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Article

Coulomb Solutions from Improper Pseudo-Unitary Free Gauge Field Operator Translations

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Abstract: Fundamental problems of quantum field theory related to the representation problem of canonical commutation relations are discussed within a gauge field version of a van Hove-type model. The Coulomb field generated by a static charge distribution is described as a formal superposition of time-like pseudo-photons in Fock space with a Krein structure. In this context, a generalization of operator gauge transformations is introduced to generate coherent states of Abelian gauge fields interacting with a charged background.

Keywords: model quantum field theories; canonical quantization of gauge theories; infrared divergences; gauge symmetries

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1. Introduction

There is now some interest in the occupation numbers of micro-states in classical field configurations in the context of the entropy of black holes. Some recent works discussed Coulomb fields as a toy model that connects classical and quantum concepts [1,2]. However, many of the hitherto presented approaches have a formal character and neglect mathematical facts and insights that are deeply rooted in the fundamental aspects of quantum field theory. There is a problem when one wants to count particles in an interacting theory if the particle notion is based on a Fock space concept and the interaction picture, as it is expressed by Haag's theorem [3]. However, Haag's theorem relies on translation invariance and does not directly apply to the Coulomb field. Invoking it in the case of [1,2] to draw any conclusions might therefore be inappropriate. However, it seems that interacting entities are not simply composed of non-interacting entities. Still, there is an urgent need for the human mind to deconstruct and count the parts of the surrounding world.

Another aspect of this insight is related to the classification problem of canonical (anti-)commutation relations and the concept of myriotic fields, since in the quantum field theoretical case of infinitely many degrees of freedom von Neumann's uniqueness theorem breaks down [4–7]. It can be shown under natural requirements that the *formal* canonical commutation relations (CCR) for the position coordinate and conjugate momentum operators of a physical system with F degrees of freedom

$$[q_l, q_m] = 0\,, \quad [p_l, p_m] = 0\,, \quad [p_l, q_m] = -i\delta_{lm}\,, \quad l, m = 1, \dots F \tag{1}$$

fix the representations of the self-adjoint operators p_l, q_m under mild natural requirements as generators of unitary transformations on a Hilbert space up to unitary equivalence, provided F is finite. Already for the case $F = 1$ it is straightforward to show that an algebra fulfilling the commutation relations Equation (1) cannot be represented by operators defined on a finite-dimensional Hilbert space $\mathcal{H}_f$, since ($\hbar = 1$)

$$tr[q, p] = tr(qp) - tr(pq) = 0 \neq i \cdot tr(1) = i \cdot dim(\mathcal{H}_f) \tag{2}$$

By substituting

$$a_l = (p_l - iq_l)/\sqrt{2}\,, \quad a_l^\dagger = (p_l + iq_l)/\sqrt{2} \tag{3}$$

to obtain creation and destruction operators, one easily derives that the eigenvalues of the occupation operator $N_l = a_l^\dagger a_l$ are non-negative integers. Choosing an occupation number distribution $\{n\}$, which is an infinite sequence of such integers in the case $F = \infty$

$$\{n\} = \{n_1, n_2, \dots\} \tag{4}$$

one may divide the set of such sequences into classes such that $\{n\} \sim \{n'\}$ are in the same class iff they differ only in a finite number of places. In the Fock space $\mathcal{F}$, only normalized state vectors

$$N_l \Psi_{\{n^{\mathcal{F}}\}} = n_l^{\mathcal{F}} \Psi_{\{n^{\mathcal{F}}\}} \tag{5}$$

corresponding to an occupation number distribution $\{n^{\mathcal{F}}\}$ with

$$\sum_k n_k^{\mathcal{F}} < \infty \tag{6}$$

are allowed to form a complete orthonormal basis in $\mathcal{F}$. However, an occupation number distribution from a different class $\{n\} \nsim \{n^{\mathcal{F}}\}$ also spans a representation space of the a_l, $a_l^\dagger$ and it is evident that representations belonging to different classes cannot be unitarily equivalent since the creation and destruction operators change $\{n\}$ only in one place. An explicit physical example for this problem will be constructed in this paper.

According to a systematic study concerning the classification of irreducible representations of canonical (anti-)commutation relations by Garding and Wightman [8,9], a complete and practically usable list of representations appears to be inaccessible. Some interesting comments on the position and momentum operators in wave mechanics can be found in the Appendix.

In flat classical space-time, the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ which is a semidirect product of the Abelian group of space-time translations $T_{1,3}$ and the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$

$$\mathcal{P}_+^\uparrow = T_{1,3} \rtimes \mathcal{L}_+^\uparrow \cong T_{1,3} \rtimes SO^+(1,3) \cong T_{1,3} \rtimes SO(3,\mathbb{C}) \tag{7}$$

is the internal symmetry group of the theory. Relative state phases play an important role in quantum theory, but since the global phase of a physical system, represented by a ray in a Hilbert space, is not observable, the Poincaré group ray representations underlying a relativistic quantum field theory can be realized by necessarily infinite dimensional representations of the covering group $\bar{\mathcal{P}}_+^\uparrow \cong T_{1,3} \rtimes SL(2,\mathbb{C})$ due to a famous theorem by Wigner [10,11]. The actual definition of a particle in non-gravitating flat space-time becomes a non-trivial task when charged particles coupling to massless (gauge) fields become involved. Based on the classical analysis of Wigner on the unitary representations of the Poincaré group, a one-particle state describing a particle of mass m alone in the world is an element of an irreducible representation space of the double cover of the Poincaré group in a physical Hilbert space, *i.e.*, some irreducible representations should occur in the discrete spectrum of the mass-squared operator $M^2 = P_\mu P^\mu$ of a relativistic quantum field theory describing interacting fields. One should note here that the particles in the present sense like, e.g., a neutron or an atom, can be viewed as composite objects, and the notion *elementary system* might be more appropriate. Then, objects like quark and gluons can be viewed as elementary particles, although they do not appear in the physical spectrum of the Standard Model. The job of the corresponding elementary fields as carriers of charges is rather to implement the principle of causality and to allow for a kind of coordinatization of an underlying physical theory and to finally extract the algebra of observables. The type and number of the elementary fields appearing in a theory is rather unrelated to the physical spectrum of empirically observable particles, *i.e.*, elementary systems.

Furthermore, (idealized) objects like the electron are accompanied by a long range field that leads an independent life at infinite spatial distance, to give an intuitive picture. It has been shown in [12] that a discrete eigenvalue of M^2 is absent for states with an electric charge as a direct consequence of Gauss' law, and one finds that the Lorentz symmetry is not implementable in a sector of states with nonvanishing electric charge, an issue that also will be an aspect of the forthcoming discussion. Such problems are related to the fact that the Poincaré symmetry is an overidealization related to global considerations of infinite flat space-time, whereas physical measurements have a local character. The expression *infraparticle* has been coined for charged particles like the electron accompanied by a dressing field of massless particles [13].

Still, concentrating on Wigner's analysis of the representations that make sense from a physical point of view, *i.e.*, singling out tachyonic or negative energy representations and ignoring infraparticle aspects, the unitary and irreducible representations of $\bar{\mathcal{P}}_+^\uparrow$ can be classified in the massive case, loosely speaking, by a real mass parameter $m^2 > 0$ and a (half-)integer spin parameter s. In the massless case, the unitary irreducible representations of $\bar{\mathcal{P}}_+^\uparrow$, which have played an important rôle in quantum field theory so far, are those that describe particles with a given non-negative (half-)integer helicity.

However, one should not forget that there exist so-called infinite spin representations $V_{\Xi,\alpha}$ of $\bar{\mathcal{P}}_+^\uparrow$ [14], which are related to the so-called string-localized quantum fields [15]. These representations can be labeled by two parameters: $0 < \Xi < \infty$ and $\alpha \in \{0, \frac{1}{2}\}$. The representations describe massless objects with a spin operator along the momentum having the unbounded spectrum $\{0, \pm 1, \pm 2, \ldots\}$ for $\alpha = 0$ and $\{\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots\}$ for $\alpha = \frac{1}{2}$. There are still ongoing investigations in order to find out whether string-localized quantum fields will have any direct application in future quantum field theories [16]. Since the infinite spin representations can be distinguished by the continuous parameter Ξ, they are also called *continuous spin* representations, a naming that sometimes leads to some confusion about the helicity spectrum, which is quantized but infinite.

2. The Electromagnetic Field

In order to fix some notational conventions, we shortly mention the well-known fact that Maxwell's equations in pre-relativistic vector notation

$$div\vec{E} = 0 \tag{8}$$

$$rot\vec{B} - \dot{\vec{E}} = 0 \tag{9}$$

$$div\vec{B} = 0 \tag{10}$$

$$rot\vec{E} + \dot{\vec{B}} = 0 \tag{11}$$

describing the dynamics of the real classical electromagnetic fields

$$\vec{E} = (E^1, E^2, E^3)\,, \quad \vec{B} = (B^1, B^2, B^3) \tag{12}$$

in vacuo can be written by the help of the electromagnetic field strength tensor F with contravariant components

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{13}$$

such that Equations (8) and (9), which become the inhomogeneous Maxwell equations in the presence of electric charges, read

$$\partial_\mu F^{\mu\nu}(x) = 0 \tag{14}$$

whereas the homogeneous Equations (10) and (11) can be written by the help of the completely antisymmetric Lorentz-invariant Levi-Civita pseudo-tensor ϵ in four dimensions with $\epsilon^{0123} = 1 = -\epsilon_{0123}$

$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}(x) = 0 \tag{15}$$

Cartesian Minkowski coordinates x have been introduced above where the speed of light is equal to one such that $x = (t, \vec{x}) = (x^0, x^1, x^2, x^3) = (x_0, -x_1, -x_2, -x_3)$ and $\partial_\mu = \partial/\partial x^\mu$.

Introducing the gauge vector field or four-vector potential A containing the electrostatic potential Φ and the magnetic vector potential $\vec{A}$ and skipping space-time arguments for notational simplicity again

$$A^\mu = (\Phi, \vec{A}) \tag{16}$$

the electric and magnetic fields can be represented via

$$\vec{E} = -grad\,\Phi - \dot{\vec{A}} = -\vec{\nabla}\Phi - \partial_0 \vec{A}\,, \quad \vec{B} = rot\vec{A} = \vec{\nabla} \times \vec{A} \tag{17}$$

or

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{18}$$

Now, Equations (10) and (11) are automatically satisfied by the definitions in Equation (17), since $rot\,grad \equiv 0$, $div\,rot \equiv 0$

$$rotE = -rot\,grad\,\Phi - rot\dot{\vec{A}} = -\dot{\vec{B}}\,, \quad div\vec{B} = div\,rot\vec{A} = 0 \tag{19}$$

and Equations (8) and (9) become

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0 \tag{20}$$

Adding the gradient of an arbitrary real analytic scalar field χ to the gauge field according to the gauge transformation

$$A^\mu \;\to\; A^\mu_g = A^\mu + \partial^\mu \chi \tag{21}$$

leaves $F^{\mu\nu}$ invariant since

$$F^{\mu\nu}_g = \partial^\mu (A^\nu + \partial^\nu \chi) - \partial^\nu (A^\mu + \partial^\mu \chi) = F^{\mu\nu} \tag{22}$$

One may assume that all fields are analytic and vanish at spatial or temporal infinity rapidly or reasonably fast. This would exclude global gauge transformations where $0 \neq \chi = const$. A strong requirement like rapid decrease also implicitly dismisses infrared problems. Still, the possibility to perform a gauge transformation according to Equation (21) makes it obvious that Equation (20) does not fix the dynamics of the gauge field A^μ. Since for a pure gauge $A^\mu_{pg} = \partial^\mu \chi$

$$\Box \partial^\nu \chi - \partial^\nu \partial_\mu \partial^\mu \chi = 0 \tag{23}$$

the gauge field can be modified in a highly arbitrary manner by the gradient of a scalar function, irrespective of the initial conditions which define the gauge field on, e.g., a spacelike hyperplane, where the scalar field can be set to zero. That is, the zeroth component of Equation (20) reads

$$\partial_\mu F^{\mu 0} = \Box A^0 - \partial^0 \partial_\nu A^\nu = -\Delta A^0 - div\dot{A} = div(-grad\Phi - \dot{A}) = divE = 0 \tag{24}$$

so there is no equation describing the dynamic evolution of the electrostatic potential $A^0 = \Phi$.

The standard way out of this annoying situation in quantum field theory, where the gauge field is an operator valued distribution, is to modify Equation (20) by coupling the four-divergence of the electromagnetic field strength tensor to an unphysical current term j_{unph}, which in the case of the so-called Feynman gauge is chosen according to

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu \partial_\mu A^\mu = -\partial^\nu \partial_\mu A^\mu = j^\nu_{unph} \tag{25}$$

such that the equations governing the dynamics of the gauge field A^μ describing a non-interacting massless spin-1 field from a more general point of view become

$$\Box A^\mu = 0 \tag{26}$$

On the classical level, such a modification can be easily justified by the argument that the four-divergence of the gauge field A^μ can be gauged away by a suitable scalar χ that solves

$$\Box \chi = -\partial_\mu A^\mu \tag{27}$$

such that for the gauge transformed field $A^\mu_g = A^\mu + \partial^\mu \chi$ one has

$$\partial_\mu A^\mu_g = \partial_\mu (A^\mu + \partial^\mu \chi) = 0 \tag{28}$$

Using the retarded propagator Δ_0^{ret} defined by

$$\Delta_0^{ret}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 + ik^0 0} = -\frac{1}{2\pi} \Theta(x^0) \delta(x^2) \tag{29}$$

fulfilling the inhomogeneous wave equation

$$\Box \Delta_0^{ret}(x) = -\delta^{(4)}(x) \tag{30}$$

χ in Equation (27) is given by

$$\chi(x) = \int d^4x' \, \Delta_0^{ret}(x - x') \partial_\mu A^\mu(x') + \chi_0(x) \tag{31}$$

with any χ_0 fulfilling $\Box \chi_0(x) = 0$. The formal strategy described above works well even after quantization for QED. However, when gauge fields couple to themselves, special care is needed.

In the presence of a conserved four-current j^ν

$$\partial_\mu F^{\mu\nu} = j^\nu \,, \quad \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0 \tag{32}$$

holds, and invoking the Lorenz condition $\partial_\mu A^\mu = 0$ leads to

$$\Box A^\mu = j^\mu \tag{33}$$

The main motivation for the introduction of gauge fields is to maintain explicit locality and manifest covariance in the quantum field theoretical description of their corresponding interactions. An inversion of Equation (18) up to a pure gauge is given by

$$A^\mu(x) = \int_0^1 d\lambda \, \lambda F^{\mu\nu}(\lambda x) x_\nu \tag{34}$$

but such a term would look rather awkward when substituted in an elegant expression like the Dirac equation. The Ehrenberg–Siday–Aharonov–Bohm effect [17,18] also indicates that the gauge vector field A may play a rather fundamental role in the description of elementary particle interactions. Many physicists feel that the classical or quantum degrees of freedom of the gauge field A are somehow physical, despite the fact that they are only virtual. Still, the observable Coulomb field generated by a spherically symmetric charge distribution cannot be composed of real, asymptotic photons, since such states rather allow for the construction of Glauber states with an electric field perpendicular to the field momentum. One also should be cautious to consider a gauge field less physical than the field strength tensor, since the latter also is no longer gauge invariant in the interacting, non-Abelian case. Finally, the quantum field theoretical Ehrenberg–Siday–Aharonov–Bohm effect is not completely understood as long as no non-trivial interacting quantum field theory in four space-time dimensions has been constructed at all.

An elegant way to describe the two helicity states of a massless photon is obtained from combining the electric and magnetic field into a single photon wave function [19]

$$\Psi = \frac{1}{\sqrt{2}}(\vec{E} + i\vec{B}) \,, \quad i^2 = -1 \tag{35}$$

Hence, the Maxwell–Faraday equation and Ampère's circuital law *in vacuo* can be cast into the equation of motion

$$\frac{\partial \Psi}{\partial t} = -i \cdot \nabla \times \Psi \tag{36}$$

This was already recognized in lectures by Riemann in the nineteenth century [20]. Taking the divergence of Equation (35)

$$\nabla \cdot \dot{\Psi} = -i \cdot \nabla \cdot (\nabla \times \Psi) = 0 \tag{37}$$

readily shows that the divergence of the electric and magnetic field is conserved. Therefore, if the analytic condition

$$div\, \vec{E} = div\, \vec{B} = 0 \tag{38}$$

holds due to the absence of electric or magnetic charges on a space-like slice of space-time, it holds everywhere.

The Field Equation (36) and condition (38) single out the helicity eigenstates of the photon wave function that are admissible for massless particles according to Wigner's analysis of the unitary representations of the Poincaré group, e.g., a circularly polarized (right-handed) plane wave moving in positive x^3-direction is given by

$$\Psi_R(x) = N(k^0) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{ik^3x^3 - ik^0x^0} = N(k^0)(\hat{e}_1 + i\hat{e}_2) e^{ik^3x^3 - ik^0x^0} \,, \quad k^0 = k^3 > 0 \tag{39}$$

where $N(k^0)$ is a normalization factor, whereas the corresponding left-handed plane wave is given by

$$\Psi_L(x) = N(k^0) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{-ik^3x^3 + ik^0x^0} \,, \quad k^0 = k^3 > 0 \tag{40}$$

If the right-handed wave moves in negative x^3-direction ($k^3 < 0$), one has

$$\Psi_R(x) = N(k^0) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{ik^3x^3 - ik^0x^0} \,, \quad k^0 = |k^3| > 0 \tag{41}$$

The presence of electric charges and the absence of magnetic charges breaks the gauge symmetry of Equation (36)

$$\Psi \mapsto e^{i\alpha}\Psi \,, \quad \alpha \in \mathbb{R} \tag{42}$$

Introducing antisymmetric matrices Σ_1, Σ_2, and Σ_3 defined by the totally antisymmetric tensor in three dimensions $\varepsilon_{lmn} = \frac{1}{2}(l-m)(m-n)(n-l)$

$$(\Sigma_l)_{mn} = i\varepsilon_{lmn} \tag{43}$$

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \,, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \,, \quad \Sigma_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{44}$$

Equation (36) can be written in the form ($\partial_j = \partial/\partial x^j$, $j = 1, 2, 3$)

$$\frac{\partial \Psi}{\partial t} = \Sigma_j \partial_j \Psi \tag{45}$$

or, defining matrices Γ^μ by $\Gamma^0 = \mathbb{1}_3$, where $\mathbb{1}_3$ denotes the 3×3 identity matrix, and $\Gamma_j = \Sigma_j = -\Gamma^j$ for $j = 1, 2, 3$, Equation (36) finally reads

$$i\Gamma^\mu \partial_\mu \Psi = 0 \tag{46}$$

The field components of Ψ covariantly transform under the representation of $SO^+(1,3)$ by the isomorphic complex orthogonal group $SO(3, \mathbb{C})$, preserving the conditions imposed by Equation (38).

It has been shown in [21] that a mass term for the Ψ-field like

$$i\Gamma^\mu \partial_\mu \Psi - m\Psi = 0 \tag{47}$$

is incompatible with the relativistic invariance of the field equation. As a more general approach one may introduce an (anti-)linear operator S and make the ansatz

$$i\Gamma^\mu \partial_\mu \Psi - mS\Psi = 0 \tag{48}$$

which also fails. Already on the classical level, one should be cautious to consider a massless theory as the limit of a massive theory, which in the case above even does not exist in a naive sense.

What remains in the quantized versions of the classical approaches touched above is the problem that the use of point-like localized gauge fields is in conflict with the positivity and unitarity of the Hilbert space and leads to the introduction of Krein structures within a BRS formalism, whereas positivity of the Hilbert space avoiding unphysical degrees of freedom like in non-covariant Coulomb gauges necessitates the introduction of a rather awkward non-local formalism.

3. Lorentz-Covariant Quantization of the Free Gauge Field

We quantize the free gauge field as four independent scalar fields in Feynman gauge according to the canonical commutation relations

$$A^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left[a^\mu(\vec{k}) e^{-ikx} + a^\mu(\vec{k})^K e^{ikx} \right] = A^\mu(x)^K \tag{49}$$

where $kx = k_\mu x^\mu = k^0 x^0 - \vec{k}\vec{x} = g_{\mu\nu} k^\mu x^\nu$ and $k^0 = k_0 = \omega(\vec{k}) = |\vec{k}|$ with creation and annihilation operators satisfying

$$[a^\mu(\vec{k}), a^\nu(\vec{k}')^\dagger] = (2\pi)^3 2\omega(\vec{k}) \delta^{\mu\nu} \delta^{(3)}(\vec{k} - \vec{k}') \tag{50}$$

$$[a^\mu(\vec{k}), a^\nu(\vec{k}')] = [a^\mu(\vec{k})^\dagger, a^\nu(\vec{k}')^\dagger] = 0 \tag{51}$$

and all annihilation operators acting on the unique Fock–Hilbert vacuum $|0\rangle$ according to

$$a^\mu(\vec{k})|0\rangle = 0 \tag{52}$$

The K-conjugation introduced above is necessary due to relativistic covariance and is related to Hermitian conjugation by

$$\begin{aligned} a_0(\vec{k})^K &= -a_0(\vec{k})^\dagger \,, \quad a_{1,2,3}(\vec{k})^K = a_{1,2,3}(\vec{k})^\dagger \\ a_0^\dagger(\vec{k})^K &= -a_0(\vec{k}) \,, \quad a_{1,2,3}^\dagger(\vec{k})^K = a_{1,2,3}(\vec{k}) \end{aligned} \tag{53}$$

such that the operator valued distributions $A^\mu(x)$ are acting on a Fock–Hilbert space $\mathcal{F}$ with positive-definite norm, and since the free field

$$A^0(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left[a^0(\vec{k}) e^{-ikx} - a^0(\vec{k})^\dagger e^{ikx} \right] = -A^0(x)^\dagger \tag{54}$$

is anti-Hermitian and due to the commutation relations

$$[a^\mu(\vec{k}), a^\nu(\vec{k}')^K] = -(2\pi)^3 2\omega(\vec{k}) g^{\mu\nu} \delta^{(3)}(\vec{k} - \vec{k}') \tag{55}$$

$$[a^\mu(\vec{k}), a^\nu(\vec{k}')] = [a^\mu(\vec{k})^K, a^\nu(\vec{k}')^K] = 0 \tag{56}$$

the gauge field has Lorentz-invariant commutators given by the (positive- and negative-) frequency Pauli–Jordan distributions $\Delta_0^{(\pm)}$

$$[A^\mu(x),\, A^\nu(y)] = -ig^{\mu\nu} \Delta_0(x - y) \tag{57}$$

with the commutators of the absorption and emission parts alone

$$[A_-^\mu(x),\, A_+^\nu(y)] = -ig^{\mu\nu} \Delta_0^+(x - y) \tag{58}$$

$$[A_+^\mu(x),\, A_-^\nu(y)] = -ig^{\mu\nu} \Delta_0^-(x - y) \tag{59}$$

The massless Pauli–Jordan distributions in configuration space are

$$\Delta_0(x) = -\frac{1}{2\pi} \mathrm{sgn}(x^0) \delta(x^2) \tag{60}$$

$$\Delta_0^{\pm}(x) = \pm \frac{i}{4\pi^2} \frac{1}{(x_0 \mp i0)^2 - \vec{x}^{\,2}} \tag{61}$$

Defining the involutive, unitary and Hermitian time-like photon number parity operator η defined via the densely defined unbounded photon number operator

$$N_0 = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \, a_0^{\dagger}(\vec{k}) a_0(\vec{k}) \tag{62}$$

by

$$\eta = (-1)^{N_0} = e^{i\pi N_0} = e^{-i\pi N_0} = \eta^{-1} = \eta^{\dagger} \tag{63}$$

one notes that η anticommutes with $a_0(\vec{k})$ and $a_0^{\dagger}(\vec{k})$, since the creation and annihilation operators change the time-like particle number by one, and the K-conjugation can be defined for an operator A via

$$A^K = \eta A^{\dagger} \eta \tag{64}$$

η can be used to define a Krein space $\mathcal{F}_K$ by introducing the indefinite inner product [22,23]

$$(\Phi, \Psi) = \langle \Phi | \eta \Psi \rangle \,, \quad \Phi,\, \Psi \in \mathcal{F} \tag{65}$$

on $\mathcal{F}$, where $\langle \cdot | \cdot \rangle$ denotes the positive definite scalar product on the Hilbert space $\mathcal{F}$.

4. Charge and Gauge Transformations as Field Translations

Defining the self-adjoint field translation operator Q with four test functions $q_\mu(\vec{k})$, $\mu = 0, 1, 2, 3$, in the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^3)$ according to

$$Q = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left[q_\nu^*(\vec{k}) a^\nu(\vec{k}) - q_\nu(\vec{k}) a^\nu(\vec{k})^{\dagger} \right] \tag{66}$$

leads to the non-covariant commutation relations

$$\begin{aligned} \left[Q, a^\mu(\vec{k}) \right] &= \frac{i}{(2\pi)^3} \int \frac{d^3k'}{2k'^0} \left[q_\nu^*(\vec{k}') a^\nu(\vec{k}') - q_\nu(\vec{k}') a^\nu(\vec{k}')^{\dagger}, a^\mu(\vec{k}) \right] \\ &= i \int d^3k' q_\nu(\vec{k'}) \delta^{\mu\nu} \delta^{(3)}(\vec{k} - \vec{k}') = i g^{\mu\mu} q^\mu(\vec{k}) \end{aligned} \tag{67}$$

and

$$\begin{aligned} \left[Q, a^\mu(\vec{k})^{\dagger} \right] &= \frac{i}{(2\pi)^3} \int \frac{d^3k'}{2k'^0} \left[q_\nu^*(\vec{k}') a^\nu(\vec{k}') - q_\nu(\vec{k}') a^\nu(\vec{k}')^{\dagger}, a^\mu(\vec{k})^{\dagger} \right] \\ &= i \int d^3k' q_\nu^*(\vec{k'}) \delta^{\mu\nu} \delta^{(3)}(\vec{k} - \vec{k}') = i g^{\mu\mu} q^\mu(\vec{k})^* \end{aligned} \tag{68}$$

The K-symmetric field translation operator $\tilde{Q}$ defined by

$$\tilde{Q} = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left[q_\nu^*(\vec{k}) a^\nu(\vec{k}) - q_\nu(\vec{k}) a^\nu(\vec{k})^K \right] \tag{69}$$

has the commutators

$$\left[\tilde{Q}, a^\mu(\vec{k})\right] = \frac{i}{(2\pi)^3}\int \frac{d^3k'}{2k^0}\left[q^*_\nu(\vec{k'})a^\nu(\vec{k'}) - q_\nu(\vec{k'})a^\nu(\vec{k'})^K, a^\mu(\vec{k})\right]$$
$$= -i\int d^3k' q_\nu(\vec{k'})g^{\mu\nu}\delta^{(3)}(\vec{k}-\vec{k'}) = -iq^\mu(\vec{k}) \tag{70}$$

and

$$\left[\tilde{Q}, a^\mu(\vec{k})^K\right] = \frac{i}{(2\pi)^3}\int \frac{d^3k'}{2k^0}\left[q^*_\nu(\vec{k'})a^\nu(\vec{k'}) - q_\nu(\vec{k'})a^\nu(\vec{k'})^K, a^\mu(\vec{k})^K\right]$$
$$= -i\int d^3k' q^*_\nu(\vec{k'})g^{\mu\nu}\delta^{(3)}(\vec{k}-\vec{k'}) = -iq^\mu(\vec{k})^* \tag{71}$$

Accordingly, one has

$$[Q, A_0(x)] = \frac{i}{(2\pi)^3}\int \frac{d^3k}{2k^0}[q_0(\vec{k})e^{-ikx} - q^*_0(\vec{k})e^{ikx}] \tag{72}$$

$$[Q, A_j(x)] = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{2k^0}[q_k(\vec{k})e^{-ikx} + q^*_k(\vec{k})e^{ikx}]\,, \quad j = 1,2,3 \tag{73}$$

but

$$[\tilde{Q}, A_0(x)] = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{2k^0}[q_0(\vec{k})e^{-ikx} + q^*_0(\vec{k})e^{ikx}] \tag{74}$$

$$[\tilde{Q}, A_j(x)] = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{2k^0}[q_k(\vec{k})e^{-ikx} + q^*_k(\vec{k})e^{ikx}]\,, \quad j = 1,2,3 \tag{75}$$

The operators Q and $\tilde{Q}$ generate unitary and pseudo-unitary transformations U and $\tilde{U}$, respectively

$$U = e^{iQ}\,, \quad \tilde{U} = e^{i\tilde{Q}} \tag{76}$$

with

$$U^\dagger = U^{-1}\,, \quad \tilde{U}^K = \tilde{U}^{-1} \tag{77}$$

The creation and annihilation operators transform according to

$$Ua^\mu(\vec{k})U^{-1} = a^\mu(\vec{k}) + i[Q, a^\mu(\vec{k})] = a^\mu(\vec{k}) - g^{\mu\mu}q^\mu(\vec{k}) \tag{78}$$

$$Ua^\mu(\vec{k})^\dagger U^{-1} = a^\mu(\vec{k})^\dagger + i[Q, a^\mu(\vec{k})^\dagger] = a^\mu(\vec{k})^\dagger - g^{\mu\mu}q^\mu(\vec{k})^* \tag{79}$$

since higher commutator terms vanish in the equations above, and furthermore

$$\tilde{a}^\mu(\vec{k}) = \tilde{U}a^\mu(\vec{k})\tilde{U}^{-1} = a^\mu(\vec{k}) + i[\tilde{Q}, a^\mu(\vec{k})] = a^\mu(\vec{k}) + q^\mu(\vec{k}) \tag{80}$$

$$\tilde{a}^\mu(\vec{k})^K = \tilde{U}a^\mu(\vec{k})^K\tilde{U}^{-1} = a^\mu(\vec{k})^K + i[\tilde{Q}, a^\mu(\vec{k})^K] = a^\mu(\vec{k})^K + q^\mu(\vec{k})^* \tag{81}$$

The vector potential transforms according to

$$A'^\mu(x) = UA^\mu(x)U^{-1} = A^\mu(x) + i[Q, A^\mu(x)] \tag{82}$$

and

$$\tilde{A}^\mu(x) = \tilde{U}A^\mu(x)\tilde{U}^{-1} = A^\mu(x) + i[\tilde{Q}, A^\mu(x)] \tag{83}$$

i.e., $\tilde{A}^0(x)$ acquires a real expectation value $q^0(x)$ on the Fock vacuum $|0\rangle$ since

$$\tilde{A}^0(x) = A^0(x) + \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0}[q_0(\vec{k})e^{-ikx} + q_0^*(\vec{k})e^{ikx}] = A^0(x) + q^0(x) = \tilde{A}^0(x)^K \tag{84}$$

whereas the unitary transformation $A^0(x) \to A'^0(x)$ preserves the skew-adjointness of A^0.

One may notice that in the case where $q^\mu(x) = \partial^\mu\chi(x)$ with a smooth scalar χ rapidly decreasing in space-like directions and fulfilling the wave equation $\Box\chi(x) = 0$, $\tilde{Q}$ becomes a BRST-generator $\tilde{Q}_g$ of free field gauge transformations [24]. Introducing emission and absorption operators for unphysical photons, which are combinations of time-like and longitudinal states, according to

$$b_{1,2} = (a_\parallel \pm a_0)/\sqrt{2}\,, \quad a_\parallel = k_j a^j/|\vec{k}| \tag{85}$$

or

$$b_1 = \frac{k_\mu a^\mu}{\sqrt{2}k_0}\,, \quad b_2^\dagger = \frac{k^\mu a_\mu^K}{\sqrt{2}k_0} \tag{86}$$

satisfying ordinary commutation relations

$$[b_i(\vec{k}), b_j^\dagger(\vec{k})] = (2\pi)^3 2k^0\delta_{ij}\delta^{(3)}(\vec{k} - \vec{k}') \tag{87}$$

one has

$$\partial_\mu A^\mu(x) = -\frac{i}{\sqrt{2}(2\pi)^3}\int d^3k\big[b_1(\vec{k})e^{-ikx} - b_2(\vec{k})^\dagger e^{ikx}\big] \tag{88}$$

and

$$b_{1,2}^K = (a_\parallel^\dagger \mp a_0^\dagger)/\sqrt{2} = b_{2,1}^\dagger\,, \quad \partial^\mu A_\mu^K = \partial^\mu A_\mu \tag{89}$$

The free physical sector $\mathcal{F}_{phys} \subset \mathcal{F}$ contains no free unphysical photons

$$|\Phi\rangle \in \mathcal{F}_{phys} \;\Leftrightarrow\; b_1(\vec{k})|\Phi\rangle = b_2(\vec{k})|\Phi\rangle = 0\ \forall\vec{k} \tag{90}$$

A quantum gauge transformation

$$A_g^\mu(x) = A^\mu(x) + \partial^\mu\chi(x) \tag{91}$$

with

$$\chi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0}\big[\chi(\vec{k})e^{-ikx} + \chi^*(\vec{k})e^{ikx}\big] \tag{92}$$

such that $\chi(x)$ fulfills the wave equation $\Box\chi(x) = 0$ and

$$\partial_\mu\chi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0}\big[-ik_\mu\chi(\vec{k})e^{-ikx} + (-ik_\mu\chi(\vec{k}))^* e^{ikx}\big] \tag{93}$$

is generated by

$$\tilde{Q}_g = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0}\big[\chi(\vec{k})^* k_\nu(\vec{k})a^\nu(\vec{k}) + \chi(\vec{k})k_\nu(\vec{k})a^\nu(\vec{k})^K\big] \tag{94}$$

or

$$\tilde{Q}_g = -\frac{1}{\sqrt{2}(2\pi)^3}\int d^3k\big[\chi(\vec{k})^* b_1(\vec{k}) + \chi(\vec{k})b_2^\dagger(\vec{k})\big] \tag{95}$$

$$\tilde{Q}_g = -\frac{1}{\sqrt{2}(2\pi)^3}\int d^3k\big[\chi(\vec{k})^* b_1(\vec{k}) + \chi(\vec{k})b_1^K(\vec{k})\big] \tag{96}$$

where $q_\nu(\vec{k})$ has been replaced by $-ik_\nu\chi(\vec{k})$ in Equation (69). Furthermore, introducing the gauge current

$$j_g^\mu(x) = \chi(x)\overleftrightarrow{\partial^\mu}\partial_\nu A^\nu(x) \tag{97}$$

satisfying the continuity equation

$$\partial_\mu j_g^\mu(x) = \partial_\mu(\chi(x)\partial^\mu\partial_\nu A^\nu(x) - \partial^\mu\chi(x)\partial_\nu A^\nu(x)) = 0 \tag{98}$$

the conserved gauge charge $\tilde{Q}_g$ can be expressed by [25]

$$\tilde{Q}_g = \int\limits_{x^0=const.} d^3x\, j_g^0(x^0,\vec{x}) = \int\limits_{x^0=const.} d^3x\, \chi(x)\overleftrightarrow{\partial^0}\partial_\nu A^\nu(x) \tag{99}$$

A generalization of the gauge transformations generated by $\tilde{Q}_g$ to non-Abelian gauge theories including ghost fields has been used in [26] to derive the classical Lie-structure of gauge theories like QCD from pure quantum principles. A further generalization to massive QED can be found in [27]; the Standard Model with a special focus on the electroweak interaction and the Higgs field mechanism is discussed in detail in [28].

5. Static Fields

The field translation operators introduced above modify the free field $A^\mu(x)$ by additional classical fields $q^\mu(x)$, which are solutions of the wave equation. This minor defect if one wants to describe static fields can be remedied by adding a time-dependence to the classical q^μ-fields, which become $\tilde{q}^\mu(x^0,\vec{k}) = q^\mu(\vec{k})e^{ik^0x^0}$. With the sometimes more suggestive notation $t = x^0$, $\omega = k^0 = |\vec{k}|$ and the definitions

$$\tilde{Q}(t) = \frac{i}{(2\pi)^3}\int\frac{d^3k}{2k^0}[q_\nu^*(\vec{k})a^\nu(\vec{k})e^{-i\omega t} - q_\nu(\vec{k})a^\nu(\vec{k})^K e^{+i\omega t}] \tag{100}$$

$$\tilde{U}(t) = e^{i\tilde{Q}(t)} \tag{101}$$

follows

$$\tilde{A}^0(x) = A^0(x) + \frac{1}{(2\pi)^3}\int\frac{d^3k}{2k^0}[q_0(\vec{k})e^{i\vec{k}\vec{x}} + q_0^*(\vec{k})e^{-i\vec{k}\vec{x}}] = A^0(x) + q^0(\vec{x}) \tag{102}$$

The well-known distributional (Fourier transform) identities related to the Coulomb field of a point-like charge

$$\int d^3x\,\frac{e^{\pm i\vec{k}\vec{x}}}{|\vec{x}|} = \frac{4\pi}{|\vec{k}|^2}\,, \quad \int d^3x\, e^{\pm i\vec{k}\vec{x}}\Delta\frac{1}{|\vec{x}|} = -4\pi\,, \quad \Delta\frac{1}{|\vec{x}|} = -4\pi\delta^{(3)}(\vec{x}) \tag{103}$$

and

$$V_C(\vec{x}) = \frac{e}{4\pi}\frac{1}{|\vec{x}|} = \frac{1}{(2\pi)^3}\int d^3k\,\frac{e^{\pm i\vec{k}\vec{x}}}{|\vec{k}|^2} \tag{104}$$

can be used to construct a field operator containing a Coulomb field centered at $\vec{x} = 0$ as an expectation value ($k^0 = |\vec{k}|$)

$$A_c^\mu(x) = A^\mu(x) + \delta_0^\mu\frac{e}{4\pi}\frac{1}{|\vec{x}|} = A^\mu(x) + \delta_0^\mu\frac{1}{(2\pi)^3}\int\frac{d^3k}{k^0}\frac{e^{i\vec{k}\vec{x}}}{|\vec{k}|}$$

$$= A^\mu(x) + \delta_0^\mu \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left\{ \frac{1}{|\vec{k}|} e^{+i\vec{k}\vec{x}} + \frac{1}{|\vec{k}|} e^{-i\vec{k}\vec{x}} \right\} \tag{105}$$

fulfilling the inhomogeneous wave equation

$$\Box A_c^\mu(x) = e\delta^{(3)}(\vec{x}) \tag{106}$$

i.e., one has

$$\tilde{q}^\mu(t,\vec{k}) = (\tilde{q}^0(t,\vec{k}),\vec{0}) \,, \quad \tilde{q}^0(t,\vec{k}) = \frac{e^{i\omega t}}{|\vec{k}|} \tag{107}$$

The time-dependence of $\tilde{q}^\mu(t,\vec{k})$ could be interpreted as originating from a kind of binding energy, which reduces the energy of non-interacting time-like pseudo-photons from $\hbar|\vec{k}|c$ to zero when they are bound in a Coulomb field generated by a point-like charge e. In fact, in order to have the correct dynamical time evolution, the Hamiltonian for non-interacting photons must be

$$H = \frac{1}{(2\pi)^3} \sum_{\mu=0}^{3} \int \frac{d^3k}{2k^0} \, \omega(\vec{k}) a_\mu^\dagger(\vec{k}) a_\mu(\vec{k}) = \frac{1}{2(2\pi)^3} \sum_{\mu=0}^{3} \int d^3k \, a_\mu^\dagger(\vec{k}) a_\mu(\vec{k}) \tag{108}$$

and the improper wave function of a free time-like one-photon state $|\vec{k},0\rangle = a_0^\dagger(\vec{k})|0\rangle$ is given by

$$\varphi_{\vec{k}}^0(x) = \langle 0|A^0(x)|\vec{k},0\rangle = \langle 0|A^0(x)a_0^\dagger(\vec{k})|0\rangle = e^{-ikx} \tag{109}$$

normalized according to

$$\langle \vec{k},0|\vec{k}',0\rangle = i \int d^3x \, \varphi_{\vec{k}}^0(x)^* \overset{\leftrightarrow}{\partial_0} \varphi_{\vec{k}'}^0(x) = (2\pi)^3 2k^0 \delta^{(3)}(\vec{k}-\vec{k}') \tag{110}$$

6. Particle Numbers

The field operator $A_c^\mu(x)$ represents a solution of the field equations for the electromagnetic field interacting with an infinitely heavy point-like charged spinless particle residing at $\vec{x}=\vec{0}$. However, the time-like pseudo-photon number operators

$$N_0 = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} a_0^\dagger(\vec{k}) a_0(\vec{k}) \,, \quad \tilde{N}_0(t) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \tilde{a}_0^\dagger(\vec{k}) \tilde{a}_0(\vec{k}) \tag{111}$$

can be written in terms of the untransformed operators as

$$(2\pi)^3 \tilde{N}_0(t) = \int \frac{d^3k}{2k^0} (a_0(\vec{k})^\dagger + \tilde{q}_0(t,\vec{k})^*)(a_0(\vec{k}) + \tilde{q}_0(t,\vec{k}))$$

$$= (2\pi)^3 N_0 + \int \frac{d^3k}{2k^0} a_0(\vec{k})^\dagger \tilde{q}_0(t,\vec{k}) + \int \frac{d^3k}{2k^0} a_0(\vec{k}) \tilde{q}_0(t,\vec{k})^* + \int \frac{d^3k}{2k^0} |\tilde{q}_0(t,\vec{k})|^2 \tag{112}$$

Alternatively, one may write

$$(2\pi)^3 N_0 = (2\pi)^3 \tilde{N}_0(t) - \int \frac{d^3k}{2k^0} \tilde{a}_0(t,\vec{k})^\dagger \tilde{q}_0(t,\vec{k}) - \int \frac{d^3k}{2k^0} \tilde{a}_0(t,\vec{k}) \tilde{q}_0(t,\vec{k})^* + \int \frac{d^3k}{2k^0} |\tilde{q}_0(t,\vec{k})|^2 \tag{113}$$

The displaced vacuum $|\tilde{0}(t)\rangle = \tilde{U}(t)|0\rangle$, which is time-dependent and permanently modified by the charge, is formally annihilated by the pseudo-unitarily displaced destruction operators $\tilde{a}_0(\vec{k})$

$$\tilde{a}_0(t,\vec{k})|\tilde{0}(t)\rangle = \tilde{U}(t)a_0(\vec{k})\tilde{U}(t)^{-1}\tilde{U}(t)|0\rangle = \tilde{U}(t)a_0(\vec{k})|0\rangle = 0 \tag{114}$$

but it is not Poincaré invariant. It contains infinitely many non-interacting time-like photons, since Equation (113) implies

$$\langle\tilde{0}(t)|N_0|\tilde{0}(t)\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0}|\tilde{q}_0(t,\vec{k})|^2 \tag{115}$$

which is clearly divergent in the presence of a charge $e \neq 0$. However, one should be cautious about the calculations presented above. In fact, the $\tilde{U}(t)$ are improper pseudo-unitary transformations, since they are not properly defined on the originally introduced Fock space $\mathcal{F}$. The pseudo-unitarily inequivalent representations (PUIR) of the canonical quantum field commutation relations induced by the $\tilde{U}(t)$ relate different spaces at different times. This also becomes clear if one realizes that the $a_\mu(\vec{k})^\dagger$ are operator valued distributions [29], such that Equation (69) defines an operator in the sense of a linear operator densely defined on $\mathcal{F}$ if the q_μ are Schwartz test functions, *i.e.*, when the $a_\mu(\vec{k})^\dagger$ are smeared with smooth functions of rapid decrease. Coulomb fields do not belong to this class of functions.

Screening the Coulomb according to

$$V_C^{scr}(\vec{x}) = \frac{e}{4\pi|\vec{x}|}e^{-\mu|\vec{x}|}(1-e^{-(m-\mu)|\vec{x}|})\,, \quad m \gg \mu > 0 \tag{116}$$

does help, but all divergences reappear in the limit $\mu \to 0$ or $m \to \infty$. Smearing the point-like charge only solves the short-distance (ultraviolet) problems and is related to renormalization issues in quantum field theory.

It is interesting to note that

$$\langle 0|\tilde{N}_0(t)|0\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0}|\tilde{q}_0(t,\vec{k})|^2 \tag{117}$$

i.e., from the point of view of the theory where the gauge field interacts with an infinitely heavy charge e, the free Fock vacuum $|0\rangle$ contains infinitely many "$\tilde{a}^\dagger$-particles". Additionally, $\langle\tilde{0}(t)|0\rangle = 0$ holds for $t \neq 0$.

Questions concerning the vacuum structure as a ground state in a new physical sector and a potential non-canonical behavior of the formal construction above shall not be discussed here. Still, it should be taken into account that charge screening *is* physical. Considering quantum electrodynamics restricted to a sector of neutral states with an electromagnetic field decaying faster than the Coulomb field of a charge distribution with non-zero total charge is fully sufficient to describe the physics of the photon and charged particles interaction. The scattering process of two electrons is not really affected by two positrons located very far away, rendering the whole system neutral. A problem with the description of charged states by local physical operator valued distributions can be highlighted by the following formal calculation. An operator C carries an elementary charge e, if

$$[\mathcal{Q}, C] = eC \tag{118}$$

where $\mathcal{Q}$ denotes the electric charge operator, since if physical states always carry integer multiples of the elementary charge, they are eigenstates of $\mathcal{Q}$ and C increases the charge of a state ψ with charge e_0 by e

$$\mathcal{Q}\psi = e_0\psi\,, \quad \mathcal{Q}C\psi = (e+e_0)C\psi \tag{119}$$

hence

$$\mathcal{Q}C\psi - C\mathcal{Q}\psi = (e+e_0-e_0)C\psi = eC\psi \tag{120}$$

From the charge density operator $j_0(x^0,\vec{x})$ one has formally

$$\mathcal{Q} = \lim_{R\to\infty} \int\limits_{|\vec{x}|<R} j_0(x^0,\vec{x})\, d^3x = \lim_{R\to\infty} \mathcal{Q}_R \tag{121}$$

To put it more correctly, one may consider $j_0(x^0,\vec{x})$ as an operator valued distribution ($x^0 = ct, c = 1$)

$$\mathcal{Q}_R = \int j_0(x^0,\vec{x}) f_R(\vec{x})\alpha(t)\, d^3x\, dt \tag{122}$$

with test functions ($\epsilon > 0$)

$$f_R(\vec{x}) = f(|\vec{x}|/R) \in \mathcal{D}(\mathbb{R}^3)\,, \quad f(x) = \begin{cases} 1 & : \ |x| < 1 \\ 0 & : \ |x| > 1+\epsilon \end{cases} \tag{123}$$

$$\alpha(t) \in \mathcal{D}(\mathbb{R})\,, \quad \int\limits_{-\infty}^{\infty} \alpha(t)dt = 1 \tag{124}$$

Insisting on the Gauss' law for local physical operator-valued distributions describing electric currents and electromagnetic fields

$$j_\mu(x) = \partial^\nu F_{\nu\mu}(x) \tag{125}$$

implies by partial integration

$$\begin{aligned} \lim_{R\to\infty} [\mathcal{Q}_R, C] &= \lim_{R\to\infty} \int d^3x\, dt\, f_R(\vec{x})\alpha(t)[\partial^j F_{j0}(t,\vec{x}), C] \\ &= \lim_{R\to\infty} \int d^3x\, dt\, \partial_j f_R(\vec{x})\alpha(t)[F_{j0}(t,\vec{x}), C] \end{aligned} \tag{126}$$

However, $\vec{\nabla} f_R(\vec{x}) \neq 0$ only holds for $R < |\vec{x}| < R(1+\epsilon)$. For local field operators

$$C = \int d^4x\, C(x)g(x)\,, \quad g \in \mathcal{D}(\mathbb{R}^4) \tag{127}$$

one has a for sufficiently large R a space-like separation of the supports supp$(\vec{\nabla} f_R(\vec{x})\alpha(t))$ and supp(g). Due to causality, one has from the vanishing commutators ($\vec{x} \in$ supp$(\vec{\nabla} f_R)$) and Equation (126)

$$[F_{j0}(t,\vec{x}), C] \stackrel{R\to\infty}{\to} 0\,, \quad \textit{i.e.}, \quad [\mathcal{Q}, C] = \lim_{R\to\infty} [\mathcal{Q}_R, C] = 0 \tag{128}$$

hence C is uncharged. The argument above also works for test functions of rapid decrease in Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$. An electron alone in the world cannot be created, and an accompanying infinitely extended Coulomb field does not exist. Many problems in the quantum field theory stem from the overidealization that by translation invariance extended systems are considered over infinite space and time regions. However, counting (unphysical) photons in a restricted space region might make sense.

7. Conclusions

Already in 1952, van Hove investigated a model where a neutral scalar field interacts with a source term describing an infinitely heavy, recoilless or static point-like nucleon [30]. There he showed that the Hilbert space of the free scalar field is "orthogonal" to the Hilbert space of states of the field interacting with the point source. His finding finally lead to what is called today Haag's theorem. This theorem has been formulated in different versions, but basically it states that there is no proper unitary operator that connects the Fock representation of the CCR of a non-interacting quantum field theory with the Hilbert space of a corresponding theory that includes a non-trivial interaction. Furthermore, the interacting Hamiltonian is not defined on the Hilbert space on which the non-interacting Hamiltonian is defined. The present paper generalizes van Hove's model by including gauge fields. One should remark that gauge transformations generated by an operator like $\tilde{Q}_g$ in Equation (94) can be implemented on one Fock–Hilbert space since smearing free field operators with test functions defined on a three-dimensional space-like plane already gives some well-defined operators, although in principle the fields are operator-valued distributions on four-dimensional Minkowski space.

A formal way out of the lost cause of unitary inequivalent representations (UIR) is possibly provided by causal perturbation theory introduced in a classic paper by Epstein and Glaser [31]. In the traditional approach to quantum field theory, one starts from classical fields and a Lagrangian that includes distinguished interaction terms. The formal free field part of the theory gets quantized and perturbative S-matrix elements or Greens functions are constructed with the help of the Feynman rules based on a Fock space description. For instance, a typical model theory often used in theoretical considerations is the (massless) Φ^3-theory, where the interaction Hamiltonian density is given by the normally ordered third order monomial of a free uncharged (massless) scalar field and a coupling constant λ

$$\mathcal{H}_{int}(x) = \frac{\lambda}{3!} : \Phi(x)^3 : \tag{129}$$

The perturbative S-Matrix is then constructed according to the expansion

$$S = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1^4 ... dx_n^4 T\{\mathcal{H}_{int}(x_1)\mathcal{H}_{int}(x_2) \cdot ... \cdot \mathcal{H}_{int}(x_n)\} \tag{130}$$

where T is the time-ordering operator. It must be pointed out that the perturbation series Equation (130) is formal and it is difficult to make any statement about the convergence of this series, but it is erroneously hoped that S reproduces the full theory.

On the perturbative level, two problems arise in the expansion given above. First, the time-ordered products

$$T_n(x_1, x_2, ..., x_n) = (-i)^n T\{\mathcal{H}_{int}(x_1)\mathcal{H}_{int}(x_2) \cdot ... \cdot \mathcal{H}_{int}(x_n)\} \tag{131}$$

are usually plagued by ultraviolet divergences. However, these divergences can be removed by regularization to all orders if the theory is renormalizable, such that the operator-valued distributions T_n can be viewed as well-defined, already regularized expressions. Second, infrared divergences are also present in Equation (130). This is not astonishing, since the T_n's are operator-valued distributions, and therefore must be smeared out by test functions in $\mathcal{S}(\mathbb{R}^{4n})$. One may therefore introduce a test function

$g(x) \in \mathcal{S}(\mathbb{R}^4)$ that plays the role of an "adiabatic switching" and provides a cutoff in the long-range part of the interaction, which can be considered as a natural infrared regulator [31,32]. Then, according to Epstein and Glaser, the infrared regularized S-matrix is given by

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1^4 ... dx_n^4 T_n(x_1, ..., x_n) g(x_1) \cdot ... \cdot g(x_n) \tag{132}$$

and an appropriate adiabatic limit $g \to 1$ must be performed at the end of actual calculations in the right quantities (like cross sections) where this limit exists. This is not one of the standard strategies usually found in the literature, but it is the most natural one in view of the mathematical framework used in perturbative quantum field theory. From a non-perturbative point of view, one may hope that taking matrix elements in the right quantities allows to reconstruct the full interacting Hilbert space.

Dütsch, Krahe and Scharf performed perturbative calculations for electron scattering off an electrostatic potential in the framework of causal perturbation theory [33]. It was found that in the adiabatic limit $g \to 1$ the electron scattering cross section is unique only if in the soft bremsstrahlung contributions from all four photon polarizations are included. Summing over two physical polarizations only, non-covariant terms survive in the physical observables. The adiabatic switching in the causal approach has the unphysical consequence that electrons lose their charge in a distant space-time region. This switching is moved from our local reality to infinity in the limit $g \to 1$. As long as $g \neq 1$, the decoupled photon field is no longer transversal, but also consists of scalar and longitudinal photons. In reality, these photons are confined to the charged particles and make them charged. Hence, there is some kind of confinement problem in QED. One must conclude that counting unphysical objects is a delicate task.

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Appendix

Position and Momentum Operators

In the case $F = 1$, the position and conjugate momentum operators q and p cannot be both bounded linear operators defined everywhere on a separable Hilbert space $\mathcal{H}$ [34,35], since in this case operator norms defined by the Hilbert space norm $|| \cdot ||$ induced by the scalar product on $\mathcal{H}$

$$||q|| = \sup_{\Psi \in \mathcal{H}} \frac{||q\Psi||}{||\Psi||}, \quad ||p|| = \sup_{\Psi \in \mathcal{H}} \frac{||p\Psi||}{||\Psi||} \tag{133}$$

would exist and by induction

$$[q, p^2] = p[q, p] + [q, p]p = 2ip \tag{134}$$

$$[q, p^3] = p[q, p^2] + [q, p]p^2 = 3ip^2 , \ldots \tag{135}$$

$$[q, p^n] = inp^{n-1} \tag{136}$$

would hold for $n \geq 1$. Since the operator norm is submultiplicative (e.g., $||qp|| \leq ||q||\,||p||$) and due to the Cauchy–Schwarz inequality, Equation (136) would imply

$$n||p^{n-1}|| = ||[q, p^n]|| \leq 2||q||\,||p^n|| \leq 2||q||\,||p||\,||p^{n-1}|| \leq C||p^{n-1}|| \tag{137}$$

for some constant C. For $n > C$ follows $p^{n-1} = 0$, and finally one is successively lead to a contradiction

$$[q, p^{n-1}] = 0 = i(n-1)p^{n-2} \Rightarrow p^{n-2} = 0\,, \ldots, p = 0 \text{ and } [q, p] = 0 \neq i \tag{138}$$

The representation problem of infinitely many dimensions encountered above does not show up in the case of the angular momentum algebra $\mathfrak{su}(2)$, where one has

$$[J_l, J_m] = i\varepsilon_{lmn}J_n\,, \quad \varepsilon_{lmn} = \frac{1}{2}(l-m)(m-n)(n-l) \tag{139}$$

since these relations can be realized by the help of the Pauli matrices $\{\sigma_l\}_{l=1,2,3}$ by setting $J_l = \frac{1}{2}\sigma_l$ acting as linear operators on $\mathbb{C}^2_{\mathbb{C}}$.

Fortunately, there exists a so-called Weyl form of the CCR [36], which uses unitary, *i.e.*, bounded and everywhere defined operators, only. Considering the unitary translation operator T_β acting on Lebesgue square integrable wave functions $\Psi \in \mathcal{L}^2(\mathbb{R})$ according to

$$T_\beta \Psi(q) = \Psi(q - \beta) \stackrel{!}{=} e^{-i\beta p}\Psi(q) \tag{140}$$

the phase operator $e^{-i\alpha q}$ is also unitary and therefore

$$T_\beta e^{-i\alpha q}\Psi(q) = e^{-i\alpha(q-\beta)}\Psi(q-\beta) = e^{-i\alpha q}e^{i\alpha\beta}T_\beta\Psi(q) \tag{141}$$

or

$$T_\beta e^{-i\alpha q} = e^{i\alpha\beta} \cdot e^{-i\alpha q}T_\beta \tag{142}$$

represents the Weyl form of the CCR for $F = 1$, which is mathematically much more robust than the better known form given by Equation (1). The self-adjoint momentum operator $p = -i\frac{d}{dq}$ is defined on a dense set $\mathcal{D}_p$ in the Hilbert space of wave functions $\mathcal{L}^2(\mathbb{R})$, where the expression

$$p = -i \lim_{\beta \to 0} \frac{id - T_\beta}{\beta} \tag{143}$$

makes sense, *i.e.*,

$$\mathcal{D}_p = \{\Psi \text{ absolutely continuous},\ d\Psi/dq \in \mathcal{L}^2(\mathbb{R})\} \tag{144}$$

and the originally formal exponential expression in Equation (140) becomes well-defined on the whole Hilbert space $\mathcal{L}^2(\mathbb{R})$. The canonical commutation relation

$$[q, p] = i \tag{145}$$

cannot hold on the whole Hilbert space $\mathcal{H}$, and Equation (145) represents a dubious statement as long the domain where it is defined is not discussed.

Conflicts of Interest

The author declares no conflict of interest.

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