/3}\big(\boldsymbol{\sigma}(j)\big)^{1/3} - k - v$$

and then use familiar incentive compatibility arguments to solve (38) for

$$\mathbf{u}(i) = \frac{3}{10}\big(\boldsymbol{\beta}(i)\big)^{2/3} - k/2 - \theta, \tag{39}$$

$$\mathbf{v}(j) = \frac{3}{10}\big(\boldsymbol{\sigma}(j)\big)^{2/3} - k/2 + \theta, \tag{40}$$

where θ is a constant. Because these utility schedules are strictly increasing in names, the individual rationality condition is satisfied if and only if $\mathbf{u}(0) \geq 0$ and $\mathbf{v}(0) \geq 0$ holds. Recalling that we have defined $\boldsymbol{\beta}(0) = \boldsymbol{\sigma}(0) = \gamma$, individual rationality thus requires

$$\frac{3}{5}\gamma^{2/3} + 2\theta \geq k \quad \text{and} \quad \frac{3}{5}\gamma^{2/3} - 2\theta \geq k. \tag{41}$$

In particular, a fully matched ex ante equilibrium exists if and only if $\frac{3}{5}\gamma^{2/3} \geq k$ holds. If this inequality holds strictly, all ex ante equilibria are fully matched.[10]

When is the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ with $J(i) = i$ and investments given by (37) an ex post equilibrium? Given that the allocation is positive assortative (and thus fully matched), we can rewrite the incentive conditions for ex post equilibrium in a manner analogous to (21)–(22) to obtain

$$\mathbf{u}(i) = \breve{\phi}\big(i, i, \mathbf{s}(i), \mathbf{v}(i)\big) = \max_{0\leq j\leq 1} \breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big), \tag{42}$$

$$\mathbf{v}(j) = \breve{\psi}\big(j, j, \mathbf{b}(j), \mathbf{u}(j)\big) = \max_{0\leq i\leq 1} \breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big). \tag{43}$$

Solving the maximization problem embedded in the definition of the functions $\breve{\phi}$ and $\breve{\psi}$ for the investments given by (37), delivers

$$\breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big) = \frac{4}{5}\big(\boldsymbol{\beta}(i)\big)^{1/4}\big(\boldsymbol{\sigma}(j)\big)^{5/12} - \frac{1}{5}\big(\boldsymbol{\sigma}(j)\big)^{2/3} - k - \mathbf{v}(j),$$

$$\breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big) = \frac{4}{5}\big(\boldsymbol{\beta}(i)\big)^{5/12}\big(\boldsymbol{\sigma}(j)\big)^{1/4} - \frac{1}{5}\big(\boldsymbol{\beta}(i)\big)^{2/3} - k - \mathbf{u}(j).$$

Using these expressions to solve (42)–(43) shows that these conditions are satisfied if and only if (39)–(40) hold. We can thus conclude that the set of fully matched ex post equilibria coincides with the set of fully matched ex ante equilibria. As we have noted above, there exists an additional, Pareto inefficient, ex post equilibrium, namely the autarchy allocation.

EXAMPLE 2: Consider the model of Example 1, but with $k = 0$, as in Cole, Mailath, and Postlewaite (2001b, p. 338). Then there exists a collection of exchange inefficient ex post equilibria that are payoff equivalent to the autarchic allocation, in which some agents match but choose zero investments. Ex post equilibria that are not exchange efficient are not pathological. Another instance appears in Example 3. Nöldeke and Samuelson (2014) provided further examples.

REMARK 4: When condition (41) holds as a strict inequality, a continuum of fully matched equilibria arises out of the ability to split the value $\frac{3}{5}\gamma^{2/3} - k > 0$ between the two bottom types in any way that respects their individual rationality conditions. This multiplicity arises from our assumption that there are equal masses of buyers and sellers. If we generalized the model to allow there to be more sellers than buyers (for example), then the shortage of buyers would

[10]In the case $\frac{3}{5}\gamma^{2/3} = k$, the fully matched ex ante equilibrium is unique, but there exists one additional ex ante equilibrium. This differs from the fully matched one only in that agents $i = j = 0$ choose to stay unmatched.

push surplus toward buyers, and there would be a unique ex ante equilibrium in which θ is determined by the condition $\mathbf{v}(0) = 0$. We are thus dealing with a nongeneric case, but nothing in our analysis exploits this nongenericity.

3. EFFICIENCY PROPERTIES OF EQUILIBRIA

Section 3.1 shows that a feasible allocation is an ex ante equilibrium if and only if it satisfies *pairwise efficiency*, a refinement of Pareto efficiency that we define in Section 3.1. This gives us the counterparts of the standard welfare theorems for ex ante equilibrium, as one would expect of a competitive economy with complete markets. Section 3.2 shows that a feasible allocation is an ex post equilibrium if and only if it satisfies *pairwise conditional efficiency*. As the name suggests, pairwise conditional efficiency is weaker than pairwise efficiency, with the difference between the two concepts reflecting the possibility of coordination failures in the choice of investments. Section 3.3 shows that the failure of pairwise efficiency in ex post equilibria can alternatively be interpreted as reflecting the existence of too few prices.

Section 3.4 introduces a *pairwise constrained efficiency* notion, stronger than pairwise conditional efficiency but weaker than pairwise efficiency,[11] and a property of the agents' preferences that we refer to as separability. Section 3.4.3 presents one of our main results: if preferences are separable, then every ex post equilibrium is pairwise constrained efficient. This generalizes a corresponding result of Cole, Mailath, and Postlewaite (2001b, Lemma 2). As we discuss in Section 3.4.4, constrained efficiency links the inefficiencies that can arise in ex post equilibrium to the (lack of) heterogeneity of equilibrium investment choices.

Figure 1 presents a summary of these results that may be helpful as the analysis proceeds.

3.1. *Ex ante Equilibrium and Pairwise Efficiency*

Our point of departure is a notion of Pareto efficiency, requiring that it is not possible to construct a Pareto improvement by changing the allocation for a finite set of agents.[12]

[11] Our terminology here follows Cole, Mailath, and Postlewaite (2001b, p. 356), who described a corresponding property as "efficient in a constrained sense." In contrast, Felli and Roberts (2012) said that an investment is "constrained efficient" if it maximizes the value available in a match, conditional on holding fixed the identities of the agents in the match and the investment of the other agent. The counterpart of this notion in our terminology is conditional exchange efficiency.

[12] Our formulation is similar in spirit to notions examined by Kaneko and Wooders, who explored core concepts for economics with an infinite number of agents based on finite blocking coalitions (e.g., Kaneko and Wooders (1996)). An alternative approach to Pareto efficiency with infinite sets of agents is to follow Aumann (1964) in requiring a Pareto superior allocation to make a positive measure of agents better off.

$$
\begin{aligned}
\text{Ex ante equilibrium} \quad &\Longleftrightarrow \quad \text{Pairwise efficiency} \\
&\Longrightarrow \quad \text{Pairwise constrained efficiency} \\
&\Longrightarrow \quad \text{Pairwise conditional efficiency} \\
&\Longleftrightarrow \quad \text{Ex post equilibrium.}
\end{aligned}
$$

FIGURE 1.—Summary of Propositions 1–2 and Corollaries 1 and 3. Proposition 4 shows that if we also impose separability, then ex post equilibria are pairwise constrained efficient.

DEFINITION 4: A feasible allocation $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ is a *finite Pareto improvement* on the feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ if both allocations agree except for a finite set of agents and

$$
\begin{aligned}
&\mathbf{u}'(i) \geq \mathbf{u}(i) \quad \forall i \in N, \\
&\mathbf{v}'(i) \geq \mathbf{v}(j) \quad \forall j \in N,
\end{aligned}
$$

with a strict inequality for at least one i or j. A feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is Pareto efficient if it allows no finite Pareto improvements.

It is immediate from this definition that Pareto efficient allocations are exchange efficient.

If the sets of buyers and sellers are finite, the restriction to allocations that differ only for finitely many agents has no effect and Pareto efficiency as defined here is the standard definition. One might consider simply applying the standard definition of Pareto efficiency—without the finiteness restriction—to cases with infinitely many buyers and sellers, but one can then exploit the continuum to obtain counterintuitive results.[13]

There may exist Pareto efficient, individually rational allocations in which the matching differs from the matching of any ex ante equilibrium. For example, suppose that half of the buyers have high types and half have low types, with a similar division for sellers. There are no investments, utility is perfectly transferable, and outside options are zero. A match between two low agents produces a zero value, a match between a low and a high agent produces

[13]For example, consider the fully matched ex ante equilibrium of Example 1. Consider the set $\{\dots, \frac{1}{2} - \frac{7}{16}, \frac{1}{2} - \frac{3}{8}, \frac{1}{2} - \frac{1}{4}, \frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{3}{8}, \frac{1}{2} + \frac{7}{16}, \dots\}$. In the equilibrium, each buyer with a name from this set is matched with a seller whose name is identical. Now suppose that we leave the equilibrium matching unchanged, except that each buyer from this set is matched with a seller whose name is the next higher name from this set. Transfers can then be arranged so that every agent in this set is better off (and no other agent worse off), ensuring that the equilibrium fails the standard test of Pareto efficiency. However, this is not a finite Pareto improvement, and the equilibrium satisfies our definition of Pareto efficiency.

value 1, and a match between two high agents produces value 4. Then the allocation in which low buyers are matched with high sellers and high buyers with low sellers, with the value shared equally within each partnership, is Pareto efficient.[14] However, ex ante equilibrium requires that high buyers match with high sellers.

We examine the following refinement of Pareto efficiency:

DEFINITION 5: A feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is *pairwise efficient* if it is individually rational and

$$\mathbf{u}(i) \geq \phi(i, j, \mathbf{v}(j)) \quad \forall (i, j) \in N^2, \tag{44}$$

$$\mathbf{v}(j) \geq \psi(j, i, \mathbf{u}(i)) \quad \forall (i, j) \in N^2. \tag{45}$$

Pairwise efficiency again obviously implies exchange efficiency, and we can view pairwise efficiency as augmenting the conditions for exchange efficiency (placing restrictions on the payoffs of matched pairs of agents) with a stability requirement (imposing restrictions on the payoffs attainable from matching with some other agent) that is familiar from the literature on matching problems without investments (Gale and Shapley (1962), Roth and Sotomayor (1990)).

REMARK 5: To see that pairwise efficiency is a refinement of Pareto efficiency, suppose that the individually rational feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is not Pareto efficient. Then there exists an alternative feasible allocation $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ with $\mathbf{u}'(i) \geq \mathbf{u}(i)$ and $\mathbf{v}'(j) \geq \mathbf{v}(j)$ for all i and j and (we can assume, with the case of a seller being analogous) a buyer i' such that $\mathbf{u}'(i') > \mathbf{u}(i')$. Because $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is individually rational, buyer i' is matched in allocation $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$. Let $j' = J'(i')$. Then we have

$$\phi(i', j', \mathbf{v}(j')) \geq \phi(i', j', \mathbf{v}'(j')) \geq \mathbf{u}'(i') > \mathbf{u}(i'),$$

where the first inequality follows from Lemma 1 and $\mathbf{v}'(j') \geq \mathbf{v}(y')$, and the second inequality holds because $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ is feasible. We thus have $\phi(i', j', \mathbf{v}(j')) > \mathbf{u}(i')$, ensuring that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ fails (44) and hence is not pairwise efficient.

The following is straightforward:

PROPOSITION 1: *Let Assumption 1 hold. Then a feasible allocation is pairwise efficient if and only if it is an ex ante equilibrium.*

[14]The Pareto efficiency of this allocation hinges on our assumptions that only transfers within a match are feasible. If unrestricted transfers were possible, then every Pareto efficient allocation features the same matching as the ex ante equilibrium.

PROOF: First, let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be an ex ante equilibrium. Then the second component of (19) implies (44) and the second component of (20) implies (45). As ex ante equilibria are feasible and individually rational, it follows that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is pairwise efficient.

Conversely, let the feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be pairwise efficient. Then individual rationality holds by definition, while conditions (44)–(45) give

$$\mathbf{u}(i) \geq \sup_{j \in N} \phi\big(i, j, \mathbf{v}(j)\big),$$

$$\mathbf{v}(j) \geq \sup_{i \in N} \psi\big(j, i, \mathbf{u}(j)\big).$$

Conditions (5)–(8) in the definition of feasibility ensure that, for each of these inequalities, either (i) the supremum is attained and the condition holds with equality, or (ii) the supremum is not attained and the agent in question is unmatched. This implies the incentive constraints (19)–(20), ensuring that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is an ex ante equilibrium. *Q.E.D.*

3.2. *Ex post Equilibrium and Pairwise Conditional Efficiency*

We will link ex post equilibria to the following efficiency notion.

DEFINITION 6: A feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is pairwise conditionally efficient if it is individually rational and

$$\mathbf{u}(i) \geq \breve{\phi}\big(i, j, \mathbf{s}(j), \mathbf{v}(j)\big) \quad \forall (i, j) \in N^2, \tag{46}$$

$$\mathbf{v}(j) \geq \breve{\psi}\big(j, i, \mathbf{b}(i), \mathbf{u}(i)\big) \quad \forall (i, j) \in N^2. \tag{47}$$

The modifier "conditional" captures the idea that each agent's payoff satisfies an efficiency criterion *given* the investments of the agents on the other side of the market. We can view pairwise conditional efficiency as the coupling of conditional exchange efficiency with a stability requirement. The (omitted) proof of the following is analogous to the proof of Proposition 1.

PROPOSITION 2: *Let Assumption* 1 *hold*. *Then a feasible allocation is pairwise conditionally efficient if and only if it is an ex post equilibrium*.

Upon observing that conditions (44)–(45) in the definition of pairwise efficiency can be rewritten as

$$\mathbf{u}(i) \geq \breve{\phi}\big(i, j, s, \mathbf{v}(j)\big) \quad \forall s \in S, (i, j) \in N^2, \tag{48}$$

$$\mathbf{v}(j) \geq \breve{\psi}\big(j, i, b, \mathbf{u}(j)\big) \quad \forall b \in B, (i, j) \in N^2, \tag{49}$$

it is immediate that pairwise efficiency implies pairwise conditional efficiency. Combining this observation with Propositions 1 and 2, we obtain the following:

COROLLARY 1: *Let Assumption 1 hold. Then:*
(i) *Every pairwise efficient allocation is also pairwise conditionally efficient.*
(ii) *Every ex ante equilibrium* $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ *is also an ex post equilibrium.*

Corollary 1 gives us the counterpart of one of the welfare theorems for the relationship between pairwise efficient allocations and ex post equilibria, namely that pairwise efficient allocations are ex post equilibria. This ensures that whenever a pairwise efficient allocation exists, then a pairwise efficient ex post equilibrium also exists. Combining Corollary 1 with Remark 5, we see that incomplete markets, arising here out of the fact that investments must be chosen before matches are formed, do not preclude pairwise or Pareto efficiency.

Pairwise inefficient ex post equilibria may also exist. We may view such equilibria as arising from coordination failures in the choice of investments: If pairwise efficiency fails, there is a pair (i, j), perhaps matched to each other or perhaps not, and an exchange (b, s, t) that would make both i and j better off. However, realizing the increased payoffs promised by the exchange (b, s, t) requires that *both* agents choose different investments.

The interpretation of pairwise inefficiencies as coordination failures can be vividly illustrated by considering the case of one-sided investment. Suppose that S is a singleton and hence only buyers make investments (the argument similarly applies to the case in which B is a singleton). Then ϕ and $\breve{\phi}$ are identical, in the sense that for all (i, j, v), we have $\phi(i, j, v) = \breve{\phi}(i, j, s, v)$, where s is the sole element of S. Consequently, (44) and (46) are equivalent. As (44) is in turn equivalent to (45), it follows that every ex post equilibrium is an ex ante equilibrium. There is no coordination to be done in this case, and hence no coordination failures. We thus have the following:

COROLLARY 2: *Let Assumption 1 hold. If either B or S is a singleton, then every ex post equilibrium is pairwise efficient.*

3.3. *Missing Prices*

This section provides an alternative interpretation of our equilibrium notions along the lines suggested by the literature on hedonic pricing.[15] Agents face prices, specifying transfers, and an equilibrium price function causes the quantity demanded for each possible match to equal the quantity supplied. This allows us to provide an alternative interpretation of the coordination failures that lie behind ex post equilibria that are not pairwise efficient, this time as

[15]The literature on hedonic pricing, with early contributions by Becker (1965), Houthakker (1952), Lancaster (1966), and Muth (1966) and a classic exposition by Rosen (1974), is centered around the idea that goods can be defined as bundles of attributes. Hedonic equilibria in competitive matching models with multidimensional types and perfectly transferable utility have been studied by Ekeland (2010a).

a reflection of incomplete markets. For convenience, we focus on fully matched allocations.

Prices are given by a function $t(b, \beta, s, \sigma)$, with the interpretation that buyer i with investment b and type $\boldsymbol{\beta}(i)$ can buy any match $(b, \boldsymbol{\beta}(i), s, \sigma)$ by paying $t(b, \boldsymbol{\beta}(i), s, \sigma)$, with a similar provision for sellers. We make no assumptions about the sign of $t(b, \beta, s, \sigma)$.

We say that a feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ can be supported by prices $t(b, \beta, s, \sigma)$ if an auctioneer or market maker could post such prices, offering to buy or sell a match to any agent at the posted price, and have the resulting optimizations on the part of the agents yield the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$. A hedonic equilibrium is a feasible allocation that can be supported by prices, together with its supporting prices.

We must pay some attention to the domain of the price function $t(b, s, \beta, \sigma)$. Following Mailath, Postlewaite, and Samuelson (2013b, p. 547), we say that prices are *complete* if t is defined on the domain $B \times \mathfrak{B} \times S \times \mathfrak{S}$. For a given allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, we say that a function $t(b, \beta, s, \sigma)$ is a specification of *ex post* prices if the domain of this function is $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$, where

$$\mathbb{S} = \{(s, \sigma) \in S \times \mathfrak{S} : s = \mathbf{s}(j), \sigma = \boldsymbol{\sigma}(j) \text{ for some } j \in N\},$$

$$\mathbb{B} = \{(b, \beta) \in B \times \mathfrak{B} : b = \mathbf{b}(i), \beta = \boldsymbol{\beta}(i) \text{ for some } i \in N\}.$$

DEFINITION 7: A fully matched feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is supported by complete prices $t : B \times \mathfrak{B} \times S \times \mathfrak{S} \to \mathbb{R}$ if, for all buyers $i \in N$,

$$(50) \qquad \big(\mathbf{b}(i), \mathbf{s}(J(i)), \sigma(J(i))\big) \in \operatorname*{arg\,max}_{(b,s,\sigma)\in B\times S\times\mathfrak{S}} U\big(b, s, \boldsymbol{\beta}(i), \sigma, t(b, \boldsymbol{\beta}(i), s, \sigma)\big),$$

$$(51) \qquad \mathbf{u}(i) = \max_{(b,s,\sigma)\in B\times S\times\mathfrak{S}} U\big(b, s, \boldsymbol{\beta}(i), \sigma, t(b, \boldsymbol{\beta}(i), s, \sigma)\big) \geq \underline{u}(i),$$

with an analogous condition holding for all sellers $j \in N$.

A fully matched feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is supported by ex post prices $t : (B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S}) \to \mathbb{R}$ if, for all buyers $i \in N$,

$$\big(\mathbf{b}(i), \mathbf{s}(J(i)), \boldsymbol{\sigma}(J(i))\big) \in \operatorname*{arg\,max}_{(b,s,\sigma)\in B\times\mathbb{S}} U\big(b, s, \boldsymbol{\beta}(i), \sigma, t(b, \boldsymbol{\beta}(i), s, \sigma)\big),$$

$$\mathbf{u}(i) = \max_{(b,s,\sigma)\in B\times\mathbb{S}} U\big(b, s, \boldsymbol{\beta}(i), \sigma, t(b, \boldsymbol{\beta}(i), s, \sigma)\big) \geq \underline{u}(i),$$

with an analogous condition holding for all sellers $j \in N$.

Complete prices attach a price to every possible combination (b, β, s, σ) of investments and types. A price function defined on the restricted domain $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$ gives us just enough prices to evaluate the maximization problems that appear in the definition of ex post equilibrium. For example, given a candidate equilibrium $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, a buyer of type β can consider any investment $b \in B$, but can consider matches only with seller investments s and types σ satisfying $(s, \sigma) \in \mathbb{S}$. In order to attach prices to such choices, we need prices defined on $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$.

PROPOSITION 3: *Let Assumption 1 hold.*

(i) *A fully matched feasible allocation can be supported by complete prices if and only if it is a fully matched ex ante equilibrium.*

(ii) *A fully matched feasible allocation can be supported by ex post prices if and only if it is a fully matched ex post equilibrium.*

PROOF: (i) Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be a fully matched ex ante equilibrium. We can in general expect an allocation supported by prices to be supported by a variety of price functions, with the individual rationality constraints identifying the bounds placed on such functions by the option of not participating in the market. We construct one such price function. For every (b, β, s, σ) with the property that there exists a buyer i' with $\boldsymbol{\beta}(i') = \beta$, we let the price $t(b, \beta, s, \sigma) = t(b, \boldsymbol{\beta}(i'), s, \sigma)$ satisfy (the existence of a solution to the following equation is implied by Assumption 1):

$$U\big(b, s, \boldsymbol{\beta}(i'), \sigma, t\big(b, \boldsymbol{\beta}(i'), s, \sigma\big)\big) = \mathbf{u}(i').$$

This price is well defined: if there are buyers i and i' with $\beta = \boldsymbol{\beta}(i) = \boldsymbol{\beta}(i')$, then the incentive constraints imply that, in equilibrium, we must have $\mathbf{u}(i) = \mathbf{u}(i')$. For those (b, β, s, σ) for which there exists no i with $\boldsymbol{\beta}(i) = \beta$, let $t(b, \beta, s, \sigma)$ satisfy

$$V\big(s, b, \sigma, \beta, t(b, \beta, s, \sigma)\big) < \underline{V}(s, \sigma).$$

The existence of such a price is again ensured by Assumption 1.

This formulation of prices ensures that every buyer i receives payoff $\mathbf{u}(i)$ no matter what (b, s, σ) he chooses, which in turn ensures that (50)–(51) hold. Next, every seller can choose any (s, b, β) with the property that $\beta = \boldsymbol{\beta}(i')$ for some i' at a price that gives buyer i' a utility of $\mathbf{u}(i')$, whereas choosing any other (s, b, β) results in less than the seller's outside option. Hence, the optimization problem faced by seller j is equivalent to

$$\max_{i \in N} \psi\big(j, i, \mathbf{u}(i)\big),$$

which duplicates the incentive constraint (22), ensuring that the optimal choice of seller j is $(\mathbf{s}(j), \mathbf{b}(I(j)), \boldsymbol{\beta}(I(j)))$.

Conversely, let the fully matched feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be supported by complete prices. Then (50)–(51) and the corresponding seller conditions immediately give the individual rationality constraint (9). Suppose that one of the incentive constraints (19)–(20) fails, say (19), so that there exist i' and j' with

$$\mathbf{u}(i') < \phi(i', j', \mathbf{v}(j')).$$

This implies that there exist (b', s', t') for which

$$\mathbf{u}(i') < U(b', s', \boldsymbol{\beta}(i'), \boldsymbol{\sigma}(j'), t'),$$
$$\mathbf{v}(j') \leq V(s', b', \boldsymbol{\sigma}(j'), \boldsymbol{\beta}(i'), t').$$

This in turn ensures that there is no $t(b', \boldsymbol{\beta}(i'), s', \boldsymbol{\sigma}(j'))$ at which both (51) and the corresponding seller condition can be satisfied, contradicting the assumption that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is supported by complete prices.

(ii) The proof for ex post equilibria is identical, except that $\phi(i, j, \mathbf{v}(j))$ is replaced by $\breve{\phi}(i, j, \mathbf{s}(j), \mathbf{v}(j))$ and $\psi(j, i, \mathbf{u}(i))$ is replaced by $\breve{\psi}(j, i, \mathbf{b}(i), \mathbf{u}(i))$. *Q.E.D.*

The inability to support an ex post equilibrium which is not pairwise efficient with complete prices arises out of the fact that if a pair of agents with types (β', σ') strictly prefers an exchange (b', s', t') to what they obtain at equilibrium, then there is no price one could post that would discourage both sides of the market from trying to demand (resp. supply) $(b', s', \beta', \sigma')$. A seller of type σ' will be willing to choose investment s' and sell to buyer type β' with investment b' at a high price, while a buyer of type β' would like to choose b' and buy from seller σ' with investment s' at a low price. We can thus interpret a failure of pairwise efficiency in an ex post equilibrium as a problem of missing markets. Markets are "complete enough" only to ensure pairwise conditional efficiency.

Why might we have ex post rather than complete prices? Prices from the set $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$ allow a market maker to answer any inquiry from a buyer (with sellers being analogous) of the form "what if I bring b to the market, am of type β, and attempt to buy (b, β, s, σ) for some (s, σ) in the market?" A good outside the set $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$ requires a doubly counterfactual inquiry, and hence might be viewed as sufficiently unlikely as to not warrant a price. Alternatively, in a decentralized market, trades involving a departure from equilibrium behavior on the part of only a single player may be salient enough or happen often enough in the process leading to equilibrium as to generate common price expectations, but the same may not be true for doubly counterfactual goods.

The coordination-failure and missing-prices interpretations of inefficient ex post equilibria are related. For an allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ to fail pairwise efficiency, there must be a pair (i, j) and an exchange (b, s, t) that makes both better off than under the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$. The coordination difficulty is that buyer i can entertain exchange $(b, \mathbf{s}(j), t)$ and seller j can entertain

$(\mathbf{b}(i), s, t)$, but there is no way (under the ex post equilibrium concept) for them to *coordinate* on the exchange (b, s, t). The allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ can then be an ex post equilibrium but fail pairwise efficiency if neither of the exchanges $(b, \mathbf{s}(j), t)$ or $(\mathbf{b}(i), s, t)$ can make buyer i and seller j both better off, even though (b, s, t) does so.

If both agents in the pair (i, j) would be better off making the exchange (b, s, t) than they are under the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, then the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ cannot be supported by complete prices. In effect, the existence of the price $t(b, \boldsymbol{\beta}(i), s, \boldsymbol{\sigma}(j))$ solves the coordination problem for agents i and j by allowing either one of them to demand the coordinated deviation to the exchange $(b, s, t(b, \boldsymbol{\beta}(i), s, \boldsymbol{\sigma}(j)))$. However, $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ might be supported by ex post prices, because such prices specify a price $t(b, \boldsymbol{\beta}(i), \mathbf{s}(j), \boldsymbol{\sigma}(j))$ and specify a price $t(\mathbf{b}(i), \boldsymbol{\beta}(i), s, \boldsymbol{\sigma}(j))$, but do not specify a price $t(b, \boldsymbol{\beta}(i), s, \boldsymbol{\sigma}(j))$. Because the latter price is missing from the market, the "coordinated" exchange (b, s, t) is out of the agents' reach.

3.4. *Pairwise Constrained Efficiency and Separability*

3.4.1. *Pairwise Constrained Efficiency*

Pairwise efficiency and pairwise conditional efficiency both require that there be no pair of agents who could match and improve their payoffs. The notions differ in terms of the sets of investments for the agents on the other side of the market that an agent can contemplate when calculating the payoffs from a match. As indicated by condition (48), pairwise efficiency allows buyer i to consider any seller investment $s \in S$ when assessing the payoff from a match with seller j, whereas condition (46) indicates that, under pairwise conditional efficiency, buyer i can only consider investment $\mathbf{s}(j)$. Our next efficiency concept lies between those two notions. Condition (52) in the following definition requires that buyer i cannot gain by matching with seller j, given that the seller's investment must be drawn from the set of investments $\mathbf{S}$ that are chosen by some seller and hence are "in the market." Cole, Mailath, and Postlewaite (2001b, p. 356) referred to equilibria with this property as "efficient in a constrained sense," and so we refer to this notion as pairwise constrained efficiency.

DEFINITION 8: A feasible allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is pairwise constrained efficient if it is individually rational and

$$\mathbf{u}(i) \geq \breve{\phi}\big(i, j, s, \mathbf{v}(j)\big) \quad \forall s \in \mathbf{S}, (i, j) \in N^2, \tag{52}$$

$$\mathbf{v}(j) \geq \breve{\psi}\big(j, i, b, \mathbf{u}(j)\big) \quad \forall b \in \mathbf{B}, (i, j) \in N^2, \tag{53}$$

where $\mathbf{B}$ is the image of N under $\mathbf{b}$ and $\mathbf{S}$ is the image of N under $\mathbf{s}$.

The following is immediate from the definitions:

COROLLARY 3:
(i) *Pairwise efficient allocations are pairwise constrained efficient.*
(ii) *Pairwise constrained efficient allocations are pairwise conditionally efficient.*

We have summarized Propositions 1–2 and Corollaries 1 and 3 in Figure 1.

Ex post equilibria can be pairwise constrained efficient without being pairwise efficient. The pairwise inefficient autarchy equilibrium in Example 1 provides an illustration. It is constrained pairwise efficient because all buyers and all sellers make identical investments, in which case pairwise conditional efficiency implies pairwise constrained efficiency. It is straightforward to construct examples of ex post equilibria that are not pairwise constrained efficient (cf. Nöldeke and Samuelson (2014)). Hence, pairwise constrained efficiency lies strictly between pairwise conditional efficiency and pairwise efficiency—both converses that are not asserted by Figure 1 fail.

3.4.2. *Separability*

As we have indicated in Section 2.1.3, the literature on investment-and-matching problems (Cole, Mailath, and Postlewaite (2001a, 2001b), Dizdar (2012), Acemoglu (1996), Iyigun and Walsh (2007)) has focused on models with additively separable preferences.[16] The following definition of separability does not impose the additive structure appearing in (10)–(13). Intuitively, preferences are separable if agents' utilities depend on the investments their partners have chosen, but not on the types of the partners choosing those investments. A marriage market may be separable because a man (for example) may care about the wealth with which his spouse has been endowed by her parents, but not the cost at which her parents amassed such wealth. A labor market may be nonseparable because firms are willing to hire software engineers who have invested relatively little in learning the relevant programming languages, but who nonetheless have great natural talent for programming.

DEFINITION 9: Preferences are separable if there exist continuous functions $\hat{f}: B\times S\times\mathbb{R}\to\mathbb{R}$, $\hat{g}: S\times B\times\mathbb{R}\to\mathbb{R}$, $\underline{f}: B\to\mathbb{R}$, $\underline{g}: S\to\mathbb{R}$, $\hat{U}:\mathbb{R}\times B\times\mathfrak{B}\to\mathbb{R}$, and $\hat{V}:\mathbb{R}\times S\times\mathfrak{S}$ such that

$$U(b,s,\beta,\sigma,t)=\hat{U}\big(\hat{f}(b,s,t),b,\beta\big), \tag{54}$$

$$V(s,b,\sigma,\beta,t)=\hat{V}\big(\hat{g}(s,b,t),s,\sigma\big), \tag{55}$$

[16]Preferences are also additively separable in the models with perfectly transferable utility considered in Mailath, Postlewaite, and Samuelson (2013a, 2013b). Separability is less evident in Felli and Roberts (2012), but holds in an (equivalent) version of their model in which what they call the "quality" of an agent is interpreted as the agent's investment choice.

(56) $\underline{U}(b, \beta) = \hat{U}\big(\underline{f}(b), b, \beta\big),$

(57) $\underline{V}(s, \sigma) = \hat{V}\big(\underline{g}(s), s, \sigma\big),$

where $\hat{U}$ and $\hat{V}$ are strictly increasing in their first arguments.

The buyer condition (54) (for example) indicates that (i) the buyer's utility does not depend on the seller's type σ, and (ii) if we can find one buyer who prefers matching with a seller on terms (b, s', t') to matching on terms (b, s, t), then every buyer has this preference; that is, writing buyer β's preferences as $\succsim_\beta$, we have that, for all $\beta, \beta' \in \mathfrak{B}$,[17]

(58) $$(b, s, t) \succsim_\beta \big(b, s', t'\big) \iff (b, s, t) \succsim_{\beta'} \big(b, s', t'\big).$$

Notice that the buyer's investment is the same across these two pairs, so the important content of (58) is that the buyer's trade-off between s and t does not depend on the buyer's type.

It is trivial to verify that additively separable preferences are indeed separable. Further, when utility is perfectly transferable, separability implies additive separability: the representation given by (12)–(15) is not only sufficient but also necessary for separability.[18]

3.4.3. *Separability and Pairwise Constrained Efficiency*

Separability of preferences implies the pairwise constrained efficiency of ex post equilibria. The proof of the following result also shows that for fully matched ex post equilibria, we need only the first part of the definition of separability, namely (54)–(55), to obtain this conclusion.

[17]Condition (54) obviously implies (58). To see the converse, suppose (58) holds. Then we can omit σ as an argument of U, and can choose an arbitrary $\beta^* \in \mathfrak{B}$ and define $\hat{f}(b, s, t) = U(b, s, \beta^*, t)$. Now, for any triple (b, s', t'), let $y' = \hat{f}(b, s', t')$ and then define $\hat{U}(y', b, \beta) := U(b, s', \beta, t')$. To confirm that this construction is well defined, we note that if $\hat{f}(b, s', t') = \hat{f}(b, s'', t'')$, then by definition $(b, s', t') \sim_{\beta^*} (b, s'', t'')$, with (58) then ensuring that $(b, s', t') \sim_\beta (b, s'', t'')$ for any $\beta \in \mathfrak{B}$, and hence $U(b, s', \beta, t') = U(b, s'', \beta, t'')$.

[18]To see this, consider the buyers. Suppose that (54) holds and that utility is perfectly transferable. We can then omit σ as an argument of $\tilde{U}$. Choose some $s^* \in S$ and let $\tilde{U}(b, s^*, \beta) =: -\mathfrak{f}(b, \beta)$. Then choose some $\beta^* \in \mathfrak{B}$ and define $\tilde{f}(b, s) := \tilde{U}(b, s, \beta^*) - \tilde{U}(b, s^*, \beta^*)$. Using separability for the second of the following equalities, we then have

$$\begin{aligned}\tilde{U}\big(b', s', \beta'\big) &= \tilde{U}\big(b', s', \beta'\big) - \tilde{U}\big(b', s^*, \beta'\big) + \tilde{U}\big(b', s^*, \beta'\big)\\ &= \tilde{U}\big(b', s', \beta\big) - \tilde{U}\big(b', s^*, \beta\big) - \mathfrak{f}\big(b', \beta'\big)\\ &= \tilde{f}\big(b', s'\big) - \mathfrak{f}\big(b', \beta'\big),\end{aligned}$$

yielding (14). Defining $\underline{f}(b) := \underline{U}(b, \beta^*) - \tilde{U}(b, s^*, \beta^*)$ and using an analogous argument gives (12).

PROPOSITION 4: *Let Assumption 1 hold and let preferences be separable. Then ex post equilibria are pairwise constrained efficient.*

PROOF: Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be an ex post equilibrium, and suppose that it is not pairwise constrained efficient. Then there exists a pair of agents (i, j) for whom (52)–(53) fail. Suppose it is (52) that fails (with the case in which (53) fails being analogous). Then there exists a pair of investments (b, s) with $s = \mathbf{s}(j')$ for some j' and a transfer t such that

$$\hat{U}\big(\hat{f}(b, s, t), b, \boldsymbol{\beta}(i)\big) > \mathbf{u}(i), \tag{59}$$

$$\hat{V}\big(\hat{g}(s, b, t), s, \boldsymbol{\sigma}(j)\big) \geq \mathbf{v}(j). \tag{60}$$

Suppose first that seller j' is matched, and let i' be the buyer matched with seller j' and let their exchange be (b', s, t'). One of the possibilities buyer i can contemplate is to match with seller j', with exchange (b, s, t). Condition (59) ensures that the exchange (b, s, t) with seller j provides buyer i with more than his equilibrium utility, and so separability ensures that such a match with seller j' does likewise. The incentive constraints (27)–(28) for ex post equilibrium ensure that the exchange (b, s, t) decreases j''s utility, or

$$\hat{V}\big(\hat{g}(s, b, t), s, \boldsymbol{\sigma}(j')\big) < \mathbf{v}(j') = \hat{V}\big(\hat{g}(s, b', t'), s, \boldsymbol{\sigma}(j')\big). \tag{61}$$

Next, by separability, the fact that buyer i' is willing to consummate an equilibrium match featuring exchange (b', s, t') with seller j' ensures that buyer i' would also be willing to make this exchange with seller j. The incentive constraints (27)–(28) for ex post equilibrium ensure that this does not increase j's utility, or

$$\hat{V}\big(\hat{g}(s, b', t'), s, \boldsymbol{\sigma}(j)\big) \leq \mathbf{v}(j). \tag{62}$$

From (60) and (62), we have

$$\hat{V}\big(\hat{g}(s, b, t), s, \boldsymbol{\sigma}(j)\big) \geq \hat{V}\big(\hat{g}(s, b', t'), s, \boldsymbol{\sigma}(j)\big),$$

whereas (61) together with separability implies the reverse strict inequality. Hence, we have obtained a contradiction to the assumption that (59)–(60) hold.

Now suppose that seller j' is not matched. Then

$$\underline{V}\big(s, \boldsymbol{\sigma}(j')\big) = \mathbf{v}(j') > \hat{V}\big(\hat{g}(s, b, t), s, \boldsymbol{\sigma}(j')\big) \tag{63}$$

holds, where the equality is from feasibility and the strict inequality follows from separability: if it failed, buyer i and seller j' could match with exchange (b, s, t) with seller j' receiving at least her equilibrium utility $\mathbf{v}(j')$ and buyer

i receiving more than his equilibrium utility (from (59)), contradicting the incentive constraints for ex post equilibrium. By (57), the outer inequality in (63) implies

$$\underline{V}(s, \boldsymbol{\sigma}(j)) > \hat{V}(\hat{g}(s, b, t), s, \boldsymbol{\sigma}(j)),$$

whereas (60) in conjunction with the incentive constraint $\mathbf{v}(j) \geq \underline{V}(s, \boldsymbol{\sigma}(j))$ implies the reverse weak inequality. This contradiction finishes the proof. *Q.E.D.*

3.4.4. *Separability, Coordination Failures, and Heterogeneity*

The link between separability and pairwise constrained efficiency is important for two reasons. We postpone one of these to Section 4, where separability and pairwise constrained efficiency play a central role in establishing conditions for positive assortative matching. This section highlights the second reason, the role of separability in limiting the scope of coordination failures.

Section 3.2 observed that failures of pairwise efficiency of ex post equilibria can be interpreted as coordination failures in investment choices—an ex post equilibrium can only fail pairwise efficiency if (44)–(45) are violated, which means that there exist agents i and j and investments $b \neq \mathbf{b}(i)$ and $s \neq \mathbf{s}(j)$ that (when accompanied by an appropriate transfer t) would make both agents strictly better off when matching with each other. When preferences are separable, the pairwise constrained efficiency conditions (52)–(53) imply that the only coordination failures that can arise are those in which both agents in a pair (i, j) could be made better off by choosing a pair of investments (b', s') (and an appropriate transfer t') with the property that neither b' nor s' is in the market. Formally, Proposition 4 leads immediately to the following result, generalizing a corresponding result for perfectly transferable utility in Cole, Mailath, and Postlewaite (2001b, Proposition 4).

COROLLARY 4: *Let Assumption 1 hold and let preferences be separable. Suppose* $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ *is an ex post equilibrium and there exist agents i and j and an exchange* (b', s', t') *such that*

$$U(b', s', \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t') \geq \mathbf{u}(i),$$
$$V(s', b', \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t') \geq \mathbf{v}(j),$$

with at least one equality strict. Then there exists no i' for which $\mathbf{b}(i') = b'$ *and no j' with* $\mathbf{s}(j') = s'$.

The scope for coordination failures in ex post equilibrium is thus limited by the heterogeneity of investments that are actually chosen in equilibrium. The richer the sets $\mathbf{B}$ and $\mathbf{S}$ of equilibrium investments, the fewer exchanges

(b', s', t') there are with $b' \notin \mathbf{B}$ and $s' \notin \mathbf{S}$, and hence the fewer opportunities for a failure of pairwise efficiency. In particular, if the sets $\mathbf{B}$ and $\mathbf{S}$ include every investment that is chosen by some agent in a pairwise efficient allocation, the ex post equilibrium in question must be pairwise efficient. In essence, it is enough to ensure the right investments are in the market, at which point the market will ensure that they are chosen by the right agents.

For example, it is immediate from Corollary 4 that with separable preferences, any ex post equilibrium satisfying $\mathbf{B} = B$ (or $\mathbf{S} = S$, with the following discussion focusing on the first of these cases) is pairwise efficient. When might this condition be satisfied? Suppose that utility is perfectly transferable. Then fully matched ex post equilibria will satisfy $\mathbf{B} = B$ if, for every $b' \in B$, there exists a buyer i for whom choosing that investment is a dominant strategy in the full appropriation game, that is, for all $s \in S$ the investment b' is the unique investment satisfying (32). If b' is also the unique autarchy investment of buyer i or all ex post equilibria are fully matched (conditions ensuring this are discussed in Section 4.2), the pairwise efficiency of ex post equilibria follows. This dominant strategy condition is stringent, but is satisfied, for example, in Chiappori, Iyigun, and Weiss (2009).

Whether the dominant strategy condition of the previous paragraph holds can depend upon whether the agents in the economy are sufficiently heterogeneous. The following example illustrates.

EXAMPLE 3: Let $N = [a, 1]$ with $0 < a < 1/6$ and $\mathfrak{B} = \mathfrak{S} = [a, 3]$. Utility is perfectly transferable and preferences are separable. Let there be two possible investments on each side of the market, so that $B = \{L, H\}$ and $S = \{L, H\}$. The return functions $\tilde{f}(b, s)$ and $\tilde{g}(s, b)$ in (14)–(15) are given by

	L	H
L	1	2
H	2	5

,

while investment costs are 0 for an L investment and $\frac{1}{\beta}$ or $\frac{1}{\sigma}$ in the case of an H investment. The return functions for unmatched agents satisfy $\underline{f}(b) = \underline{g}(s) = 0$, so that, for all types, outside options are zero and L is the autarchy investment.

Suppose first that $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are the identity functions, so that the sets of buyer and seller types in the economy are both $[a, 1]$. Pairwise efficiency for this economy calls for all agents with types less than 1/2 to choose L investments, and all agents with types greater than 1/2 to choose H investments. Agents who choose L match with one another, as do agents who choose H, with the matching being arbitrary within these constraints. However, there is also an ex post equilibrium in which every agent chooses L. Agents with names (and hence types) greater than 1/2 are not choosing exchange efficient investments, but the equilibrium is pairwise constrained efficient. Pairwise constrained efficiency does not imply exchange efficiency in this case because the sets of investments in the market, $\mathbf{B} = \mathbf{S} = \{L\}$, are too sparse.

Suppose now that $\boldsymbol{\beta}(i) = 2i$ and $\boldsymbol{\sigma}(j) = 2j$, and so the sets of buyer and seller types in the economy are both $[2a, 2]$. As before, pairwise efficiency for this economy calls for all agents with types less than $1/2$ to choose L investments, and all agents with types greater than $1/2$ to choose H investments. Every ex post equilibrium must be fully matched (as the existence of an unmatched pair of agents leads to an immediate contradiction of the pairwise conditional efficiency of ex post equilibria). For buyers and sellers with names above $1/2$ (and thus types above 1), the ex post exchange efficiency conditions (32)–(33) now imply $\mathbf{b}(i) = \mathbf{s}(j) = H$, irrespective of the partner they are matched with and the investment chosen by that partner. Similarly, for matched buyers and sellers with names below $1/6$ (and thus types below $1/3$), every ex post equilibrium satisfies $\mathbf{b}(i) = \mathbf{s}(j) = L$. Consequently, in every ex post equilibrium all investments are in the market, that is, $\mathbf{B} = B$ and $\mathbf{S} = S$, ensuring that every pairwise constrained efficient allocation is pairwise efficient. Hence, Proposition 4 implies that every ex post equilibrium is pairwise efficient.

REMARK 6: Section 3.3 showed that we can alternatively formulate the notions of ex ante and ex post equilibria in terms of prices attached to quadruples (b, β, s, σ) of investments and types. If preferences are separable, then we can write prices simply as a function $t(b, s)$, as do Mailath, Postlewaite, and Samuelson (2013b). To see this, suppose there exist values (b, β, s) and distinct values σ and σ' such that the prices supporting an ex post equilibrium feature $t(b, \beta, s, \sigma) > t(b, \beta, s, \sigma')$. Then the good (b, β, s, σ) does not trade in equilibrium, because buyer preferences are independent of σ and hence no buyer will buy the more expensive good (b, β, s, σ). We can thus reduce the price of (b, β, s, σ) to $t(b, \beta, s, \sigma')$ without disrupting the equilibrium, and hence can assume that prices take the form $t(b, \beta, s)$. An analogous argument now ensures that prices also need not depend on β. Mailath, Postlewaite, and Samuelson (2013a, 2013b), continuing with separable preferences, explored the circumstances under which prices in the ex post market can be written as a function of the seller's investment s only. This is not an implication of separability, and requires additional conditions.

4. CHARACTERIZATION OF EX POST EQUILIBRIA

This section develops conditions under which we can refine the characterization of ex post equilibria we have obtained in Propositions 2 and 4. Our goal is to establish conditions under which ex post equilibria will be Pareto efficient, a task we complete in Section 5, and so we organize our discussion around three obvious sources of inefficiency.

First, exchange efficiency is a necessary condition for both pairwise efficiency and Pareto efficiency. Every ex ante equilibrium is thus exchange efficient. In contrast, Examples 2 and 3 each exhibit ex post equilibria in which some (and in Example 2, all) pairs of agents choose investments that are exchange inefficient, that is, that place them strictly inside their utility frontiers. Section 4.1 establishes conditions under which an ex post equilibrium will be exchange efficient.

Second, as in the ex post equilibrium of Example 1 in which all agents choose their autarchy investments and remain unmatched, there may be too few agents participating in the market. Section 4.2 identifies conditions ensuring that every ex post equilibrium (and hence every ex ante equilibrium) is fully matched. These conditions are stronger than the ones required to ensure that ex ante equilibria are fully matched, but they are quite straightforward. For fully matched equilibria, assuming continuity of the maps from names into types has important implications that we also record here.

Third, agents may be matched with the "wrong" partners.[19] Section 4.3 identifies conditions under which all ex post equilibria, and hence also all ex ante equilibria, feature (essentially) identical matchings. In doing so, we focus on the case which has been most prominent in the literature, in which all ex ante equilibria are positive assortative (e.g., Cole, Mailath, and Postlewaite (2001b), Iyigun and Walsh (2007), and Peters and Siow (2002)).[20] This requires strong but familiar assumptions.

4.1. *Exchange Efficiency of ex post Equilibria*

In models with perfect transferability and separable preferences, two approaches to establishing exchange efficiency have been considered. The first, suggested by Dizdar (2012), is to seek conditions on preferences under which, for any pair of types (i, j), conditional efficiency of an exchange for that pair implies (unconditional) exchange efficiency for that pair. The second approach, pioneered by Cole, Mailath, and Postlewaite (2001b, Section 6), uses assumptions about the distribution of types in the economy and about the relationship between the preferences of various types. This approach seeks to "leverage" the pairwise conditional efficiency conditions (46)–(47) to infer exchange efficiency of ex post equilibria, even when conditional exchange efficiency for a given pair of types does not imply exchange efficiency for that pair. In Example 3, for instance, we showed that if the set of types is sufficiently rich, then all investments are chosen by some agents, at which point pairwise conditional efficiency implies exchange efficiency.

[19]Dizdar (2012, Section 5.2) and Nöldeke and Samuelson (2014, Sections 3.3.3 and G.3) provide examples.

[20]Positive assortment rather than negative assortment is not critical to our argument. Replacing Definition 10 appearing in Section 4.3 by the corresponding generalized condition for negative assortative matching from Legros and Newman (2007b) will give results for negative assortative matching equivalent to the ones we obtain here.

This section pursues the approach of Dizdar (2012), assuming neither perfect transferability nor separability. We do so with the help of convexity assumptions on the preferences over exchanges between given pairs of types, but require no assumptions about the distribution of types in the economy or about how the preferences of various types are related to one another.[21]

Suppose we are given a pair of types (β, σ) and an exchange (b, s, t) solving

$$(64) \qquad (b, t) \in \arg\max_{(b', t') \in B \times \mathbb{R}} U(b', s, \beta, \sigma, t') \quad \text{s.t.} \quad V(s, b', \sigma, \beta, t') \geq V(s, b, \sigma, \beta, t),$$

$$(65) \qquad (s, t) \in \arg\max_{(s', t') \in S \times \mathbb{R}} V(s', b, \sigma, \beta, t') \quad \text{s.t.} \quad U(b, s', \beta, \sigma, t') \geq U(b, s, \beta, \sigma, t).$$

Does it follow that (b, s, t) also solves

$$(66) \qquad (b, s, t) \in \arg\max_{(b', s', t') \in B \times S \times \mathbb{R}} U(b', s', \beta, \sigma, t') \quad \text{s.t.} \quad V(s', b', \sigma, \beta, t') \geq V(s, b, \sigma, \beta, t)?$$

If the answer is positive for all $(\beta, \sigma) \in \mathfrak{B} \times \mathfrak{S}$, it follows from the definitions of the utility frontier functions $\breve{\phi}$, $\breve{\psi}$, and ϕ that (30)–(31) imply (24) for all $(i, j) \in N^2$. Because (24) and (25) are equivalent, it follows that the conditional exchange efficiency of an allocation implies its (unconditional) exchange efficiency, ensuring the exchange efficiency of every ex post equilibrium.

The natural approach to establishing a connection between (64)–(65) and (66) is to consider the Kuhn–Tucker conditions for the solutions to these problems. This requires differentiability assumptions which strengthen the continuity and monotonicity requirements from Assumption 1. Convexity of the set of feasible investments in conjunction with quasiconcavity of the utility functions then implies that the Kuhn–Tucker conditions developed in Arrow and Enthoven (1961) are applicable and hence that conditional exchange efficiency implies exchange efficiency.[22]

PROPOSITION 5: *Let Assumption 1 hold. Let B and S be convex and let U and V be quasiconcave and differentiable in (b, s, t) for all $(\beta, \sigma) \in \mathfrak{B} \times \mathfrak{S}$, with the*

[21] Nöldeke and Samuelson (2014, Section 4.1.2 and Appendix E) discussed the leveraging approach of Cole, Mailath, and Postlewaite (2001b) in more detail, noting (as one would expect from Corollary 4) that it is more powerful when preferences are separable, and presenting an example adapted from Dizdar (2012) illustrating its potential limitations.

[22] Quasiconcavity does not depend upon the sign convention we adopt for transfers, so the assumptions that U and V are quasiconcave are symmetric, despite the fact that transfers enter these functions with different signs.

partial derivatives with respect to t satisfying $U_t < 0$ and $V_t > 0$. Then every ex post equilibrium is exchange efficient.

PROOF: As explained above, it suffices to show that (64)–(65) imply (66).

Using the strict Pareto property, we can exchange the role of the objective function and the constraint in (65) to obtain that an exchange (b, s, t) satisfies (65) if and only if (s, t) solves

$$\max_{(s',t')\in S\times\mathbb{R}} U\big(b, s', \beta, \sigma, t'\big) \quad \text{s.t.} \quad V\big(s', b, \sigma, \beta, t'\big) \geq V(s, b, \sigma, \boldsymbol{\beta}, t). \tag{67}$$

Using U_b, U_s, V_b, and V_s to denote the vectors of partial derivatives of the utility functions with respect to the corresponding variables, the Kuhn–Tucker–Lagrange conditions for (64) are (Arrow and Enthoven (1961, p. 790)) that there exists $\lambda \geq 0$ satisfying, for all $(b', t') \in B \times \mathbb{R}$,

$$\begin{aligned}&\big(U_b(b, s, \beta, \sigma, t) + \lambda V_b(s, b, \sigma, \beta, t)\big) \cdot \big(b' - b\big)\\&\quad + \big(U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t)\big)\big(t' - t\big) \leq 0.\end{aligned} \tag{68}$$

Similarly, the Kuhn–Tucker–Lagrange conditions for (67) are that there exists $\mu \geq 0$ satisfying, for all $(s', t') \in S \times \mathbb{R}$,

$$\begin{aligned}&\big(U_s(b, s, \beta, \sigma, t) + \mu V_s(s, b, \sigma, \beta, t)\big) \cdot \big(s' - s\big)\\&\quad + \big(U_t(b, s, \beta, \sigma, t) + \mu V_t(s, b, \sigma, \beta, t)\big)\big(t' - t\big) \leq 0.\end{aligned} \tag{69}$$

Because (i) both U and V are quasiconcave in (b, s, t), (ii) t is unconstrained, and (iii) $V_t > 0$ holds, these Kuhn–Tucker–Lagrange conditions are necessary for (64) and (67) (Arrow and Enthoven (1961, p. 791)). Further, setting $b' = b$ in (68) and $s' = s$ in (69), we obtain

$$\begin{aligned}&U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t) = 0,\\&U_t(b, s, \beta, \sigma, t) + \mu V_t(s, b, \sigma, \beta, t) = 0.\end{aligned}$$

Because $U_t < 0$ and $V_t > 0$ hold, these equalities imply $\mu = \lambda > 0$, so that (68) and (69) imply the existence of $\lambda \geq 0$ such that

$$\begin{aligned}&\big(U_b(b, s, \beta, \sigma, t) + \lambda V_b(s, b, \sigma, \beta, t)\big) \cdot \big(b' - b\big)\\&\quad + \big(U_s(b, s, \beta, \sigma, t) + \lambda V_s(s, b, \sigma, \beta, t)\big) \cdot \big(s' - s\big)\\&\quad + \big(U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t)\big)\big(t' - t\big) \leq 0\end{aligned}$$

holds for all $(b', s', t') \in B \times S \times \mathbb{R}$. These are the Kuhn–Tucker–Lagrange conditions for (66). Because t is unconstrained and $U_t < 0$ holds, condition (a) in Theorem 3 from Arrow and Enthoven (1961) is satisfied and these conditions are then sufficient for (66). Hence, (b, s, t) solves (66). *Q.E.D.*

As indicated by (30)–(31), conditional exchange efficiency for a pair may be understood as the requirement that the "conditional utility frontiers" $\breve{\phi}$ and $\breve{\psi}$ both pass through the point (u, v) in utility space induced by the exchange (b, s, t). The first part of the proof of Proposition 5 establishes that the two conditional utility frontiers are tangent to one another at (u, v). The second part then shows that this equal slope condition is sufficient to imply that (u, v) lies on the unconditional utility frontier. The convexity and differentiability assumptions imposed in Proposition 5 play an essential role in this argument, by ensuring that local considerations suffice to evaluate whether there is any scope to increase both agents' utilities by adjusting their exchange.

When utility is perfectly transferable, the question we address in this section reduces to the question of whether conditions (32)–(33) imply condition (26). Recall that (32)–(33) are the conditions for a pair of investments (b, s) to be a Nash equilibrium in the full appropriation game in which buyer i chooses $b \in B$, seller j chooses $s \in S$, and both agents have the value $Z(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j))$ as a payoff function, whereas (26) states that (b, s) maximizes this value. Hence, in the perfectly transferable case, we are asking for conditions under which all Nash equilibria of the full appropriation game solve the value maximization problem. As Dizdar (2012) has noted, any solution to the value maximization problem is a Nash equilibrium in the full appropriation game, so assuming the existence of a unique equilibrium in the full appropriation game is clearly sufficient for such a result. Proposition 5 provides a complementary result, showing that all Nash equilibria in the full appropriation game solve the value maximization problem whenever the value function is differentiable and concave in (b, s) on the convex domain $B \times S$.[23] While our approach generalizes to the case of imperfectly transferable utility, Dizdar's observation has no natural counterpart with imperfectly transferable utility as, in general, different solutions to the exchange efficiency problems (24)–(25) feature distinct investments.

A value function Z can be both supermodular in (b, s) and concave in (b, s), so that the conditions appearing in Proposition 5 are applicable in the case of a supermodular value function. When investments are unidimensional and Z twice differentiable, we simply need the supermodularity requirement $Z_{bs}(b, s, \beta, \sigma) > 0$, along with standard concavity conditions $Z_{bb}(b, s, \beta, \sigma) \leq 0$ and $Z_{bs}(b, s, \beta, \sigma)^2 \leq Z_{bb}(b, s, \beta, \sigma) Z_{ss}(b, s, \beta, \sigma)$, with the last condition ensuring that the complementarities giving rise to $Z_{bs} > 0$ are not so strong as to overwhelm the "partial concavity" of the value function in each of b and s (as they do in Example 1).

[23]Concavity and differentiability of Z implies that both $U(b, s, \beta, \sigma, t) = Z(b, s, \beta, \sigma) - t$ and $V(s, b, \sigma, \beta, t) = t$ are quasiconcave and differentiable. Proposition 5 then implies that Nash equilibria of the full appropriation game solve the value maximization problem. The same result could be obtained by noting that the full appropriation game is a potential game with the potential $Z(b, s, \beta, \sigma)$ and applying the observation from footnote 4 in Monderer and Shapley (1996).

EXAMPLE 4: Consider Example 2. The value function $Z(b, s, \beta, \sigma) = bs - b^5/5\beta - s^5/5\sigma$ is supermodular but is not concave (it is convex in a neighborhood of the origin). Proposition 5 thus does not apply. Indeed, as we have noted in Example 2, there exists a fully matched zero-investment ex post equilibrium that is not exchange efficient. We could replace $\mathfrak{f}$ and $\mathfrak{g}$ with functions that are increasing in b and s (with positive derivatives at zero) for which the value function Z would be concave. However, the zero-investment equilibrium would remain, and so Proposition 5 would then imply that zero investments are exchange efficient.

4.2. *Full Matching*

4.2.1. *Sufficient Conditions for Full Matching*

A simple sufficient condition to ensure that all pairwise efficient allocations and, hence, all ex ante equilibria, are fully matched is to assume that for every $(i, j) \in N^2$, there exists some exchange (b, s, t) with

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) > \underline{u}(i) \quad \text{and} \quad V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) > \underline{v}(j). \tag{70}$$

Any feasible allocation in which there exists a pair of unmatched agents (i, j) is then Pareto dominated by an otherwise unchanged allocation in which these two agents match with an exchange (b, s, t) satisfying (70).[24] This condition can be interpreted as the requirement that *all* possible matches are productive, allowing the matching partners to achieve utilities strictly higher than their outside options.

Assuming all matches to be productive, however, does not suffice to ensure that all ex post equilibria are fully matched. This is evident from Example 1, in which unmatched agents choose their autarchy investments of zero and no match involving an agent who has chosen an investment of zero can generate any strictly positive surplus. A condition sufficient to ensure that all ex post equilibria are fully matched is that matches are productive even when one of the agents in the match has chosen an autarchy investment. The following is immediate from the pairwise conditional efficiency of ex post equilibria:

PROPOSITION 6: *Let Assumption 1 hold. Suppose that for all $(i, j) \in N^2$, either* (i) *for all autarchy investments b of buyer i, there exists (s, t) with $U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) > \underline{u}(i)$ and $V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) > \underline{v}(j)$, or* (ii) *an analogous condition holds for the autarchy investments of seller j. Then every ex post equilibrium $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is fully matched.*

[24] In the extension of our model to the case in which the masses of buyers and sellers may differ, (70) suffices to ensure that the short side of the market is fully matched.

Suppose that preferences are separable. Then the conditions appearing in Proposition 6 will hold if we have

$$\hat{f}(b, s, 0) \geq \underline{f}(b) \quad \text{and} \quad \hat{g}(s, b, 0) \geq \underline{g}(s), \tag{71}$$

with at least one inequality strict, whenever b is an autarchy investment for some type of buyer and s an autarchy investment for some type of seller. These conditions will, in turn, be satisfied if all autarchy investments are strictly positive, the functions $\hat{f}$ and $\hat{g}$ are strictly increasing in the partner's investment, and being unmatched is equivalent to being matched to a partner with a zero investment. The conditions appearing in (71) are also satisfied in Iyigun and Walsh (2007).

4.2.2. *Full Matching, Continuity, and Separability*

So far we made no assumptions on the functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. To make further progress, we require some regularity of the map from names into types to ensure that we can link assumptions on the agents' utility functions (which are expressed in terms of types) to properties of the utility frontiers (which are expressed in terms of names).

ASSUMPTION 2: *The functions* $\boldsymbol{\beta}$ *and* $\boldsymbol{\sigma}$ *are continuous*.

Assumptions 1 and 2 imply continuity of the utility frontiers and, as a consequence, the continuity of equilibrium utility schedules for fully matched equilibria. Appendix B proves the following:

LEMMA 2: *Let Assumptions* 1 *and* 2 *hold*. *Then*

(i) *The functions* ϕ, ψ, $\breve{\phi}$, *and* $\breve{\psi}$ *are continuous*.

(ii) *In any fully matched ex ante or ex post equilibrium* $(J, I, b, s, \mathbf{u}, \mathbf{v})$, *the functions* $\mathbf{u}$ *and* $\mathbf{v}$ *are continuous*.

The intuition for the second part of this result is standard: if the utility schedule $\mathbf{u}$ (for example) took a jump at i^*, then some buyer with a name very close to i^* and with a utility on the lower side of the jump could increase his utility by matching with seller $J(i)$ currently matched with another buyer i who is also close to i^* but on the upper side of the jump. Of course, Lemma 2 trivially holds in the finite case.

Our next result exploits separability to show that fully matched ex post equilibria are ex ante equilibria in an economy in which the investment opportunities are restricted in a particular way. In light of the constrained efficiency result from Proposition 4, this is not surprising. To state the result, we introduce some notation and terminology that we also require in Section 4.3.

For any pair of nonempty closed sets $\tilde{B} \subseteq B$ and $\tilde{S} \subseteq S$, define $\phi_{\tilde{B},\tilde{S}} : N \times N \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi_{\tilde{B},\tilde{S}}(i, j, v) = \max_{b\in\tilde{B}, s\in\tilde{S}, t\in\mathbb{R}} U(b, s, \beta, \sigma, t) \quad \text{s.t.} \quad V(s, b, \sigma, \beta, u) \geq v.$$

Define $\psi_{\tilde{S},\tilde{B}}$ analogously. We assume that $\tilde{B}$ and $\tilde{S}$ are nonempty and closed to ensure that the utility frontiers $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ are well defined and Lemmas 1 and 2 are applicable. Given such sets $\tilde{B}$ and $\tilde{S}$, consider an allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ satisfying $\mathbf{b}(i) \in \tilde{B}$ and $\mathbf{s}(j) \in \tilde{S}$. We say that this allocation is *individually rational on* $(\tilde{B}, \tilde{S})$ if $\mathbf{u}(i) \geq \max_{b\in\tilde{B}} \underline{U}(b, \boldsymbol{\beta}(i))$ and $\mathbf{v}(j) \geq \max_{s\in\tilde{S}} \underline{V}(s, \sigma(j))$ hold for all i and j. If, in addition, the pairwise efficiency conditions (44)–(45) from Definition 5 hold for $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$, then $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is *pairwise efficient on* $(\tilde{B}, \tilde{S})$. If we let $\tilde{B} = B$ and $\tilde{S} = S$, we have $\phi_{B,S} = \phi$ and $\psi_{S,B} = \psi$ and recover the standard definition of pairwise efficiency. Note that we may apply Proposition 1 to conclude that an allocation that is pairwise efficient on some sets $\tilde{B}$ and $\tilde{S}$ is an ex ante equilibrium in an economy in which $\tilde{B}$ and $\tilde{S}$ are the sets of available investments.

LEMMA 3: *Let Assumptions 1 and 2 hold, let preferences be separable, let* $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ *be a fully matched ex post equilibrium, and let* $\overline{\mathbf{B}}$ *and* $\overline{\mathbf{S}}$ *be the closures of the sets* $\mathbf{B}$ *and* $\mathbf{S}$ *of investments chosen by buyers and sellers. Then* $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ *is pairwise efficient on* $(\overline{\mathbf{B}}, \overline{\mathbf{S}})$.

PROOF: Individual rationality of $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ on $(\overline{\mathbf{B}}, \overline{\mathbf{S}})$ is immediate. Applying the definition of pairwise efficiency on $(\overline{\mathbf{B}}, \overline{\mathbf{S}})$, it thus suffices to show that, for all i and j,

$$\begin{aligned}
\mathbf{u}(i) &\geq \sup_{s\in\mathbf{S}} \breve{\phi}\big(i, j, s, \mathbf{v}(j)\big) = \max_{s\in\overline{\mathbf{S}}} \breve{\phi}\big(i, j, s, \mathbf{v}(j)\big) \\
&= \phi_{B,\overline{\mathbf{S}}}\big(i, j, \mathbf{v}(j)\big) \geq \phi_{\overline{\mathbf{B}},\overline{\mathbf{S}}}\big(i, j, \mathbf{v}(j)\big), \\
\mathbf{v}(j) &\geq \sup_{b\in\mathbf{B}} \breve{\psi}\big(j, i, b, \mathbf{u}(i)\big) = \max_{b\in\overline{\mathbf{B}}} \breve{\psi}\big(j, i, b, \mathbf{u}(i)\big) \\
&= \psi_{\overline{\mathbf{B}},S}\big(j, i, \mathbf{u}(i)\big) \geq \psi_{\overline{\mathbf{S}},\overline{\mathbf{B}}}\big(j, i, \mathbf{u}(i)\big).
\end{aligned}$$

The first inequality in each case follows from Proposition 4 and the definition of pairwise constrained efficiency (cf. (52)–(53)), the subsequent equality is implied by continuity of $\breve{\phi}$ and $\breve{\psi}$ (and the continuity of $\mathbf{u}$ and $\mathbf{v}$) established in Lemma 2, the second equality follows from the relationship between $\breve{\phi}$ and ϕ and between $\breve{\psi}$ and ψ (cf. (48)–(49)), and the final inequality follows from the observation that restricting agents to a smaller set of investments cannot increase the utility possibilities open to them. *Q.E.D.*

4.3. *Positive Assortative Matching*

Throughout this section, we consider fully matched allocations. We seek conditions under which all fully matched ex post equilibria are payoff equivalent to positive assortative allocations. Assuming unidimensional names is a prerequisite for such an investigation. The following assumption directs our attention to the two most commonly studied cases.

ASSUMPTION 3: *The set $N \subset \mathbb{R}$ is either finite or an interval.*

Section 4.3.1 shows that familiar single crossing conditions on the restricted utility frontiers $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ introduced in Section 4.2.2 ensure payoff equivalence to positive assortative equilibria. Section 4.3.2 considers assumptions on the underlying utility functions U and V (and set of types and investments) that, when coupled with the natural monotonicity requirements on the maps from names to types, imply the requisite single crossing properties of the utility frontiers.

4.3.1. *A Single Crossing Condition for Positive Assortative Matching*

In their study of matching models with imperfectly transferable utility, Legros and Newman (2007b) have introduced the concept of generalized increasing differences. For an economy with a finite number of agents, they showed that generalized increasing differences ensures payoff equivalence of equilibrium matchings to positive assortative matching, and that a strict version of this property yields positive assortative equilibrium matchings. Generalized increasing differences is a property on the functions describing the utility frontiers. As noted by Legros and Newman (2007b), the property of (strict) generalized increasing differences is equivalent to the (strict) single crossing condition that appears in the following definition. The interpretation is that higher buyers have a comparative advantage in matching with higher sellers, and vice versa.

DEFINITION 10: Let $\tilde{B} \subseteq B$ and $\tilde{S} \subseteq S$ be closed sets. Then $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ satisfy single crossing if, for all $\bar{i} > \underline{i}$ and $\bar{j} > \underline{j}$,

$$\phi_{\tilde{B},\tilde{S}}(\underline{i}, \bar{j}, v_1) \geq \phi_{\tilde{B},\tilde{S}}(\underline{i}, \underline{j}, v_2) \implies \phi_{\tilde{B},\tilde{S}}(\bar{i}, \bar{j}, v_1) \geq \phi_{\tilde{B},\tilde{S}}(\bar{i}, \underline{j}, v_2), \tag{72}$$

$$\psi_{\tilde{S},\tilde{B}}(\underline{j}, \bar{i}, u_1) \geq \psi_{\tilde{S},\tilde{B}}(\underline{j}, \underline{i}, u_2) \implies \psi_{\tilde{S},\tilde{B}}(\bar{j}, \bar{i}, u_1) \geq \psi_{\tilde{S},\tilde{B}}(\bar{j}, \underline{i}, u_2). \tag{73}$$

If the inequalities in the consequents of (72)–(73) are strict, then single crossing is said to be strict.

By Lemma 1(ii), the functions $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ appearing in Definition 10 are inverse. This implies that conditions (72)–(73) are not independent, but equivalent to each other.

The following key lemma asserts the payoff equivalence of fully matched ex ante equilibria to positive assortative ex ante equilibria when the utility frontiers ϕ and ψ satisfy single crossing. Further, with strict single crossing the equivalence is exact. For the finite case, this result is the counterpart to Proposition 1 in Legros and Newman (2007b). Extending the result to infinite sets of agents is straightforward when the single crossing is strict, but raises a number of technical issues otherwise. We resolve these with the help of the continuity result in Lemma 2. The proof is in Appendix C.

LEMMA 4: *Let Assumptions 1–3 hold and assume that ϕ and ψ satisfy single crossing. Then every fully matched ex ante equilibrium is payoff equivalent to a positive assortative ex ante equilibrium. If ϕ and ψ satisfy strict single crossing, then every fully matched ex ante equilibrium is positive assortative.*

In general, the result in Lemma 4 has no obvious counterpart for ex post equilibria, as there is no natural generalization of the single crossing conditions to the conditional utility frontiers $\breve{\phi}$ and $\breve{\psi}$. However, when preferences are separable, we can use Lemma 3 to infer that a fully matched ex post equilibrium is an ex ante equilibrium in an economy in which buyers are restricted to choose investments in $\overline{\mathbf{B}}$ and sellers are restricted to choose investments in $\overline{\mathbf{S}}$. Provided that the functions $\phi_{\overline{\mathbf{B}},\overline{\mathbf{S}}}$ and $\psi_{\overline{\mathbf{S}},\overline{\mathbf{B}}}$ satisfy single crossing, we can then apply Lemma 4 to obtain a positive assortment result. However, it is an immediate consequence of the separability of preferences that $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ must fail the strict single crossing condition when $\tilde{B}$ or $\tilde{S}$ is a singleton, or more generally, whenever two types of agent choose the same investment—if $\mathbf{s}(j)=\mathbf{s}(j')$, then one buyer will be indifferent between sellers j and j' only if all buyers are indifferent. Consequently, the following result does not contain a counterpart of the strict single crossing result from Lemma 4.

PROPOSITION 7: *Let Assumptions 1–3 hold, let preferences be separable, and assume that $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ satisfy single crossing for all nonempty closed $\tilde{B}\subseteq B$ and $\tilde{S}\subseteq S$. Then every fully matched ex post equilibrium is payoff equivalent to a positive assortative ex post equilibrium.*

PROOF: Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be a fully matched ex post equilibrium. By Lemma 3, the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is pairwise efficient on $\overline{\mathbf{B}}$ and $\overline{\mathbf{S}}$ and thus a fully matched ex ante equilibrium in the corresponding economy in which the sets of feasible investments are given by $\overline{\mathbf{B}}$ and $\overline{\mathbf{S}}$. Because these sets are compact, Assumptions 1–3 hold in the restricted economy. We can then apply Lemma 4 to infer the existence of a payoff equivalent, positive assortative ex ante equilibrium $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ in the restricted economy.

It remains to show that $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ is an ex post equilibrium in the original economy. First, $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ is clearly feasible in the original economy. Second, the allocations $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ and $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ have identical payoffs and the only new investments we have possibly added when moving from $\mathbf{b}$ and $\mathbf{s}$ to $\mathbf{b}'$ and $\mathbf{s}'$ are contained in the closures of the sets $\mathbf{B}$ and $\mathbf{S}$. Noting that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is individually rational and satisfies the pairwise constrained efficiency conditions (52)–(53) in the original economy, we can then conclude that $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ also has these properties. From Corollary 3 and Proposition 2, this implies that $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}, \mathbf{v})$ is an ex post equilibrium in the original economy, giving the result. *Q.E.D.*

4.3.2. *Sufficient Conditions for Single Crossing*

The single crossing properties in Definition 10 are not written in terms of the primitives of the problem. We have formulated Proposition 7 in terms of this single crossing property for two reasons. First, Definition 10 succinctly and intuitively identifies what is needed to ensure positive assortative matching, namely a single crossing condition on utility frontiers. Second, as Legros and Newman (2007b) discussed, it is difficult to find general sufficient conditions, ensuring that single crossing is satisfied. In this section, we exploit separability to identify conditions on utility functions guaranteeing single crossing of $\phi_{\tilde{B}\tilde{S}}$ and $\psi_{\tilde{B}\tilde{S}}$ for all closed subsets of B and S, ensuring the applicability of Proposition 7. As is common in the literature (cf. Section 2.1.3), we restrict attention to the case of unidimensional types and investments. In addition, we assume that agents with higher names have higher types.

ASSUMPTION 4:
(i) *The sets* $\mathfrak{B}$, $\mathfrak{S}$, B, *and* S *are subsets of* $\mathbb{R}$.
(ii) *The functions* $\boldsymbol{\beta}$ *and* $\boldsymbol{\sigma}$ *are increasing*.

Recall that with separable preferences we have (cf. (54)–(55))

$$U(b, s, \beta, \sigma, t) = \hat{U}\big(\hat{f}(b, s, t), b, \beta\big),$$
$$V(s, b, \sigma, \beta, t) = \hat{V}\big(\hat{g}(s, b, t), s, \sigma\big).$$

We say that separable preferences satisfy *outer single crossing* if

$$\hat{U}(x_1, \overline{b}, \underline{\beta}) \geq \hat{U}(x_2, \underline{b}, \underline{\beta}) \quad \Longrightarrow \quad \hat{U}(x_1, \overline{b}, \overline{\beta}) \geq \hat{U}(x_2, \underline{b}, \overline{\beta}), \tag{74}$$
$$\hat{V}(y_1, \overline{s}, \underline{\sigma}) \geq \hat{V}(y_2, \underline{s}, \underline{\sigma}) \quad \Longrightarrow \quad \hat{V}(y_1, \overline{s}, \overline{\sigma}) \geq \hat{V}(y_2, \underline{s}, \overline{\sigma}) \tag{75}$$

hold whenever $\overline{b} > \underline{b}$, $\overline{\beta} > \underline{\beta}$, $\overline{s} > \underline{s}$, and $\overline{\sigma} > \underline{\sigma}$. The interpretation of the outer single crossing properties is obvious: given the returns associated with the different investments, higher types are (weakly) more inclined to choose higher investments.

Now define

$$\rho(b,s,y)=\max_{t\in\mathbb{R}}\hat{f}(b,s,t)\quad\text{s.t.}\quad\hat{g}(s,b,t)\geq y,$$

$$\sigma(s,b,x)=\max_{t\in\mathbb{R}}\hat{g}(s,b,t)\quad\text{s.t.}\quad\hat{f}(b,s,t)\geq x,$$

for all b, s, y, and x. The functions ρ and σ are the utility frontiers for an economy in which pairs of agents are described by their investments (b,s) and the utility functions for a match between such agents with a transfer t are given by the return functions $\hat{f}$ and $\hat{g}$.

We say that separable preferences satisfy *inner single crossing* if

$$\rho(\underline{b},\overline{s},y_1)\geq\rho(\underline{b},\underline{s},y_2)\quad\Longrightarrow\quad\rho(\overline{b},\overline{s},y_1)\geq\rho(\overline{b},\underline{s},y_2),\tag{76}$$

$$\sigma(\underline{s},\overline{b},x_1)\geq\sigma(\underline{s},\underline{b},x_2)\quad\Longrightarrow\quad\sigma(\overline{s},\overline{b},x_1)\geq\sigma(\overline{s},\underline{b},x_2)\tag{77}$$

hold whenever $\overline{b}>\underline{b}$ and $\overline{s}>\underline{s}$. The interpretation of inner single crossing is again obvious: agents which have chosen higher investments are more eager to match with agents who have chosen high investments. Outer single crossing ensures that, in equilibrium, agents with higher types choose higher investments, whereas inner single crossing implies positive assortment of investments. Because we have assumed that types are increasing in names, this suffices to imply positive assortment in ex post equilibrium. Appendix D proves the following:

COROLLARY 5: *Let Assumptions 1–4 hold. Suppose preferences are separable and satisfy outer and inner single crossing. Then every fully matched ex post equilibrium is payoff equivalent to a fully matched allocation satisfying positive assortative matching.*

The proof proceeds by showing that outer and inner single crossing of preferences imply the single crossing conditions appearing in Proposition 7, and then applying this proposition.

Corollary 5 still leaves us with the task of determining when the outer and inner single crossing conditions (74)–(77) hold. We focus on additively separable preferences. In this case, outer single crossing is equivalent to the submodularity of the cost functions $\mathfrak{f}$ and $\mathfrak{g}$. It remains to identify conditions on the return functions $\hat{f}$ and $\hat{g}$ ensuring the inner single crossing conditions (76)–(77) for the case of imperfectly transferable utility. (With transferable utility, supermodularity of the surplus function z is necessary and sufficient.) The following result, proven in Appendix D, does so.

COROLLARY 6: *Let Assumptions 1–4 hold, and let preferences be additively separable with submodular cost functions $\mathfrak{f}$ and $\mathfrak{g}$. Suppose further that there exist* *continuous functions* $F:\mathbb{R}\to\mathbb{R}$, $G:\mathbb{R}\to\mathbb{R}$, $f:B\times S\to\mathbb{R}$, $g:S\times B\to\mathbb{R}$, *and* $h:\mathbb{R}\to\mathbb{R}$ *such that*

$$\hat{f}(b,s,t)=F\big(f(b,s)-t\big), \tag{78}$$

$$\hat{g}(s,b,t)=G\big(g(s,b)+h(t)\big), \tag{79}$$

where F *and* G *are strictly increasing*, f *and* g *are supermodular and increasing in their second argument*, *and* h *is increasing and concave*. *Then every fully matched ex post equilibrium is payoff equivalent to a positive assortative ex post equilibrium*.

The proof shows that under the stated assumptions, (78)–(79) imply inner single crossing, at which point the result follows from Corollary 5.

The assumptions on the return functions in the statement of Corollary 6 are patterned after the ones in the example studied by Iyigun and Walsh (2007) that we have discussed in Section 2.1.3. We can think of investments as determining an amount of a second period consumption good, given by $f(b,s)+g(s,b)$, and a baseline division of this consumption good across the two agents, given by $(f(b,s),g(s,b))$. When $h(t)=t$ is the identity function (as in Iyigun and Walsh (2007)), the division of the consumption good can be changed without cost; the case of concave h allows for the possibility that there are increasing costs in transferring the consumption good from one agent to the other.

5. EFFICIENT EX POST EQUILIBRIA

Sections 4.2.1 and 4.3.1 have identified conditions, namely condition (70) and the conditions appearing in Lemma 4, under which all ex ante equilibria are positive assortative. It is clear that under these conditions, being positive assortative and exchange efficient is necessary for the Pareto (and pairwise) efficiency of ex post equilibria. The following result shows that having the correct, positive assortative matching and being exchange efficient are then also sufficient for Pareto efficiency of ex post equilibria.

PROPOSITION 8: *Let Assumptions* 1 *and* 3 *hold*, *let condition* (70) *hold*, *and assume* ϕ *and* ψ *satisfy single crossing*. *Then every positive assortative ex post equilibrium that is exchange efficient is also Pareto efficient*.

PROOF: Let $(J,I,\mathbf{b},\mathbf{s},\mathbf{u},\mathbf{v})$ be positive assortative and exchange efficient. We show that there exists no $(J',I',\mathbf{b}',\mathbf{s}',\mathbf{u}',\mathbf{v}')$ which is a finite Pareto improvement on $(J,I,\mathbf{b},\mathbf{s},\mathbf{u},\mathbf{v})$.

Suppose, contrariwise, that $(J',I',\mathbf{b}',\mathbf{s}',\mathbf{u}',\mathbf{v}')$ is a finite Pareto improvement on the allocation $(J,I,\mathbf{b},\mathbf{s},\mathbf{u},\mathbf{v})$. As $(J,I,\mathbf{b},\mathbf{s},\mathbf{u},\mathbf{v})$ is fully matched, the allocation $(J',I',\mathbf{b}',\mathbf{s}',\mathbf{u}',\mathbf{v}')$ has at most a finite number of unmatched agents, with identical numbers of unmatched buyers and sellers. From condition (70), these agents can be matched with each other in a way that is still a finite improvement on $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$. Hence, we may assume without loss of generality that $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ is fully matched. Let n be the cardinality of the set $\{i \mid J'(i) \neq J(i)\} = \{j \mid I'(j) = I(j)\}$ (where the equality of these sets is from the fact that $J(i) \neq i$ is equivalent to $I(i) \neq i$). Because $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is exchange efficient, we must have $n > 0$, ensuring that there is a lowest type (cf. Assumption 3), $\underline{i}$, such that $J'(i) \neq i$ holds. Let $\bar{j} = J'(\underline{i}) > \underline{i}$ and $\bar{i} = I'(\underline{i}) > \underline{i}$ (where the strict inequalities hold because $\underline{i}$ is also the lowest type for whom $I'(j) \neq j$ holds).

Because both allocations are fully matched, if one buyer has a different partner, then there must be at least one other buyer with a different partner, and hence we cannot have $n = 1$. Next, consider the case $n = 2$. Then we have $\bar{j} = \bar{i}$. From exchange efficiency of $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, we have

$$\mathbf{u}(\underline{i}) = \phi(\underline{i}, \underline{i}, \mathbf{v}(\underline{i})),$$
$$\mathbf{u}(\bar{i}) = \phi(\bar{i}, \bar{i}, \mathbf{v}(\bar{i})).$$

From feasibility of $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$, we have

$$\mathbf{u}'(\underline{i}) \leq \phi(\underline{i}, \bar{i}, \mathbf{v}'(\bar{i})),$$
$$\mathbf{u}'(\bar{i}) \leq \phi(\bar{i}, \underline{i}, \mathbf{v}'(\underline{i})).$$

Because $\mathbf{v}'(\bar{i}) \geq \mathbf{v}(\bar{i})$ and $\mathbf{v}'(\underline{i}) \geq \mathbf{v}(\underline{i})$ hold, the latter two inequalities imply

$$\mathbf{u}'(\underline{i}) \leq \phi(\underline{i}, \bar{i}, \mathbf{v}(\bar{i})),$$
$$\mathbf{u}'(\bar{i}) \leq \phi(\bar{i}, \underline{i}, \mathbf{v}(\underline{i})).$$

Because $\mathbf{u}'(\underline{i}) \geq \mathbf{u}(\underline{i})$ and $\mathbf{u}'(\bar{i}) \geq \mathbf{u}(\bar{i})$ hold, the exchange efficiency equalities then yield

$$\phi(\underline{i}, \underline{i}, \mathbf{v}(\underline{i})) \leq \phi(\underline{i}, \bar{i}, \mathbf{v}(\bar{i})),$$
$$\phi(\bar{i}, \bar{i}, \mathbf{v}(\bar{i})) \leq \phi(\bar{i}, \underline{i}, \mathbf{v}(\underline{i})).$$

These inequalities contradict the single crossing property unless they both hold with equality. But equality in both of these inequalities can only hold if $\mathbf{u}'(i) = \mathbf{u}(i)$ and $\mathbf{v}'(i) = \mathbf{v}(i)$ hold for $i = \underline{i}, \bar{i}$, contradicting the assumption that $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ is a finite Pareto improvement on $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$.

Now consider the case $n > 2$. We argue that if there exists such a finite Pareto improvement, then there also exists a finite Pareto improvement with $n' < n$. Repeating this argument a finite number of times then yields the existence of a finite Pareto improvement with $n = 2$, which we have already shown to be impossible. We consider two cases, namely $\bar{j} = \bar{i}$ and $\bar{j} \neq \bar{i}$.

In the first case, we can apply the argument from the case $n = 2$ to conclude that $\mathbf{u}'(i) = \mathbf{u}(i)$ and $\mathbf{v}'(i) = \mathbf{v}(i)$ hold for $i = \underline{i}, \bar{i}$. Consequently, if $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ is a finite Pareto improvement on $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$, so will be the allocation which coincides with $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ except that the buyers and sellers with types $\underline{i}$ and $\bar{i}$ are assigned their original partners and exchanges from the allocation $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$. This new finite Pareto improvement has cardinality $n' = n - 2$.

Suppose now that we have $\bar{j} \neq \bar{i}$. As in the case $n = 2$, exchange efficiency gives us

$$\mathbf{u}(\underline{i}) = \phi\big(\underline{i}, \underline{i}, \mathbf{v}(\underline{i})\big),$$

$$\mathbf{u}(\bar{i}) = \phi\big(\bar{i}, \bar{i}, \mathbf{v}(\bar{i})\big).$$

Feasibility gives us

$$\mathbf{u}'(\underline{i}) \leq \phi\big(\underline{i}, \bar{j}, \mathbf{v}'(\bar{j})\big),$$

$$\mathbf{u}'(\bar{i}) \leq \phi\big(\bar{i}, \underline{i}, \mathbf{v}'(\underline{i})\big).$$

Using $\mathbf{v}'(\bar{j}) \geq \mathbf{v}(\bar{j})$ and $\mathbf{v}'(\underline{i}) \geq \mathbf{v}(\underline{i})$, this yields

$$\mathbf{u}'(\underline{i}) \leq \phi\big(\underline{i}, \bar{j}, \mathbf{v}(\bar{j})\big),$$

$$\mathbf{u}'(\bar{i}) \leq \phi\big(\bar{i}, \underline{i}, \mathbf{v}(\underline{i})\big).$$

Combining the first of these with the exchange efficiency condition and $\mathbf{u}'(\underline{i}) \geq \mathbf{u}(\underline{i})$ yields

$$\phi\big(\underline{i}, \underline{i}, \mathbf{v}(\underline{i})\big) \leq \phi\big(\underline{i}, \bar{j}, \mathbf{v}(\bar{j})\big).$$

Because $\bar{i} > \underline{i}$ and $\bar{j} > \underline{i}$, the single crossing property implies

$$\phi\big(\bar{i}, \underline{i}, \mathbf{v}(\underline{i})\big) \leq \phi\big(\bar{i}, \bar{j}, \mathbf{v}(\bar{j})\big).$$

From previous inequalities, we have

$$\phi\big(\bar{i}, \bar{i}, \mathbf{v}(\bar{i})\big) \leq \phi\big(\bar{i}, \underline{i}, \mathbf{v}(\underline{i})\big),$$

so that we can infer

$$\mathbf{u}(\bar{i}) \leq \phi\big(\bar{i}, \bar{j}, \mathbf{v}(\bar{j})\big).$$

If this last inequality is strict, we can change $(J', I', \mathbf{b}', \mathbf{s}', \mathbf{u}', \mathbf{v}')$ by (i) "rematching" buyer and seller $\underline{i}$ with each other and having them make the exchange from the original allocation and (ii) matching buyer $\bar{i}$ and seller $\bar{j}$ with each other and fixing an exchange for them such that both of them strictly improve on their utility in the original allocation. Hence, we have found a finite Pareto improvement with cardinality $n' = n - 1$. If, on the other hand, we have

$$\mathbf{u}(\bar{i}) = \phi(\bar{i}, \bar{j}, \mathbf{v}(\bar{j})),$$

then it must have been the case that $\mathbf{u}'(i) = \mathbf{u}(i)$ and $\mathbf{v}'(j) = \mathbf{v}(j)$ must have held for $i = \underline{i}, \bar{i}$ and $j = \underline{i}, \bar{j}$, so that performing the same rematching as described above again generates a finite Pareto improvement with cardinality $n' = n - 1$. *Q.E.D.*

We can combine Proposition 8 with previous results to obtain conditions under which all ex post equilibria are Pareto efficient. In particular, Proposition 5 provides sufficient conditions for ex post equilibria to be exchange efficient, Proposition 6 offers sufficient conditions for ex post equilibria to be fully matched, and Proposition 7 gives sufficient conditions for positive assortment of fully matched ex post equilibria. Together, the conditions appearing in Propositions 5–7, which imply the conditions from Proposition 8, thus preclude coordination failures in ex post equilibrium:

COROLLARY 7: *Let the conditions from Propositions 5–7 hold. Then every ex post equilibrium is Pareto efficient.*

Nöldeke and Samuelson (2014, Section 4.4) showed that the conditions in Proposition 8 do not imply pairwise efficiency of positive assortative ex post equilibria. Ex ante and ex post equilibria give rise to fundamentally different incentive constraints. Buyer and seller investments are both up for grabs when agents consider the alternative matchings that give rise to the pairwise efficiency conditions, and this imposes tighter incentive constraints on equilibrium payoffs than do the corresponding pairwise conditional efficiency considerations. We thus cannot in general expect ex post equilibria to be pairwise efficient, even if they are exchange efficient and the matching is unambiguously "correct."

6. DISCUSSION

6.1. *Existence of Equilibrium*

The existence of ex post equilibria is implied by the existence of ex ante equilibria (Corollary 1), but we have not addressed the question of when the latter exist. As we have noted, the pairwise efficiency conditions characterizing ex ante equilibria are equivalent to the stability conditions from the literature on matching and assignment models, which contains a number of existence results. As long as the functions ϕ and ψ emerging from our investment-choice problem satisfy the conditions from these results, we can apply them to infer existence of ex ante equilibria in our model.

In the finite case, our Assumption 1 ensures that the continuity and monotonicity assumptions of Alkan and Gale (1990, Theorem 1) are met (cf. our Lemma 1). As long as the agents' outside options are feasible within each match (i.e., the full matching condition (70) holds), this suffices for the existence of an ex ante equilibrium in our model.

With an infinite number of agents, the case most commonly considered in the literature is that in which types are continuously distributed and utility is perfectly transferable. Conditions ensuring the existence of pairwise efficient allocations are provided by Chiappori, McCann, and Nesheim (2010) and Ekeland (2010b). These results allow for multidimensional types, but require restrictions on utility functions reminiscent of the supermodularity conditions in Cole, Mailath, and Postlewaite (2001b), who proved existence for their unidimensional model. Legros and Newman (2007a) studied matching models with imperfectly transferable utility and a continuum of types under assumptions akin to the ones we impose in Section 4.3. They identified conditions (including the continuous differentiability of ϕ and ψ, which we could obtain from an appropriate strengthening of Assumption 2) under which the existence of equilibrium follows from the existence of the solution to a differential equation.

The literature has obtained more general existence results for models with an infinite number of agents than the ones cited above, but these results use a notion of feasibility different from the one we employ. Kaneko and Wooders (1996) presented a general existence result for stable allocations in a model with either perfectly or imperfectly transferable utility, but their notion of an f-core considers any allocation to be feasible which lies in the closure of our set of feasible allocations. To make their result applicable to our setting would then require the identification of additional conditions ensuring feasibility of a stable outcome. That this is a nontrivial task becomes clear when considering the case of perfectly transferable utility in which stable allocations coincide with the solutions of an optimal transport problem (e.g., Gretsky, Ostroy, and Zame (1992), Ekeland (2010b)). In particular, our existence problem is analogous to the existence of solutions to the so-called Monge problem, which is a notoriously difficult problem, whereas general existence results have been obtained for the so-called Kantorovich problem which considers an enlarged set of feasible allocations (Villani (2009)).[25]

[25]The problem of finding a solution to the pairwise efficiency conditions in our model is a Monge problem because we specify a matching as a map from names on one side of the market into names on the other side. In the Kantorovich problem, the set of feasible matchings is identified with a joint probability measure over $N \times N$, with the constraint that the induced marginal distributions, over the sets of buyers and sellers, match the distributions of buyer and seller names. The interpretation is that the probability attached to any subset of $N \times N$ is the probability that agents from this subset are drawn to match. Again, see Villani (2009).

6.2. *Foundations for Competitive Matching*

We have focused on investment decisions in competitive matching environments by building the assumption that agents behave competitively into our equilibrium notions. In particular, all agents solve a maximization problem that takes prices (whether in monetary or utility terms) as fixed at the candidate equilibrium level. Cole, Mailath, and Postlewaite (2001b), Dizdar (2012), and Peters and Siow (2002) adopted a similar approach.

Makowski (2004, pp. 19–20), building on work by Gretsky, Ostroy, and Zame (1992, 1999), argued that one should be leery of simply assuming the matching market to be competitive, even when dealing with a continuum of agents, because by "accepting this point of view, one runs the danger of making continuum analysis totally unconnected with the analysis of large but finite economies. . . ."[26] Cole, Mailath, and Postlewaite (2001a) showed that allocations in a finite model satisfying a "double overlap" condition will satisfy a constrained efficiency condition analogous to the constrained efficiency condition that characterizes ex post equilibria (when preferences are separable) in our model. It would be important to investigate similar conditions in our setting. Bhaskar and Hopkins (2011, Appendix B) showed that their competitive matching market, with a continuum of agents, is the limit of a sequence of models with finite numbers of agents. Hadfield (1999) also offered such a limiting analysis. However, Peters (2007, 2011) examined models whose equilibria do not exhibit convergence to competitive equilibrium as the number of agents grows arbitrarily large. Investigating the conditions under which matching markets with a large numbers of agents will be competitive remains an important area for further work.

APPENDIX A: PROOF OF LEMMA 1 (SECTION 2.2.1)

We first confirm that ϕ (and similarly ψ) is well defined on $N \times N \times \mathbb{R}$. Fix a pair $(i, j) \in N \times N$. Then for any $v \in \mathbb{R}$, we can fix a pair (b, s) and then use Assumption 1(iii) to infer that there is some t for which $V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) \geq v$. This ensures that the maximization problem in (17) is feasible, and the existence of the maximum then follows from the continuity assumed in Assumption 1(i), the fact that B and S are compact, and the fact that U and V move in opposite dictions in t (Assumptions 1(ii)–(iii)).

(i) We provide the proof for the function ϕ, with the case of ψ being similar. It is immediate from (17) that ϕ is weakly decreasing in v. To see that it is strictly decreasing, fix (i, j) and let $\overline{v} > \underline{v}$. Then there exists an exchange

[26] Makowski (2004) defined the post-investment matching market as being perfectly competitive if the equilibrium price vector is a continuous function of the measures describing the investments present in the ex post market, so that an investment deviation by a small group of agents can have only a small effect on equilibrium prices. An individual member of the continuum is then "viewed as the limit of a small group of individuals," and may or may not have market power.

$(\overline{b}, \overline{s}, \overline{t})$ with $\phi(i, j, \overline{v}) = U(\overline{b}, \overline{s}, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), \overline{t})$ and $V(\overline{s}, \overline{b}, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), \overline{t}) \geq \overline{v}$. By Assumption 1(i) and Assumption 1(iii), there exists $\varepsilon > 0$ such that $V(\overline{s}, \overline{b}, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), \overline{t} - \varepsilon) \geq \underline{v}$. Using Assumption 1(ii), we then have $\phi(i, j, \underline{v}) \geq U(\overline{b}, \overline{s}, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), \overline{t} - \varepsilon) > U(\overline{b}, \overline{s}, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), \overline{t}) = \phi(i, j, \overline{v})$, giving the result.

(ii) We establish that $u = \phi(i, j, \psi(j, i, u))$. Fix $(i, j) \in N \times N$ and $u \in \mathbb{R}$. Then $\psi(j, i, u)$ exists (as established in our opening remarks) and we can let $v := \psi(j, i, u)$. The definition of ψ (cf. (18)) ensures that there exist b, s and t such that

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) \geq u,$$
$$V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) = v.$$

This implies that $\phi(i, j, v) \geq u$. To complete the argument by showing that this is in fact an equality, suppose $\phi(i, j, v) > u$. Then there exist b, s and t with

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big) > u,$$
$$V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) \geq v.$$

From the strict Pareto property (cf. (1)–(2)), this in turn ensures that there exists t' for which

$$U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t'\big) > u,$$
$$V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t'\big) > v,$$

contradicting the definition $v := \psi(j, i, u)$.

(iii) As an implication of Lemma 1(ii), ϕ has full range as a function of v and, from Lemma 1(i), is strictly decreasing in v. Hence, ϕ is continuous in v. The same argument gives continuity of ψ in u. *Q.E.D.*

APPENDIX B: PROOF OF LEMMA 2 (SECTION 4.2.2)

(i) We show that $\breve{\phi}$ is continuous. Because S is compact, this implies the continuity of $\phi(i, j, v) = \max_{s \in S} \breve{\phi}(i, j, s, v)$. The argument for the continuity of $\breve{\psi}$ and ψ is analogous.

Define the function $\tau : S \times B \times \mathfrak{S} \times \mathfrak{B} \times \mathbb{R} \to \mathbb{R}$ by

$$V\big(s, b, \sigma, \beta, \tau(s, b, \sigma, \beta, v)\big) = v.$$

To confirm that the function τ is well defined, we note that for each (s, b, σ, β), the function V has $\mathbb{R}$ as its range (Assumption 1(iii)), ensuring that there exists a value t satisfying $V(s, b, \sigma, \beta, t) = v$, and the fact that V is strictly increasing in t ensures that this value is unique. Moreover, because V is continuous and strictly increasing in its last argument, τ is continuous. Now define

$$\overline{\tau}(i, j, s, v) = \max_{b\in B} \tau\big(s, b, \boldsymbol{\sigma}(i), \boldsymbol{\beta}(j), v\big),$$
$$\underline{\tau}(i, j, s, v) = \min_{b\in B} \tau\big(s, b, \boldsymbol{\sigma}(i), \boldsymbol{\beta}(j), v\big).$$

Berge's maximum theorem (Ok (2007, p. 306)) ensures that $\overline{\tau}$ and $\underline{\tau}$ are continuous. Then we have

$$\breve{\phi}(i, j, s, v) = \max_{(b,t)\in B\times[\underline{\tau}(i,j,s,v),\overline{\tau}(i,j,s,v)]} U\big(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t\big)$$
$$\text{s.t.} \quad V\big(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t\big) \geq v.$$

This maximization problem again satisfies the conditions of Berge's maximum theorem, giving the result.

(ii) Suppose $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is a fully matched ex post equilibrium and $\mathbf{u}$ is not continuous (the case of $\mathbf{v}$ is similar). Then there exist a value $\delta > 0$ and sequences $\{\underline{i}_n\}_{n=1}^{\infty}$ and $\{\bar{i}_n\}_{n=1}^{\infty}$ such that

$$\mathbf{u}(\bar{i}_n) - \delta > \mathbf{u}(\underline{i}_n),$$
$$|\bar{i}_n - \underline{i}_n| < \frac{1}{n}.$$

The conditional exchange efficiency condition (29) for $\bar{i}_n$ gives us

$$\mathbf{u}(\bar{i}_n) = \breve{\phi}\big(\bar{i}_n, J(\bar{i}_n), \mathbf{s}\big(J(\bar{i}_n)\big), \mathbf{v}\big(J(\bar{i}_n)\big)\big).$$

The fact that $\breve{\phi}(i, j, s, v)$ is continuous in i on the compact set N (and hence uniformly continuous) then enures that, for sufficiently large n,

$$\mathbf{u}(\underline{i}_n) < \mathbf{u}(\bar{i}_n) - \delta = \breve{\phi}\big(\bar{i}_n, J(\bar{i}_n), \mathbf{s}\big(J(\bar{i}_n)\big), \mathbf{v}\big(J(\bar{i}_n)\big)\big) - \delta$$
$$< \left[\breve{\phi}\big(\underline{i}_n, J(\bar{i}_n), \mathbf{s}\big(J(\bar{i}_n)\big), \mathbf{v}\big(J(\bar{i}_n)\big)\big) + \frac{\delta}{2}\right] - \delta,$$

with the outside two terms then giving

$$\mathbf{u}(\underline{i}_n) < \breve{\phi}\big(\underline{i}_n, J(\bar{i}_n), \mathbf{s}\big(J(\bar{i}_n)\big), \mathbf{v}\big(J(\bar{i}_n)\big)\big) - \frac{\delta}{2},$$

contradicting the incentive constraint (27) for $\underline{i}_n$. *Q.E.D.*

APPENDIX C: APPENDIX FOR SECTION 4.3.1

Section C.1 provides simple necessary and sufficient conditions for (strict) single crossing of $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$, stated as Lemma 5. A similar result appears in Legros and Newman (2007b). Building on Lemma 5, Section C.2 gives the proof of Lemma 4.

C.1. *Cross Matched Agents*

Let Assumption 1 hold. The functions $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ then satisfy the properties noted in Lemma 1 for any choice of nonempty closed sets $\tilde{B} \subset B$ and $\tilde{S} \subset S$. As the following argument only uses these properties, we may then simplify notation by considering the case $\phi = \phi_{B,S}$ and $\psi = \psi_{S,B}$.

Let $\underline{i} < \overline{i} \in N$ and $\underline{j} < \overline{j} \in N$. If there exist utility levels $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in \mathbb{R}$ such that

$$\underline{u} = \phi(\underline{i}, \overline{j}, \overline{v}) \geq \phi(\underline{i}, \underline{j}, \underline{v}), \tag{80}$$

$$\overline{u} = \phi(\overline{i}, \underline{j}, \underline{v}) \geq \phi(\overline{i}, \overline{j}, \overline{v}), \tag{81}$$

then we say that the pairs $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ are a *cross match*. We may apply the inverse and monotonicity relationships in Lemma 1 to obtain that $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ are a cross match if and only if there exist utility levels $\underline{u}, \overline{u}, \underline{u}, \overline{u} \in \mathbb{R}$ such that

$$\underline{v} = \psi(\underline{j}, \overline{i}, \overline{u}) \geq \psi(\underline{j}, \underline{i}, \underline{u}), \tag{82}$$

$$\overline{v} = \psi(\overline{j}, \underline{i}, \underline{u}) \geq \psi(\overline{j}, \overline{i}, \overline{u}). \tag{83}$$

To motivate the terminology of a cross match, observe, first, that the equalities in the above conditions indicate that the utility levels are chosen in such a way that they are consistent with the agents in the pairs $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ matching with each other and choosing exchange efficient exchanges. Second, the inequalities indicate that if the agents under consideration were matched in this way, then no agent has an incentive to switch partners.

We say that a cross match can be *uncrossed* if the inequalities in (80)–(81) (or, equivalently, the inequalities in (82)–(83)) can only hold as equalities, indicating that the agents in the cross match can be reassigned to form matches $(\underline{i}, \underline{j})$ and $(\overline{i}, \overline{j})$ without changing their payoffs. If a cross match cannot be uncrossed, then the strict Pareto property (or, more formally, Lemma 1) implies that the pairs $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ are a *strict cross match*, meaning that u_1, u_2, v_1, v_2 can be chosen such that the inequalities in (80)–(83) hold strictly.

LEMMA 5: *Let Assumption 1 hold. Then the functions ϕ and ψ satisfy strict single crossing if and only if there exist no cross matches. They satisfy single crossing if and only if every cross match can be uncrossed.*

PROOF: The result for strict single crossing is immediate from the definitions.

Suppose there exists a cross match that cannot be uncrossed. Then, as noted above, there exists a strict cross match $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ with

$$\phi(\underline{i}, \bar{j}, \bar{v}) > \phi(\underline{i}, \underline{j}, \underline{v}),$$

$$\phi(\bar{i}, \underline{j}, \underline{v}) > \phi(\bar{i}, \bar{j}, \bar{v}),$$

contradicting the single crossing condition (72). Hence, if single crossing holds, then every cross match can be uncrossed. To prove the reverse implication, suppose the single crossing condition (72) fails. Then there exist $\underline{i} < \bar{i}$, $\underline{j} < \bar{j}$ and $\underline{v}$, $\bar{v}$ such that

$$\phi(\underline{i}, \bar{j}, \bar{v}) \geq \phi(\underline{i}, \underline{j}, \underline{v}),$$

$$\phi(\bar{i}, \bar{j}, \bar{v}) < \phi(\bar{i}, \underline{j}, \underline{v}).$$

Upon setting $\underline{u} = \phi(\underline{i}, \bar{j}, \bar{v})$ and $\bar{u} = \phi(\bar{i}, \underline{j}, \underline{v})$, we then have a cross match that cannot be uncrossed. *Q.E.D.*

C.2. *Proof of Lemma 4*

Let $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ be a fully matched ex ante equilibrium. Suppose J is strictly increasing. Because J is a measure-preserving bijection, this implies that J is the identity function, ensuring that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is positive assortative.

We may thus suppose that J is not strictly increasing or, equivalently, that there exist $\underline{i} < \bar{i}$ and $\underline{j} < \bar{j}$ such that $(\underline{i}, \bar{j}) \in M$ and $(\bar{i}, \underline{j}) \in M$ hold. Using the incentive constraints (21), every such pair of matches satisfies

$$\mathbf{u}(\underline{i}) = \phi\big(\underline{i}, \bar{j}, \mathbf{v}(\bar{j})\big) \geq \phi\big(\underline{i}, \underline{j}, \mathbf{v}(\underline{j})\big), \tag{84}$$

$$\mathbf{u}(\bar{i}) = \phi\big(\bar{i}, \underline{j}, \mathbf{v}(\underline{j})\big) \geq \phi\big(\bar{i}, \bar{j}, \mathbf{v}(\bar{j})\big), \tag{85}$$

so that $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ are a cross match. We refer to a cross match in which (84)–(85) hold as an *equilibrium cross match*.

From Lemma 5, the existence of an equilibrium cross match contradicts strict single crossing. Hence, if strict single crossing holds, $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is positive assortative and the proof of the strict single crossing result in Lemma 4 is finished.

Suppose N is finite. Then the conclusion of Lemma 4 is immediate from the ability to uncross any given cross match asserted in Lemma 5: We can start with the lowest buyer–seller pair and proceed upward until we find a pair that is not matched to each other. This pair must then be part of an equilibrium cross match, which we can uncross. We can repeat this exercise, doing so at most finitely many times, until arriving at a payoff-equivalent ex ante equilibrium featuring the identity matching.

To finish the proof, it remains to consider the case in which N is an interval and show that $(J, I, \mathbf{b}, \mathbf{s}, \mathbf{u}, \mathbf{v})$ is payoff equivalent to a positive assortative allocation when single crossing holds. Without loss of generality we let $N = [0, 1]$. The incentive constraints (21)–(22) imply

$$\mathbf{u}(i) \geq \phi\big(i, i, \mathbf{v}(i)\big),$$
$$\mathbf{v}(j) \geq \psi\big(j, j, \mathbf{u}(j)\big),$$

for all i and j. If all these inequalities hold as equalities, then it is clear that the equilibrium is payoff equivalent to an equilibrium satisfying positive assortative matching. We accordingly suppose there exists $i \in [0, 1]$ such that buyer i and seller i cannot achieve their equilibrium payoffs when matched to each other, that is,

$$\mathbf{u}(i) > \phi\big(i, i, \mathbf{v}(i)\big). \tag{86}$$

We show that this leads to a contradiction.

The inequality in (86) implies $J(i) \neq i$ and $I(i) \neq i$. If i were part of an equilibrium cross match, then Lemma 5 implies that this cross match could be uncrossed, contradicting (86). Hence, we must either have $J(i) < i < I(i)$ or the reverse chain of inequalities. We focus on the first of these cases throughout the following (with the case $I(i) < i < J(i)$ following from an analogous argument, swapping the roles of buyers and sellers throughout the following). This gives us the configuration illustrated in Figure 2.

If $J(i') > i$ holds for some $i' < i$, then, because $i > J(i)$ holds, we have an equilibrium cross match with pairs $(\underline{i}, \bar{j}) = (i', J(i'))$ and $(\bar{i}, \underline{j}) = (i, J(i))$. We can uncross to match i with $J(i')$ while preserving payoffs. This gives us an equilibrium cross match with pairs $(\underline{i}, \bar{j}) = (i, J(i'))$ and $(\bar{i}, \underline{j}) = (I(i), i)$ which we can uncross to obtain a contradiction to (86). Hence, we have that $i' < i$ implies $J(i') < i$.

As the equilibrium matching is measure preserving, $J(i') < i$ for all $i' < i$ implies that $I(j) < i$ holds for almost all sellers $j < i$. We can thus choose a sequence $\{j_n\}_{n=1}^{\infty}$ of sellers with $j_n > J(i)$ and $j_n \nearrow i$ and $i_n = I(j_n) \leq i$ for all n. As (i_n, j_n) are matched, the equilibrium feasibility conditions (23) and the incentive constraints (21)–(22) imply

$$\mathbf{u}(i_n) = \phi\big(i_n, j_n, \mathbf{v}(j_n)\big) \geq \phi\big(i_n, J(i), \mathbf{v}\big(J(i)\big)\big).$$

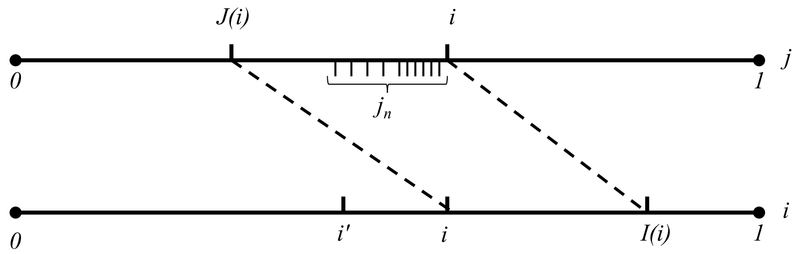

FIGURE 2.—Illustration for the proof of Lemma 4 when $N = [0, 1]$. We hypothesize the existence of a buyer i matched with seller $J(i) < i$, and suppose that buyer i and seller i cannot achieve their equilibrium payoffs when matched with one another. We first invoke the cross match argument of Section C.1 to conclude that every buyer $i' < i$ is matched with a seller $J(i') < i$, and hence feasibility requires that (almost) all sellers $j < i$ must be matched with buyers less than i. This allows us to consider a sequence $\{j_n\}$ of sellers whose types converge to i. Each such seller must be matched with a buyer less than i. We use these converging sequences of matched pairs and the continuity of ϕ to derive a contradiction.

Because $i \geq i_n$, and $j_n > J(i)$ holds, the single crossing property (72) implies that the above weak inequality also holds for i. We can then use $\mathbf{u}(i) = \phi(i, J(i), \mathbf{v}(J(i)))$ to obtain $\mathbf{u}(i) \leq \phi(i, j_n, \mathbf{v}(j_n))$, and the equilibrium incentive constraints then imply

$$\mathbf{u}(i) = \phi\big(i, j_n, \mathbf{v}(j_n)\big)$$

for all n. The continuity of ϕ and $\mathbf{v}$, established in Lemma 2, along with $j_n \nearrow i$, then ensures

$$\lim_{n\to\infty} \phi\big(i, j_n, \mathbf{v}(j_n)\big) = \phi\big(i, i, \mathbf{v}(i)\big) = \mathbf{u}(i).$$

The second of these equalities contradicts (86), finishing the proof. *Q.E.D.*

APPENDIX D: APPENDIX FOR SECTION 4.3.2

Let Assumption 1 hold and let preferences be separable. Then the functions ρ and σ appearing in (76)–(77) satisfy the counterparts to the properties established for ϕ and ψ in Lemma 1.

We define a *cross match in investments* in analogy to the cross matches introduced in Appendix C.1, namely as a pair of investment choices $(\underline{b}, \overline{s})$ and $(\overline{b}, \underline{s})$ with $\underline{b} < \overline{b}$ and $\underline{s} < \overline{s}$ such that there exist $\underline{f}, \overline{f}, \underline{g}, \overline{g} \in \mathbb{R}$ satisfying

$$\underline{f} = \rho(\underline{b}, \overline{s}, \overline{g}) \geq \rho(\underline{b}, \underline{s}, \underline{g}), \tag{87}$$

$$\overline{f} = \rho(\overline{b}, \underline{s}, \underline{g}) \geq \rho(\overline{b}, \overline{s}, \overline{g}). \tag{88}$$

As in Appendix C.1, we say that a cross match in investments can be uncrossed if the inequalities in (87)–(88) can only hold as equalities and observe that a cross match in investments cannot be uncrossed if and only if it is strict, that is, there exist $\underline{f}$, $\overline{f}$, $\underline{g}$, and $\overline{f}$ such that both inequalities in (87)–(88) hold strictly. As the proof of Lemma 5 relied solely on Lemma 1, the following is then immediate:

LEMMA 6: *Let Assumption 1 hold and let preferences be separable. Then preferences satisfy inner single crossing if and only if every cross match in investments can be uncrossed.*

D.1. *Proof of Corollary 5*

We fix a pair of nonempty closed sets $\tilde{B} \subset B$ and $\tilde{S} \subset S$ and, to simplify notation, let $\phi = \phi_{\tilde{B},\tilde{S}}$ and $\psi = \psi_{\tilde{B},\tilde{S}}$. From Lemma 5 in Appendix C.1, it suffices to show that, for these utility frontiers, every cross match can be uncrossed or, equivalently, that the existence of a strict cross match leads to a contradiction.

Suppose that the pairs $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ with $\underline{i} < \overline{i}$ and $\underline{j} < \overline{j}$ are a strict cross match. That is, there exist $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in \mathbb{R}$ such that

$$\underline{u} = \phi(\underline{i}, \overline{j}, \overline{v}) > \phi(\underline{i}, \underline{j}, \underline{v}), \tag{89}$$

$$\overline{u} = \phi(\overline{i}, \underline{j}, \underline{v}) > \phi(\overline{i}, \overline{j}, \overline{v}). \tag{90}$$

Let $\underline{\beta} = \boldsymbol{\beta}(i)$, $\overline{\beta} = \boldsymbol{\beta}(i)$, $\underline{\sigma} = \boldsymbol{\sigma}(\underline{j})$, and $\overline{\sigma} = \boldsymbol{\sigma}(\overline{j})$. From Assumption 4(ii), we have $\underline{\beta} < \overline{\beta}$ and $\underline{\sigma} < \overline{\sigma}$. Consider any pair of exchanges $(\underline{b}, \overline{s}, t_1)$ and $(\overline{b}, \underline{s}, t_2)$ such that

$$\underline{u} = \hat{U}\big(\hat{f}(\underline{b}, \overline{s}, t_1), \underline{b}, \underline{\beta}\big), \quad \overline{u} = \hat{U}\big(\hat{f}(\overline{b}, \underline{s}, t_2), \overline{b}, \overline{\beta}\big),$$
$$\underline{v} = \hat{V}\big(\hat{g}(\underline{s}, \overline{b}, t_2), \underline{s}, \underline{\sigma}\big), \quad \overline{v} = \hat{V}\big(\hat{g}(\overline{s}, \underline{b}, t_1), \overline{s}, \overline{\sigma}\big)$$

hold (the existence of such exchanges is assured by the definition of ϕ). Let $\underline{f} = \hat{f}(\underline{b}, \overline{s}, t_1)$, $\overline{f} = \hat{f}(\underline{b}, \overline{s}, t_2)$, $\underline{g} = \hat{g}(\underline{s}, \overline{b}, t_2)$, and $\overline{g} = \hat{g}(\overline{s}, \underline{b}, t_1)$. By definition of ρ, we have

$$\underline{f} = \rho(\underline{b}, \overline{s}, \overline{g}) \quad \text{and} \quad \overline{f} = \rho(\overline{b}, \underline{s}, \underline{g}).$$

Using separability of the sellers' preferences, we have the inequalities

$$\phi(\underline{i}, \underline{j}, \underline{v}) \geq \hat{U}\big(\rho(\overline{b}, \underline{s}, \underline{g}), \overline{b}, \underline{\beta}\big), \tag{91}$$

$$\phi(\overline{i}, \overline{j}, \overline{v}) \geq \hat{U}\big(\rho(\underline{b}, \overline{s}, \overline{g}), \overline{b}, \overline{\beta}\big), \tag{92}$$

$$\phi(\underline{i}, \underline{j}, \underline{v}) \geq \hat{U}\big(\rho(\underline{b}, \underline{s}, \underline{g}), \underline{b}, \underline{\beta}\big), \tag{93}$$

$$\phi(\overline{i}, \overline{j}, \overline{v}) \geq \hat{U}\big(\rho(\overline{b}, \overline{s}, \overline{g}), \overline{b}, \overline{\beta}\big). \tag{94}$$

Combining the strict inequalities in (89)–(90) with (91)–(92), we obtain

$$\hat{U}(\underline{f}, \underline{b}, \underline{\beta}) > \hat{U}(\overline{f}, \overline{b}, \underline{\beta}),$$
$$\hat{U}(\overline{f}, \overline{b}, \overline{\beta}) > \hat{U}(\underline{f}, \underline{b}, \overline{\beta}),$$

so that the outer single crossing property (74) implies $\underline{b} \leq \overline{b}$. Because $\hat{U}$ is strictly increasing in its first argument, $\underline{b} = \overline{b}$ is inconsistent with the above two inequalities holding simultaneously. We thus have $\underline{b} < \overline{b}$. We can repeat this argument using the equivalent restatement of (89)–(90) for ψ and the outer single crossing condition (75) for the seller to obtain the inequality $\underline{s} < \overline{s}$.

Combining the strict inequalities in (89)–(90) with (93)–(94), we obtain

$$\hat{U}\big(\rho(\underline{b}, \overline{s}, \overline{g}), \underline{b}, \underline{\beta}\big) > \hat{U}\big(\rho(\underline{b}, \underline{s}, \underline{g}), \underline{b}, \underline{\beta}\big),$$
$$\hat{U}\big(\rho(\overline{b}, \underline{s}, \underline{g}), \overline{b}, \overline{\beta}\big) > \hat{U}\big(\rho(\overline{b}, \overline{s}, \overline{g}), \overline{b}, \overline{\beta}\big).$$

Because $\hat{U}$ is strictly increasing in its first argument, this implies

$$\rho(\underline{b}, \overline{s}, \overline{g}) > \rho(\underline{b}, \underline{s}, \underline{g}),$$
$$\rho(\overline{b}, \underline{s}, \underline{g}) > \rho(\overline{b}, \overline{s}, \overline{g}).$$

Hence, $(\underline{s}, \overline{b})$ and $(\overline{b}, \underline{s})$ are a strict cross match in investments. From Lemma 6, this contradicts the inner single crossing condition (76). *Q.E.D.*

D.2. *Proof of Corollary 6*

It suffices to show that the inner single crossing condition (77) holds. As single crossing is an ordinal property and F and G are strictly increasing, we may assume that F and G are the identity functions. We then have

$$\sigma(s, b, x) = g(s, b) + h\big(f(b, s) - x\big).$$

Let $\underline{s} < \overline{s}$, $\underline{b} < \overline{b}$, and $x_1, x_2 \in \mathbb{R}$ satisfy

$$g(\underline{s}, \overline{b}) + h\big(f(\overline{b}, \underline{s}) - x_1\big) = g(\underline{s}, \underline{b}) + h\big(f(\underline{b}, \underline{s}) - x_2\big). \tag{95}$$

We show that this implies

$$g(\overline{s}, \overline{b}) + h\big(f(\overline{b}, \overline{s}) - x_1\big) \geq g(\overline{s}, \underline{b}) + h\big(f(\underline{b}, \overline{s}) - x_2\big), \tag{96}$$

which (because of continuity and monotonicity in x) suffices for σ as given above to satisfy the inner single crossing condition (77).

From (95), we have

$$g(\underline{s},\overline{b})-g(\underline{s},\underline{b})=h\big(f(\underline{b},\underline{s})-x_2\big)-h\big(f(\overline{b},\underline{s})-x_1\big)\geq 0, \tag{97}$$

where the inequality holds because g is increasing in b. As h is increasing, this implies

$$f(\underline{b},\underline{s})-x_2\geq f(\overline{b},\underline{s})-x_1.$$

Because f is supermodular, we have

$$\big[f(\underline{b},\underline{s})-x_2\big]-\big[f(\overline{b},\underline{s})-x_1\big]\geq\big[f(\underline{b},\overline{s})-x_2\big]-\big[f(\overline{b},\overline{s})-x_1\big],$$

and because f is increasing in s, we have

$$f(\overline{b},\overline{s})-x_1\geq f(\overline{b},\underline{s})-x_1,$$
$$f(\underline{b},\overline{s})-x_2\geq f(\underline{b},\underline{s})-x_2.$$

From the concavity of h, these inequalities imply

$$h\big(f(\underline{b},\underline{s})-x_2\big)-h\big(f(\overline{b},\underline{s})-x_1\big)$$
$$\geq h\big(f(\underline{b},\overline{s})-x_2\big)-h\big(f(\overline{b},\overline{s})-x_1\big).$$

Using supermodularity of g and (97), this suffices to give (96), finishing the proof. *Q.E.D.*

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Manuscript received March, 2014; final revision received April, 2014.