Compositions and relations in the Cremona groups

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Contents

Contents

Introduction

The Cremona group is the group of algebraic symmetries of the affine n-dimensional space. More mathematically, the Cremona group is the group of birational transformations of the n-dimensional affine space $\mathbb{A}^n$ which are defined over some field k, i.e. "maps" of the form

$$f\colon \mathbb{A}^n \dashrightarrow \mathbb{A}^n, \quad (x_1,\dots,x_n) \dashrightarrow \left(\frac{f_1(x_1,\dots,x_n)}{g_1(x_1,\dots,x_n)},\dots,\frac{f_n(x_1,\dots,x_n)}{g_n(x_1,\dots,x_n)}\right)$$

for some polynomials $f_1,\dots,f_n,g_1,\dots,g_n \in \mathrm{k}[x_1,\dots,x_n]$, $g_1,\dots,g_n \neq 0$, such that there exists a "map" g of the same form and $f\circ g = g\circ f = \mathrm{Id}_{\mathbb{A}^n}$. These "maps" are not maps at all, since they are not defined at the points where all g_i vanish. However, they are well defined outside the common zero set of the g_i and there exist Zariski-open dense sets $U,V\subset\mathbb{A}^n$ such that $f|_U\colon U \dashrightarrow V$ is an isomorphism. These sets are in fact the open sets where the determinant of the differential of f (resp. g) does not vanish.

By homogenising, we obtain birational transformations of the n-dimensional projective space $\mathbb{P}^n$. Depending on the situation when studying such a birational transformation, it is useful to work with affine or projective coordiantes. We denote the Cremona group by

$$\text{Cremona group} = \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n),$$

although $\mathrm{Cr}_n(\mathrm{k})$, $\mathrm{Cr}_{\mathrm{k}}(n)$, $\mathrm{Bir}(\mathbb{P}^n_{\mathrm{k}})$ or $\mathrm{Bir}_{\mathrm{k}}(\mathbb{A}^n)$ are common notations as well.

Being the symmetry group of the simplest type of variety, the Cremona group is quite large and its group theoretic properties are closely related to the geometric properties of its elements. To work out properties of transformations, one has to study the geometric behaviour of the transformation on $\mathbb{P}^n$. The study of the Cremona group thus combines group theory and algebraic geometry. One big aim of algebraic geometry is to classify all algebraic varieties. Two varieties whose groups of birational self-maps are not isomorphic are not birational. Exploring the groups of birational transformations is therefore one way to check that two varieties are not in the same birational class. Studying large groups of birational self-maps is challenging, and the Cremona group is the most accessible large group of birational self-maps because one can use projective coordiantes. It is thus not surprising that it has been studied almost continuously for over hundred years; the Cremona group has become an object of its own interest and many questions are still open. For instance, no non-trivial generating set is known for $n\geq 3$. The Cremona groups can be endowed with the Zariski-topology, which allows to define morphisms from varieties to the Cremona group [Dem1970, Ser2008]. This opens the path to study the Cremona group in a topological setting. If the field is a local field (e.g. $\mathbb{R}$, $\mathbb{C}$), the Zariski topology can be refined to the Euclidean topology, which makes the Cremona group a Hausdorff topological group, and which restricted to any linear algebraic subgroup is the Euclidean topology [BlaFur2013]. This opens the path to study the Cremona group from the point of view of geometric group theory.

This thesis explores the plane Cremona group from the view point of generating sets and relations. Writing the Cremona group as quotient of a free group may make way to find quotients of the Cremona group itself or to study it from a geometric group theoretical aspect. Many generating sets and generating relations have been presented, the most fundamental one being the Noether-Castelnuovo theorem that first yielded a generating set of the plane Cremona group [Cas1901]: If k is algebraically closed, then $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and the standard Cremona involution. Presentations can for instance be found in [Giz1983, Isk1985, Isk1991, Wri1992, Bla2012], which may even come in the form of a structure theorem involving amalgamated products of two or three groups (see overview in Chapter II).

In this thesis, presentations of two plane Cremona groups are given; one for the field of complex numbers and one for the field of real numbers. The first is a presentations at the end of a long list of presentations and solely serves the purpose to show that $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is compactly presented when endowed with the Euclidean topology, which is a property of Lie groups (see Chapter III, corresponding to [Zim2016]). It shows that, although $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is not finite dimensional in any sense (see Example I.0.4), it is not far from being a Lie group. The second presentation is rather technical and is cooked up to find quotients of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$; it allows in fact to find the abelianisation homomorphism $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, from which one deduces that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ cannot be generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and countably many transformations, and obtains an infinite number of non-trivial proper normal subgroups for free (see Chapter IV, corresponding to [Zim2015]). That any plane Cremona group contains non-trivial proper normal subgroups had been a long open question and was recently proven in [CanLam2013, ShB2013, Lon2015], for respectively algebraically closed, finite and any fields, the last reference also giving explicit examples. The questions is still open for higher dimensions.

For $n \geq 3$, no non-trivial generating set of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is known, although it is known that it cannot be generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ and a countable number of elements, or any subset of bounded degree [Pan1999]. Currently, the only option to perhaps obtain information about the whole group is to study large families of transformations, specific subgroups or transformations whose properties stand out among the general throng of blurriness.

In this spirit, the last chapter leaves the plane and studies the family of punctual in $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$, $n \geq 2$, which are geometrically similar to plane Cremona transformations and for which there exist easy formulae for the degree and multiplicities of compositions, just like for $n = 2$. Any plane Cremona transformation is punctual, and for $n \geq 3$, the family of punctual transformation is a very small subset of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$. Their similarity to plane transformations makes it seems plausible that for $n \geq 3$, any punctual transformation is the composition of linear maps and the standard Cremona involution, as is the case for $n = 2$, and as was claimed in [Kan1897], although with an incomplete proof. The collection of properties listed in the last chapter might be a step towards proving or disproving the conjecture, and a tentative step towards understanding the geometry of birational maps of $\mathbb{P}^n$, $n \geq 3$.

The thesis is organised as follows: In Chapter I, a we remind of a few basic techniques to study birational transformations are recalled. Chapter II then reviews what is known about generating sets and relations of Cremona groups. The third chapter consists of the article [Zim2016] describing that the plane Cremona group over the field of complex numbers is compactly presented when endowed with the Euclidean topology. The fourth chapter consists of the article [Zim2015] that presents the abelianisation of the plane Cremona group over the field of real numbers. Chapter V then studies the set of punctual transformations and lists a few of their properties.

I Preliminaries

Throughout this chapter, k is any field, and any variety and rational map will be defined over k unless stated otherwise. By $\mathbf{k}$ we denote the algebraic closure of k.

Most definitions and lemmata in this chapter are classical and can be found in almost any introduction to algebraic geometry or surfaces.

Definition I.0.1. The group $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is the group of birational transformations of the n-dimensional projective space $\mathbb{P}^n$.

An element of $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is by definition given by

$$f\colon [x_0 : \cdots : x_n] \dashrightarrow [f_0(x_0, \ldots, x_n) : \cdots : f_n(x_0, \ldots, x_n)]$$

for some homogeneous polynomials $f_0, \ldots, f_n \in \mathrm{k}[x_0, \ldots, x_n]$ of equal degree with no common factors, such that there exists a transformation

$$g\colon [x_0 : \cdots : x_n] \dashrightarrow [g_0(x_0, \ldots, x_n) : \cdots : g_n(x_0, \ldots, x_n)]$$

where $g_1, \ldots, g_n \in \mathrm{k}[x_0, \ldots, x_n]$ are homogenous of equal degree without common factors, and $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{P}^n}$ is the identity map. We write $g = f^{-1}$ and define the degree of f to be

$$\deg(f) := \deg(f_i), \quad i = 0, \ldots, n.$$

The subvariety of $\mathbb{P}^n$ given by $f_0 = \cdots = f_n = 0$ is called the *indeterminacy-locus* of f. It is invariant by $\mathrm{Gal}(\mathbf{k}/\mathrm{k})$ and of codimension ≥ 2 [Sha1998, Vol. 1, Chapter II, §3.1, Theorem 3]. (The reference proves this for the algebraic closure $\mathbf{k}$ of k. However, codimension does not change when descending to k.) Composition of transformations makes $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ a group.

To obtain properties of the group and its elements, we study the associated linear system of an element f (see definition in Chapter I.2).

Example I.0.2. Any linear element of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is given by an element of $\mathrm{PGL}_{n+1}(\mathrm{k})$ and, vice versa, any element of PGL_{n+1} yields a linear transformation of $\mathbb{P}^n$:

$$\left([x_0 : \cdots : x_n] \mapsto [\sum_{j=0}^{n} a_{0j}x_j : \cdots : \sum_{j=0}^{n} a_{nj}x_j]\right) \longleftrightarrow (a_{ij})_{i,j=0}^{n} \in \mathrm{PGL}_{n+1}.$$

They are defined everywhere on $\mathbb{P}^n$ and are thus contained in the automorphism group

$$\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n) := \{f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n) \mid f, f^{-1} \text{ are defined everywhere}\}$$

of $\mathbb{P}^n$. On the other hand, any element of $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ has empty base-locus, which means that it is given by linear polynomials. In other words,

$$\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n) \simeq \mathrm{PGL}_{n+1}(\mathrm{k}).$$

If $n = 1$, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^1) = \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathrm{k})$ because the base-locus of a birational transformation of $\mathbb{P}^1$ is of codimension≥ 2 and thus empty.

Example I.0.3. The most simple non-linear transformation is the *standard Cremona involution*

$$[x_0 : \cdots : x_n] \dashrightarrow [\frac{1}{x_0} : \cdots : \frac{1}{x_n}] = [x_1 x_2 \ldots x_n : \cdots : x_0 \ldots \hat{x}_i \ldots x_n : \cdots : x_0 \ldots x_{n-1}]$$

It is of degree n and its base-locus is the union of the zero sets $x_i = x_j = 0$, $i \neq j$. Further, it contracts the hyperplane given by $x_i = 0$ onto the i-th coordinate point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ which has zero everywhere except at the i-th coordinate.

Example I.0.4. For any $p \in \mathrm{k}[x_2, \ldots, x_n]$, the transformation

$$(x_1, \ldots, x_n) \mapsto (x_1 + p(x_2, \ldots, x_n), x_2, \ldots, x_n)$$

is an automorphism of $\mathbb{A}^n$ and therefore contained in $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$. In particular, if $n \geq 2$, we have an injection $\mathrm{k}[x_2, \ldots, x_n] \hookrightarrow \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ and so $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is not finite dimensional for $n \geq 2$.

I.1 Blowing up and intersecting

We present some classical definitions and lemmata used throughout the thesis.

Let $\pi_X : X \to \mathbb{P}^n$ and $\pi_Y : Y \to \mathbb{P}^n$ be sequences of blow-ups of points.

Remark I.1.1. There exist sequences of blow-ups of points $\eta_X : Z \to X$ and $\eta_Y : Z \to Y$ such that the following diagram is commutative.

$$\begin{tikzcd} & Z \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\ X \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathbb{P}^n & \end{tikzcd}$$

We define an equivalence relation on the set of points of blow-ups of $\mathbb{P}^n$.

Definition I.1.2 ((Punctual) bubble space)**.** Two triples (p, X, π_X) and (q, Y, π_Y), where $p \in X$ and $q \in Y$, are equivalent if the birational map $\eta_Y(\eta_X)^{-1}$ is an isomorphism in a neighbourhood of p that sends p onto q.

We call the space of equivalence classes *(punctual) bubble space* of $\mathbb{P}^n$ and denote it by $\mathcal{B}(\mathbb{P}^n)$.

A point in $\mathcal{B}(\mathbb{P}^n)$ is simply a point in a blow-up of points of $\mathbb{P}^n$. Actually, it would be more general to define an equivalence relation on the set of points in blow-ups of $\mathbb{P}^n$

along varieties of codimension≥ 2, and to define the bubble space of $\mathbb{P}^n$ to be the set of these equivalence classes. For $n = 2$, this is exactly what we just defined, but for $n \geq 3$ the more general version is much larger than the punctual bubble space. However, the punctual bubble space is all we need in this thesis.

We use the following conventions (cf. [AC2002]):
Let $\pi\colon X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_1} X_0 = \mathbb{P}^n$ be a sequence of blow-ups of points $p_1, \ldots, p_k \in \mathcal{B}(\mathbb{P}^n)$, where $p_1 \in \mathbb{P}^n$ and $p_i \in \pi_i^{-1}(p_{i-1}) \subset X_i$.

Definition I.1.3.

1. A point in $\mathcal{B}(\mathbb{P}^n)$ is called *proper point* of X if it is equivalent to a point of X.
2. We say that points in $(\pi_k \cdots \pi_{i+1})^{-1}(p_i)$ are *infinitely near* p_i or *in the* $(k-i)$*th neighbourhood* of p_i.
3. A point in the strict transform of the exceptional divisor of p_i is called *proximate to* p_i.
4. Let $\pi_W\colon W \to X$ be a sequence of blow-ups of points $q_1, \ldots, q_m$ and $D \subset X$ an irreducible hypersurface. We denote by $I \subset \{q_1, \ldots, q_m\}$ the set of proper points of X and
$$\overline{D}^{\pi_W} := (\pi_W)^*(D) \subset W, \quad \widetilde{D}^{\pi_W} := \overline{(\pi_W)^{-1}(D \setminus I)} \subset W$$
the *total transform* and the *strict transform* of D. Analogously, we define for $D = \sum a_i D_i \in \mathrm{Pic}(X)$ the strict and total transform to be
$$\overline{D}^{\pi_W} := \sum a_i \overline{D_i}^{\pi_W}, \quad \widetilde{D}^{\pi_W} := \sum a_i \widetilde{D_i}^{\pi_W}.$$
5. For a curve $c \subset X$, we denote by
$$\widetilde{c}^{\pi_W} := \overline{(\pi_W^{-1})(c \setminus \{q_1, \ldots, q_k\})} \subset W$$
the *strict transform* of c.

Definition I.1.4. Let $n \geq 2$ and $0 \in S \subset \mathbb{A}^n$ a hypersurface given by the equation $g = 0$. We write $g = g_d + g_{d-1} + \cdots + g_e$, where $g_i \in \mathrm{k}[x_1, \ldots, x_n]$ are homogeneous of degree $\deg(g_i) = i$, $e \leq i \leq d$ and $g_e \neq 0$. We define
$$e =: \text{ multiplicity of } S \text{ in } 0 =: m_0(S).$$

Suppose that $\pi_X\colon X := X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_1} X_0 = \mathbb{P}^n$ is the blow-up of $q_1, q_2, \ldots, q_k \in \mathcal{B}(\mathbb{P}^n)$ and $E_i \subset X$ the total transform of the exceptional divisor of q_i. The Picard group of X is the group of divisors on X up to linear equivalence and is isomorphic to
$$\mathrm{Pic}(X) = \overline{H}^{\pi_X}\mathbb{Z} \oplus E_1\mathbb{Z} \oplus \cdots \oplus E_k\mathbb{Z},$$
where $H \subset \mathbb{P}^n$ is a hyperplane not passing through any q_i. Similarly, the group of 1-cycles on X (formal finite sums of curves up to numberical equivalence) is isomorphic to
$$N_1(X) = \bar{l}^{\pi_X}\mathbb{Z} \oplus e_1\mathbb{Z} \oplus \ldots e_k\mathbb{Z},$$

where $\bar{l}^{\pi_X} \subset X$ is the pre-image of a line $l \subset \mathbb{P}^n$ not passing through any of the q_i and $e_i \subset \tilde{E}_i$ is a general line in the strict transform $\tilde{E}_i$ of the exceptional divisor of q_i.

The projection formula

$$(\pi_i \cdots \pi_j)^*(D) \cdot c = D \cdot (\pi_i \cdots \pi_j)_*(c), \qquad \forall\ D \in \mathrm{Pic}(\mathbb{P}^n),\ \forall\ \ c \in N_1(X)$$

states how to intersect divisors and curves on blow-ups [Deb2001, §1.2.1.9].

The following classical statement explains the geometrical relation between the strict and the total transform of a divisor.

Lemma I.1.5. *Let $S \subset \mathbb{P}^n$ be hypersurface. Then $\overline{S}^{\pi_X}$ is linearly equivalent to*

$$\overline{S}^{\pi_X} \sim \widetilde{S}^{\pi_X} + \sum_{i=1}^{k} m_{p_i}(S) E_i.$$

Further, for any general line $l \subset \mathbb{P}^n$ and general hyperplane $H \subset \mathbb{P}^n$, we have

$$\bar{H}^{\eta_1} e_i = 0, \quad E_i e_j = 0, \quad E_i e_i = -1$$

for all $i, j = 1, \ldots, n$ and $i \neq j$.

Proof. We look at the first blown-up point in local coordinates: The blow-up $\eta\colon Y \to \mathbb{A}^n$ of $0 \in \mathbb{A}^n$ is given by

$$\eta\colon (u_1, \ldots, u_n) \mapsto (u_1, u_1 u_2, \ldots, u_1 u_n).$$

Let S be given by the equation $g = 0$, where $g \in \mathrm{k}[x_1, \ldots, x_n]$. We write $g = g_d + g_{d-1} + \cdots + g_e$, where $g_i \in \mathrm{k}[x_1, \ldots, x_n]$ is homogenous of degree $\deg(g_i) = i$ with $e \leq i \leq d$ and $g_e \neq 0$. Then the pull-back $\eta^*(S) \subset Y$ of S is given by the equation

$$u_1^e \left(u_1^{d-e} g_d(1, u_2, \ldots, u_n) + u_1^{d-e-1} g_{d-1}(1, u_2, \ldots, u_n) + \cdots + g_e(1, u_2, \ldots, u_n) \right) = 0$$

Therefore, since we defined $e = m_0(S)$, we obtain that $\eta^*(S)$ is linearly equivalent to the divisor

$$\eta^*(S) \sim \widetilde{S}^{\pi_X} + m_0(S) E_1.$$

Proceeding like this for all points blown up by π_X, we obtain the claimed equivalence.

The first intersection follows from the projection formula. We prove the other two by induction. Let $H \subset \mathbb{A}^n$ be a hyperplane through 0. With the above, the projection formula implies

$$0 = \overline{H}^{\eta} e_1 = (\widetilde{H}^{\eta} + E_1) e_1 = \widetilde{H}^{\eta} e_1 + E_1 e_1 = 1 + E_1 e_1.$$

Let $\eta_k\colon Y_N \to Y$ be the blow-up of q_k. We obtain that for $i < k$, the general lines $e_i \subset \tilde{E}_i$ do not intersect E_k, hence $E_k e_i = 0$ for $i < k$. The projection formula implies that for all $i < k$, $j = 1, \ldots, k$

$$E_i e_j = (\eta_k)^*((\eta_k)_*(E_i)) e_j = (\eta_k)_*(E_i) \cdot (\eta_2)_*(e_j),$$

which implies $E_i e_i = 1$ for $i < k$ and $E_i e_j = 0$ for $i < k$, $j = 1, \ldots, k$, $i \neq j$. Suppose that q_k is a proper point of the exceptional divisor of q_{k-1}. Then E_k and $\tilde{E}_{k-1}$ intersect in a hyperplane of E_k and so every line in E_k intersects $\tilde{E}_{k-1}$ in one point, hence

$$0 = E_{k-1}e_k = (\tilde{E}_{k-1} + E_k)e_k = \tilde{E}_{k-1}e_k + E_ke_k = 1 + E_ke_k.$$

□

I.2 Linear systems

Let X be a smooth projective variety and D a divisor on it. We define

$$\mathcal{L}(D) = \{D' \in \mathrm{Pic}(X) \mid D' \sim D,\ D' \geq 0\} \cup \{0\},$$

the set of effective divisors linearly equivalent to D, which is a finite dimensional vector space and isomorphic to $\{f \in \mathrm{K}(X) \mid f = 0, \text{ or } (f) + D \geq 0\}$ [Mum1976, §6]. If $\mathcal{L} \neq 0$, its projectivisation exists and is called the *linear system* of D and is denoted by $|D|$. A *complete linear system* is the linear system of some divisor D, and a *linear system* Λ is a linear subspace of a complete linear system. We call $\mathrm{Ind}(\Lambda) := \cap_{D\in\Lambda}\mathrm{supp}(D) \subset \mathbb{P}^n$ the set of *indeterminacy point* of the linear system Λ.

For $X = \mathbb{P}^n$ and $D = H$ a hyperplane, $|H|$ is the projective variety of all hyperplanes in $\mathbb{P}^n$, which is the dual space $\hat{\mathbb{P}}^n$ and isomorphic to $\mathbb{P}^n$, and its set of indeterminacy points is empty.

Definition I.2.1 (Linear system of a transformation)**.** Let X be a projective variety and $f\colon X \dashrightarrow \mathbb{P}^n$ a rational map. The *linear system of f* is defined as closure of the set of pre-images by f of general hyperplanes $H \subset \mathbb{P}^n$.

We call $\mathrm{Ind}(f) := \mathrm{Ind}(\Lambda_f) \subset \mathbb{P}^n$ the set of *indeterminacy points* of f.

It is a linear system but in general not a complete linear system.

Remark I.2.2. Let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be the transformation given by

$$f\colon [x_0 : \cdots : x_n] \dashrightarrow [f_0 : \cdots : f_n]$$

for some homogenous $f_0, \ldots, f_n \in \mathrm{k}[x_0, \ldots, x_n]$ without common factors and of equal degree.

Denote by $H_i \subset \mathbb{P}^n$ the hyperplane given by $x_i = 0$. Then $f^{-1}(H_i)$ is given by $f_i = 0$. More generally, the pre-image of the hyperplane $H_{[a_0:\cdots:a_n]}$ given by $\sum_{i=0}^n a_ix_i = 0$ is the hypersurface $S_{[a_0:\cdots:a_n]}$ given by $\sum_{i=0}^n a_if_i = 0$. In other words, any general element of Λ_f is a hypersurface of degree $\deg(f)$ passing through $\mathrm{Ind}(f)$.

Definition I.2.3. We denote by $\mathrm{Base}(f) \subset \mathcal{B}(\mathbb{P}^n)$ the set of points in $\mathcal{B}(\mathbb{P}^n)$ where all f_i simultanously vanish, which is the set of points where f is not defined, and call it the set of *base-points* of f. Further, we define

$$\deg(\Lambda_f) := \deg(f).$$

Definition I.2.4. Let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ and $p \in \mathcal{B}(\mathbb{P}^n)$. Any $S \in \Lambda_f$ is given by $a_0f_0 + \cdots + a_nf_n = 0$ for some $[a_0 : \cdots : a_n] \in \mathbb{P}^n$. Then there exists $m \in \mathbb{N}_{>0}$ and an open dense subset $U \subset \Lambda_f$ such that any element of U has multiplicity m in p. For $p \in \mathcal{B}(\mathbb{P}^n)$, we define m to be the *multiplicity of f* in p, and denote it by $m_p(\Lambda_f)$.

Let $\eta\colon X_1 \to \mathbb{P}^n$ be the sequence of blow-ups of $p_1, \dots, p_{l-1}, p_l$ and E_i the total transform of the exceptional divisor of p_i. For a general element $S \in \Lambda_f$ Lemma I.1.5

$$\overline{S}^\eta = \tilde{S}^\eta + \sum_{i=1}^{l} m_{p_i}(\Lambda_f)E_i.$$

Now, let Λ be any linear system in $\mathbb{P}^n$ and $S \in \Lambda$ a general element. Then $\overline{S}^\eta = \widetilde{S}^\eta + \sum_{i=1}^{l} m_i E_i$ for some $m_1, \dots, m_l \in \mathbb{N}$ that do not depend on S. We write $m_p(\Lambda) := m_l$ and call it the *multiplicty* of Λ in p.

Next, we define the image of a variety or a linear system by a birational transformation.

Definition I.2.5 (Image by transformation)**.** For a birational transformation $f\colon X \dashrightarrow Y$ between smooth projective varieties and $Z \subset X$ a subvariety, we call

$$f(Z) := \overline{f(Z \setminus \mathrm{Ind}(f))}$$

the *image* of Z by f.

The following well-known theorem presents a base to dealing with plane Cremona transformations and is the reason why we defined the linear system associated to a birational transformation in the first place.

Theorem I.2.6 ([Sha1998, Vol. 1, Chapter IV, §3.4,Theorem 4])**.** *Let $f\colon X \dashrightarrow Y$ be a birational map between smooth, projective surfaces defined over some field* k*. Then there exist two sequences of blow-ups $\eta\colon Z \to Y$ and $\pi\colon Z \to X$ of points defined over* $\mathbf{k}$ *such that the following diagram is commutative*

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

Remark I.2.7. The proof of the theorem is done in two steps:

First, we blow up the base-points of f and show that we arrive at a birational morphism $\eta\colon Z \to Y$ [Sha1998, Vol. 1, Chapter IV, §3.3, Theorem 3].

Then, one shows that any birational morphism $\eta\colon Z \to X$ between smooth projective surfaces decomposes into a sequence of blow-ups [Sha1998, Vol. 1, Chapter IV, §3.4, Theorem 5].

Remark I.2.8. If k is perfect, then for any base-point $p \in \mathcal{B}(\mathbb{P}^2)$ of f, also all its Galois-conjugates are base-points of f. By grouping the blow-ups of the Galois conjugates of p, we obtain a sequence of blow-ups defined over k.

Remark I.2.9. In general, the theorem is false in higher dimensions. For a birational transformation $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined over a field k of $\mathrm{char}(\mathrm{k}) = 0$, there still exists a sequence of blow-ups $\pi\colon Z \to \mathbb{P}^n$ of varieties of codimension≥ 2 such that $f \circ \pi\colon Z \to \mathbb{P}^n$ is a morphism because a resolution of singularities can still be found [Hir1964], but the birational morphism $f \circ \pi$ is in general not a sequence of blow-ups of subvarieties of codimension≥ 2.

Example I.2.10. Lets look at the linear system of the standard Cremona involution of $\mathbb{P}^2$, which is given by

$$\sigma\colon [x_0 : x_1 : x_2] \dashrightarrow [x_1x_2 : x_0x_2 : x_0x_1]$$

Its base-points are $p_0 = [1 : 0 : 0], p_1 = [0 : 1 : 0], p_2 = [0 : 0 : 1]$ and it contracts the line l_i given by $x_i = 0$ onto p_i. The blow-up of the three points p_0, p_1, p_2 is

$$X := \{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid x_0y_0 = x_1y_1 = x_2y_2\} \xrightarrow{\mathrm{pr}_1} \mathbb{P}^2.$$

As for general elements of X, we have

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([x_0 : x_1 : x_2], [x_1x_2 : x_0x_2 : x_0x_1])$$

hence σ lifts to the isomorphism $\hat{\sigma}$

$$\hat{\sigma}\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \overset{\simeq}{\mapsto} ([y_0 : y_1 : y_2], [x_0 : x_1 : x_2])$$

which exchanges the the exceptional divisor of p_i with $\widetilde{l_i}^{\mathrm{pr}_1}$. Take a general line $l \subset \mathbb{P}^2$ (thin, dotted in Figure I.1). Its strict transforms intersect once each l_i, which are the exceptional divisors of p_0, p_1, p_2. Therefore, $\sigma^{-1}(l)$ passes through p_0, p_1, p_2. This way, we see geometrically that Λ_f consists of all conics passing through p_0, p_1, p_2.

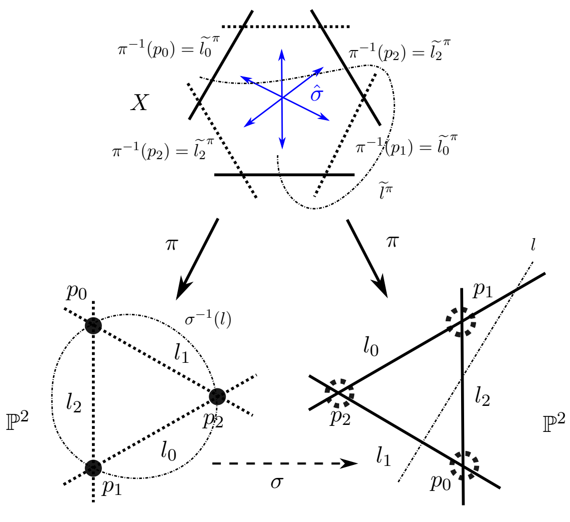

Figure I.1: The resolution of $[x : y : z] \dashrightarrow [yz : xz : xy]$.

Definition I.2.11. Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ and $p \in \mathcal{B}(\mathbb{P}^2)$ not a base-point of f. Let $\nu_1\colon S \to \mathbb{P}^2$ and $\nu_2\colon S' \to \mathbb{P}^2$ respectively be the blow-ups of the base-points of f and f^{-1}. Then f lifts to an isomorphism $\hat{f}\colon S \overset{\simeq}{\to} S'$, making the following diagram commutative.

$$\begin{tikzcd} S \arrow[r, "\hat{f}"] \arrow[d, "\nu_1"] & S' \arrow[d, "\nu_2"'] \\ \mathbb{P}^2 \arrow[r, dashed, "f"] & \mathbb{P}^2 \end{tikzcd}$$

The point p corresponds via ν_1 to a proper or infinitely near point of S. Its image via $\hat{f}$ is a point of S', proper or infinitely near, which corresponds via ν_2 to a point $f_\bullet(p) \in \mathcal{B}(\mathbb{P}^n)$.

Lets look at an example to understand $f_\bullet(p)$ and $f(p)$:

Example I.2.12. Consider the standard quadratic involution $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$ and $q = [0:1:1]$. Let l be the line given by $x = 0$, which is contracted by σ onto the point $[1:0:0]$, i.e. $\sigma(q) = [1:0:0]$. The line h given by $y = z$ passes through q and $[1:0:0]$, and $\sigma(h) = h$.

By definition, $\sigma_\bullet(p)$ is the point in the first neighbourhood of $[1:0:0]$ corresponding to the tangent direction h at $[1:0:0]$. In conclusion, $\sigma(p)$ is a proper point of $\mathbb{P}^2$, whereas $\sigma_\bullet(p)$ is not. Figure I.2 illustrates the situation; the dotted and undotted lines in X are the exceptional divisors of the dotted and undotted points respectively (compare Figure I.1).

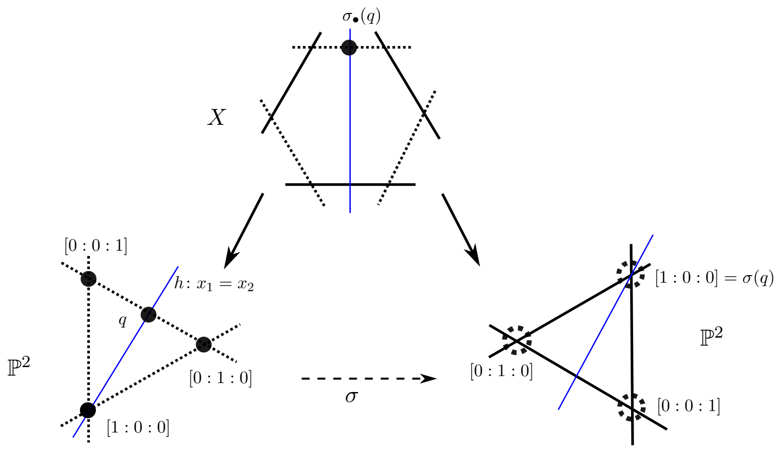

Figure I.2: The points $\sigma(q)$ and $\sigma_\bullet(q)$.

Remark I.2.13. Note that $f_\bullet$ is a one-to-one correspondence between the sets

$$\mathcal{B}(\mathbb{P}^2) \setminus \{\text{base-points of } f\} \xleftrightarrow{f_\bullet} \mathcal{B}(\mathbb{P}^2) \setminus \{\text{base-points of } f^{-1}\}.$$

I.3 Composition of transformations

In this chapter, we recall the classical formulae for degree and multiplicities of compositions of plane Cremona transformations.

Lemma I.3.1 ([AC2002, Proposition 2.1.12], [Hud1927, §I.1.3]). *For any $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$, f and f^{-1} have the same degree.*

The proof given in the reference works over any field because the resolution of a birational map of $\mathbb{P}^2$ exists for any field (cf. Theorem I.2.6).

Remark I.3.2. The above lemma is false in general for birational maps of $\mathbb{P}^n$, $n \geq 3$: For any $n \geq 1$, the inequalities

$$\sqrt[n-1]{\deg(f)} \leq \deg(f^{-1}) \leq \deg(f)^{n-1}$$

hold [BCW1982, Theorem 1.5, p. 292]. For any $n \geq 3$, any $d \in \mathbb{N}$ and any $\sqrt{d} \leq D \leq d^{n-1}$, there exist examples of birational maps of degree d with inverse of degree D. Examples can be found in [Pan2000, Pan2013].

The following classical formulae are called *Noether equations* or *equations of condition* and relate the degree of a transformation to its multiplicities (see for instance [AC2002, §2] or [Hud1927, §I.6]).

Lemma I.3.3 (Noether equations). *Let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ of degree $\deg(f) = d$. Then*

$$d^2 - 1 = \sum_{p \in \mathcal{B}(\mathbb{P}^2)} m_p(\Lambda_f)^2, \quad 3(d-1) = \sum_{p \in \mathcal{B}(\mathbb{P}^2)}^{n} m_p(\Lambda_f)$$

Note that $m_p(\Lambda_f) \neq 0$ if and only if $p \in \mathrm{Base}(f)$.

Proof. By Theorem I.2.6 there exist two sequences of blow-ups $\pi\colon Z \to \mathbb{P}^2$, $\eta\colon Z \to \mathbb{P}^2$ defined over $\mathbf{k}$ such that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

and which blow-up the base-points of f and f^{-1}.

Pick a general line $l \subset \mathbb{P}^2$, i.e. a line that does not contain any base-points of f^{-1}. Lemma I.1.5 implies that

$$\bar{l}^\eta = \widetilde{l}^\eta \sim d\bar{l}^\pi - \sum m_p(\Lambda_f) E_p$$

on Z. The intersection formula in Lemma I.1.5 implies that

$$\begin{aligned} 1 = l^2 = (\bar{l}^\eta)^2 &= (d\bar{h}^\pi - \sum m_p(\Lambda_f)E_p)^2 \\ &= d^2 - \sum m_p(\Lambda_f)^2 \end{aligned}$$

Further, we have $K_{\mathbb{P}^2} \sim -3l$ and $K_Z \sim \eta^*(K_{\mathbb{P}^2}) + \sum_{i=1}^n E_i$. Hence

$$\begin{aligned} -3 = K_{\mathbb{P}^2} \cdot l = \eta^*(K_{\mathbb{P}^2}) \cdot \bar{l}^\eta &= (K_Z - \sum E_p) \cdot \widetilde{l}^\eta \\ &= K_Z \cdot \widetilde{l}^\eta \\ &= (\pi^*(K_{\mathbb{P}^2}) + \sum E_p)(d\bar{l}^\pi - \sum m_p(\Lambda_f)E_p) \\ &= -3d + \sum m_p(\Lambda_f) \end{aligned}$$

As $m_p(\Lambda_f) \neq 0$ if and only if $p \in \mathrm{Base}(f)$, we can safely sum over all points in $\mathcal{B}(\mathbb{P}^2)$. □

To study relations among Cremona transformations by exploring their linear systems, it is essential to be able to deduce information about the linear system of a composition of two transformations from the two factors. What follows are the classical formulae for degree and multiplicity of compositions (see for instance [AC2002, Corollary 4.2.12]).

Lemma I.3.4 (Composition). *Let $f, g \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$. Then*

$$\deg(fg) = \deg(f)\deg(g) - \sum_{p \in \mathcal{B}(\mathbb{P}^n)} m_p(\Lambda_f) m_p(\Lambda_{g^{-1}})$$

and $\mathrm{Base}(fg) \subset \mathrm{Base}(g) \cup (g^{-1})_\bullet(\mathrm{Base}(f) \setminus \mathrm{Base}(g^{-1}))$.

If $p \in \mathrm{Base}(fg) \cap (g^{-1})_\bullet(\mathrm{Base}(f) \setminus \mathrm{Base}(g^{-1}))$*, then* $m_p(\Lambda_{fg}) = m_{g_\bullet(p)}(\Lambda_f)$.

Proof. By Theorem I.2.6 and Remark I.2.7 we find sequences of blow-ups $\pi_1, \eta_1'\colon Z_1 \to \mathbb{P}^2$ and $\pi_1', \eta_1\colon Z_2 \to \mathbb{P}^2$ such that $f\pi_1' = \eta_1$ and $g\pi_1 = \eta_1'$. Again for $h := \eta_1^{-1} f g \pi_1\colon Z_1 \dashrightarrow Z_2$

we find sequences of blow-ups $\pi_2\colon W \to Z_1$ and $\eta_2\colon W \to Z_2$ such that $h\pi_2 = \eta_2$. The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
& & W \arrow[dl, "\pi_2"'] \arrow[dr, "\eta_2"] & & \\
& Z_1 \arrow[rr, dashed, "h"] \arrow[dl, "\pi_1"'] \arrow[dr, "\eta_1'"] & & Z_2 \arrow[dl, "\pi_1'"'] \arrow[dr, "\eta_1"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "g"] & & \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2
\end{tikzcd}$$

In fact, π_2 blows up the base-points of f, viewed on Z_1, which are not blown up by η_1', and η_2 blows up the base-points of g^{-1}, viewed on Z_2, which are not blown up by π_1'.

Let $p \in \mathrm{Base}(fg)$. If p is not a base-point of g, then $q := g_\bullet(p)$ is not blown up by η_1 and is a base-point of f. Let $l \subset \mathbb{P}^2$ be a general line and. Then $f^{-1}(l)$ passes through q with multiplicity $m_q(f)$. As η_1' does not blow up q, the $\widetilde{f^{-1}(l)}^{\eta_1'}$ has multiplicity $m_q(\Lambda_f)$ in $(\eta_1'^{-1})_\bullet(q)$. The map π_1 does not contract any curve onto the point p (else p would be a base-point of g), hence π_1 sends $\widetilde{f^{-1}(l)}^{\eta_1'}$ onto a curve passing through p with multiplicity $m_p(\Lambda_f)$.

The degree of fg is equal to the degree of a general element $S \in \Lambda_{fg}$, which is the intersection of S with a general line $l \subset \mathbb{P}^2$. Furthermore, S is the the pre-image by fg of a general line $h \subset \mathbb{P}^2$, i.e. $S = g^{-1}(f^{-1}(h))$, and $g(l) \in \Lambda_{g^{-1}}$. With Lemma I.1.5 and the intersection formula in Lemma I.1.5 we obtain

$$\begin{aligned}
\deg(fg) = \deg(S) = S \cdot l &= \overline{S}^{\pi_2\pi_1} \cdot \overline{l}^{\pi_2\pi_1} \\
&= \widetilde{S}^{\pi_2\pi_1} \cdot \widetilde{l}^{\pi_2\pi_1} \\
&= (\widetilde{f^{-1}(h)})^{\pi_2\eta_1'} \cdot \widetilde{g(l)}^{\pi_2\eta_1'} \\
&= \left(\deg(f)\bar{l}^{\eta_2\eta_1} - \sum m_p(\Lambda_f)E_p\right)\left(deg(g)\bar{l}^{\pi_2\eta_1'} - \sum m_p(\Lambda_{g^{-1}})E_p\right) \\
&= \deg(f)\deg(h) - \sum m_p(\Lambda_f)m_p(\Lambda_{g^{-1}})
\end{aligned}$$

where $E_p \subset W$ is the total transform of the exceptional divisor of p. As $m_p(\Lambda_f) \neq 0$ if and only if p is a base-point of f, we can safely sum over all points in $\mathcal{B}(\mathbb{P}^2)$. $\square$

II Generating sets and relations

To work with the Cremona groups it is helpful to know a generating set that is easy to work with. A generating set of the plane Cremona group over an algebraically closed field has been known for over hundred years, whereas for the Cremona groups of higher dimensional projective spaces, no non-trivial generating set is known.

In this chapter, we recall some theorems about generating sets of the Cremona groups of the plane.

Recall that we denote by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n) \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ the group of transformations that are defined at every point of $\mathbb{P}^2$. It is the group of linear transformations of $\mathbb{P}^n$ and is isomorphic to $\mathrm{PGL}_{n+1}(\mathrm{k})$.

The following definition specifies what is meant by the terms generating set, generating relations and presentation of a group.

Definition II.0.1. Let G be a group. A *presentation* $\langle S \mid R\rangle$ of G is a triple made up of a set S, a surjective homomorphism $\pi\colon F_S \twoheadrightarrow G$ of the free group F_S on S onto G and a subset R of F_S generating $\ker(\pi)$ as a normal subgroup.

The *relations* of the presentation are the elements of $\ker(\pi)$ and the elements of R are the *relators* (or *generating relations*) of the presentation. The set S is called *generating set* of G. We write $G \simeq \langle S \mid R\rangle$.

II.1 The plane Cremona groups

II.1.1 Algebraically closed fields

Suppose that k is an algebraically closed field. Then we know a superbly nice generating set of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$:

Theorem II.1.1 (Noether-Castelnuovo theorem, [Cas1901]). *Let k be an algebraically closed field. Then $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and the standard Cremona involution.*

See [Sha1967, §V.5, Theorem 2, p.100] for a proof working over any algebraically closed field.

The theorem implies that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by the two algebraic groups $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and the group of order 2 generated by the standard Cremona involution. It does not give any information about the generating relations.

The Noether-Castelnuovo theorem implies that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by the set of all linear and quadratic transformations and a first presentation was given using this generating set:

Theorem II.1.2 ([Giz1983, Theorem 10.7, p.267]). *Let* k *be an algebraically closed field and denote by* $\mathcal{Q} \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *the set of quadratic transformations. Then* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \simeq \langle \mathcal{Q} \mid R\rangle$ *and all relators* $r \in R$ *are of the form*

$$r = q_1q_2q_3.$$

The standard Cremona involution preserves the pencil of lines through $[0:0:1]$ and this leads to the following definition:

Definition II.1.3 (de Jonquières transformations)**.** By $\mathcal{J}_* \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ we denote the subgroup of elements preserving the pencil of lines through $[1:0:0]$. In other words,

$$\mathcal{J}_* = \{f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \mid \exists \alpha \in \mathrm{PGL}_2(\mathrm{k}) \ : \ \pi_* f = \alpha\pi_*\}$$

where $\pi_*\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x:y:z] \mapstochar\dashrightarrow [y:z]$, whose fibres are the lines through $[1:0:0]$. An element of $\mathcal{J}_*$ is called *de Jonquières transformation*.

Writing the de Jonquières tranformations in local coordinates, we see that $\mathcal{J}_*$ is given by

$$\begin{aligned}\mathcal{J}_* &= \left\{ (x,y) \mapstochar\dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{\alpha(x)y+\beta(x)}{\gamma(x)y+\delta(x)}\right) \middle| \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{PGL}_2(\mathrm{k}), \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x)\end{pmatrix} \in \mathrm{PGL}_2(\mathrm{k}[x]) \right\} \\ &\simeq \mathrm{PGL}_2(\mathrm{k}(x)) \rtimes \mathrm{PGL}_2(\mathrm{k})\end{aligned}$$

and is not an algebraic group as $\mathrm{PGL}_2(\mathrm{k}(x))$ is not an algebraic group over k.

As the standard Cremona involution is contained in $\mathcal{J}_*$, the Noether-Castelnuovo theorem implies that for algebraically closed fields, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and $\mathcal{J}_*$. Of course, this is a much weaker statement than the Noether-Castelnuovo theorem but allows the following structure theorem:

Theorem II.1.4 ([Bla2012, Theorem 1])**.** *Let* k *be an algebraically closed field. Then* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *is the amalgamated product of* $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ *and* $\mathcal{J}_*$ *along their intersection, divided by one relation, which is*

$$\sigma\tau = \tau\sigma$$

where $\tau \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ *is given by* $\tau([x:y:z]) = [y:x:z])$ *and* σ *is the standard Cremona involution.*

The birational transformation

$$\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1,\ [x:y:z] \mapstochar\dashrightarrow ([x:z],[y:z]), \quad ([u_0:u_1],[v_0:v_1]) \overset{\psi^{-1}}{\dashleftarrow\!\shortmid} [u_0v_1 : u_1v_0 : u_1v_1]$$

is blow-up of the two point $[1:0:0]$ and $[0:1:0]$ followed by the contraction of the line given by $z=0$. Further, it conjugates $\mathcal{J}_*$ to the subgroup of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^1 \times \mathbb{P}^1)$ that preserves the projection onto the second factor. The above theorem was preceded by a similar statement on $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem II.1.5 ([Isk1985, Theorem])**.** *Let* k *be an algebraically closed field. Then the group* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *is the amalgamated product of* $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *and* $\mathcal{J}_*$ *along their intersection, divided by the relation*

$$(\rho\tau)^3 = \psi\sigma\psi^{-1},$$

where $\rho\colon (x,y) \mapstochar\dashrightarrow (x, x/y)$ *and* $\tau\colon (x,y) \mapsto (x,y)$*, and* σ *the standard Cremona involution.*

Both statements yield an almost amalgamated structure of the plane Cremona group and it is as close as one can get, as $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is not isomorphic to a non-trivial amalgam if k is algebraically closed [CanLam2013, Appendix]. However, it is isomorphic to the generalised amalgamated product of three groups, meaning that it is isomorphic to the free product of three groups amalamated along all pairwise intersections.

Theorem II.1.6 ([Wri1992, Theorem 3.13])**.** *Let* k *be an algebraically closed field. Then* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *is isomorphic to the free group of the three groups*

$$\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2), \quad \mathrm{PGL}_2(\mathrm{k}) \times \mathrm{PGL}_2(\mathrm{k}), \quad \mathcal{J}_*,$$

amalgamated along their pairwise intersections in $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$*, where the second group is the group of automorphisms of* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *respecting the projections onto the two factors and is embedded into* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *via the birational map* $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$*.*

All these presentations have in common that they do not use linear algebraic groups as generating groups. On the other hand, the Noether-Castelnuovo theorem states that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ can be generated by two linear algebraic groups, although without giving a presentation. The following theorem combines the idea of Theorem II.1.6 with linear algebraic generating groups, having to make a compromise by modding one further relation.

Theorem II.1.7 ([Zim2016, Theorem B])**.** *Let* k *be algebraically closed. Then* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *is isomorphic to the free product of the linear algebraic groups*

$$\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2), \quad \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^1 \times \mathbb{P}^1), \quad \mathrm{Aut}_{\mathrm{k}}(\mathbb{F}_2)$$

amalgamated along their pairwise intersections and divided by the relation

$$\tau_{13}\sigma\tau_{13}\sigma = 1$$

where $\tau_{13}\colon [x:y:z] \mapsto [z:y:x]$ *and* σ *is the standard Cremona involution.*

This structure theorem does not stand out among the presentations given in this chapter. However, it allows to approach the plane Cremona group from a topological point of view. Endowed with the Euclidean topology as defined in [BlaFur2013, Theorem 3, §5] the Cremona group becomes a Hausdorff topological group and the restriction of the topology to any linear algebraic subgroup is the Euclidean topology on it. For $\mathrm{k} = \mathbb{C}$ and $\mathrm{k} = \mathbb{R}$, any linear algebraic group endowed with the Euclidean topology is a Lie group and as such compactly generated, i.e. it has a compact generating set. The Noether-Castelnuovo theorem then implies that $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is compactly generated as well. Theorem II.1.7 enables us to prove that we can even find a presentation $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2) = \langle S \mid R \rangle$ where S is compact and R has bounded length with respect to the word length given by S.

More generally, a Hausdorff topological group G is called *compactly presented* if there exists a presentation $\langle S \mid R \rangle$ where $S \subset G$ is compact and R is of bounded word length. Being compactly presented is a property usually associated to Lie groups with finitely many connected components (cf. Chapter III, §6). Although the Cremona group is connected [Bla2010], it is not a Lie group, as it is not finite dimensional in any sense (see Example I.0.4).

Theorem II.1.8 ([Zim2016, Theorem A]). *Endowed with the Euclidean topology,* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ *is compactly presented by any compact generating set of* $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2)$ *and the standard Cremona involution.*

Theorem II.1.17 implies that $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is not so far from being a Lie group, albeit not being finite dimensional.

Note that Theorem II.1.2 yields a presentation with all relators of length three but the generating set is not compact because the set of quadratic transformations is not closed [BlaCal2016, Theorem 1] and hence not compact.

Presentations do not only exist for $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ but also for some of its subgroups, as for instance the classical presentation of $\mathrm{Aut}_{\mathrm{k}}(\mathbb{A}^2) \simeq \mathrm{Aff}_{\mathrm{k}}(\mathbb{A}^2) *_{\mathrm{Aff}_{\mathrm{k}}(\mathbb{A}^2)\cap E} E$, where k is any field and $\mathrm{Aff}_{\mathrm{k}}(\mathbb{A}^2) \subset \mathrm{Aut}_{\mathrm{k}}(\mathbb{A}^2)$ is the subgroup of affine automorphisms and $E \subset \mathrm{Aut}_{\mathrm{k}}(\mathbb{A}^2)$ is the subgroup of elementary automorphisms (automorphisms of the form $(x, y) \mapsto (ax + P(y), by + c)$ for some $P \in \mathrm{k}[x]$, $a, b, c \in \mathrm{k}^*$) [VdK1053]. Further, non-trivial generating sets are known for the decomposition groups of plane curves $c \subset \mathbb{P}^2$, that is $\mathrm{Dec}(c) = \{f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \mid f|_c \colon c \dashrightarrow c \text{ is birational}\}$; [HedZim2016] shows that, just like $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ itself, the decomposition group of a line is generated by its linear subgroup and one quadratic element, and that it does not have the structure of a non-trivial amalgam. The decomposition group of curves of genus≥ 1 have been closely studied in [BPV2009].

II.1.2 Non algebraically-closed fields

For fields that are not algebraically closed the Noether-Castelnuovo theorem never holds. The standard Cremona involution has three base-points, each defined over k, and contracts three lines, also each defined over k. In fact, the group generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and the standard involution is equal to the subgroup of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ consisting of elements that contract only k-rational curves, which is equal to the subgroup of transformations having all base-points defined over k [BlaHed2014, Proposition 7.4]. However, if k is not algebraically closed, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ always contains transformations contracting non-rational curves. For instance, let $p \in \mathrm{k}[X]$ be irreducible and of degree $d > 1$. The de Jonquières transformation

$$T\colon [x : y : z] \dashrightarrow [z^d x : yz^d p(\frac{x}{z}) : z^{d+1}]$$

contracts the curve given by $z^d p(\frac{x}{z}) = 0$, which is not rational over k. (Over $\mathbf{k}$, it is a union of lines.)

Even more, the following statement holds:

Lemma II.1.9. *Let* k *be a field whose algebraic closure* $\mathbf{k}$ *does not have finite degree over* k*. Then* $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ *is not generated by a set of bounded degree.*

Proof. The idea of the proof is the same as in [Can2015, Proposition 3.6] where it is shown that for any field, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is not finitely generated.

Let $d \in \mathbb{N}$ and denote by $S \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ the set of transformations of degree $\leq d$ and by $\langle S \rangle \subseteq \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ the subgroup generated by S. An element of S is of the form

$$[x : y : z] \dashrightarrow [s_0(x, y, z) : s_1(x, y, z) : s_2(x, y, z)]$$

for some homogenous $s_0, s_1, s_2 \in \mathrm{k}[x, y, z]$ of equal degree $\deg(s) = \deg(s_i) \leq d$. Let k_s be the smallest field extension of k over which the base-points of s and s^{-1} are defined. The Galois group $\mathrm{Gal}(\mathrm{k}_s/\mathrm{k})$ permutes the coefficients of the base-points, whose minimal polynomials have degree at most d. Therefore, $[\mathrm{k}_s : \mathrm{k}] \leq |\mathrm{Gal}(\mathrm{k}_2/\mathrm{k})| \leq d! =: D$ for all $s \in S$. Let $s, s' \in S$. By Lemma I.3.4, the base-points of $s's$ are contained in $\mathrm{Base}(s) \cup ((s^{-1})_\bullet(\mathrm{Base}(s')))$. As s is defined over k, it follows that every base-point of $s's$ is defined over k_s or $\mathrm{k}_{s'}$. Inductively, we obtain that for any $s_1, \dots, s_n \in S$, every base-point of the composition $s_n \cdots s_1$ is defined over one of the k_{s_i}.

Let $p > D$ be a prime number and $q(x) \in \mathrm{k}[x_0, \dots, x_n]$ an irreducible polynomial of degree p, and consider the de Jonquières transformation

$$T\colon [x : y : z] \dashrightarrow [xq(\frac{x}{z})z^d : yz^d : q(\frac{x}{z})z^{d+1}]$$

Its base-points are $[0 : 1 : 0]$ and $[\alpha_i : 0 : 1]$, where the α_i are the roots of $q(x)$ in the algebraic closure $\mathbf{k}$ of k, and hence have degree $p > D$. Therefore, the $[\alpha_i : 1 : 0]$ are not defined over k_s for any $s \in S$. It follows that T is not contained in $\langle S\rangle$. $\square$

For certain fields k, non-trivial generating sets of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ are known. The involution

$$\sigma'\colon [x : y : z] \dashrightarrow [xz : yz : x^2 + y^2]$$

is not contained in the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the standard Cremona involution σ, as it has two non-real conjugate base-points. The following theorem presents a generating set of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Theorem II.1.10 ([BlaMan2014, Theorem 1.1]). *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ, σ' and the (uncountable) family of standard quintic transformations of $\mathbb{P}^2$ (see Definition II.1.11).*

In particular, $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by a set of bounded degree.

Definition II.1.11. We define a type of real birational transformation called *standard quintic tranformation*.

Let $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{p}_3 \in \mathbb{P}^2$ be three pairs of non-real conjugate points of $\mathbb{P}^2$, not lying on the same conic. Denote by $\pi\colon X \to \mathbb{P}^2$ the blow-up of these points, which induces an isomorphism $X(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$. The set of strict transforms of the conics passing through five of the six points correspond to three pairs of non-real conjugate (-1)-curves. Their contraction yields a birational morphism $\eta\colon X \to \mathbb{P}^2$, inducing an isomorphism $X(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$, which contracts the curves onto three pairs of non-real points $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3 \in \mathbb{P}^2$. We choose the order so that r_i is the image of the conic not passing through q_i. The map $\psi := \eta\pi^{-1}$ is a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ inducing an isomorphism $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.

Let $L \subset \mathbb{P}^2$ be a general line of $\mathbb{P}^2$. The strict transform of L on X by π^{-1} has self-intersection 1 and intersects the six curves contracted by η in two points (because these are conics). The image of $\psi(L)$ has then six singular points of multiplicty 2 and self-intersection 25; it is thus a quintic passing through the r_i with multiplicity 2. The construction of ψ^{-1} being symmetric to the one of ψ, the linear system of ψ consists of quintics of $\mathbb{P}^2$ having multiplicity 2 at $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$.

One can moreover check that ψ sends the pencil of conics through $q_1, \bar{q}_1, q_2, \bar{q}_2$ onto the pencil of conics through $r_1, \bar{r}_1, r_2, \bar{r}_2$ (and the same holds for the two other real pencils of conics, through $q_1, \bar{q}_1, q_3, \bar{q}_3$ and through $q_2, \bar{q}_2, q_3, \bar{q}_3$).

The generating set in Theorem II.1.10 is not far from being minimal in the sense that it is not possible to generate $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and only countably many elements [Zim2015, Theorem 1].

A more general statement of Theorem II.1.10 can be found in [Isk1991, Theorem] where a generating set of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ for any perfect field k is given. As the statement is much more general, the list of non-linear generators is much more diverse than the one given in Theorem II.1.10. In [IKT94], generating relations are given for the generating set found in [Isk1991].

The standard quintic transformations send a real conic bundle onto a real conic bundle and this leads to the following definition:

Definition II.1.12. Fixing two points $p_1 = [1 : i : 0], p_2 = [0 : 1 : i]$, we denote by $\mathcal{J}_\circ \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup of transformations preserving the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$. In other words,

$$\mathcal{J}_\circ = \{f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid \exists \alpha \in \mathrm{PGL}_2(\mathbb{R}) \ : \ \pi_\circ f = \alpha \pi_\circ\},$$

where $\pi_\circ \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x : y : z] \mapstochar\dashrightarrow [y^2 + (x+z)^2 : y^2 + (x-z)^2]$, whose fibres are the conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$.

Remark II.1.13 ([Zim2015, Lemma 2.5])**.** For any standard quintic transformation $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ there exist $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ such that $\beta f \alpha \in \mathcal{J}_\circ$.

Extending the scalars to $\mathbb{C}$, the two groups $\mathcal{J}_\circ$ and $\mathcal{J}_*$ are conjugate; for instance, by any quadratic transformation having base-points $p_1, \bar{p}_1, p_2$ and sending $\bar{p}_2$ onto $[1 : 0 : 0]$ (cf. Chapter II.4.2). It is not true over $\mathbb{R}$ [Zim2015, Remark 4.11].

Denote by $\eta_1 \colon X_5 \to \mathbb{P}^2$ the blow-up of $p_1, \bar{p}_1, p_2, \bar{p}_2$. Then, because no three of the four points are collinear, X_5 is a real del Pezzo surface of degree 5, and $\pi_\circ \eta_1 : X_5 \to \mathbb{P}^1$ is a real conic bundle whose fibres are the strict transforms of the conics through $p_1, \ldots, \bar{p}_2$ and which has exactly three singular fibers, which are the strict transforms of the three reducible conics $C_1 := L_{p_1,\bar{p}_2} \cup L_{\bar{p}_1,p_2}$, $C_2 := L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}$, $C_3 := L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2}$. The only singular fibre having real components is C_1 and the contraction of one of its components, for instance the strict transform of $L_{p_2,\bar{p}_2}$, yields a birational morphism $\eta_2 \colon X_5 \to X_6$ onto a real del Pezzo surface of degree 6, and $\pi'_\circ := \pi_\circ \eta_1 (\eta_2)^{-1} \colon X_6 \to \mathbb{P}^1$ is a minimal real conic bundle. The group $\mathcal{J}_\circ$ is conjugate via $(\eta_2 \eta_1^{-1})$ to the group of birational transformations of X_6 preserving the conic bundle structure. This is summarised in Figure II.1; the exceptional divisors of $p_2, \bar{p}_2$ are not drawn, the circle in X_6 is the image of $\widetilde{L}_{p_2,\bar{p}_2}$, and the numbers in the square brackets are self-intersection numbers.

From this point of view, a standard quintic transformation is conjugate via $(\eta_2 \eta_1^{-1})$ to an elementary link of X_6 defined over $\mathbb{R}$; it is the composition of the blow-up of two non-real conjugate points on X_6 contained in two non-real conjugate fibres and the contraction of the strict transform of these fibres.

The image of the exceptional divisors of $p_1, \bar{p}_1$ are non-real conjugate (-1)-sections on the real conic bundle $X_6 \to \mathbb{P}^1$. Contracting these, we obtain the rational fibration $\mathcal{Q}_{3,1} \dashrightarrow \mathbb{P}^1$, where $\mathcal{Q}_{3,1} \subset \mathbb{P}^3$ is given by $w^2 = x^2 + y^2 + z^2$ and whose real chapter $\mathcal{Q}_{3,1}(\mathbb{R}) \simeq \mathbb{S}^2$ is diffeomorphic to the real 2-sphere.

Lemma II.1.14 ([Zim2015, Corollary 2.6])**.** *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is generated by its subgroups* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$.

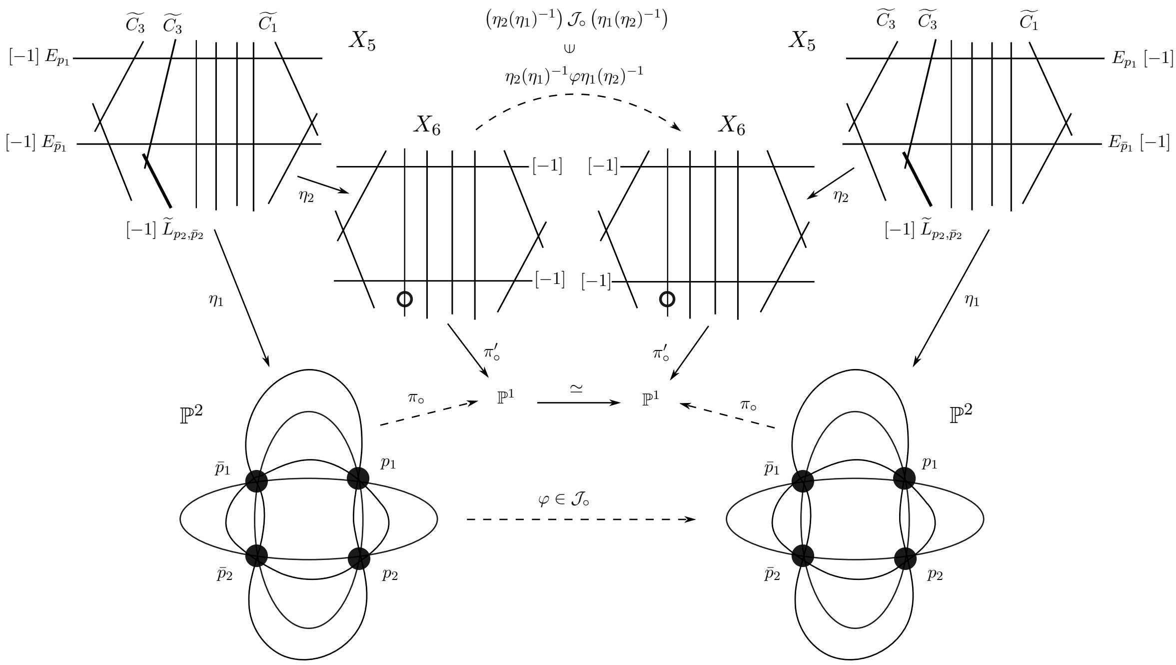

Figure II.1: The birational action of $\mathcal{J}_\circ$ on the real conic bundle $\pi'_\circ\colon X_6 \to \mathbb{P}^1$.

Proof. By Theorem II.1.10 and Remark II.1.13, $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ, σ' and the family of standard quintic transformations contained in $\mathcal{J}_\circ$. Observing that $\sigma \in \mathcal{J}_*$ and $\sigma' \in \mathcal{J}_\circ$, the claim follows. □

We also obtain a structure theorem similar to Theorem II.1.6 and Theorem II.1.7.

Define $S := \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ and let F_S be the free group generated by S. Let $w\colon S \to F_S$ be the canonical word map.

Definition II.1.15. We denote by $\mathcal{G}$ be the following group:

$$F_S/\left\langle \begin{array}{ll} w(f)w(g)w(h), & f,g,h \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2),\ fgh = 1 \text{ in } \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_*,\ fgh = 1 \text{ in } \mathcal{J}_* \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_\circ,\ fgh = 1 \text{ in } \mathcal{J}_\circ \\ \text{the relations in the list below} & \end{array} \right\rangle$$

1. Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \text{ in } \mathcal{G} \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$

2. Let $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_\circ$ both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \text{ in } \mathcal{G} \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$

3. Let $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$ all of degree 2, or τ_1, τ_2 of degree 2 and τ_3 of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \text{ in } \mathcal{G} \quad \text{if} \quad \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

Note that the group $\mathcal{G}$ is isomorphic to the free product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$ amalgamated along all pairwise intersections and divided by the relations in the above list.

Since $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ (Lemma II.1.14), there exists a natural surjective group homomorphism $F_S \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ which gives rise to a group homomorphism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, since all relations above hold in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Theorem II.1.16 ([Zim2015, Proposition 2.9])**.** *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to $\mathcal{G}$.*

This structure theorem is rather technical and ugly to look at, but it allows us to prove the following theorem.

Theorem II.1.17 ([Zim2015, Theorem 2])**.** *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is not perfect: its abelianisation is isomorphic to*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)] \simeq \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the commutator subgroup $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$, and contains all elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree ≤ 4.

The situation for $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is quite different: The group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is perfect [CerDes2013, Corollary 5.15], i.e $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2) = [\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2),\ \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)]$, and the normal subgroup generated by any non-trivial element of degree ≤ 4 is the whole group [Giz1994, Remark on Lemma 2, p. 42] (see also [Zim2015, Lemma 4.13]).

Remark II.1.18 ([Zim2015, §3.2, §4])**.** Let $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ be the abelianisation homomorphism from Theorem II.1.17. Then the image of the set of standard quintic transformations in $\mathcal{J}_\circ$ is a generating set of $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$.

For a deeper discussion of this theorem and its implications, see Chapter IV, §4, §5.

II.2 Higher dimensions

For $n \geq 3$ and any field k, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is not generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ and the standard Cremona involution. In fact, we have even more than just the Noether-Castelnuovo theorem being false:

Theorem II.2.1 ([Pan1999, Théorème 1])**.** *For any field k and $n \geq 3$, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is not generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ and a countable number of elements of degree $d > 1$.*

The proof of the theorem presents a family of counter examples, which relies on the following: Every irreducible curve is birational to a plane curve [Har1977, §IV.3.11], and $C \times \mathbb{P}^2$ is birational to $C' \times \mathbb{P}^2$ if and only if C and C' are birational [Kan1987, Theorem 3].

Let $p \in \mathrm{k}[X,Y]$ be irreducible and of degree $d > 1$. It defines a curve C in $\mathbb{A}^2$. The de Jonquières map

$$T\colon [x_0 : \cdots : x_n] \dashrightarrow [x_0 p(\frac{x_1}{x_3}, \frac{x_2}{x_3}) x_3^d : x_1 x_3^d : \cdots : x_n x_3^d]$$

contracts the hypersurface given by $p(\frac{x_1}{x_3}, \frac{x_2}{x_3})x_3^d = 0$, which is birational to $C \times \mathbb{P}^{n-2}$. Thus we need at least as many generators $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ of degree> 1 as birational classes of curves.

Until now, no non-trivial generating set of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is known for $n \geq 3$.

II.3 Birational diffeomorphisms

Recall that every automorphism of $\mathbb{P}^n$ defined over k is linear and vice versa. However, if k is not algebraically closed, the Cremona group might contain elements that are defined at every k-point of $\mathbb{P}^n$ and yet is not linear.

Definition II.3.1. For a variety X, we denote by $X(\mathrm{k})$ its set of k-points and define

$$\mathrm{Aut}(X(\mathrm{k})) = \{f \in \mathrm{Bir}_{\mathrm{k}}(X) \mid f, f^{-1} \text{ are defined at every point of } X(\mathrm{k})\} \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2).$$

For $\mathrm{k} = \mathbb{R}$, the group also called the group of *birational diffeomorphisms* of X.

We always have the inclusion $\mathrm{Aut}_{\mathrm{k}}(X) \subset \mathrm{Aut}(X(\mathrm{k}))$, which is strict in general, as we see from the following theorems presenting the generating sets of the groups of birational diffeomorphisms of the three minimal real rational surfaces $\mathbb{P}^2$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{Q}_{3,1} = \{[w : x : y : z] \in \mathbb{P}^3 \mid w^2 = x^2 + y^2 + z^2\}$. Their real points are respectively $\mathbb{P}^2(\mathbb{R})$, the real torus $\mathbb{S}^1 \times \mathbb{S}^1$ and the real 2-sphere $\mathbb{S}^2$.

Theorem II.3.2 ([RonVus2005, Teorema II],[BlaMan2014, Theorem 1.2])**.** *The group* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \simeq \mathrm{PGL}_2(\mathbb{R})$ *and the family of standard quintic transformations (see Definition* II.1.11*).*

Theorem II.3.3 ([KolMan2009, Theorem 1],[BlaMan2014, Theorem 1.3])**.** *The group* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1}) = \mathrm{PO}_3(\mathbb{R})$ *and the standard cubic transformations (see definition below).*

Let us quickly recall the definition of standard cubic transformations on $\mathcal{Q}_{3,1}$, which is quite similar to the definition of standard quintic transformations on $\mathbb{P}^2$.

Definition II.3.4. Let $p_1, \bar{p}_1, p_2, \bar{p}_2 \in \mathcal{Q}_{3,1} \subset \mathbb{P}^3$ be two conjugate pairs of non-real points, not all lying on the same plane of $\mathbb{P}^3$. Let $\pi \colon X \to \mathcal{Q}_{3,1}$ be the blow-up of these points. The non-real plane of $\mathbb{P}^3$ passing through $p_1, \bar{p}_1, p_2$ intersects $\mathcal{Q}_{3,1}$ onto a conic, having self-intersection 2: two general different conics on $\mathcal{Q}_{3,1}$ are the trace of hyperplanes, and intersect then into two points, being on the line of intersection of the two planes. The strict transform of this conic on X is thus a (-1)-curve. Doing the same for the other conics passing through three of the points $p_1, \bar{p}_1, p_2, \bar{p}_2$, we obtain four disjoint (-1)-curves on X, that we can contract in order to obtain a birational morphism $\eta \colon X \to \mathcal{Q}_{3,1}$; note that the target is $\mathcal{Q}_{3,1}$ because it is a smooth projective rational surface of Picard rank 1. We obtain then a birational map $\psi = \eta\pi^{-1} \colon \mathcal{Q}_{3,1} \dashrightarrow \mathcal{Q}_{3,1}$ inducing an isomorphism $\mathcal{Q}_{3,1}(\mathbb{R}) \to \mathcal{Q}_{3,1}(\mathbb{R})$.

Denote by $H \subset \mathcal{Q}_{3,1}$ a general hyperplane section. The strict transform of H on X by π^{-1} has self-intersection 2 and has intersection 2 with the four curves contracted by η. The image $\psi(H)$ has thus multiplicity 2 and self-intersection 18; it is then the trace of a cubic section. The construction of ψ and ψ^{-1} being similar, the linear system of ψ consists of cubic sections with multiplicity 2 at $p_1, \bar{p}_1, p_2, \bar{p}_2$.

Theorem II.3.5 ([BlaMan2014, Theorem 1.4])**.** *The group* $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0) \simeq \mathrm{PGL}_2(\mathbb{R})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ *and by the involution*

$$\tau_0 \colon ([x_0 : x_1], [y_0 : y_1]) \mapsto\!\dashrightarrow ([x_0 : x_1], [x_0y_0 + x_1y_1 : x_1y_0 - x_0y_1])$$

Remark II.3.6. The generating sets of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ are not far from being minimal; they are not generated by their automorphism groups countably many transformations [Zim2016, Corollary 1.4]. This follows from the fact that the abelianisation homomorphism $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ in Theorem II.1.17 restricted to these groups is surjective and that the image of their generators is a generating set of $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$.

For $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ no non-trivial generating set known. Yet, we have the following statement: Every standard quintic transformation in $\mathcal{J}_\circ$ is contained in $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$, so, by Remark II.1.18, the abelianisation homomorphism restricted to $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ is surjective. Hence also $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ cannot be generated by its linear elements and countably many elements [Zim2015, Corollary 1.4].

II.4 Relations in the plane Cremona group

To study the plane Cremona transformations over an algebraically closed field, it is quite useful to have certain relations between quadratic transformations at hand. For $\mathrm{k} = \mathbb{R}$ it is useful to know some relations among elements of $\mathcal{J}_\circ$. In this chapter, we present the most obvious and most commonly used relations in Chapter III and Chapter IV.

II.4.1 Quadratic transformations

If k is algebraically closed, then $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by the standard Cremona involution, which is of degree 2, and the linear group $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$. Therefore, to study relations in the Cremona group, we just have to know what happens to the linear system of a transformation when we compose it with a quadratic transformation.

The quadratic transformations of $\mathbb{P}^2$ are the easiest transformations to deal with and there are just three types of them, their prototypes being the involutions

$$\begin{aligned}
\sigma_3 &\colon [x:y:z] \dashrightarrow [yz:xz:xy]\\
\sigma_2 &\colon [x:y:z] \dashrightarrow [xy:z^2:yz]\\
\sigma_1 &\colon [x:y:z] \dashrightarrow [-xy+z^2:y^2:yz],
\end{aligned}$$

The first is the standard Cremona involution. The second has two base-points $[1:0:0]$, $[0:1:0]$ and the point p on the exceptional divisor of $[1:0:0]$ that corresponds to the line given by $y=0$. The third has base-points $[1:0:0], p$ and a point on the exceptional divisor of p that is not on the intersection of the strict transform of the exceptional divisor of $[1:0:0]$ and the exceptional divisor of p.

Lemma II.4.1 ([AC2002, §2.8]). *Let k be algebraically closed and $\tau \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ a quadratic transformation. Then there exists $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and $i \in \{1,2,3\}$ such that*

$$\tau = \beta\sigma_i\alpha.$$

Of course, if k is not algebraically closed, such α, β exist but might only be defined over the algebraic closure $\mathbf{k}$ of k. For instance, the involution

$$\sigma_3'\colon [x:y:z] \dashrightarrow [xz:xz:x^2+y^2]$$

is defined over $\mathrm{k}=\mathbb{R}$ and has base-points $[0:0:1], [1:\mathbf{i}:0], [1:-\mathbf{i}:1]$, but there is no linear transformation defined over $\mathbb{R}$ sending these points onto $[1:0:0], [0:1:0], [0:0:1]$. However, since any quadratic element $\tau\in\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with three proper base-points has either three real base-points or two non-real conjugate base-points and one real one, we can find $\alpha,\beta\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\tau=\beta\sigma_3\alpha$ or $\tau=\beta\sigma_3'\alpha$. If τ has at least one infinitely near base-point, then all its base-points are real points and we can find $\alpha,\beta\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $i\in\{1,2\}$ such that $\tau=\beta\sigma_i\alpha$.

The following lemma tells us just how easy it is to calculate the degree and multiplicities of the composition of any transformation with a quadratic transformation.

Lemma II.4.2 ([AC2002, Proposition 4.2.5])**.** *Let $\tau, f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ be transformations of degree 2 and d respectively. Let p_1,p_2,p_3 be the base-points of τ and q_1,q_2,q_3 the base-points of τ^{-1}. Define*

$$\varepsilon = m_{q_1}(\Lambda_f)+m_{q_2}(\Lambda_f)+m_{q_3}(\Lambda_f).$$

Then

$$\deg(f\tau)=2d-\varepsilon,\quad m_{p_i}(\Lambda_{f\tau})=d-\varepsilon+m_{q_i}(\Lambda_f)$$

and $m_{(\tau^{-1})_\bullet(r)}(\Lambda_{f\tau})=m_r(\Lambda_f)$ if r is not a base-point of τ^{-1}.

Remark II.4.3. We get the following consequence of the above lemma: Let $\tau,\tau'\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ be two quadratic transformations. If τ^{-1},τ' have $3,2,1$ or zero common base-points, then $\deg(\tau'\tau)$ is respectively $1,2,3$ or 4.

Remark II.4.4. By writing $\sigma_3=\tau_{12}\sigma_2\tau_{12}\sigma_2$, we see that $\mathrm{Bir}(\mathbb{P}^2)$ is also generated by $\mathrm{Aut}(\mathbb{P}^2)$ and σ_2.

The following proposition emerged form a discussion with Isac Hedén.

Proposition II.4.5 (cf. [Giz1999, p. 122] for $n=3$)**.** *For any field k of $\mathrm{char}(\mathrm{k})=0$ and any $n>1$, the set*

$$\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)} := \{f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)\mid \mathrm{Jac}(f)=ap^{n+1},\quad p\in k[x_0,\dots,x_n], a\in\mathrm{k}^*\}$$

is a proper subgroup of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$. In particular, $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is not generated by σ_1 and $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$.

Proof. Let $f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be given by

$$f\colon [x_0:\cdots:x_n]\dashrightarrow [f_0(x_0,\dots,x_n):\cdots:f_n(x_0,\dots,x_n)]$$

for some $f_0,\dots,f_n\in\mathrm{k}[x_0,\dots,x_n]$ homogenous without common factors. We define the Jacobian of f to be

$$\mathrm{Jac}(f) := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^{n+1}\in\mathrm{k}[x_0,\dots,x_n]$$

For the composition of two elements $f,g\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$, we get

$$\mathrm{Jac}(fg)=\frac{\mathrm{Jac}(f)(g_0,\dots,g_n)\cdot\mathrm{Jac}(g)}{h^{n+1}},$$

where h is the largest factor of $f_0(g_0,\dots,g_n),\dots,f_n(g_0,\dots,g_n)$ [BlaHed2014, Lemma 2.3]. If $\mathrm{Jac}(f)=ap^{n+1}$ and $\mathrm{Jac}(g)=bq^{n+1}$ for some $p,q\in\mathrm{k}[x_0,\dots,x_n]$ and $a,b\in\mathrm{k}^*$, then

$$\mathrm{Jac}(fg)=\frac{ap^{n+1}(g_0,\dots,g_n)\cdot bq^{n+1}}{h^{n+1}}=ab\left(\frac{p(g_0,\dots,g_n)\cdot q}{h}\right)^{n+1}\in\mathrm{k}[x_0,\dots,x_n]$$

and hence $fg\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)}$. Similarly, one shows that if $f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)}$, then $f^{-1}\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)}$. The group $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)}$ is a proper subgroup of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ because by [BlaHed2014, Corollary 2.4], the Jacobian of the standard Cremona involution σ is

$$\mathrm{Jac}(\sigma)=(-1)^n n\prod_{i=0}^{n}(x_i)^{n-1},$$

which is not a $(n+1)$-th power of a polynomial. The quadratic transformation σ_1 has Jacobian $\mathrm{Jac}(\sigma_1)=-2y^3$, hence $\sigma_1\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)^{(3)}$. Further, the Jacobian of any element in $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ is a non-zero constant, hence $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)\subset\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)^{(n+1)}$. □

II.4.2 Standard quintic transformations

For $\mathrm{k}=\mathbb{R}$, the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ_3,σ_3' and the family of standard quintic transformations (Theorem II.1.10, see Definition II.1.11). Therefore, we have to study the image of linear systems of transformations by standard quintic transformations.

Lemma II.4.6. *Let $\theta,f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ be a standard quintic transformation and a transformation of degree d respectively. Let $p_1,\bar{p}_1,p_2,\bar{p}_2,p_3,\bar{p}_3$ be the base-points of θ and $q_1,\dots,\bar{q}_3$ the base-points of θ^{-1}. Define*

$$\varepsilon=2m_{q_1}(\Lambda_f)+2m_{q_2}(\Lambda_f)+2m_{q_3}(\Lambda_f).$$

Then

$$\deg(f\theta)=5d-2\varepsilon,\quad m_{p_i}(\Lambda_{f\theta})=2d-\varepsilon+m_{q_i}(\Lambda_f)$$

and $m_{(\theta^{-1})_\bullet(r)}(\Lambda_{f\theta})=m_r(\Lambda_f)$ if r is not a base-point of θ^{-1}.

Proof. Recall that $m_{\bar{q}_i}(\Lambda_{\theta^{-1}})=m_{q_i}(\Lambda_{\theta^{-1}})=2$ and $m_{q_i}(\Lambda_f)=m_{\bar{q}_i}(\Lambda_f)$ for any $i=1,2,3$. Then the formula for $\deg(f\theta)$ follows from Lemma I.3.4.

Lemma I.3.4 also states that any base-point of $f\theta$ is contained in $\mathrm{Base}(\theta)\cup(\theta^{-1})_\bullet(\mathrm{Base}(f)\setminus\mathrm{Base}(\theta^{-1}))\subset\mathcal{B}(\mathbb{P}^2)$, and if $p\in\mathrm{Base}(\Lambda_{f\theta})\setminus\mathrm{Base}(\theta)$, then $m_p(\Lambda_{f\theta})=m_{(\theta^{-1})_\bullet(p)}(\Lambda_f)$.

For $i=1,2$, let $\eta_i\colon X_i\to\mathbb{P}^2$ respectively be the blow-ups of the points $p_1,\dots,\bar{p}_3$ and $q_1,\dots,\bar{q}_3$. By definition of the standard quintic transformation, there exists an isomorphism $\alpha\colon X_1\to X_2$ such that the following diagram is commutative

$$\begin{tikzcd} X_1 \arrow[r, "\alpha", "\simeq"'] \arrow[d, "\eta_1"'] & X_2 \arrow[d, "\eta_2"] \\ \mathbb{P}^2 \arrow[r, dashed, "\theta"] & \mathbb{P}^2 \end{tikzcd}$$

If $p=p_i\in\mathrm{Base}(\theta)$, then p is the image by θ^{-1} of the conic $C_{q_i}\subset\mathbb{P}^2$ passing through all of $q_1,\dots,\bar{q}_3$ except through q_i (see Definition II.1.11). Let $c\in\Lambda_f$ be a general element. As $\eta_1\alpha$ contracts only C_{q_i} onto p, the multiplicity of the general element $\theta^{-1}(c)\in\Lambda_{f\theta}$ is exactly

$$m_p(f\theta) = \widetilde{C_{q_i}}^{\eta_2} \cdot \widetilde{c}^{\eta_2} = 2\deg(f) - \varepsilon + m_{q_i}(\Lambda_f).$$

□

It is no surprise that the formulae for composing with a standard quintic transformation look so similar to the ones for composing with a quadratic transformation. By Lemma II.4.1 there exist quadratic transformations $\tau, \tau' \in \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ with base-points $p_1, \bar{p}_1, p_2$ and $q_1, \bar{q}_1, q_2$ respectively. Denote by r_1, r_2, r_3 and s_1, s_2, s_3 the base-points of their respective inverse and consider the following commutative diagram, where the points are the base-point of the corresponding map

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "{[p_1,\dots,\bar{p}_3]\quad \theta \quad [q_1,\dots,\bar{q}_3]}"] \arrow[d, dashed, "\tau", "{[p_1,\bar{p}_1,p_2]}"', "{[r_1,r_2,r_3]}"'] & \mathbb{P}^2 \arrow[d, dashed, "\tau'"', "{[q_1,\bar{q}_1,q_2]}", "{[s_1,s_2,s_3]}"] \\ \mathbb{P}^2 \arrow[r, dashed, "\tau'\theta\tau^{-1}"] & \mathbb{P}^2 \end{tikzcd}$$

Using the formula for degree and multiplicity in Lemmata I.3.4 and II.4.2, we get $\deg(\theta\tau^{-1}) = 4$ and has base-points $q_1, \bar{q}_1, q_2$ of multiplicity 2 and $\bar{q}_2, q_3, \bar{q}_3$ of multiplicity 1. Then, by using the same formulae, $\deg(\tau'(\theta\tau^{-1})) = 2$ with base-points $\tau_\bullet(\bar{q}_2)$, $\tau_\bullet(q_3)$, $\tau_\bullet(\bar{q}_3)$.

Over $\mathrm{k} = \mathbb{R}$ this is not possible, because by the construction of the abelianisation $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ (see Chapter IV, §3.2), the standard quintic transformations are sent onto generators of $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, whereas all quadratic maps are in the kernel of φ.

It is possible to compute the elements of $\mathcal{J}_\circ$ in an explicit way, although, of course, the calculations get complicated very fast. What follows now is a manual at whose end are the formulas needed for the explicit calculation.

Definition II.4.7. Let $\pi\colon X \to \mathbb{P}^1$ be a real conic bundle. We denote by

$$\mathrm{Bir}_{\mathbb{R}}(X,\pi) = \{f \in \mathrm{Bir}_{\mathbb{R}}(X) \mid \exists \alpha \in \mathrm{PGL}_2(\mathbb{R}) \text{ such that } \pi f = \alpha\pi\}$$

the group of real birational transformations of X that respect the conic bundle structure.

We fix $p_1 = [1 : i : 0], p_2 = [0 : 1 : i]$. Let $X_5 \to \mathbb{P}^2$ be the blow-up of $p_1, \bar{p}_1, p_2, \bar{p}_2$, which is the real conic bundle $\tilde{\pi}_\circ\colon X_5 \to \mathbb{P}^1$, whose fibres are the strict transforms of the conics passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$. Via this blow-up, $\mathcal{J}_\circ$ is conjugate to $\mathrm{Bir}_{\mathbb{R}}(X_5, \tilde{\pi}_\circ)$. The real conic bundle $\tilde{\pi}_\circ\colon X_5 \to \mathbb{P}^1$ is not minimal because the strict transforms of the conic $L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2}$ is a singular fibre with two real components. Contracting one of these components, we obtain the minimal real conic bundle $\tilde{\pi}_\circ\colon X_6 \to \mathbb{P}^1$, the surface X_6 being a del Pezzo surface of degree 6. Via this morphism, $\mathcal{J}_\circ$ is conjugate to $\mathrm{Bir}_{\mathbb{R}}(X_6, \tilde{\pi}_\circ)$. The conic bundle X_6 is in fact the blow-up of the sphere $\mathrm{Q}_{3,1} = \{[w : x : y : z] \in \mathbb{P}^3 \mid w^2 = x^2 + y^2 + z^2\}$ in a pair of non-real conjugate points $p, \bar{p}$, as indicated in Figure II.2.

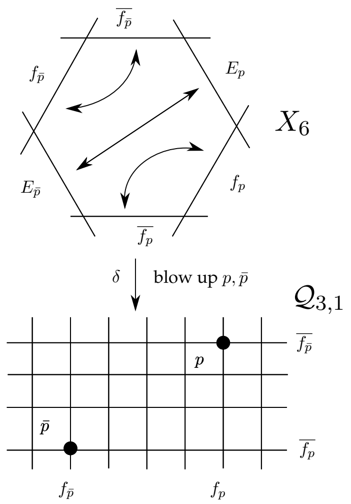

Figure II.2: The real conic bundle $\tilde{\pi}_\circ\colon X_6 \to \mathbb{P}^1$ obtained by blowing up $p, \bar{p} \in \mathcal{Q}_{3,1}$.

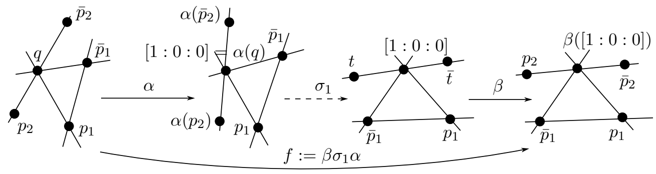

Figure II.3: The map $\varepsilon\colon X_6 \to \mathbb{P}^1 \times \mathbb{P}^1$.

The real surface $Q_{3,1}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ endowed with the anti-holomorphic involution

$$\sigma_S\colon ([u_0 : u_1], [v_0 : v_1]) \mapsto ([\bar{v}_0 : \bar{v}_1], [\bar{u}_0 : \bar{u}_1]).$$

Lemma II.4.8 ([RobZim2016, Definition 4.1]).

1. *The real conic bundle X_6 is isomorphic to*

$$X_6 \simeq (\{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid x_0y_0 = x_1y_1 = x_2y_2\}, \sigma_{X_6})$$

where $\sigma_{X_6}\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([\bar{y}_1 : \bar{y}_0 : \bar{y}_2], [\bar{x}_1 : \bar{x}_0 : \bar{x}_2])$ is the anti-holomorphic involution inherited from $\mathcal{Q}_{3,1}$.

2. *The morphism $\delta\colon (X_6, \sigma_{X_6}) \to (\mathbb{P}^1 \times \mathbb{P}^1, \sigma_S)$,*

$$([x_0 : x_1 : x_2], [y_0 : y_1 : x_2]) \overset{\delta}{\mapsto} ([x_0 : x_2], [x_2 : x_1]) = ([y_2 : y_0], [y_1 : y_2])$$
$$([u_0v_0 : u_1v_1 : u_1v_0], [u_1v_1 : u_0v_0 : u_0v_1] \overset{\delta^{-1}}{\dashleftarrow} ([u_0 : u_1], [v_0 : v_1])$$

is the blow-up of the two non-real points $p = ([0 : 1], [1 : 0])$, $\bar{p} = ([1 : 0], [0 : 1])$ on $(\mathbb{P}^1 \times \mathbb{P}^1, \sigma_S)$.

Definition II.4.9 (Descending to a non-real surface)**.** Contracting the (-1)-curves $f_{\bar{p}}$ and f_p on X_6 indicated in Figure II.2 yields a non-real birational map $\varepsilon\colon X_6 \to \mathbb{P}^1 \times \mathbb{P}^1$

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \overset{\varepsilon}{\mapsto} ([x_0 : x_1], [x_2 : x_0]) = ([y_1 : y_0], [y_0 : y_2])$$
$$([u_0v_1 : u_1v_1 : u_0v_0], [u_1v_0 : u_0v_0 : u_1v_1]) \overset{\varepsilon^{-1}}{\dashleftarrow} ([u_0 : u_1], [v_0 : v_1])$$

which respects the conic bundle structure of either surface (cf. [RobZim2016, Definition 4.4]). The two contracted components are contracted onto $([0 : 1], [1 : 0])$ and $([1 : 0], [0 : 1])$, as indicated in Figure II.3. The anti-holomorphic involution σ_{X_6} on X_6 descends via ε to the rational involution

$$\sigma_C\colon ([u_0 : u_1], [v_0 : v_1]) \dashrightarrow ([\bar{u}_0 : \bar{u}_1], [\bar{u}_1\bar{v}_1 : \bar{u}_0\bar{v}_0]).$$

Here, on $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_C)$ it is rather straight forward to do any calculations. Then, conjugating with $\varepsilon\delta^{-1}$, we can get explicit formulas on the sphere $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)$.

$$\begin{tikzcd} X_6 \arrow[d, "\delta"] \arrow[dr, "\varepsilon"] & \\ (\mathbb{P}^1\times\mathbb{P}^1,\sigma_S) \arrow[r, dashed, "\varepsilon\delta^{-1}"] & (\mathbb{P}^1\times\mathbb{P}^1,\sigma_C) \end{tikzcd}$$

More explicitly:

Manual II.4.10 (Computation of elements of $\mathcal{J}_\circ$)**.**

- Pick an element of $\mathcal{J}_\circ$
- Compute its explicit form on $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_C)$
- Conjugate with $\varepsilon\delta^{-1}\colon (\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)\dashrightarrow(\mathbb{P}^1\times\mathbb{P}^1,\sigma_C)$, which is given by

$$([u_0:u_1],[v_0:v_1]) \overset{\varepsilon\delta^{-1}}{\mapstochar\dashrightarrow} ([u_0v_0:u_1v_1],[u_1:u_0])$$
$$([v_1:v_0],[u_0v_0:u_1v_1]) \overset{\delta\varepsilon^{-1}}{\dashleftarrow\!\shortmid} ([u_0:u_1],[v_0:v_1])$$

to obtain a real birational transformation of the real surface $(\mathbb{P}^1\times\mathbb{P}^1,\sigma S)\simeq Q_{3,1}$.

The following question was asked by I. CHELTSOV during one of my talks about the real plane Cremona group.

Question: Is the real plane Cremona group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and involutions, just like the complex plane Cremona group is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and involutions?

The answer is yes and given in Corollary II.4.12.

Lemma II.4.11. *If a standard quintic involution induces the identity map on $\mathbb{P}^1$, it is conjugate to an involution of the sphere $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)$ of the form*

$$([x_0:x_1],[y_0:y_1])\mapstochar\dashrightarrow$$
$$([x_0(\mu x_1y_0-A(x_0y_0,x_1y_1)):x_1(A(x_0y_0,x_1y_1)+\lambda x_0y_1)],[y_0(A(x_0y_0,x_1y_1)+\lambda x_0y_1):y_1(\mu x_1y_0-A(x_0y_0,x_1y_1))])$$

where $A\in\mathbb{C}[u,v]$ is linear homogeneous and $|\lambda|=|\mu|$, $\bar{A}=-\frac{\bar{\mu}}{\lambda}A$.

Proof. As explained above, we can see a standard quintic transformation as elementary link θ of the real surface X_6 (see Definition II.4.8). By Definition II.1.11, the pair of non-real conjugate points that is blown up by the link is not contained in any of the (-1)-curves of X_6. The link θ descends to a link θ' of the surface $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_C)$ preserving the fibration of the projection onto the first component, and its base-points $q,\bar{q}$ are not contained in any of the lines $x_0x_1y_0y_1=0$. [Zim2015, Lemma 3.7] states that a standard quintic transformation in $\mathcal{J}_\circ$ induces the identity map or $[x_0:x_1]\mapsto[x_1:x_0]$ on $\mathbb{P}^1$. Furthermore, the image of a fiber of the second projection is mapped by θ' onto a curve intersecting a general fibre of the second projection in one point. Therefore, θ' is of the form

$$\theta'\colon([x_0:x_1],[y_0:y_1])\mapstochar\dashrightarrow([x_i:x_j],[A(x_0,x_1)y_0+B(x_0,x_1)y_1:C(x_0,x_1)y_0+D(x_0,x_1)y_1])$$

where $\{i,j\}=\{0,1\}$ and $A,B,C,D\in\mathbb{C}[x_0,x_1]$ are homogeneous of degree 1. We focus on the case $(i,j)=(0,1)$. The property $\bar{\theta}=\mathrm{Id}^2$ imposes that $A^2=D^2$ and $B(A+D)=$

$C(A+D)=0$. The option $B=C=0$ yields an automorphism, so θ' is of one of the two forms

$$\theta'\colon ([x_0:x_1],[y_0:y_1]) \dashrightarrow ([x_0:x_1],[B(x_0,x_1)y_1 : C(x_0,x_1)y_0])$$
$$\theta'\colon ([x_0:x_1],[y_0:y_1]) \dashrightarrow ([x_0:x_1],[A(x_0,x_1)y_0+B(x_0,x_1)y_1 : C(x_0,x_1)y_0-A(x_0,x_1)y_1]), \quad ABC\neq 0.$$

Further, θ' commutes with σ_C. For the first option, it imposes that $q,\bar{q}$ are on the lines $x_0x_1y_0y_1=0$, which contradicts our assumptions on θ'. For the second option, we get $B=\lambda x_1$, $C=\mu x_0$, $|\lambda|=|\mu|$ and $\bar{A}=-\frac{\bar{\mu}}{\lambda}A$.

$$\theta'\colon ([x_0:x_1],[y_0:y_1]) \dashrightarrow ([x_0:x_1],[Ay_0+\lambda x_1y_1 : \mu x_0y_0-Ay_1]), \quad |\lambda|=|\mu|, \bar{A}=-\frac{\bar{\mu}}{\lambda}A.$$

Conjugating as instructed in Manual II.4.10, we get the involution

$$([x_0:x_1],[y_0:y_1]) \dashrightarrow$$
$$([x_0\,(\mu x_1y_0-A(x_0y_0,x_1y_1)) : x_1\,(A(x_0y_0,x_1y_1)+\lambda x_0y_1)],[y_0\,(A(x_0y_0,x_1y_1)+\lambda x_0y_1) : y_1\,(\mu x_1y_0-A(x_0y_0,x_1y_1))])$$

of $(\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)$. □

Corollary II.4.12. *For each non-real conic $C\subset\mathbb{P}^2$ through $p_1=[1:i:0],\bar{p}_1,p_2=[0:1:i],\bar{p}_2$, there exists a standard quintic involution in $\mathcal{J}_\circ$ with a base-point on $C\setminus\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$.*

In particular, $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, the quadratic involutions

$$\sigma\colon [x:y:z]\dashrightarrow [yz:xz:xy], \quad \sigma'\colon [x:y:z]\dashrightarrow [xz:yz:x^2+y^2]$$

and the set of standard quintic involutions.

Proof. Finding such a standard quintic involution is equivalent to finding for each non-real $a\in\mathbb{C}^*$ an involution as in Lemma II.4.11 that contracts the fibres of $[a:1],[\bar{a}:1]$ of the conic bundle $\tilde{\pi}_\circ\delta^{-1}\colon(\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)\dashrightarrow\mathbb{P}^1$.

Let $a\in\mathbb{C}^*$ non-real and $A\in i\mathbb{R}[u,v]$ linear and homogeneous such that $A(a,1)\neq -\overline{A(a,1)}$. Plugging $\lambda=\bar{\mu}=-\frac{A(a,1)}{a}$ into the involution in Lemma II.4.11, we see that it has base-points

$$q:=([a:1],[-A(a,1)\bar{a}:\overline{A(a,1)}]),\bar{q},$$

which is a pair of non-real conjugate points because of $A(a,1)\neq A(\bar{a},1)$. Furthermore, the condition $A\bar{\mu}+\bar{A}\lambda=0$ is satisfied, so we have obtained an involution of the conic bundle $\tilde{\pi}_\circ\delta^{-1}\colon(\mathbb{P}^1\times\mathbb{P}^1,\sigma_S)\dashrightarrow\mathbb{P}^1$ that contracts the fibres of $[a:1],[\bar{a}:1]$. In particular, for any non-real conic $C\subset\mathbb{P}^2$ through $p_1=[1:i:0],\bar{p}_1,p_2=[0:1:i],p_2$ there exists a standard quintic involution in $\mathcal{J}_\circ$ with a base-point on $C\setminus\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$.

The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ, σ' and the family of standard quintic transformations. Any quadratic transformation in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is contained in the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ and σ'. [Zim2015, Lemma 5.6] implies that any two standard quintic transformations in $\mathcal{J}_\circ$ that have the same image in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\langle\langle\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ can be obtained form one another by composing with linear and/or quadratic transformations from the left and the right. It follows that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ, σ' and the set of standard quintic involutions. □

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The Cremona group of the plane is compactly presented

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Abstract

This article shows that the Cremona group of the plane is compactly presented. To do this, we prove that it is a generalized amalgamated product of three of its algebraic subgroups (automorphisms of the plane and Hirzebruch surfaces) divided by one relation.

1. *Introduction*

Let k be a field. The Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is the group of birational transformations of the projective space $\mathbb{P}^n_k = \mathbb{P}^n$. It corresponds to a very intensively studied topic in algebraic geometry (see [7, 10, 15] and references therein).

A birational transformation of $\mathbb{P}^n$ is simply a birational change of coordinates, so $\mathrm{Bir}(\mathbb{P}^n)$ is a natural generalization of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(k)$, and in many aspects the Cremona group behaves like semi-simple groups, but also in many aspects it does not. Some analogies between the Cremona groups and semi-simple groups have been presented by Serre in the 1000th Bourbaki seminar [15], and by Cantat [6].

For k a local field and endowed with the Euclidean topology, constructed in [5], the Cremona group becomes a Hausdorff topological group. For $k = \mathbb{C}$ and $k = \mathbb{R}$, the restriction to its subgroup $\mathrm{PGL}_{n+1}(k)$ of linear coordinate changes of $\mathbb{P}^n$ is the Euclidean topology. This not only opens the path to study the geometric properties of the Cremona group coming from the Euclidean topology, but also presents the opportunity to study the Cremona group from the point of view of geometric group theory and raises the question of analogies to Lie groups.

In this article, we will present one of these analogies, namely the property of being compactly presented (see Definition 6.1).

Let us take a closer look at the Cremona group endowed with the Euclidean topology:

The group $\mathrm{Bir}(\mathbb{P}^1_{\mathbb{C}}) = \mathrm{PGL}_2(\mathbb{C})$ is compactly presented by any neighbourhood of 1 because it is a connected complex algebraic group (see, for example, [1, Satz 3.1]).

For $n \geqslant 2$ and k any local field, the group $\mathrm{Bir}(\mathbb{P}^n_k)$ is not locally compact [5, Lemma 5.15], though the topology is the inductive topology given by the family of closed sets $\mathrm{Bir}(\mathbb{P}^n)_{\leqslant d} = \{f \in \mathrm{Bir}(\mathbb{P}^n) \mid \deg(f) \leqslant d\}$, which are locally compact [5, Proposition 2.10, Lemma 5.4]. Furthermore, any compact subset of $\mathrm{Bir}(\mathbb{P}^n)$ is of bounded degree.

For $n \geqslant 3$, the group $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is not compactly generated [5, Lemma 5.17], hence not compactly presented.

The group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}_2(\mathbb{C})$ and the standard quadratic transformation $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$ (see [8]). Its subgroup $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$, being a connected complex algebraic group, is compactly presented by any neighbourhood of 1. Hence $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is compactly generated by any compact neighbourhood of 1 in $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ and σ.

The aim of this article is to show that, even though it is neither an algebraic group nor locally compact, $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is moreover compactly presented.

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Theorem A (Corollary 6.8). *Endowed with the Euclidean topology, the Cremona group* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *is compactly presented by* $\{\sigma\} \cup K$, *where* K *is any compact neighbourhood of* 1 *in* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ *and* $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$ *is the standard involution of* $\mathbb{P}^2_{\mathbb{C}}$.

For algebraically closed fields, the generating sets and generating relations of $\mathrm{Bir}(\mathbb{P}^2)$ have been studied thoroughly: The famous Noether–Castelnuovo theorem [8] states that if k is algebraically closed, then $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and the standard quadratic involution $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$, that is, the generating set is the union of two complex linear algebraic groups.

A presentation was given in [11], where the generating set consists of all quadratic transformations of $\mathbb{P}^2$ and the generating relations are of the form $q_1q_2q_3 = 1$, where q_i are quadratic transformations. Another presentation was given in [14], where it is shown that $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ (isomorphic to $\mathrm{Bir}(\mathbb{P}^2)$) is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the de Jonquières group of birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ preserving the first projection along their intersection modulo one relation. In [4], a similar result is presented; the group $\mathrm{Bir}(\mathbb{P}^2)$ is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and the de Jonquières group $J_{[1:0:0]}$ of birational maps of $\mathbb{P}^2$ preserving the pencil of lines through $[1 : 0 : 0]$ along their intersection modulo one relation.

Since neither $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(k)$ nor the set of quadratic transformations nor the de Jonquières group are compact in the Euclidean topology, these presentations yield no compact presentation. However, all three presentations yield bounded presentations (the length of the generating relations are universally bounded).

In [18], using [14], a presentation of $\mathrm{Bir}(\mathbb{P}^2)$ is given by the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $J_{[0:1:0]}$ (as subgroups of $\mathrm{Bir}(\mathbb{P}^2)$) along their pairwise intersection, where $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is viewed as a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ via a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by the pencils of lines through $[0 : 1 : 0]$ and $[1 : 0 : 0]$. Again, since $J_{[0:1:0]}$ is not compact, this does not yield a compact but only a bounded presentation, but it gives rise to the following idea, which is the key step in the proof of Theorem A.

Theorem B (Theorem 5.5). *Let* k *be algebraically closed. Then the Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is isomorphic to the amalgamated product of* $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ *(as subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$*) along their pairwise intersection in* $\mathrm{Bir}(\mathbb{P}^2)$ *modulo the relation* $\tau_{13}\sigma\tau_{13}\sigma$, *where* $\tau_{13} \in \mathrm{Aut}(\mathbb{P}^2)$ *is given by* $\tau_{13} \colon [x : y : z] \mapsto [z : y : x]$.

Here the inclusion of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ into $\mathrm{Bir}(\mathbb{P}^2)$ is the same as before, $\mathbb{F}_2$ is the second Hirzebruch surface and the inclusion of $\mathrm{Aut}(\mathbb{F}_2)$ into $\mathrm{Bir}(\mathbb{P}^2)$ is given by a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{F}_2$ given by the system of lines through $[1 : 0 : 0]$, and the point infinitely near corresponding to the tangent direction $\{y = 0\}$.

The method used to prove Theorem B is, like in [4, 14], to study linear systems and their base-points. The difference here is that our maps have bounded degree. This rigidifies the situation and changes the possibilities for simplifications. The proof of Theorem B does not use [18].

For $k = \mathbb{C}$, the three groups $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ are locally compact algebraic groups. Using this and Theorem B, we prove Theorem A.

The plan of the article is as follows:

In Sections 2 and 3, we give basic definitions and results on $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{Aut}(\mathbb{F}_2)$. Section 4 is devoted to relations in the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ modulo the relation $\tau_{13}\sigma\tau_{13}\sigma$. These are the backbone of the proof of Theorem B, which will be given in Section 5. In Section 6, we visit some facts about compactly presented groups and then finally prove Theorem A.

In Sections 2–5, we work over any algebraically closed field k and Section 6 restricts to $k = \mathbb{C}$.

2. *Description of* $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ *and* $\mathrm{Aut}(\mathbb{F}_2)$ *inside the Cremona group*

This section is devoted to the description of the subgroups $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{Aut}(\mathbb{F}_2)$ of $\mathrm{Bir}(\mathbb{P}^2)$.

Remember that the nth Hirzebruch surface $\mathbb{F}_n$, $n \in \mathbb{N}$, is given by

$$\mathbb{F}_n = \{([x:y:z],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\}.$$

Observe that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and that $\mathbb{F}_1$ is isomorphic to the blow-up of one point in $\mathbb{P}^2$.

Consider the birational maps $\varphi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_0$ and $\varphi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_2$ given as follows: The map φ_0 is given by the blow-up of the points $[1:0:0]$ and $[0:1:0]$ followed by the contraction of the line passing through them. The map φ_2 is given by the blow-up of $[1:0:0]$ and the point infinitely near $[1:0:0]$ lying on the strict transform of $\{y=0\}$, followed by the contraction of the strict transform of $\{y=0\}$. The birational maps φ_0 and φ_2 are only defined up to automorphism of $\mathbb{F}_0$ and $\mathbb{F}_2$. They induce homomorphisms of groups

$$\begin{aligned} \mathrm{Aut}(\mathbb{F}_0) &\longrightarrow \mathrm{Bir}(\mathbb{P}^2), \quad \psi \longmapsto \varphi_0^{-1}\psi\varphi_0, \\ \mathrm{Aut}(\mathbb{F}_2) &\longrightarrow \mathrm{Bir}(\mathbb{P}^2), \quad \psi \longmapsto \varphi_2^{-1}\psi\varphi_2 \end{aligned}$$

whose image is uniquely determined by the choice of points blown up in $\mathbb{P}^2$. We will denote the image of $\mathrm{Aut}(\mathbb{F}_i)$ also by $\mathrm{Aut}(\mathbb{F}_i)$ for $i = 0,2$ since no confusion occurs.

Remark 2.1 (and Notation). (i) We can check that

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_0, \quad [x:y:z] \mapstochar\dashrightarrow ([x:z],[y:z])$$

with inverse $([u_0:u_1],[v_0:v_1]) \mapstochar\dashrightarrow [u_0v_1 : v_0u_1 : u_1v_1]$, and

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_2, \quad [x:y:z] \mapstochar\dashrightarrow ([xy:y^2:z^2],[y:z])$$

with inverse $([u:v:w],[a:b]) \mapstochar\dashrightarrow [ua:va:vb]$ are examples for φ_0 and φ_2.

(ii) For $i = 0,2$, the map $(\varphi_i)^{-1}$ has exactly one base-point, which we denote by p_i (Figure 1).

(iii) The image of the linear system of lines of $\mathbb{P}^2$ by φ_i has a unique base-point, namely p_i.

(iv) We denote by C_1 the curve of self-intersection 0 in $\mathbb{F}_0$ which is contracted by $(\varphi_0)^{-1}$ onto $[1:0:0]$, and by C_2 the curve of self-intersection 0 which is contracted onto $[0:1:0]$. Remark that $p_0 = \varphi_0(\{z=0\})$ and that $\{p_0\} = C_1 \cap C_2$.

(v) We denote by E the exceptional curve of self-intersection -2 in $\mathbb{F}_2$. It is contracted onto $[1:0:0]$ by $(\varphi_2)^{-1}$. Denote by C the curve of self-intersection 0 in $\mathbb{F}_2$ which is contracted by $(\varphi_2)^{-1}$ onto the point infinitely near $[1:0:0]$ corresponding to the tangent $\{y=0\}$. Remark that $p_2 = \varphi_2(\{y=0\})$ and $p_2 \in C \setminus E$.

(vi) Let $L \subset \mathbb{P}^2$ be a general line. Then $C_j \cdot \varphi_0(L) = 1$, $j=1,2$, and $C \cdot \varphi_2(L) = 1$, $E \cdot \varphi_2(L) = 0$.

The following picture illustrates for $i = 0,2$ the transformation $(\varphi_i)^{-1}\psi_i\varphi_i$, where ψ_i is some automorphism of $\mathbb{F}_i$. At the same time it shows the blow-up diagram of $(\varphi_i)^{-1}\psi_i\varphi_i$.

Consider the birational transformations of $\mathbb{P}^2$ given by

$$\begin{aligned} \sigma_1 &: [x:y:z] \mapstochar\dashrightarrow [-xy+z^2 : y^2 : yz], \\ \sigma_2 &: [x:y:z] \mapstochar\dashrightarrow [xy : z^2 : yz], \\ \sigma_3 &: [x:y:z] \mapstochar\dashrightarrow [yz : xz : xy]. \end{aligned}$$

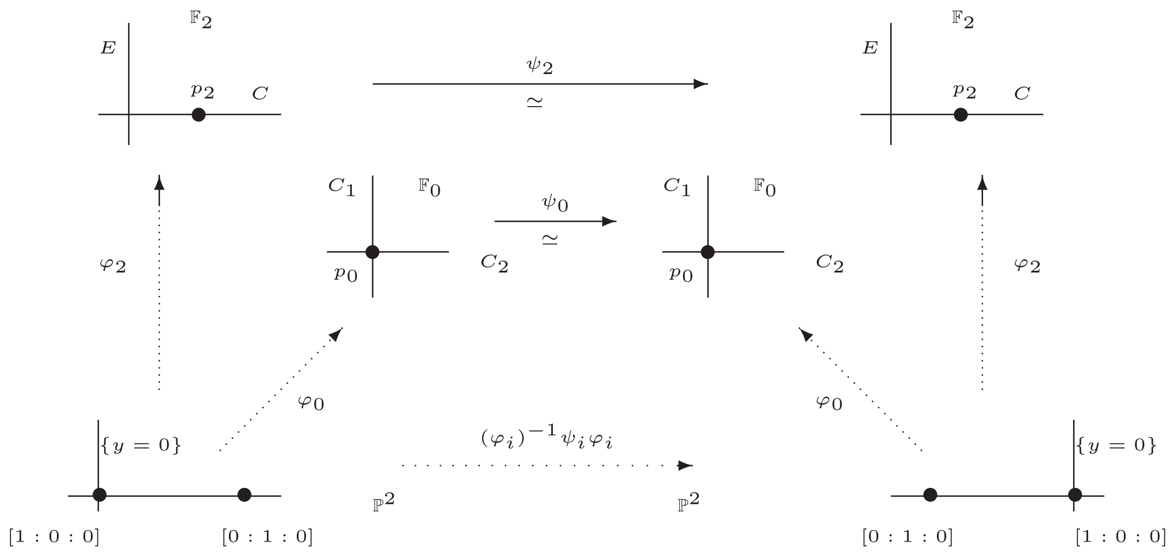

FIGURE 1. *The transformation* $(\varphi_i)^{-1}\psi_i\varphi_i$ *for* $i=0,2$.

They are three quadratic involutions of $\mathbb{P}^2$ with, respectively, exactly one, two and three proper base-points in $\mathbb{P}^2$. The map σ_3 is usually referred to as standard quadratic involution of $\mathbb{P}^2$.

The map σ_3 has base-points $[1:0:0],[0:1:0],[0:0:1]$, the map σ_2 has base-points $[1:0:0],[0:1:0]$ and the point p infinitely near $[1:0:0]$ corresponding to the direction $\{y=0\}$, and the map σ_1 has base-points $[1:0:0],p,q$, where q is a point infinitely near p not contained in the intersection of the strict transform of the exceptional divisor of $[1:0:0]$.

REMARK 2.2. For any quadratic map $\tau\in\mathrm{Bir}(\mathbb{P}^2)$ we can find $i=1,2,3$ and $\alpha,\beta\in\mathrm{Aut}(\mathbb{P}^2)$ such that α sends the base-points of τ onto the base-points of σ_i and β sends the base-points of τ^{-1} onto the base-points of σ_i. We can then write $\tau=\beta^{-1}\sigma_i\alpha$ (see [2, Subsections 2.1 and 2.8]). It follows that the linear system of τ is the image of the linear system of σ_i by α^{-1}, and that τ and σ_i have the same amount of proper base-points in $\mathbb{P}^2$. Since $\sigma_1,\sigma_2,\sigma_3$ have, respectively, one, two and three proper base-points in $\mathbb{P}^2$, the amount of proper base-points of τ determines i.

The following is the description of the groups $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ as subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ given by the above inclusions:

LEMMA 2.3. (i) *For* $i=0,2$, *the group* $\mathcal{A}_i:=\mathrm{Aut}(\mathbb{F}_i)\cap\mathrm{Aut}(\mathbb{P}^2)$ *is the group of automorphisms of* $\mathbb{P}^2$ *fixing the set of base-points of* φ_i, *that is, the set* $\{[1:0:0],[0:1:0]\}$ *if* $i=0$, *and the point* $[1:0:0]$ *and the line* $\{y=0\}$ *if* $i=2$.

For each $i\in\{0,2\}$, $\mathcal{A}_i$ *corresponds via* φ_i *to the set of automorphisms of* $\mathbb{F}_i$ *that fix* p_i.

(ii) *The set* $\mathrm{Aut}(\mathbb{F}_0)\setminus\mathrm{Aut}(\mathbb{P}^2)$ *consists of all elements of the form* $\beta\sigma_i\alpha$, *where* $i=2,3$ *and* $\alpha,\beta\in\mathrm{Aut}(\mathbb{F}_0)\cap\mathrm{Aut}(\mathbb{P}^2)$.

(iii) *The set* $\mathcal{A}_0\cup\mathcal{A}_0\sigma_2\mathcal{A}_0$ *corresponds via* φ_0 *to the set of automorphisms of* $\mathbb{F}_0$, *sending* p_0 *into* $C_1\cup C_2$.

(iv) *The set* $\mathrm{Aut}(\mathbb{F}_2)\setminus\mathrm{Aut}(\mathbb{P}^2)$ *consists of all elements of the form* $\beta\sigma_i\alpha$, *where* $i=1,2$ *and* $\alpha,\beta\in\mathrm{Aut}(\mathbb{F}_2)\cap\mathrm{Aut}(\mathbb{P}^2)$.

(v) *The set* $\mathcal{A}_2\cup\mathcal{A}_2\sigma_1\mathcal{A}_2$ *corresponds via* φ_2 *to the set of automorphisms of* $\mathbb{F}_2$ *that send* p_2 *into* C.

Proof. For $i = 0, 2$, let ψ_i be an automorphism of $\mathbb{F}_i$ and consider the following commutative diagram

$$\begin{tikzcd} & \mathbb{F}_i \arrow[r, "\psi_i"] & \mathbb{F}_i \arrow[dr, dashed, "(\varphi_i)^{-1}"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi_i"] \arrow[rrr, dashed, "(\varphi_i)^{-1}\psi_i\varphi_i"] & & & \mathbb{P}^2 \end{tikzcd}$$

(i) The map $(\varphi_i)^{-1}\psi_i\varphi_i$ is an automorphism if and only if it does not have any base-points, which is equivalent to ψ_i preserving the union of the curves contracted by $(\varphi_i)^{-1}$ and fixing the point p_i blown up by $(\varphi_i)^{-1}$. This is equivalent to $(\varphi_i)^{-1}\psi_i\varphi_i$ being an automorphism preserving the set of base-points of φ_i.

(ii)–(v) Let Δ be the linear system of lines in $\mathbb{P}^2$. We will determine the linear system $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta)$. Note that (i) shows that $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta) = \Delta$ if and only if ψ_i preserves p_i and the union of lines contracted by $(\varphi_i)^{-1}$, which is equivalent to ψ_i fixing p_i.

Assume that $\psi_i(p_i) \neq p_i$ holds. From this, it follows that $(\varphi_i)^{-1}\psi_i\varphi_i$ has at least one and at most three base-points, hence is a quadratic map. In particular, $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta)$ is a linear system of conics.

(ii) and (iii) If $i = 0$, then we can check that σ_2, σ_3 are elements of $\mathrm{Aut}(\mathbb{F}_0)$. In fact, if we take φ_0 as in Remark 2.1(i), then they are given by the automorphisms $([u_0 : u_1], [v_0 : v_1]) \dashrightarrow [u_0v_1 : v_0u_1 : u_1v_1]$ and $([u_1 : v_1], [u_2 : v_2]) \dashrightarrow ([v_1 : u_1], [v_2 : u_2])$, respectively. It follows that the set $\mathcal{A}_0\sigma_2\mathcal{A}_0 \cup \mathcal{A}_0\sigma_3\mathcal{A}_0$ is contained in $\mathrm{Aut}(\mathbb{F}_0)$.

A general element of $\psi_0\varphi_0(\Delta)$ intersects each C_j in exactly one point different from p_0 (Remark 2.1(iii) and (vi)), which means that $[1 : 0 : 0], [0 : 1 : 0]$ are base-points of the linear system of conics $(\varphi_0)^{-1}\psi_0\varphi_0(\Delta)$. The third base-point corresponds via φ_0 to the point $\psi_0(p_0)$. In particular, it is infinitely near to $[1 : 0 : 0]$ (respectively, $[0 : 1 : 0]$) if and only if $\psi_0(p_0) \in C_1$ (respectively, $\psi_0(p_0) \in C_2$). By Remark 2.2, we can write $(\varphi_0)^{-1}\psi_0\varphi_0 = \beta\sigma_j\alpha$ for some $j = 2, 3$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$, where α, β^{-1}, respectively, send the base-points of $(\varphi_0)^{-1}\psi_0\varphi_0$, $(\varphi_0)^{-1}(\psi_0)^{-1}\varphi_0$ onto the base-points of σ_j. If $j = 2$, then it follows that α, β fix the set $\{[1 : 0 : 0], [0 : 1 : 0]\}$. If $j = 3$, then we have $(\beta\theta)\sigma_3(\theta\alpha) = \beta\sigma_3\alpha$ for any permutation θ of coordinates x, y, z, hence we can assume that α fixes the set $\{[1 : 0 : 0], [0 : 1 : 0]\}$ and it follows that $\alpha \in \mathcal{A}_0$. Since $\sigma_2, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, it follows that $\beta \in \mathcal{A}_0$.

Note that $(\varphi_0)^{-1}\psi_0\varphi_0$ has an infinitely near base-point if and only if $\psi_0(p_0) \in (C_1 \cup C_2) \setminus \{p_0\}$.

(iv) and (v) If $i = 2$, then we can check that $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathbb{F}_2)$. In fact, if we take φ_2 as in Remark 2.1(i), then they are given by the automorphisms $([u : v : w], [a : b]) \mapsto ([-u + w : v : w], [a : b])$ and $([u : v : w], [a : b]) \mapsto ([u : w : v], [b : a])$, respectively. It follows that $\mathcal{A}_2\sigma_1\mathcal{A}_2 \cup \mathcal{A}_2\sigma_2\mathcal{A}_2 \subset \mathrm{Aut}(\mathbb{F}_2)$.

A general element of $\psi_2\varphi_2(\Delta)$ does not intersect E and intersects C in exactly one point different from p_2 (Remark 2.1(iii) and (vi)). Therefore, $[1 : 0 : 0]$, the point p infinitely near to it corresponding to the tangent direction $\{y = 0\}$ are base-points of the linear system $(\varphi_2)^{-1}\psi_2\varphi_2(\Delta)$. The third base-point corresponds via φ_2 to the point $\psi_2(p_2)$. In particular, it is infinitely near to p if and only if $\psi_2(p_2) \in C$ and it is a proper point of $\mathbb{P}^2$ otherwise. It follows from Remark 2.2 that we can write $(\varphi_2)^{-1}\psi_2\varphi_2 = \beta\sigma_j\alpha$ for $j = 1, 2$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$ where α, β^{-1}, respectively, send the linear system of $(\varphi_2)^{-1}\psi_2\varphi_2$, $(\varphi_2)^{-1}(\psi_2)^{-1}\varphi_2$ onto the linear system of σ_j. It follows that α, β fix $[1 : 0 : 0], p$, and hence $\alpha, \beta \in \mathcal{A}_2$.

Note that $(\varphi_2)^{-1}\psi_2\varphi_2$ has exactly one proper base-point in $\mathbb{P}^2$ if and only if ψ_2 sends p_2 into $C \setminus \{p_2\}$. □

Lemma 2.3 allows us to present the following classical results and also describe a Zariski-open set in each $\mathrm{Aut}(\mathbb{F}_i)$, which will be useful in Section 6 when we prove that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented.

Lemma 2.4. *For $i = 0, 2$, let $\mathcal{A}_i := \mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_i)$.*

(i) *The groups $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ are linear algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.*
(ii) *The group $\mathrm{Aut}(\mathbb{F}_0)$ has two irreducible components, namely the component $\mathrm{Aut}(\mathbb{F}_0)^0$ containing 1 and $\tau_{12}\mathrm{Aut}(\mathbb{F}_0)^0$, where $\tau_{12} \in \mathrm{Aut}(\mathbb{F}_0)$ is given by $\tau_{12} \colon [x : y : z] \mapsto [y : x : z]$.*
(iii) *The group $\mathrm{Aut}(\mathbb{F}_2)$ is irreducible.*
(iv) *The set $\mathcal{A}_0\sigma_3\mathcal{A}_0$ is a Zariski-open set of $\mathrm{Aut}(\mathbb{F}_0)$.*
(v) *The set $\mathcal{A}_2\sigma_2\mathcal{A}_2$ is a Zariski-open set of $\mathrm{Aut}(\mathbb{F}_2)$.*

Proof. (i)–(iii) are classical results, which, for example, can be found in [**3**, Proposition 2.2.6, Théorème 2].

(iv) By Lemma 2.3, the set $\mathrm{Aut}(\mathbb{F}_0) \setminus (\mathcal{A}_0\sigma_3\mathcal{A}_0)$ is the set of elements of $\mathrm{Aut}(\mathbb{F}_0)$ that send the point p_0 into the curve $C_1 \cup C_2$ and is therefore closed.

(v) By Lemma 2.3, the set $\mathrm{Aut}(\mathbb{F}_2) \setminus (\mathcal{A}_2\sigma_2\mathcal{A}_2)$ is the set of elements of $\mathrm{Aut}(\mathbb{F}_2)$ that fix the curve C and is therefore closed. □

Remark 2.5. The Noether–Castelnuovo theorem states that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and σ_3 ([**8**], see also [**2**, Section 8]). Furthermore, we can write $\sigma_3 = \tau_{12}\sigma_2\tau_{12}\sigma_2$ where $\tau_{12}([x : y : z]) = ([y : x : z])$, hence the group $\mathrm{Bir}(\mathbb{P}^2)$ is also generated by $\mathrm{Aut}(\mathbb{P}^2)$ and σ_2. Therefore, for any $i = 0, 2$, the group $\mathrm{Bir}(\mathbb{P}^2)$ is generated by its subgroups $\mathrm{Aut}(\mathbb{P}^2)$ and $\mathrm{Aut}(\mathbb{F}_i)$, and thus also generated by all three subgroups $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$.

Note that Lemma 2.3 in particular implies that all elements of $\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ are linear or quadratic.

Definition 2.6. For a set S, let F_S be the free group generated by S.

For the set $S := \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2) \subset \mathrm{Bir}(\mathbb{P}^2)$, define

$$\mathfrak{G} := F_S \Big/ \left\langle \begin{array}{ll} fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{P}^2) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_0) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_2) \\ \tau_{13}\sigma_3\tau_{13}\sigma_3 & \end{array} \right\rangle,$$

where $\tau_{13} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{13} \colon [x : y : z] \mapsto [z : y : x]$.

Remark 2.7. The group $\mathfrak{G}$ is isomorphic to the free product of the three groups $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_2), \mathrm{Aut}(\mathbb{F}_2)$ amalgamated along all the pairwise intersections (generalized amalgamated product of the three groups) modulo the relation $\tau_{13}\sigma_3 = \sigma_3\tau_{13}$.

A geometric approach to generalized amalgamated products can be found in [**13**, **16**, **17**]. The generalized amalgamated product

$$F_S \Big/ \left\langle \begin{array}{ll} fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{P}^2) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_0) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_2) \end{array} \right\rangle$$

is in [**17**, Subsection 1.3] the colimit of the diagram

$$\begin{tikzcd}
 & & \mathrm{Aut}(\mathbb{P}^2) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathcal{A}_0 \arrow[r, dash] \arrow[dl, dash] & \mathcal{A}_0 \cap \mathcal{A}_2 \arrow[r, dash] \arrow[d, dash] & \mathcal{A}_2 \arrow[dr, dash] & \\
\mathrm{Aut}(\mathbb{F}_0) \arrow[rr, dash] & & \mathrm{Aut}(\mathbb{F}_0) \cap \mathrm{Aut}(\mathbb{F}_2) \arrow[rr, dash] & & \mathrm{Aut}(\mathbb{F}_2)
\end{tikzcd}$$

Equivalently, it is the fundamental group of a 2-complex of groups. The vertices are $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$, the edges are their pairwise intersection and the 2-simplex is the group $\mathrm{Aut}(\mathbb{P}^2)\cap\mathrm{Aut}(\mathbb{F}_0)\cap\mathrm{Aut}(\mathbb{F}_2)$ [**13**, Subsections 2.1 and 3.4, **16**, 4.4].

Remark 2.8. By Remark 2.5, there exists a canonical surjective homomorphism of groups

$$\pi : \mathfrak{G} \longrightarrow \mathrm{Bir}(\mathbb{P}^2)$$

and by definition of $\mathfrak{G}$ a natural map

$$w\colon \mathrm{Aut}(\mathbb{P}^2)\cup\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2)\longrightarrow \mathfrak{G}$$

which sends an element to its corresponding word. Note that $\pi\circ w$ is the identity map.

3. *Base-points, multiplicities, de Jonquières*

The methods we use mainly consist of studying linear systems of $\mathbb{P}^2$ and their base-points. In this section, we recall some definitions, notions and formulae which will be used almost constantly in Section 4 and 5, which have the aim to prove Theorem B (Theorem 5.5).

Definition 3.1. A *point over* $\mathbb{P}^2$ is a point $p\in S$, where $S := S_{n+1} \overset{\nu_n}{\to} S_{n-1} \overset{\nu_{n-1}}{\to} \cdots \overset{\nu_1}{\to} S_0 := \mathbb{P}^2$ is a sequence of blow-ups, and where we identify $p\in S$ with $p_i\in S_i$ if $\nu_{i+1}\cdots\nu_n\colon S\to S_i$ is a local isomorphism around p sending p to p_i.

A point $p\in S$ over $\mathbb{P}^2$ is *proper* if it is equivalent to a point $p'\in\mathbb{P}^2$, and *infinitely near* otherwise.

Definition 3.2. Let $f\in\mathrm{Bir}(\mathbb{P}^2)$ be a quadratic birational transformation, and call p_1,p_2,p_3 its base-points and q_1,q_2,q_3 the base-points of f^{-1}. We say that *the base-points of f are ordered consistently* if the following holds: The base-points of f and of f^{-1} are ordered such that the following conditions are satisfied.

(i) If p_1,p_2,p_3 are proper points of $\mathbb{P}^2$, then all the lines through p_1 (respectively, p_2, p_3) are sent onto lines through q_1 (respectively, q_2, q_3).

(ii) If p_1,p_2 are proper points of $\mathbb{P}^2$ and p_3 is infinitely near to p_1, then all the lines through p_1 (respectively, p_2) are sent onto lines through q_1 (respectively, q_2).

(iii) If p_1 is a proper point of $\mathbb{P}^2$, p_2 infinitely near p_1 and p_3 infinitely near p_2, then the lines through p_1 are sent onto lines through q_1 and the exceptional curve associated to p_3 is sent onto the tangent associated to q_2.

Remark 3.3. Writing down the blow-up diagram of the three quadratic involutions $\sigma_1,\sigma_2,\sigma_3$, we see that we can always order their base-points consistently (for example, for σ_3 the ordering $p_1=q_1=[1:0:0], p_2=q_2=[0:1:0], p_3=q_3=[0:0:1]$ is consistent). Since any quadratic birational transformation of $\mathbb{P}^2$ can be written $\beta\sigma_i\alpha$ for some suitable $i\in\{1,2,3\}$, $\alpha,\beta\in\mathrm{Aut}(\mathbb{P}^2)$ (Remark 2.2), it is always possible to order its base-points consistently.

Throughout the article, we will always assume that the base-points of a quadratic transformation of $\mathbb{P}^2$ are ordered consistently.

Let us remind the reader of the following formula: Let Δ be a linear system and $f\in\mathrm{Bir}(\mathbb{P}^2)$ be a quadratic transformation with base-points p_1,p_2,p_3, and q_1,q_2,q_3 the base-points of f^{-1}. Let a_i be the multiplicity of Δ in p_i and b_i be the multiplicity of $f(\Delta)$ in q_i. If the base-points of f are ordered consistently, then

$$\deg(f(\Delta)) = 2\deg(\Delta)-\varepsilon,\quad b_i=\deg(\Delta)-\varepsilon+a_i$$

for $i=1,2,3$ and $\varepsilon=a_1+a_2+a_3$ (see [**2**, Subsection 4.2]).

Definition 3.4. We define

$$J := \{f \in \mathrm{Bir}(\mathbb{P}^2) : f \text{ preserves the pencil of lines through } [1:0:0]\}.$$

The elements of J are called *de Jonquières transformations*.

A linear system Δ of $\mathbb{P}^2$ of degree $\deg(\Delta) = d$ and with base-points $p_1, \dots, p_n$ of multiplicity $a_1, \dots, a_n$ is called *de Jonquières linear system* if it has multiplicity $d-1$ at $[1:0:0]$ and satisfies the conditions $d^2 - 1 = \sum_{i=1}^n a_i^2$ and $3(d-1) = \sum_{i=1}^n a_i$.

We call a base-point of f a *simple base-point* if it is different from $[1:0:0]$, and denote the set of simple base-points by $\mathrm{sBp}(f)$.

Remark 3.5. (1) We have the following inclusions: $\mathrm{Aut}(\mathbb{F}_2) \subset J$ and $\mathrm{Aut}(\mathbb{F}_0)^0 \subset J$, where $\mathrm{Aut}(\mathbb{F}_0)^0$ is the connected component of $\mathrm{Aut}(\mathbb{F}_0)$ containing Id and which is equal to $(\mathcal{A}\sigma_2\mathcal{A}) \cup (\mathcal{A}\tau_{12}\sigma_2\tau_{12}\mathcal{A}) \cup (\mathcal{A}\sigma_3\mathcal{A}) \cup \mathcal{A}$, where $\mathcal{A} = \{\alpha \in \mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_2) \mid \alpha([1:0:0]) = [1:0:0]\}$ and $\tau_{12} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{12}\colon [x:y:z] \mapsto [y:x:z]$ (Lemma 2.4).

(2) Any element of $f \in J\backslash\mathrm{Aut}(\mathbb{P}^2)$ of degree d has $2d-1$ base-points: the base-point $[1:0:0]$ of multiplicity $d-1$ and $2d-2$ other base-points of multiplicity one (this follows from the conditions on the degree and multiplicities). Thus the definition of simple base-point of f is quite natural. If $f \in J$ is of degree 2, then it has exactly three base-points, all of multiplicity one. Its simple base-points are just the ones different from $[1:0:0]$.

(3) A de Jonquières linear system of $\mathbb{P}^2$ of degree d has $2d-1$ base-points and the multiplicity at any base-point different from $[1:0:0]$ is one. Such a point is called a *simple base-point* of Δ. Observe that, for $f \in J$ and Δ a de Jonquières linear system, $f(\Delta)$ is a de Jonquières linear system, and the linear system of f is a de Jonquières linear system.

Lemma 3.6. *For any quadratic de Jonquières transformation $f \in \mathrm{Bir}(\mathbb{P}^2)$ there exist $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$, $\tau \in \{\sigma_1, \sigma_2, \tau_{12}\sigma_2\tau_{12}, \sigma_3\} \subset \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$, where $\tau_{12} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{12} : [x:y:z] \mapsto [y:x:z]$, such that $f = \alpha_2\tau\alpha_1$.*

In particular, $(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$ generates $\mathcal{J}$.

Proof. By Remark 2.1, we can write $f = \alpha_2\sigma_i\alpha_1$ for some $\alpha_1, \alpha_2 \in \mathrm{Bir}(\mathbb{P}^2)$ and i is determined by the amount of proper base-points of f. Since f is de Jonquières, the point $[1:0:0]$ is a base-point of f.

If f has only one proper base-point in $\mathbb{P}^2$, then it has to be fixed by α_1 and α_2, which belong thus to J. This gives the result.

Suppose that f has exactly two proper base-points, namely $[1:0:0]$ and p. This implies that $\sigma_i = \sigma_2$, which has base-points $[1:0:0], [0:1:0]$ and a third one, infinitely near $[1:0:0]$. The base-point of f which is not a proper point of $\mathbb{P}^2$ is either infinitely near $[1:0:0]$ or p. If it is infinitely near $[1:0:0]$, then α_1, α_2 fix $[1:0:0]$ and are therefore de Jonquières. If it is infinitely near p, then α_1 sends p onto $[1:0:0]$ and $[1:0:0]$ onto $[0:1:0]$. We write $\alpha_1 = \tau_{12}\beta_1$, $\alpha_2 = \beta_2\tau_{12}$, for some $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{P}^2)$, which means that β_1 fixes $[1:0:0]$, that is, $\beta_1 \in J \cap \mathrm{Aut}(\mathbb{P}^2)$ and $f = \beta_2(\tau_{12}\sigma_2\tau_{12})\beta_1$. Since $f, \beta_1, \tau_{12}\sigma_2\tau_{12} \in J$, we have $\beta_2 \in J$.

Suppose that f has three proper base-points. For any $\theta \in \mathrm{Aut}(\mathbb{P}^2)$ that permutes the coordinate x, y, z, we have $\theta\sigma_3\theta = \sigma_3$. Therefore, we can assume that $\alpha_1 \in J$. Since σ_3 is de Jonquières, the map α_2 has to be de Jonquières as well.

Since every element of $\mathcal{J}$ decomposes into quadratic elements of $\mathcal{J}$ (see [**2**, Theorem 8.4.3]), Lemma 3.6 implies that $\mathcal{J}$ is generated by $(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$ generates $\mathcal{J}$. □

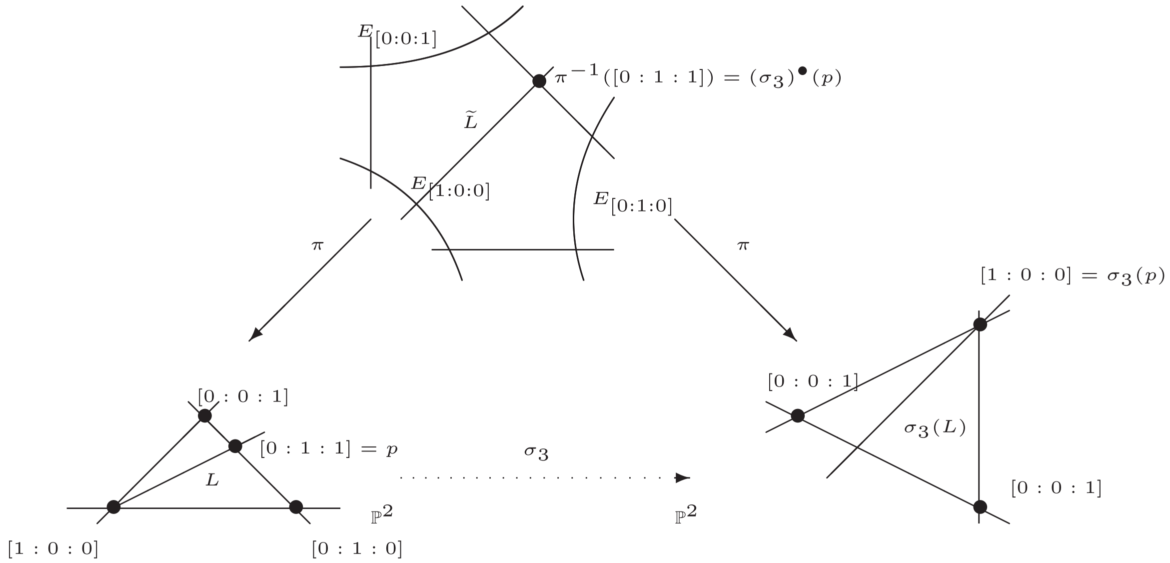

FIGURE 2. *The points* $(\sigma_3)^\bullet(p)$ *and* $\sigma_3(p)$.

REMARK 3.7. Suppose Δ is a de Jonquières linear system of degree d and f a quadratic de Jonquières transformation. We can say the following about the degree of $f(\Delta)$: Let $p_1 = [1:0:0], p_2, p_3$ be the base-points of f and a_i be the multiplicity of Δ in p_i. Then $a_1 = \deg(\Delta) - 1$ and by the formula given above, we have

$$\deg(f(\Delta)) = 2d - (d-1) - a_2 - a_3 = d + 1 - a_2 - a_3.$$

Since Δ has one base-point of multiplicity $d-1$ and all the other base-points are of multiplicity 1, we know that, for $i = 2, 3$, a_i is either zero or one. In fact, $a_i = 0$ if p_i is not a common base-point of f and Δ, and $a_i = 1$ if p_i is a common base-point of f and Δ. Thus the formula implies

$$\deg(f(\Delta)) = \begin{cases} d+1 & \text{if } f \text{ and } \Delta \text{ have no common simple base-points,} \\ d & \text{if } f \text{ and } \Delta \text{ have exactly one common simple base-point,} \\ d-1 & \text{if } f \text{ and } \Delta \text{ have exactly two common simple base-points.} \end{cases}$$

Furthermore, if p is a simple base-point of Δ that is not a base-point of f, then $f^\bullet(p)$ (see definition below) is a simple base-point of $f(\Delta)$ [**2**, Subsection 4.1].

DEFINITION 3.8. Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ and p be a point over the domain $\mathbb{P}^2$ that is not a base-point of f. Take a minimal resolution of f

$$\begin{tikzcd} & S \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

where ν_1, ν_2 are sequences of blow-ups. Let $p' \in S$ be a representative of p. We can see p' as a point over the range $\mathbb{P}^2$, and call it $f^\bullet(p)$.

Let us look at an example to understand $f^\bullet(p)$ and $f(p)$:

EXAMPLE 3.9. Consider the standard quadratic involution $\sigma_3 \in \mathrm{Bir}(\mathbb{P}^2)$ and the point $p = [0:1:1]$, which is on the line $\{x = 0\}$ contracted by σ_3 onto the point $[1:0:0]$, which means that $\sigma_3(p) = [1:0:0]$. The line $L = \{y = z\}$ passing through p and $[1:0:0]$ is sent by σ_3 onto itself. By definition, $(\sigma_3)^\bullet(p)$ is the point in the first neighbourhood of $[1:0:0]$ corresponding to the tangent direction $\{y = z\}$. In conclusion, $\sigma_3(p)$ is a proper point of $\mathbb{P}^2$, whereas $(\sigma_3)^\bullet(p)$ is not. The following picture (Figure 2) illustrates the situation.

Remark 3.10. Note that $f^\bullet$ is a one-to-one correspondence between the sets

$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f\} \quad \text{and}$$
$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f^{-1}\}.$$

4. *Basic relations in $\mathfrak{G}$*

In this section, we present basic relations that hold in $\mathfrak{G}$ and which will be the backbone of the proof of Theorem A (Theorem 5.5). We prove relations for words in $\mathfrak{G}$ of length three using properties of the elements of $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ and $\mathrm{Aut}(\mathbb{P}^2)$. (Lemmas 4.2–4.4.) They will then be used in the next section to prove that there exists an injective map $w_J\colon J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$ (Lemma 5.1, Corollary 5.2), which will enable us to prove Theorem A (Theorem 5.5) using the result that $\mathrm{Bir}(\mathbb{P}^2)$ is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and J modulo one relation [4] (Theorem 5.3).

Lemmas 4.1 and 4.2 yield that words of length three in $\mathfrak{G}$ whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is linear or quadratic behave like their images in $\mathrm{Bir}(\mathbb{P}^2)$. Lemmas 4.3 and 4.4 yield relations for words of length three whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is de Jonquières and of degree 3.

Define $\mathrm{TAut}(\mathbb{P}^2) = D \rtimes S_3$, where $S_3 \subset \mathrm{Aut}(\mathbb{P}^2)$ is the image of the permutation matrices of GL_3 and D is the image of the three-dimensional torus. We can check that the group $\mathrm{TAut}(\mathbb{P}^2)$ is normalized by σ_3, and the automorphism of $\mathrm{TAut}(\mathbb{P}^2)$ given by the conjugation of σ_3 will be denoted by ι. Note that $\iota(\alpha) = \alpha$ for $\alpha \in S_3$ and $\iota(\delta) = \delta^{-1}$ for $\delta \in D$.

As subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, we can embed $\mathrm{TAut}(\mathbb{P}^2)$ (as a set) into $\mathfrak{G}$ by the word map w. Lemma 4.1 shows that in $\mathfrak{G}$ the image of $\mathrm{TAut}(\mathbb{P}^2)$ is normalized by $w(\sigma_3)$:

Lemma 4.1. *For any $(\delta, \alpha) \in D \rtimes S_3$ the relation $w(\delta\alpha)w(\sigma_3) = w(\sigma_3)w(\iota(\delta\alpha))$ holds in $\mathfrak{G}$.*

Proof. Let $\tau_{12} : [x : y : z] \mapsto [y : x : z]$. In $\mathrm{Aut}(\mathbb{F}_0)$, the relation $\tau_{12}\sigma_3\tau_{12} = \sigma_3$ holds, hence the relation $w(\tau_{12})w(\sigma_3) = w(\sigma_3)w(\tau_{12})$ holds in $\mathfrak{G}$. By definition, $w(\tau_{13})w(\sigma_3) = w(\sigma_3)w(\tau_{13})$ is a relation in $\mathfrak{G}$, and τ_{13} and τ_{12} generate S_3. Therefore, the relation $w(\alpha)w(\sigma_3) = w(\sigma_3)w(\alpha) = w(\sigma_3)w(\iota(\alpha))$ holds in $\mathfrak{G}$ for any $\alpha \in S_3$.

Let $\delta \in D$. The relation $\delta\sigma_3\delta = \sigma_3$ holds in $\mathrm{Aut}(\mathbb{F}_0)$, hence $w(\delta)w(\sigma_3) = w(\sigma_3)w(\delta^{-1}) = w(\sigma_3)w(\iota(\delta))$ holds in $\mathfrak{G}$. □

Using Lemma 4.1, we now show that, for $f, g, h \in \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ and $\deg(fgh) \leqslant 2$, the word $w(f)w(g)w(h)$ behaves like the composition fgh.

Lemma 4.2. *Let $g \in \mathrm{Aut}(\mathbb{P}^2), h, f \in \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ such that $\deg(fgh) \in \{1, 2\}$.*

(i) *If $\deg(fgh) = 1$, then $w(f)w(g)w(h) = w(fgh)$ in $\mathfrak{G}$.*
(ii) *If $\deg(fgh) = 2$, then there exist $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$, $\tilde{g} \in \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ such that $w(f)w(g)w(h) = w(\beta)w(\tilde{g})w(\alpha)$ in $\mathfrak{G}$.*
(iii) *If $\deg(fgh) = 2$ and $f, g, h \in J$, then $\alpha, \beta, \tilde{g}$ can be chosen to be in J.*

Proof. Suppose that $f \in \mathrm{Aut}(\mathbb{P}^2)$ or $h \in \mathrm{Aut}(\mathbb{P}^2)$. The first claim follows from the definition of $\mathfrak{G}$. The second and third claim follow by putting $\beta := fg$ or $\alpha = gh$ if $\deg(f) = 1$ or $\deg(h) = 1$, respectively.

Assume that $h \in \mathrm{Aut}(\mathbb{F}_i)\backslash\mathrm{Aut}(\mathbb{P}^2)$ and $f \in \mathrm{Aut}(\mathbb{F}_j)\backslash\mathrm{Aut}(\mathbb{P}^2)$. Since $\mathrm{Aut}(\mathbb{F}_0)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)\cap\mathrm{Aut}(\mathbb{F}_0)$ and σ_2,σ_3 and $\mathrm{Aut}(\mathbb{F}_2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)\cap\mathrm{Aut}(\mathbb{F}_2)$ and σ_1,σ_2 (Lemma 2.3), we can write $f=\beta_2\sigma_k\alpha_2$ and $h=\beta_1\sigma_l\alpha_1$ for some $\alpha_1,\beta_1\in\mathrm{Aut}(\mathbb{F}_i)\cap\mathrm{Aut}(\mathbb{P}^2)$, $\alpha_2,\beta_2\in\mathrm{Aut}(\mathbb{F}_j)\mathrm{Aut}(\mathbb{P}^2)$ and $k,l\in\{1,2,3\}$. By replacing g with $\alpha_2 g\beta_1$ in $\mathrm{Aut}(\mathbb{P}^2)$, we can assume that $\alpha_2=\beta_1=\mathrm{Id}$, and hence $f=\beta_2\sigma_k$ and $h=\sigma_l\alpha_1$. It follows from Remark 2.2 that

$$\text{the base-points of } f \text{ are exactly the base-points of } \sigma_k, \tag{$*$}$$
$$\text{the base-points of } h^{-1} \text{ are exactly the base-points of } \sigma_l.$$

(i) Suppose that $\deg(fgh)=1$. Then f and $(gh)^{-1}$ have exactly the same base-points, which are, respectively, the base-points of σ_k and the image of the base-points of σ_l by g. In particular, f, $(gh)^{-1}$, and hence also σ_k, σ_l have the same amount of proper base-points in $\mathbb{P}^2$. Since σ_1, σ_2, σ_3 have exactly one, two and three proper base-points, it follows that $\sigma_k=\sigma_l$.

If $k\in\{1,2\}$, then the equation $\sigma_k=\sigma_l$, the fact that f, $(gh)^{-1}$ have the same base-points and $(*)$ imply that $g\in\mathrm{Aut}(\mathbb{F}_2)\cap\mathrm{Aut}(\mathbb{P}^2)$ and so $f,g,h\in\mathrm{Aut}(\mathbb{F}_2)$. The definition of $\mathfrak{G}$ then implies $w(f)w(g)w(h)=w(fgh)$.

If $k=3$, then the equation $\sigma_k=\sigma_l$, the fact that f, $(gh)^{-1}$ have the same base-points and $(*)$ imply that g permutes the base-points of σ_3. Lemma 4.1 states that $w(g)w(\sigma_3)=w(\sigma_3)w(\iota(g))$. We get

$$\begin{aligned} w(f)w(g)w(h) &= w(\beta_2\sigma_3)w(g)w(\sigma_3\alpha_1)\\ &= w(\beta_2)w(\sigma_3)w(\sigma_3)w(\iota(g))w(\alpha_1)\\ &= w(\beta_2)w(\iota(g))w(\alpha_1)=w(\beta_2\iota(g)\alpha_1)\\ &= w(\beta_2\sigma_3 g\sigma_3\alpha_1)=w(fgh). \end{aligned}$$

(ii) Suppose $\deg(fgh)=2$, that is, f and $(gh)^{-1}$ have exactly two common base-points s,t, at least one of them being proper. Assume that s is proper.

If t is infinitely near to s, $(*)$ implies that $\{k,l\}\subset\{1,2\}$, that is, $f,h\in\mathrm{Aut}(\mathbb{F}_2)$. Then $(*)$ and the fact that t is infinitely near s imply that $s=[1:0:0]$ and that t lies on the strict transform of $\{y=0\}$. Then s,t are base-points of both h^{-1} and $(gh)^{-1}$ and it follows that $g(\{s,t\})=\{s,t\}$, thus $g\in\mathrm{Aut}(\mathbb{F}_2)$. It follows that in $\mathrm{Aut}(\mathbb{F}_2)$ (hence also in $\mathfrak{G}$)

$$w(f)w(g)w(h)=w(\beta_2\sigma_k)w(g)w(\sigma_l\alpha_1)=w(\beta_2)w(\sigma_k g\sigma_l)w(\alpha_1).$$

Remark that any map contained in $\mathrm{Aut}(\mathbb{F}_2)$ is de Jonquières (Remark 3.5), from which claim (iii) follows for this subcase.

If s and t are both proper, $(*)$ implies that $\{k,l\}\subset\{2,3\}$, that is, $f,h\in\mathrm{Aut}(\mathbb{F}_0)$. Then $(*)$ yields that $\{s,t\}\subset\{[1:0:0],[0:1:0],[0:0:1]\}$. There exist $\alpha,\beta\in\mathrm{TAut}(\mathbb{P}^2)$ such that

$$\alpha(\{[1:0:0],[0:1:0]\})=g^{-1}(\{s,t\}),\quad \beta(\{s,t\})=\{[1:0:0],[0:1:0]\}$$
$$\beta g\alpha([1:0:0])=[1:0:0],\quad \beta g\alpha([0:1:0])=[0:1:0].$$

If $k=2$, then we may choose $\beta=\mathrm{Id}$. If $l=2$, then we may choose $\alpha=\mathrm{Id}$. We get

$$\begin{aligned} w(f)w(g)w(h) &= w(\beta_2\sigma_k)w(\beta^{-1})w(\beta)w(g)w(\alpha)w(\alpha^{-1})w(\sigma_l\alpha_1)\\ &\overset{\text{Lem 4.1}}{=} w(\beta_2\iota(\beta^{-1}))w(\sigma_k)w(\beta g\alpha)w(\sigma_l)w(\iota(\alpha^{-1})\alpha_1)\\ &= w(\beta_2\iota(\beta^{-1}))w(\sigma_k\beta g\alpha\sigma_l)w(\iota(\alpha)\alpha_1). \end{aligned}$$

The claim follows with $\alpha=\iota(\alpha^{-1})\alpha_1$, $\tilde{g}=\sigma_k(\tilde{\beta}g\tilde{\alpha})\sigma_l$, $\beta=\beta_1\iota(\beta^{-1})$. It remains to prove claim (iii) for this subcase: If f,g,h are de Jonquières, then $g([1:0:0])=[1:0:0]$ and $[1:0:0]$ is a common base-point of gf and h^{-1}. Choosing α,β above such that they fix $[1:0:0]$ (that is, are de Jonquières), it follows that $\tilde{g}=\beta g\alpha$ is de Jonquières. The maps f and h being de Jonquières implies that α_1,β_2 are de Jonquières (Remark 3.5), hence $\iota(\alpha^{-1})\alpha_1$, $\beta_1\iota(\beta^{-1})$ and $\tilde{g}$ are de Jonquières. □

The next two lemmata yield relations for words of length three whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is of degree 3.

Lemma 4.3. *Let $f \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$ be a quadratic transformation, $\alpha_1, \dots, \alpha_4 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ such that*

(i) *f is a local isomorphism at the simple base-points q_2, q_3 of $(\alpha_2\sigma_3\alpha_1)^{-1}$;*
(ii) *$\mathrm{sBp}(\alpha_4\sigma_3\alpha_3) = \{f(q_2), f(q_3)\}$.*

Then

(i) *the map $(\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)$ is quadratic de Jonquières;*
(ii) *$\mathrm{sBp}((\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)) = ((\alpha_2\sigma_3\alpha_1)^{-1})^\bullet(\mathrm{sBp}(f))$;*
(iii) *there exist $\beta_1, \beta_3 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\beta_2 \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ such that the following equation holds in $\mathfrak{G}$:*

$$w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1);$$

that is, the following diagram corresponds to a relation in $\mathfrak{G}$:

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, dashed, "f"] & \mathbb{P}^2 \arrow[d, dashed, "\alpha_4\sigma_3\alpha_3"] \\
\mathbb{P}^2 \arrow[u, dashed, "\alpha_2\sigma_3\alpha_1"] \arrow[r, dashed, "\beta_3\beta_2\beta_1"'] & \mathbb{P}^2
\end{tikzcd}$$

Proof. Define $\tau_1 := \alpha_2\sigma_3\alpha_1$ and $\tau_2 := \alpha_4\sigma_3\alpha_2$, and denote by $p_1 = [1:0:0], p_2, p_3$ the base-points of f and by $p_1, \bar{p}_2, \bar{p}_3$ the base-points of its inverse (ordered consistently, see Definition 3.2).

Since f is a local isomorphism at q_2, q_3, the map f^{-1} is a local isomorphism at $f(q_2), f(q_3)$. Hence there exist simple base-points $p_i, \bar{p}_i$ of f, f^{-1}, respectively, either proper points of $\mathbb{P}^2$ or infinitely near p_1, which do not lie on the lines contracted by $(\tau_1)^{-1}$ and τ_2. Up to order, we can assume that $p_i = p_2$. Therefore, the points $\tilde{p}_2 := (\tau_1^{-1})^\bullet(p_2)$ and $\hat{p}_2 := (\tau_2)^\bullet(\bar{p}_2)$ are proper points of $\mathbb{P}^2$.

Observe that the map $\tau_2 f \tau_1$ is de Jonquières of degree 2 having base-points $p_1, \tilde{p}_2, \tilde{p}_3 := (\tau_1^{-1})^\bullet(p_3)$ and its inverse having base-points $p_1, \hat{p}_2, \hat{p}_3 := (\tau_2)^\bullet(\bar{p}_3)$. Indeed, the map $f\tau_1$ is of degree 3 with base-points $p_1, \tilde{p}_2, \tilde{p}_3, \tilde{q}_2, \tilde{q}_3$, where $\tilde{q}_2, \tilde{q}_3$ are the simple base-points of τ_1, and its inverse having base-points $p_1, p_4, p_5, f(q_2), f(q_3)$. Thus $\tau_2 f\tau_1$ is de Jonquières of degree 2 with base-points $p_1, \tilde{p}_2, \tilde{p}_3$ and its inverse having base-points $p_1, \hat{p}_2, \hat{p}_3$ (by the formula given in Section 3).

Since $\tau_2 f \tau_1$ has at least one simple proper base-point (namely $\tilde{p}_2$), Lemmas 3.6 and 2.2 imply that there exist $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\beta_2 \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ such that $\tau_2 f \tau_1 = \beta_3\beta_2\beta_1$.

It is left to prove that $w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1)$ in $\mathfrak{G}$. We will use Lemma 4.2, and for this we fill the diagram

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, dashed, "f"] & \mathbb{P}^2 \arrow[d, dashed, "\tau_2"] \\
\mathbb{P}^2 \arrow[u, dashed, "\tau_1"] \arrow[r, dashed, "\beta_3\beta_2\beta_1"'] & \mathbb{P}^2
\end{tikzcd}$$

with triangles corresponding to relations in $\mathfrak{G}$.

The map f is a local isomorphism at q_2, q_3, hence the three points p_1, p_2, q_2 are not collinear. Since moreover p_1, q_2 are both proper points of $\mathbb{P}^2$, there exists a quadratic map $\rho \in \mathrm{Bir}(\mathbb{P}^2) \cap J$ which has base-points p_1, q_2, p_2. The maps $\rho\tau_1$ and ρf^{-1} are quadratic de Jonquières maps with base-points $p_1, \tilde{q}_2, \tilde{p}_2$ and $p_1, \bar{p}_2, f(q_2)$, respectively. It follows that also the map $\rho\tau_1(\beta_3\beta_2\beta_1)^{-1}$ is quadratic. The situation is summarized in the following diagram, where all the arrows are quadratic maps and the points in the brackets are the simple base-points of the corresponding quadratic map contained in J:

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[rr, dashed, "f", "{[p_2,p_3]}" near start, "{[\bar{p}_2,\bar{p}_3]}" near end] \arrow[dr, dashed, "{[p_2,q_2]}", "\rho"'] & & \mathbb{P}^2 \arrow[dl, dashed, "{[\bar{p}_2,f(q_2)]}"'] \arrow[dd, dashed, "\tau_2", "{[f(q_2),f(q_3)]}" near start] \\
& \mathbb{P}^2 & \\
\mathbb{P}^2 \arrow[uu, dashed, "\tau_1", "{[q_2,q_3]}" near end, "{[\tilde{q}_2,\tilde{q}_3]}" near start] \arrow[ur, dashed, "{[\tilde{q}_2,\tilde{p}_2]}"] \arrow[rr, dashed, "\beta_3\beta_2\beta_1", "{[\tilde{p}_2,\tilde{p}_3]}"' near start, "{[\hat{p}_2,\hat{p}_3]}"' near end] & & \mathbb{P}^2 \arrow[ul, dashed]
\end{tikzcd}$$

Writing $\rho = \gamma_3\gamma_2\gamma_1$ for some $\gamma_1, \gamma_2, \gamma_3 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, only γ_2 quadratic (possible by Lemma 3.6), Lemma 4.2 implies that each triangle in the above diagram corresponds to a relation in $\mathfrak{G}$, making the whole diagram correspond to a relation in $\mathfrak{G}$. □

Lemma 4.4. *Let $f, h \in \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$, $g \in \mathrm{Aut}(\mathbb{P}^2)$, $f, g, h \in J$, and let Δ be a de Jonquières linear system. Assume that*

$$\deg(fgh) = 3, \quad \deg(fgh(\Delta)) < \deg(gh(\Delta)), \quad \deg(\Delta) \leqslant \deg(gh(\Delta))$$

and that $(gh)(\Delta)$ has a proper base-point different from $[1:0:0]$. Then there exist $\alpha_1, \ldots, \alpha_7 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, $\alpha_1, \alpha_3, \alpha_5, \alpha_7 \in \mathrm{Aut}(\mathbb{P}^2)$, such that

(i) *the following equation holds in $\mathfrak{G}$:*

$$w(f)w(g)w(h) = w(\alpha_7) \cdots w(\alpha_1)$$

that is, the following diagram corresponds to a relation in $\mathfrak{G}$:

$$\begin{tikzcd}
& & (gh)(\Delta) \arrow[dll, dashed, "(gh)^{-1}"'] \arrow[drr, dashed, "f"] & & \\
\Delta \arrow[dr, dashed, "\alpha_2\alpha_1"] & & & & fgh(\Delta) \\
& \alpha_1\alpha_2(\Delta) \arrow[rr, dashed, "\alpha_5\alpha_4\alpha_3"] & & \alpha_5\cdots\alpha_1(\Delta) \arrow[ur, dashed, "\alpha_7\alpha_6"'] &
\end{tikzcd}$$

(ii) *for $i = 2, \ldots, 7$*

$$\deg(\alpha_i \cdots \alpha_1(\Delta)) < \deg((gh)(\Delta)).$$

Proof. The equality $\deg(fgh) = 3$ implies that f and $(gh)^{-1}$ have exactly one common base-point, namely $p_1 = [1:0:0]$. Denote by $\mathrm{sBp}((gh)^{-1}) = \{p_2, p_3\}$ and $\mathrm{sBp}(f) = \{p_4, p_5\}$ the simple base-points of $(gh)^{-1}$ and f, respectively, and write $d = \deg(gh(\Delta))$.

By assumption, $gh(\Delta)$ is a de Jonquières linear system that has a proper base-point s different from p_1. For any point r, let $m(r)$ be the multiplicity of $gh(\Delta)$ in r, respectively. Then $m(p_1) = d - 1$ and $m(s) = 1$ (Remark 3.5), and Remark 3.7 implies that because $\deg(\Delta) \leqslant d$ and $\deg(fgh(\Delta)) < d$, we have (up to ordering of p_2, p_3)

$$m(p_2) = 1, \quad m(p_3) \leqslant 1,$$
$$\deg(fgh(\Delta)) = d - 1, \quad m(p_4) = m(p_5) = 1.$$

We will now construct $\alpha_1, \ldots, \alpha_7$.

Assume that $s \in \{p_2, p_3, p_4, p_5\}$. If $s \in \{p_2, p_3\}$, then we choose $r \in \{p_4, p_5\}$. If $s \in \{p_4, p_5\}$, then we choose $r \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$. We choose r to be infinitely near p_1 or a proper point (this is always possible). The points p_1, s, r are not aligned, because $a_1 + m(s) + m(r) > d$, thus there exist $\rho \in \mathrm{Bir}(\mathbb{P}^2)$ quadratic de Jonquières with base-points p_1, r, s. The following commutative diagram, where the points in the brackets are the base-points of the corresponding map, summarizes the situation:

$$\begin{tikzcd}
& gh(\Delta) \arrow[dl, dashed, "{(gh)^{-1}}"', "{[p_1,p_2,p_3]}"] \arrow[d, dashed, "{[p_1,s,r]}"', "\rho"] \arrow[dr, dashed, "{[p_1,p_4,p_5]}"', "f"] & \\
\Delta \arrow[r, dashed] & \rho gh(\Delta) \arrow[r, dashed] & fgh(\Delta)
\end{tikzcd}$$

Using Remark 3.7, we obtain

$$\deg(\rho gh) = \deg(f\rho^{-1}) = 2,$$
$$\deg(\rho gh(\Delta)) = d - 1 < \deg(gh(\Delta)).$$

We write $\rho = \gamma\tilde{\rho}\delta$, $\rho gh = \alpha_3\alpha_2\alpha_1$, $f\rho^{-1} = \alpha_6\alpha_5\alpha_4$, where $\delta, \gamma, \alpha_1, \cdots, \alpha_6 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, only $\tilde{\rho}, \alpha_2, \alpha_5$ quadratic (Lemma 3.6). By Lemma 4.2, the above diagram is generated by relations in $\mathfrak{G}$. Hence $w(f)w(g)w(h) = w(\alpha_6) \cdots w(\alpha_1)$ in $\mathfrak{G}$.

Assume $s \notin \{p_2, p_3, p_4, p_5\}$; we choose $r_1 \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$, $r_2 \in \{p_4, p_5\}$ such that r_1 (respectively, r_2) is either a proper point or infinitely near p_1 (this is always possible). For $i = 1, 2$, the points p_1, s, r_i are not collinear, because $a_1 + m(s) + m(r_i) > d$. Thus there exist $\rho_1, \rho_2 \in \mathrm{Bir}(\mathbb{P}^2)$ quadratic de Jonquières with base-points p_1, s, r_1 and p_1, s, r_2, respectively. The following commutative diagram, where the brackets are the base-points of the corresponding map, summarizes the situation:

$$\begin{tikzcd}
& & gh(\Delta) \arrow[dll, dashed, "{(gh)^{-1}}"', "{[p_1,p_2,p_3]}"] \arrow[dl, dashed, "{[p_1,s,r_1]}", "\rho_1"'] \arrow[dr, dashed, "{[p_1,s,r_2]}"', "\rho_2"] \arrow[drr, dashed, "{[p_1,p_4,p_5]}", "f"'] & & \\
\Delta \arrow[r, dashed] & \rho_1 gh(\Delta) \arrow[rr, dashed] & & \rho_2 gh(\Delta) \arrow[r, dashed] & fgh(\Delta)
\end{tikzcd}$$

Using Remark 3.7, we obtain

$$\deg(\rho_1 gh) = \deg(\rho_2\rho_1^{-1}) = \deg(f\rho_2^{-1}) = 2,$$
$$\deg(\rho_1 gh(\Delta)) = d - 1 < \deg(gh(\Delta)),$$
$$\deg(\rho_2 gh(\Delta)) = d - 1 < \deg(gh(\Delta)).$$

We write $\rho_1 = \gamma_1\tilde{\rho}_1\beta_1$, $\rho_2 = \gamma_2\tilde{\rho}_2\beta_2$, $\rho_1 gh = \alpha_3\alpha_2\alpha_1$, $\rho_2\rho_1^{-1} = \alpha_6\alpha_5\alpha_4$, $f\rho_2^{-1} = \alpha_9\alpha_8\alpha_7$ for $\alpha_1, \ldots, \alpha_9, \beta_1, \beta_2, \gamma_1, \gamma_2, \tilde{\rho}_1, \tilde{\rho}_2 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, only $\alpha_2, \alpha_5, \alpha_8, \tilde{\rho}_1, \tilde{\rho}_2$ quadratic (Lemma 3.6). Lemma 4.2 implies that all triangles of the above diagram are generated by relations in $\mathfrak{G}$, and thus $w(f)w(g)w(h) = w(\alpha_9) \cdots w(\alpha_1)$ in $\mathfrak{G}$. We obtain the maps α_i in the claim by merging neighbour automorphisms of $\mathbb{P}^2$ in the product $\alpha_9 \cdots \alpha_1$. $\square$

5. *The Cremona group is isomorphic to* $\mathfrak{G}$

In this section, we prove Theorem B (Theorem 5.5). The main tool will be Lemma 5.1, which yields the existence of an injective map $w_J \colon J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$ (Corollary 5.2) and enables us to use the result (Theorem 5.3) of [4], that $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and J along their intersection modulo one relation, for the proof of Theorem B (Theorem 5.5).

Lemma 5.1. *Let $f_1,\dots,f_n \in (\mathrm{Aut}(\mathbb{P}^2)\cup\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2))\cap J$ such that $f_n\cdots f_1=\mathrm{Id}$. Then $w(f_n)\cdots w(f_1)=\mathrm{Id}$ in $\mathfrak{G}$.*

Proof. We can write $w(f_n)\cdots w(f_1)=w(\alpha_{m+1})w(g_m)w(\alpha_m)\cdots w(\alpha_2)w(g_1)w(\alpha_1)$, where $\alpha_i\in\mathrm{Aut}(\mathbb{P}^2)\cap J$ and $g_i\in(\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2))\cap J\setminus\mathrm{Aut}(\mathbb{P}^2)$ as follows: We put $g_j:=f_i$ if f_i is quadratic, $\alpha_j:=f_i$ if f_i is linear. Then we proceed by putting $\alpha_j:=\alpha_{i+1}\alpha_i$ ($w(\alpha_j)=w(\alpha_{i+1}\alpha_i)$ by Lemma 4.2). Proceeding like this, we will reach a word where no two consecutive letters both have linear image in $\mathrm{Bir}(\mathbb{P}^2)$. We then insert $\alpha_j=\mathrm{Id}$ between any two consecutive letters whose both image in $\mathrm{Bir}(\mathbb{P}^2)$ is quadratic.

We denote by Δ_0 the linear system of lines in $\mathbb{P}^2$ and define, for $i=1,\dots,m$,

$$\Delta_i:=(\alpha_i g_{i-1}\cdots g_1\alpha_1)(\Delta_0)$$

which is the linear system of the map $(\alpha_i g_{i-1}\cdots g_1\alpha_1)^{-1}$. We define $d_i:=\deg(\Delta_i)$, which is also the degree of the map $(\alpha_i g_{i-1}\cdots g_1\alpha_1)^{-1}$. Furthermore, we define

$$D:=\max\{d_i\mid i=1,\dots,m\},\quad N:=\max\{i\mid d_i=D\}.$$

If $D=1$, then it follows that $m=1$ and $\alpha_1=\mathrm{Id}$. We can therefore assume that $D>1$ and prove the result by induction over the lexicographically ordered pair (D,N).

The induction step consists of finding $\tilde\alpha_{k+1},\dots,\tilde\alpha_1\in\mathrm{Aut}(\mathbb{P}^2)\cap J$ and $\tilde g_1,\dots,\tilde g_k\in(\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2))\cap J\setminus\mathrm{Aut}(\mathbb{P}^2)$ such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N)=w(\tilde\alpha_{k+1})w(\tilde g_k)\cdots w(\tilde g_1)w(\tilde\alpha_1)$$

and such that the pair $(\tilde D,\tilde N)$ associated to the product

$$\alpha_{m+1}g_m\cdots g_{N+2}(\alpha_{N+2}\tilde\alpha_{k+1})\tilde g_k\cdots\tilde g_1(\tilde\alpha_1\alpha_N)g_{N-1}\cdots g_1\alpha_1$$

is strictly smaller than (D,N).

We look at three cases, depending on the degree of $g_{N+1}\alpha_{N+1}g_N$, and if the degree is 3, we look at two subcases, the 'good case' and the 'bad case'.

If $\deg(g_{N+1}\alpha_{N+1}g_N)=1$, then define $\mathrm{Aut}(\mathbb{P}^2)\ni\tilde\alpha:=g_{N+1}\alpha_{N+1}g_N$. It follows from Lemma 4.2 (1) that $w(g_{N+1})w(\alpha_{N+1})w(g_N)=w(\tilde\alpha)$ in $\mathfrak{G}$. We replace $g_{N+1}\alpha_{N+1}g_N$ by $\tilde\alpha$, which decreases (D,N).

If $\deg(g_{N+1}\alpha_{N+1}g_N)=2$, then it follows from Lemma 4.2 (2),(3) that there exist $\tilde\alpha,\tilde\beta\in\mathrm{Aut}(\mathbb{P}^2)\cap J$ and $\tilde g\in(\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2))\cap J\setminus\mathrm{Aut}(\mathbb{P}^2)$ such that $w(g_{N+1})w(\alpha_{N+1})w(g_N)=w(\tilde\beta)w(\tilde g)w(\tilde\alpha)$. We replace $g_{N+1}\alpha_{N+1}g_N$ by $\tilde\beta\tilde g\tilde\alpha$, which decreases (D,N).

Finally, suppose that $\deg(g_{N+1}\alpha_{N+1}g_N)=3$. By definition of N, we have

$$d_{N-1}\leqslant D,\quad d_N=D,\quad d_{N+1}<D.$$

'Good case': If Δ_N has a proper simple base-point, then it follows from Lemma 4.4 (with $\Delta=\Delta_{N-1}$) that there exist $\tilde\alpha_1,\dots,\tilde\alpha_4\in\mathrm{Aut}(\mathbb{P}^2)\cap J$, $\tilde g_1,\tilde g_2,\tilde g_3\in(\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2))\cap J$ such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N)=w(\tilde\alpha_4)w(\tilde g_3)\cdots w(\tilde\alpha_2)w(\tilde g_1)w(\tilde\alpha_1)\quad\text{in }\mathfrak{G}$$

and

$$\deg((\tilde\alpha_{i+1}\tilde g_i\cdots g_1\tilde\alpha_1)(\Delta_{N-1}))<\deg(\Delta_N)=D$$

for $i=1,\dots,4$. Replacing $g_{N+1}\alpha_{N+1}g_N$ by $\tilde\alpha_4\tilde g_3\cdots\tilde g_1\tilde\alpha_1$ decreases (D,N).

'Bad case': Assume that Δ_N has no simple proper base-points. Without changing the pair (D,N), we will replace the word $w(\alpha_{m+1})w(g_m)\cdots w(g_1)w(\alpha_1)$ in $\mathfrak{G}$ by an equivalent word $w(\hat\alpha_{m+1})w(\hat g_m)\cdots w(\hat g)w(\hat\alpha_1)$ satisfying the 'good case'.

Choose two general points p_0, q_0 in $\mathbb{P}^2$ and write $p_1 = (\alpha_2 g_1 \alpha_1)(p_0)$, $q_1 = (\alpha_2 g_1 \alpha_1)(q_0)$ and $p_i = (\alpha_{i+1} g_i)(p_{i-1})$, $q_i = (\alpha_{i+1} g_i)(q_{i-1})$ for $i = 2, \dots, m$. Note that $p_m = p_0$ and $q_m = q_0$ because $\alpha_{m+1} g_m \cdots g_1 \alpha_1 = \mathrm{Id}$.

For $i = 0, \dots, m$, we denote by $\beta_i \in \mathrm{Aut}(\mathbb{P}^2)$ an element sending $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, respectively, onto $[1:0:0], p_i, q_i$ (this is possible, because we took p_0, q_0 general), and write $\tau_i := \beta_i \sigma_3 (\beta_i)^{-1}$, which is a quadratic de Jonquières involution having base-points $[1:0:0], p_i, q_i$. We choose $\beta_m = \beta_0$ and then have $\tau_m = \tau_0$.

By Lemma 4.3, the maps $\tau_1(\alpha_2 g_1 \alpha_1)\tau_0^{-1}$, $\tau_i(g_i \alpha_i)\tau_{i-1}^{-1}$ are quadratic de Jonquières and there exist $\gamma_i, \delta_i \in \mathrm{Aut}(\mathbb{P}^2) \cap J$, $\hat{g}_i \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ such that

$$w(\beta_1)w(\sigma_3)w(\beta_1^{-1})w(\alpha_2)w(g_1)w(\alpha_1)w(\beta_0)w(\sigma_3)w(\beta_0^{-1}) = w(\delta_1)w(\hat{g}_1)w(\gamma_1),$$
$$w(\beta_i)w(\sigma_3)w(\beta_i^{-1})w(\alpha_{i+1})w(g_i)w(\beta_{i-1})w(\sigma_3)w(\beta_{i-1}^{-1}) = w(\delta_i)w(\hat{g}_i)w(\gamma_i)$$

for $i = 1, \dots, m$. We get the following diagram

$$\begin{tikzcd}
{} \arrow[r, "\alpha_2 g_1 \alpha_1"] \arrow[d, "\tau_0"'] & {} \arrow[r, "\alpha_3 g_2"] \arrow[d, "\tau_1"] & {} \arrow[d, "\tau_2"] & \cdots & {} \arrow[r, "\alpha_{i+1} g_i"] \arrow[d, "\tau_{i-1}"'] & {} \arrow[d, "\tau_i"] & \cdots & {} \arrow[r, "\alpha_{m+1} g_m"] \arrow[d, "\tau_{m-1}"'] & {} \arrow[d, "\tau_m"] \\
{} \arrow[r, "\delta_1 \hat{g}_1 \gamma_1"'] & {} \arrow[r, "\delta_2 \hat{g}_2 \gamma_2"'] & {} & \cdots & {} \arrow[r, "\delta_i \hat{g}_i \gamma_i"'] & {} & \cdots & {} \arrow[r, "\delta_m \hat{g}_m \gamma_m"'] & {}
\end{tikzcd}$$

where each square in the diagram corresponds to a relation in $\mathfrak{G}$, making the whole diagram correspond to a relation in $\mathfrak{G}$. Therefore, writing $\tilde{\alpha}_i := \delta_i \gamma_{i-1}$ for $i = 2, \dots, m$, $\tilde{\alpha}_{m+1} := \delta_m$, $\tilde{\alpha}_1 := \gamma_1$, the equality

$$w(\alpha_{m+1})w(g_m) \cdots w(g_1)w(\alpha_1) = w(\hat{\alpha}_{m+1})w(\hat{g}_m)w(\hat{\alpha}_m) \cdots w(\hat{\alpha}_2)w(\hat{g}_1)w(\hat{\alpha}_1)$$

holds in $\mathfrak{G}$. We replace $\alpha_{m+1} g_m \cdots g_1 \alpha_1$ by $\hat{\alpha}_{m+1} \hat{g}_m \hat{\alpha}_m \cdots \hat{\alpha}_2 \hat{g}_1 \hat{\alpha}_1$.

For $i = 1, \dots, m$, call $\hat{\Delta}_i := (\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1)(\Delta_0)$, which is the linear system of the map $(\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1)^{-1}$, and denote by $\hat{d}_i$ its degree. Using Remark 3.7, we get $\deg(\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1) = \deg(\alpha_i g_{i-1} \cdots g_1 \alpha_1)$ for each i, thus $\hat{d}_i = d_i$ for $i = 1, \dots, m$. Therefore, the replacement does not change the pair (D, N), that is, $(\hat{D}, \hat{N}) = (D, N)$.

It remains to show that $\hat{\alpha}_{n+1} \hat{g}_n \cdots \hat{g}_1 \hat{\alpha}_1$ satisfies the 'good case', that is, that $\hat{\Delta}_N$ has a simple proper base-point.

Since $d_{N+1} < D$, it follows from Remark 3.7 that $d_{N+1} = D - 1$ and that all the base-points of g_{N+1} are base-points of Δ_N. Since the base-points of τ_N are general, it follows from Remark 3.7 that each point in $(\tau_N)^\bullet(\mathrm{sBp}(g_N))$ is a base-point of $\hat{\Delta}_N$. Lemma 4.3 states that $(\tau_N)^\bullet(\mathrm{sBp}(g_N)) = \mathrm{sBp}(\hat{g}_N)$, and $\hat{g}_N \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ has a simple proper base-point. Hence $\hat{\Delta}_N$ has a simple proper base-point. ☐

Corollary 5.2. *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ de Jonquières such that $\alpha_n \cdots \alpha_1 = \beta_m \cdots \beta_1$. Then*

$$w(\alpha_n) \cdots w(\alpha_1) = w(\beta_m) \cdots w(\beta_1).$$

In particular, there exists a homomorphism $w_J \colon J \to \mathfrak{G}$ which sends $\alpha_n \cdots \alpha_1$ onto $w(\alpha_n) \cdots w(\alpha_1)$ and $\pi \circ w_J = \mathrm{Id}$, that is, w_J is injective.

Proof. The claim follows from applying Lemma 5.1 to $\beta_1^{-1} \cdots \beta_m^{-1} \alpha_n \cdots \alpha_1$. ☐

Proposition 5.3 [4]. *The group $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to*

$$(\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2) \cap J} J)/\langle \tau_{12}\sigma_3\tau_{12}\sigma_3 \rangle,$$

the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and J along their intersection and divided by the relation $\tau_{12}\sigma_3 = \sigma_3\tau_{12}$, where $\tau_{12}([x:y:z]) = [y:x:z]$.

REMARK 5.4. In $\mathrm{Bir}(\mathbb{P}^2)$, the three relations

(i) $\tau_{12}\sigma_3\tau_{12}\sigma_3 = \mathrm{Id}$;
(ii) $\tau_{13}\sigma_3\tau_{13}\sigma_3 = \mathrm{Id}$;
(iii) $\tau_{23}\sigma_3\tau_{23}\sigma_3 = \mathrm{Id}$

hold. Choosing two of them, the remaining relation of the three is generated by the chosen two. Relation (iii) is a relation holding in J. Thus it suffices to impose relation (i) or (ii) in Theorem 5.3.

Since $\tau_{12}, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, relation (i) holds in $\mathrm{Aut}(\mathbb{F}_0)$, so in particular it holds in the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ along all their pairwise intersections. It is *a priori* not clear whether or not one of the relations (ii), (iii) holds in the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ along all their pairwise intersections because it is not clear whether or not J embeds into it or not. Therefore, we need to impose one of the relations (ii) or (iii).

THEOREM 5.5 (Theorem B). *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is isomorphic to* $\mathfrak{G}$, *the generalized amalgamated product of* $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ *along all the pairwise intersections modulo the relation* $\tau_{13}\sigma_3\tau_{13}\sigma_3$, *where* $\tau_{13}([x:y:z]) = [z:y:x]$.

Proof. By Corollary 5.2, there exists $w_J \colon J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$, and w and w_J coincide on $\mathrm{Aut}(\mathbb{P}^2) \cap J$. Thus the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{G} & J \arrow[l, "w_J"'] \\
\mathrm{Aut}(\mathbb{P}^2) \arrow[u, "w"] & \mathrm{Aut}(\mathbb{P}^2) \cap J \arrow[l, hook', "\iota_1"'] \arrow[u, hook', "\iota_2"']
\end{tikzcd}$$

where ι, ι_1 are the canonical inclusion maps. The universal property of the amalgamated product implies the existence of a unique homomorphism $\varphi : \mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2)\cap J} J \to \mathfrak{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{G} & & \\
& \mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2)\cap J} J \arrow[ul, "\varphi", "\exists !"'] & J \arrow[l] \arrow[ull, bend right, "w_J"'] \\
& \mathrm{Aut}(\mathbb{P}^2) \arrow[u] \arrow[uul, bend left, "w"] & \mathrm{Aut}(\mathbb{P}^2) \cap J \arrow[l, hook', "\iota"'] \arrow[u, hook', "\iota"']
\end{tikzcd}$$

By Proposition 5.3, $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to $\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2)\cap J} J$ modulo the relation $\sigma_3\tau_{12} = \tau_{12}\sigma_3$, where $\tau_{12}([x:y:z]) = [y:x:z]$. Since $\tau_{12}, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, the relation $\sigma_3\tau_{12} = \tau_{12}\sigma_3$ also holds in $\mathrm{Aut}(\mathbb{F}_0)$, and hence in $\mathfrak{G}$. Thus, the homomorphism φ induces a homomorphism $\bar{\varphi} : (\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2)\cap J} J)/\langle \sigma_3\tau_{12}\sigma_3\tau_{12}\rangle \longrightarrow \mathfrak{G}$. By construction, $\bar{\varphi}$ and the canonical homomorphism $\pi : \mathfrak{G} \to \mathrm{Bir}(\mathbb{P}^2)$ are inverse to each other. □

6. *The Cremona group is compactly presented*

In this section, we restrict to case $k = \mathbb{C}$ and show that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented using Theorem 5.5 (Theorem B).

Being compactly presented is a notion reserved for Hausdorff topological groups and we consider $\mathrm{Bir}(\mathbb{P}^2)$ endowed with the Euclidean topology as constructed in [**5**, Section 5], which makes $\mathrm{Bir}(\mathbb{P}^2)$ a Hausdorff topological group [**5**, Theorem 3], which is not locally compact [**5**, Lemma 5.15].

Definition 6.1. Let G be a group.

(i) A *presentation* $\langle S \mid R\rangle$ of G is a triple made up of a set S, an epimorphism $\pi : F_S \twoheadrightarrow G$ of the free group on S onto G, a subset R of F_S generating $\ker(\pi)$ as a normal subgroup. The *relations* of the presentation are the elements of $\ker(\pi)$ and the elements of R the *relators* (or *generating relations*) of the presentation.

(ii) A *bounded presentation* of G is a presentation $\langle S \mid R\rangle$ of G with R a set of relators of bounded length.

(iii) Let G be a Hausdorff topological group. A *compact presentation* of G is a presentation $\langle S \mid R\rangle$ of G with S a compact subset of G and R a set of relators of bounded length. We say that G is *compactly presented by* S if G is given by a compact presentation $\langle S \mid R\rangle$. We also say that G is *compactly presented* if G is compactly presented by some subset.

Lemma 6.2. (i) *Let G be a group and $S_1, S_2 \subset G$ be generating subsets. If $S_1^m \subset S_2^n \subset S_1^{m'}$ for some $m, n, m' \in \mathbb{N}$, then G is boundedly presented by S_1 if and only if it is boundedly presented by S_2.*

(ii) *Any connected topological group is generated by any neighbourhood of* 1.

(iii) *If G is a locally compact Hausdorff topological group having only finitely many connected components, then it is compactly presented.*

(iv) *If G is a locally compact Hausdorff topological group that is compactly presented, then it is compactly presented by all its compact generating subsets.*

(v) *Let G be a locally compact topological group with finitely many connected components $G_0, \dots, G_n$, where $1 \in G_0$. For each i choose some $g_i \in G$ such that $G_i = g_iG_0$. Then G is generated by any compact neighbourhood of* 1 *and $g_1, \dots, g_n$. In particular, it is compactly presented by any compact neighbourhood of* 1 *and $g_1, \dots, g_n$.*

Proof. (i) is proved in the forthcoming paper 'Metric geometry of locally compact groups', by de Cornulier and de la Harpe (http://www.normalesup.org/cornulier/MetricLC.pdf), Lemma 7.A.9) and [**9**, Lemma 2.6] and (iii) in [**1**, Satz 3.2] (see also the forthcoming paper 'Metric geometry of locally compact groups', by de Cornulier and de la Harpe (http://www.normalesup.org/cornulier/MetricLC.pdf), Subsection 8.A).

(ii) Let $U \subset G$ be an open neighbourhood of 1. Then the subgroup H of G generated by U is open because $H = \bigcup_{h\in H} hU$. It is also closed because $G \setminus H = \bigcup_{g\in G\setminus H} gH$, which is an open set.

(iii) If G is compactly generated by a compact set S and $K \subset G$ is a compact set, then $K \subset S^n$ for some large n. This follows from the fact that any locally compact topological group is a Baire space and that S is compact. The claim now follows from (i).

(iv) Let $K \subset G_0$ be a compact neighbourhood of 1. By (ii), K generates G_0, and thus the compact set $K \cup \{g_1, \dots, g_n\}$ generates G. By (iii) and (iv), the locally compact group G is compactly presented by $K \cup \{g_1, \dots, g_n\}$. □

Remark 6.3. Any irreducible algebraic variety over $\mathbb{C}$ is connected with respect to the Euclidean topology [**12**, Chapter XII, Proposition 2.4]. Any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ has finitely many irreducible components in the Zariski topology, which are exactly the cosets of the component containing 1. Thus they are the connected components in the Zariski topology, and hence also the connected components in the Euclidean topology.

Furthermore, any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is a closed subset of $\mathrm{Bir}(\mathbb{P}^2)_{\leqslant d}$ for some $d \in \mathbb{N}$ (see [**5**, Lemma 2.19]), which is a locally compact set [**5**, Lemma 5.4]. Hence any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is locally compact and therefore satisfies the conditions of Lemma 6.2(v).

Remark 6.4. Any algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is a linear algebraic group [**3**, Théorème 2]. The Euclidean topology on these groups is exactly the topology inherited from the Euclidean topology on $\mathrm{Bir}(\mathbb{P}^2)$ (see [**5**, Proposition 5.11]).

The groups $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$, $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ are linear algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ (Lemma 2.4), and thus locally compact by Remark 6.3.

Corollary 6.5. (i) *The group* $\mathrm{Aut}(\mathbb{P}^2)$ *is compactly presented by any compact neighbourhood of* 1.

(ii) *The group* $\mathrm{Aut}(\mathbb{F}_0)$ *is compactly presented by the union of the linear map* $\tau_{12}\colon [x : y : z] \to [y : x : z]$ *and any compact neighbourhood of* 1.

(iii) *The group* $\mathrm{Aut}(\mathbb{F}_2)$ *is compactly presented by any compact neighbourhood of* 1.

Proof. The groups $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ are linear algebraic groups and locally compact by Remark 6.4.

The group $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(k)$ is irreducible, hence connected (Remark 6.3), and the group $\mathrm{Aut}(\mathbb{F}_2)$ is connected by Lemma 2.4(iii). By Lemma 2.4(ii), the group $\mathrm{Aut}(\mathbb{F}_0)$ has two connected components, namely $\mathrm{Aut}(\mathbb{F}_0)^0$ containing the identity element and $\tau_{12}\mathrm{Aut}(\mathbb{F}_0)^0$. The claim now follows from Remark 6.3 and Proposition 6.2(v). ☐

Using Corollary 6.5 and the fact that $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ along their pairwise intersection divided by one relation (Theorem 5.5), we prove that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presentable.

Lemma 6.6. *Let G be a group, $n \geqslant 2$ be an integer and $G_1, \ldots, G_n \subset G$ be subgroups of G such that the following hold.*

(i) *The group G admits the presentation $G = \langle \bigcup_{i=1}^n G_i \mid R_G \rangle$, where R_G is the set of all relators of the form $ab = c$, where $a, b, c \in G_i$ for some $i \in \{1, \ldots, n\}$.*
(ii) *For $i = 1, \ldots, n$, there exists a presentation $\langle K_i \mid R_i \rangle$ of G_i such that, for any subset $I \subset \{1, \ldots, n\}$, the set $\bigcap_{i \in I} K_i$ generates $\bigcap_{i \in I} G_i$.*

Then, G admits the presentation $G = \langle \bigcup_{i=1}^n K_i \mid \bigcup_{i=1}^n R_i \rangle$.

Proof. Denote by F_G the free group generated by $\bigcup_{i=1}^n G_i$ and by F_K the free group generated by $K = \bigcup_{i=1}^n K_i$; we view F_K as a subgroup of F_G.

The natural group homomorphism $\pi\colon F_K \to G$ is surjective, because G is generated by $\bigcup_{i=1}^n G_i$ and each G_i is generated by K_i. Moreover, each set of relators R_i corresponds to a subset of $\ker(\pi)$. It remains to see that $\ker \pi$ is contained in the normal subgroup generated by $\bigcup_{i=1}^n R_i$.

We take an element in $\ker(\pi)$, which in F_K is a word

$$w = s_1 s_2 \cdots s_m$$

such that each s_i belongs to K and $s_1 \cdots s_m = 1$ in G. Because G admits the presentation $G = \langle \bigcup_{i=1}^n G_i \mid R_G \rangle$, we can write w in F_G as a product

$$w = a_1 r_1 a_1^{-1} a_2 r_2 a_2^{-1} \cdots a_l r_l a_l^{-1},$$

where all the a_i, r_i are elements of F_G and $r_i \in R_G$, which means by definition of R_G that $r_i = a_i b_i c_i$ for $a_i, b_i, c_i \in G_{j(i)}$, that is, each r_i is a word in elements of $G_{j(i)}$.

The word w is equal in F_G to a reduced word, whose letters are elements of K because $w \in F_K$. Hence, each $g = \bigcup_{i=1}^n G_i \setminus K$ which appears in the word $a_1 r_1 a_1^{-1} \cdots a_l r_l a_l^{-1}$ disappears after the reduction. We can thus replace each occurrence of g with any chosen element of F_G and do not change the value of the word. We do this in the following way: if $g \in G_i \setminus K_i$, then we replace then g with a word with letters in K_i, which belongs to $\pi^{-1}(\pi(g))$ (this is possible since K_i generates G_i). If g belongs to more than one of the G_i, then we can moreover assume that the letters of the word also belong to these K_i, because of the second hypothesis.

After this replacement, we obtain an equality in F_K

$$s_1 \cdots s_m = b_1 t_1 b_1^{-1} b_2 t_2 b_2^{-1} \cdots b_l t_l b_l^{-1},$$

where each t_i is a word with letters in $K_{j(i)}$, such that $\pi(t_i) = 1$. For each $i = 1, \ldots, n$ denote by F_{K_i} the free group generated by K_i and by $\pi_i : F_{K_i} \to G_i$ the natural group homomorphism onto G_i whose kernel is generated by R_i. We consider F_{K_i} as a subgroup of F_K, which means that $\pi_i = \pi|_{F_{K_i}}$, and hence $\ker(\pi_i) = \ker(\pi) \cap F_{K_i}$. Therefore, $\pi_i(t_i) = 1$, and thus t_i is a product of conjugates of $R_{j(i)}$. This yields the result. □

Corollary 6.7. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$, $K_0 \subset \mathrm{Aut}(\mathbb{F}_0)$, $K_2 \subset \mathrm{Aut}(\mathbb{F}_2)$ be compact neighbourhoods of* 1 *in the respective groups. Then* $\mathrm{Bir}(\mathbb{P}^2)$ *is compactly presented by* $K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$.

Proof. Lemma 6.6 yields that the union of any compact generating sets of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ giving a compact presentation of the respective groups yields a compact presentation of $\mathfrak{G}$, the generalized amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ along their pairwise intersection divided by one relation. Such compact generating sets are given by Corollary 6.5: Any compact neighbourhood of 1 of the groups $\mathrm{Aut}(\mathbb{P}^2)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively, and the union of τ_{12} and any compact neighbourhood of 1 in $\mathrm{Aut}(\mathbb{F}_0)$. Since $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathfrak{G}$ are isomorphic (Theorem 5.5), the claim follows. □

Corollary 6.8. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$, $K_0 \subset \mathrm{Aut}(\mathbb{F}_0)$, $K_2 \subset \mathrm{Aut}(\mathbb{F}_2)$ be compact neighbourhoods of* 1 *in the respective groups. Then* $\mathrm{Bir}(\mathbb{P}^2)$ *is compactly presented by* $K \cup K_0 \cup K_2$.

Proof. We define $S_1 := K \cup K_0 \cup K_2$ and $S_2 := K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$. The set S_2 generates $\mathrm{Bir}(\mathbb{P}^2)$ by Corollary 6.7. The set K generates $\mathrm{Aut}(\mathbb{P}^2)$ (Corollary 6.5), hence there exists $n \in \mathbb{N}$ such that $\tau_{12} \in K^n$. It follows that also S_2 generates $\mathrm{Bir}(\mathbb{P}^2)$ and moreover that $S_1 \subset S_2 \subset (S_1)^n$. The claim now follows from Lemma 6.2(i) and Corollary 6.7. □

Lemma 6.2(i) and Corollary 6.8 imply that to prove Theorem A (Corollary 6.10), we only need to check that, for any compact neighbourhood $K \subset \mathrm{Aut}(\mathbb{P}^2)$ of 1, there exist $K_i \subset \mathrm{Aut}(\mathbb{F}_i)$, $i = 0, 2$, compact neighbourhoods of 1 and integers $m, m', n \in \mathbb{N}$ such that $(K \cup \{\sigma_3\})^m \subset (K \cup K_0 \cup K_2)^n \subset (K \cup \{\sigma_3\})^{m'}$.

Lemma 6.9. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$ be a compact neighbourhood of* 1. *Then there exists $N \in \mathbb{N}$ such that $(K \cup \{\sigma_3\})^N$ contains compact neighbourhoods of* 1 *in* $\mathrm{Aut}(\mathbb{F}_i)$ *for $i = 0, 2$.*

Proof. Let $\mathcal{A}_0 = \mathrm{Aut}(\mathbb{F}_0)^0 \cap \mathrm{Aut}(\mathbb{P}^2)$ and $\mathcal{A}_2 = \mathrm{Aut}(\mathbb{F}_2) \cap \mathrm{Aut}(\mathbb{P}^2)$, which are connected algebraic subgroups of $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively (Lemma 2.4). For $i = 0, 2$, the set $K_i = K \cap \mathcal{A}_i$ is a compact neighbourhood of 1 in $\mathcal{A}_i$. Corollary 6.5 implies that $\mathcal{A}_i =$

$\bigcup_{n\in\mathbb{N}}(K_i)^n$ for $i=0,2$. It follows that

$$\mathcal{A}_0\sigma_3\mathcal{A}_0=\bigcup_{n\in\mathbb{N}}(K_0)^n\sigma_3(K_0)^n,\quad \mathcal{A}_2\sigma_2\mathcal{A}_2=\bigcup_{n\in\mathbb{N}}(K_2)^n\sigma_2(K_2)^n.$$

The sets $\mathcal{A}_0\sigma_3\mathcal{A}_0$ and $\mathcal{A}_2\sigma_2\mathcal{A}_2$ are Zariski-open subsets of $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively (Lemma 2.4), and are thus locally compact, and hence Baire spaces. There exists then some $m\in\mathbb{N}$ such that $(K_0)^m\sigma_3(K_0)^m$ and $(K_2)^m\sigma_2(K_2)^m$ have non-empty interior in $\mathcal{A}_0\sigma_3\mathcal{A}_0$ and $\mathcal{A}_2\sigma_2\mathcal{A}_2$, and thus in $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively.

Since $(K_i)^m\sigma_j(K_i)^m\subset(K_i\cup\{\sigma_j\})^{2m+1}$, the sets $(K_0\cup\{\sigma_3\})^{2m+1}$ and $(K_2\cup\{\sigma_2\})^{2m+1}$ also have non-empty interior in $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively. Since $(K_i)^{-1}\subset K_i^{m_i}$ for some big m_i and $(\sigma_j)^{-1}=\sigma_j$, the sets $(K_0\cup\{\sigma_3\})^{m'}$ and $(K_2\cup\{\sigma_2\})^{m'}$ are neighbourhoods of 1 in the corresponding groups for some m' big enough. Since $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $K\cup\{\sigma_3\}$ (by the Noether–Castelnuovo theorem), we find m'' such that $\sigma_2\in(K\cup\{\sigma_3\})^{m''}$. A suitable power of $K\cup\{\sigma_3\}$ contains thus $(K_0\cup\{\sigma_3\})^{m'}$ and $(K_2\cup\{\sigma_2\})^{m'}$. □

Corollary 6.10 (Theorem A). *Let $K\subset\mathrm{Aut}(\mathbb{P}^2)$ be a compact neighbourhood of* 1. *Then* $\mathrm{Bir}(\mathbb{P}^2)$ *is compactly presented by* $K\cup\{\sigma_3\}$.

Proof. According to Lemma 6.9, there exist $N\in\mathbb{N}$ and compact neighbourhoods K_0,K_2 of 1 in $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$, respectively, such that $(K\cup\{\sigma_3\})^N$ contains $K_0\cup K_2$.

We define $S_1:=K\cup\{\sigma_3\}$ and $S_2:=K\cup K_0\cup K_2$. Because $\sigma_3\in\mathrm{Aut}(\mathbb{F}_0)^0$ and $\mathrm{Aut}(\mathbb{F}_0)^0$ is compactly generated by K_0 (Lemma 6.2(ii)), there exists $M\in\mathbb{N}$ such that $\sigma_3\in(K_0)^M$. It follows that $S_1\subset(S_2)^M$. Since $S_2\subset(S_1)^N$, we have $S_1\subset(S_2)^M\subset(S_1)^{MN}$. The group $\mathrm{Bir}(\mathbb{P}^2)$ being compactly presented by S_2 (Corollary 6.8), it is also compactly presented by S_1 (Lemma 6.2(i)). □

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THE ABELIANISATION OF THE REAL CREMONA GROUP

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Abstract. We present the Abelianisation of the birational transformations of $\mathbb{P}^2_{\mathbb{R}}$. Its kernel is equal to the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{R})$, and contains all elements of degree ≤ 4. The description of the quotient yields the existence of normal subgroups of index 2^n for any n and implies that any normal subgroup generated by a countable set of elements is a proper subgroup. This also holds for the group of birational diffeomorphisms respectively of $\mathbb{P}^2_{\mathbb{R}}$, $\mathbb{A}^2_{\mathbb{R}}$ and the sphere.

Contents

1. Introduction

Let $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \subset \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ be the groups of birational transformations of the projective plane defined over the respective fields of real and complex numbers, and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{R})$, $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{C})$ the respective subgroups of linear transformations.

According to the Noether-Castelnuovo Theorem [Cas1901], the group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2)$ and the standard quadratic transformation $\sigma_0\colon [x:y:z] \dashrightarrow [yz:xz:xy]$. As an abstract group, it is not simple [CL2013], i.e. there exist non-trivial, proper normal subgroups $N \subset \mathrm{Bir}(\mathbb{P}^2)$. However, all such groups have uncountable index (see Remark 4.12) and the isomorphism class of the corresponding quotients $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)/N$ is quite complicated (essentially as complicated as $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ itself). Moreover, the normal subgroup generated by any non-trivial element which preserves a pencil of lines or which has degree $d \leq 4$ is the whole group (see [Giz1994, Lemma 2] and Lemma 4.13) and the group is perfect [CD2013], which means that $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is equal to its commutator subgroup.

As we will show, the situation for the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is quite different. First of all, the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$ and σ_0 is certainly not the whole group, as all its elements have only real base-points. This is not the case for $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$; for instance the quadratic involution $\sigma_1\colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$ has non-real base-points. The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ however is generated by $\mathrm{PGL}_3(\mathbb{R})$, σ_0, σ_1, and by a family of transformations of degree 5 [BM2012]. We will show that this set

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is not far from being a minimal set of generators. In particular, we obtain the following result, similar to the case of $\mathrm{Bir}(\mathbb{P}^n)$, $n \geq 3$ [Pan1999].

Theorem 1.1. *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is not generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *and a countable set of elements.*

The proof consists in finding explicit generators and relations for the group (see Proposition 2.9). This description also allows to construct a natural quotient, and gives our main result:

Theorem 1.2. *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is not perfect: its Abelianisation is isomorphic to*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)] \simeq \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the commutator subgroup $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ *is the normal subgroup generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$*, and contains all elements of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *of degree* ≤ 4.

Corollary 1.3. *The sequence of iterated commutator subgroups of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is stationary. More precisely: Let* $H := [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$*. Then* $[H, H] = H$.

Let X be a real variety. We denote by $X(\mathbb{R})$ its set of real points of and by $\mathrm{Aut}(X(\mathbb{R})) \subset \mathrm{Bir}(X)$ the subgroup of birational transformations defined at each point of $X(\mathbb{R})$. It is also called the group of *birational diffeomorphisms* of X, and is, in general, strictly larger than the group of automorphisms $\mathrm{Aut}_{\mathbb{R}}(X)$ of X defined over $\mathbb{R}$. The group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the standard quintic transformations (see Definition 2.2) [RV2005, BM2012]. Until now no similar result has been found for $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$.

In the following, let $\mathbb{P}^3 \supset \mathcal{Q}_{3,1} = \{[y : x : y : z] \in \mathbb{P}^3 \mid x^2 + y^2 + z^2 = w^2\}$.

Corollary 1.4. *The statement in Theorem 1.1 also holds for* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$*, replacing* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *for the latter two by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{A}^2)$, $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ *respectively..*

Corollary 1.5. *There exist surjective group homomorphisms*

$$\mathrm{Aut}(\mathbb{P}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Aut}(\mathbb{A}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$$

Corollary 1.6. *For any real birational map* $\psi\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$*, the group* $\psi\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi^{-1}$ *is a subgroup of* $\ker(\varphi)$.

Corollary 1.7. *For any* $n \in \mathbb{N}$ *there is a normal subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *of index* 2^n *containing all elements of degree* ≤ 4.
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Corollary 1.8. *The normal subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *generated by any countable set of elements of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is a proper subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

The plan of the article is as follows: After giving the basic definitions and notations in Section 2, we define in Section 3 a surjective group homomorphism from the subgroup $\mathcal{J}_\circ \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of elements preserving a pencil of conics to the group $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. In Section 4, we extend the homomorphism to a surjective group homomorphism $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ and give Theorem 1.1, Corollary 1.7 and Corollary 1.5. In Section 5, we proof that its kernel is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, which will turn out to be commutator subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. We will finally be able to prove Theorem 1.2.

In the proof of the main theorems we use a technical proposition (Proposition 2.9) that gives an explicit representation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generators and relations, and which is independent of all the other results. Its proof is quite long and rather technical, so we devote the whole last section (Section 6) to proving it.

In [Pol2015] one can find another description of the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, or rather, more specifically, a description of the elementary links between real rational surfaces and relations between them. However, this description was not used in the proof of Proposition 2.9.

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2. Basic notions

We now give some basic notations and definitions. Throughout the article, every variety and rational map is defined over $\mathbb{R}$, unless stated otherwise.

Definition 2.1. We define two rational fibrations

$$\begin{aligned}\pi_*\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1\\ [x:y:z] &\mapsto [y:z]\end{aligned}$$

$$\begin{aligned}\pi_\circ\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1\\ [x:y:z] &\mapsto [y^2+(x+z)^2 : y^2+(x-z)^2]\end{aligned}$$

whose fibres are respectively the lines through $[1:0:0]$ and the conics through $p_1 := [1:i:0], p_2 := [0:1:i]$, and their conjugates $\bar{p}_1 = [1:-i:0], \bar{p}_2 = [0:1:-i]$.

We define by $\mathcal{J}_*$, $\mathcal{J}_\circ$ the subgroups of $\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ preserving the fibrations $\pi_*, \pi_\circ$:

$$\mathcal{J}_* = \{f \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2) \mid \exists \hat{f} \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\colon \hat{f}\pi_* = \pi_* f\}$$
$$\mathcal{J}_\circ = \{f \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2) \mid \exists \hat{f} \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\colon \hat{f}\pi_\circ = \pi_\circ f\}$$

Extending the scalars to $\mathbb{C}$, the analogues of these groups are conjugate in $\mathrm{Bir}_\mathbb{C}(\mathbb{P}^2)$ and are called *de Jonquières groups*. In $\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$, the groups $\mathcal{J}_\circ, \mathcal{J}_*$ are not conjugate. This can, for instance, be seen as consequence of Proposition 4.3 (see Remark 4.11).

Definition 2.2. We define a type of real birational transformation called *standard quintic tranformation.*

Let $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{p}_3 \in \mathbb{P}^2$ be three pairs of non-real conjugate points of $\mathbb{P}^2$, not lying on the same conic. Denote by $\pi\colon X \to \mathbb{P}^2$ the blow-up of these points. The strict transforms of the six conics passing through exactly five of the six points are three pairs of non-real conjugate (-1)-curves. Their contraction yields a birational morphism $\eta\colon X \to \mathbb{P}^2$ which contracts the curves onto three pairs of non-real points $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3 \in \mathbb{P}^2$. We choose the order so that r_i is the image of the conic not passing through q_i. The birational map $\eta\pi^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is contained in $\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$, is of degree 5 and is called *standard quintic transformation.*

Lemma 2.3. *Let $\theta \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ be a standard quintic transformation. Then:*

(1) *The points $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$ are the base-points of θ and $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ are the base-points of θ^{-1}, and they are all of multiplicity 2.*
(2) *For $i, j = 1, 2, 3$, $i \neq j$, θ sends the pencil of conics through $q_i, \bar{q}_i, q_j, \bar{q}_j$ onto the pencil of conics through $r_i, \bar{r}_i, r_j, \bar{r}_j$.*
(3) *We have $\theta \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$.*

Proof. (1), (2) Let $L \subset \mathbb{P}^2$ be a general line. The strict transform of L on X by π^{-1} has self-intersection 1 and intersects the six curves contracted by η in 2 points. The image $\theta(L)$ then has six singular points of multiplicity 2 and self-intersection 25. It is thus a quintic passing through the r_i with multiplicity 2. Therefore, the linear system of θ^{-1} consists of quintics in $\mathbb{P}^2$ having multiplicity 2 at $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$. The construction of θ^{-1} being symmetric to the one of θ, the linear system of θ consists of quintics having multiplicity 2 at $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$.

(3) This is shown by simply calculating the degree of the images of the conics and their multiplicities in the base-points.

(4) The birational morphisms η, π induce bijections $X(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ and hence θ, θ^{-1} are defined on each point of $\mathbb{P}^2(\mathbb{R})$. □

The family of standard quintic transformations plays an important role in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$: Let

$$\sigma_0\colon [x:y:z] \dashrightarrow [yz:xz:xy]$$
$$\sigma_1\colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$$

Theorem 2.4 ([RV2005],[BM2012])**.** *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by σ_0, σ_1, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the infinite family of standard quintic transformations.*

Lemma 2.5. *For any standard quintic transformation θ there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\beta\theta\alpha \in \mathcal{J}_\circ$*

Proof. For any two non-collinear non-real pairs of conjugate points there exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that sends the two pairs onto $p_1 := [1:i:0], p_2 := [0:1:i]$ and their conjugates $\bar{p}_1 = [1:-i:0], \bar{p}_2 = [0:1:-i]$. Let θ be a standard quintic transformation. Then there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that send q_1, q_2 (resp. r_1, r_2) onto p_1, p_2. The transformation $\beta\theta\alpha^{-1}$ preserves the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (Lemma 2.3) and is thus contained in $\mathcal{J}_\circ$. □

Corollary 2.6. *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$*

Proof. By Theorem 2.4 and Lemma 2.5, $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ_0, σ_1 and the family of standard quintic transformations contained in $\mathcal{J}_\circ$. Observing that $\sigma_0 \in \mathcal{J}_*$, $\sigma_1 \in \mathcal{J}_\circ$, the claim follows. □

Using these generating groups, we can give a representation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generating sets and relations:

Define $S := \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ and let F_S be the free group generated by S. Let $w\colon S \to F_S$ be the canonical word map.

Definition 2.7. We denote by $\mathcal{G}$ be the following group:

$$F_S / \left\langle \begin{array}{ll} w(f)w(g)w(h), & f,g,h \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2),\ fgh = 1 \text{ in } \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_*,\ fgh = 1 \text{ in } \mathcal{J}_* \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_\circ,\ fgh = 1 \text{ in } \mathcal{J}_\circ \\ \text{the relations in the list below} & \end{array} \right\rangle$$

(1) Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.
$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \text{ in } \mathcal{G} \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$
(2) Let $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_\circ$ both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.
$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \text{ in } \mathcal{G} \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$
(3) Let $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$ all of degree 2, or τ_1, τ_2 of degree 2 and τ_3 of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.
$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \text{ in } \mathcal{G} \quad \text{if} \quad \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

Remark 2.8. Note that the group $\mathcal{G}$ is isomorphic to the quotient of the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$ along all intersections by the relations in the above list.

Since $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ (Corollary 2.6), there exists a natural surjective group homomorphism $F_S \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ which gives rise to a group homomorphism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, since all relations above hold in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Proposition 2.9. *The natural surjective group homormophism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is an isomorphism.*

The proof of Proposition 2.9 is quite long and technical, and we therefore prefer to present it in the last section. The proposition (and its proof) is independent of all the other results proven in this article. The method used in the proof has been described in [Bla2012], [Isk1985] and [Zim2015], and is to study linear systems and their base-points.

We now give some further notation used throughout the article.

Definition 2.10. Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and p be a point that belongs to $\mathbb{P}^2$ as a proper or infinitely near point. Assume moreover that p is not a base-point of f. We define a point $f_\bullet(p)$, which will also be in $\mathbb{P}^2$ or infinitely near. For this, take a minimal resolution of f

$$\begin{tikzcd} & S \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

where ν_1, ν_2 are sequences of blow-ups. Since p is not a base-point of f it corresponds via ν_1 to a point of S or infinitely near. Using ν_2 we view this point on $\mathbb{P}^2$, again maybe infinitely near, and call it $f_\bullet(p)$.

Remark 2.11. Note that $f_\bullet$ is a one-to-one correspondence between the sets

$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f\} \quad \text{and}$$
$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f^{-1}\}$$

Furthermore, if p is a base-point of a linear system Λ of multiplicity m that is not base-point of f, then $f_\bullet(p)$ is a base-point of $f(\Lambda)$ of multiplicity m [AC2002, §4.1].

Definition 2.12.

(1) Let $C \subset \mathbb{P}^2$ be an irreducible (closed) curve, $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathrm{Bp}(f)$ the set of base-points of f. We denote by

$$f(C) := \overline{f(C \setminus \mathrm{Bp}(f))}$$

the (Zariski-) closure of the image by f of C minus the base-points of f, and call it *the image of C by f.*

(2) Throughout the article, we fix the notation

$$p_1 := [1 : i : 0], \quad p_2 := [0 : 1 : i]$$

for these two specific points of $\mathbb{P}^2$, because we will use them extremely often.

(3) The following definition will be used for base-points of elements of $\mathcal{J}_\circ$. Let $\eta\colon X \to \mathbb{P}^2$ be the blow-up of $p_1, \bar{p}_1, p_2, \bar{p}_2$. The morphism $\tilde{\pi}_\circ := \pi_\circ \eta\colon X \to \mathbb{P}^1$ is a real conic bundle with fibres being the strict transforms of the conics passing through $p_1, \ldots, \bar{p}_2$.

$$\begin{tikzcd} X \arrow[d, "\eta"'] \arrow[dr, "\tilde{\pi}_\circ"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\pi_\circ"'] & \mathbb{P}^1 \end{tikzcd}$$

Let $\eta'\colon Y \to X$ be a birational morphism and $q \in Y$. We define

$$C_q := \pi_\circ^{-1}(\tilde{\pi}_\circ(\eta'(q))).$$

It is the conic passing through $p_1, \bar{p}_1, p_2, \bar{p}_2, \eta'(q)$, which is irreducible or the union of two lines. The latter case corresponds to $\tilde{\pi}_\circ(\eta'(q)) \in \{[1 : 0], [0 : 1], [1 : 1]\}$.

3. A quotient of $\mathcal{J}_\circ$

We first construct a surjective group homomorphism $\varphi_\circ\colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ and then (in Section 4) use the representation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generators and relations (Proposition 2.9) to extend $\varphi_\circ$ to a homomorphism $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. Both quotients are generated by classes of standard quintic transformations contained in $\mathcal{J}_\circ$, as we will see from the construction in Subsection 3.2.

In order to construct the surjective homomorphism $\mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, we first need some additional information about the elements of $\mathcal{J}_\circ$, such as their characteristic (Lemma 3.1) and their action on the pencil of conics passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (Lemma 3.7).

3.1. **The group $\mathcal{J}_\circ$.** The next lemmata state the characteristic and some other properties of the elements of $\mathcal{J}_\circ$ (recall that for $f \in \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the characteristic of f is the sequence $(\deg(f); m_1^{e_1}, \dots, m_k^{e_k})$ where $m_1, \dots, m_k$ are the multiplicities of the base-points of f and e_i is the number of base-points of f which have multiplicity m_i (see [AC2002, Definition 2.1.7])). We will use these properties to obtain the action of $\mathcal{J}_\circ$ on the pencil of conics through $p_1, \dots, \bar{p}_2$. The information will be used to construct the quotients. In Section 6 (proof of Proposition 2.9), we will use the properties to study linear systems and their base-points in connection with the relations given in Definition 2.7.

Lemma 3.1. *Any element of $\mathcal{J}_\circ$ of degree $d > 1$ has characteristic:*

$$\left(d;\ \frac{d-1}{2}^4,\ 2^{\frac{d-1}{2}}\right), \quad \text{if } \deg(f) \text{ is odd}$$

$$\left(d;\ \frac{d}{2}^2,\ \frac{d-2}{2}^2,\ 2^{\frac{d-2}{2}},\ 1\right), \quad \text{if } \deg(f) \text{ is even}$$

and $p_1, \dots, \bar{p}_2$ are (the) base-points of multiplicity $\frac{d}{2}, \frac{d-1}{2}$ or $\frac{d-2}{2}$.

Furthermore,

1. *no two double points are contained in the same conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$,*
2. *any element of $\mathcal{J}_\circ$ exchanges or preserves the real reducible conics $C_1 := L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}$ and $C_2 := L_{p_1,\bar{p}_2} \cup L_{\bar{p}_1,p_2}$,*
3. *any element of $\mathcal{J}_\circ$ of even degree contracts one of the lines $L_{p_i,\bar{p}_i}$, $i \in \{1,2\}$ onto a point on a real conic different from C_1, C_2.*

Proof. Let $f \in \mathcal{J}_\circ$ be of degree $d > 1$. Let C be a general conic passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$. By definition of $\mathcal{J}_\circ$, the curve $f(C)$ is a conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$. Let $m(q)$ be the multiplicity of f at the point q. Computing the intersection of C on the blow-up of the base-points of f with the linear system of f gives the degree of $f(C)$:

$$2 = \deg(f(C)) = 2d - 2m(p_1) - 2m(p_2) = (d - 2m(p_1)) + (d - 2m(p_2)).$$

Applying Bézout to the line through $p_i, \bar{p}_i$, we obtain that $d \geq 2m(p_i)$, $i = 1, 2$.

If $d - 2m(p_1) = d - 2m(p_2) = 1$, then

$$m(p_1) = m(p_2) = \frac{d-1}{2}.$$

Else, $d - 2m(p_i) = 0$, $d - 2m(p_{3-i}) = 2$ for some $i \in \{1, 2\}$, and so

$$m(p_i) = \frac{d}{2},\ m(p_{3-i}) = \frac{d-2}{2}, \quad i \in \{1,2\}.$$

Let q be a base-point of f not equal to $p_1, \bar{p}_1, p_2, \bar{p}_2$ and C_q its associated conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (see Definition 2.12). Then $2 \geq \deg(f(C_q)) \geq 0$ and

$$0 \leq \deg(f(C_q)) \leq 2d - 2m(p_1) - 2m(p_2) - m(q) = 2 - m(q) \leq 2$$

In particular, $m(q) \in \{1, 2\}$. Let D be a general member of the linear system of f. The genus formula

$$0 = g(D) = \frac{(d-1)(d-2)}{2} - \sum_{q \text{ base-point of } f} \frac{m(q)(m(q)-1)}{2}$$

and $m(q) \in \{1, 2\}$ for all base-points q of f different from $p_1, \bar{p}_1, p_2, \bar{p}_2$ imply that

$$\frac{(d-1)(d-2)}{2} = 2\sum_{i=1}^{2} \frac{m(p_i)(m(p_i)-1)}{2} + |\{\text{base-points of multiplicity } 2\}|$$

and in particular that

$$|\{\text{base-points of multiplicity } 2\}| = \begin{cases} \frac{d-1}{2}, & d \text{ odd} \\ \frac{d-2}{2}, & d \text{ even} \end{cases}$$

It follows from the Noether equalities that f has exactly one simple base-point if d is even and none otherwise. This yields the characteristics. Bézout's theorem implies that no two double points are contained in the same conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$. The conics $C_1 = L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}$, $C_2 = L_{p_1,\bar{p}_2} \cup L_{\bar{p}_1,p_2}$,

$C_3 := L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2}$ are the only reducible conics through $p_1, \ldots, \bar{p}_2$, and C_1, C_2 each consist of two non-real lines while C_3 consists of two real lines. If f has even degree, it contracts the line $L_{p_i,\bar{p}_i}$, where $m(p_i) = \frac{d}{2}$, onto the base-point of f^{-1} of multiplicity 1, and no other line is contracted (because f^{-1} has only one base-point of multiplicity 1). Because of this and the multiplicities of the base-points of f, f sends $L_{p_i,p_j}, L_{p_i,\bar{p}_j}, i \neq j$, onto non-real lines. This is also true if f has odd degree (simply because of the multiplicities of its base-points). Thus f preserves or exchanges C_1, C_2. In particular, the induced automorphism $\hat{f}$ of f on $\mathbb{P}^1$ does not send $\pi_\circ(C_3)$ onto either of $\pi_\circ(C_1), \pi_\circ(C_2)$. It follows that if f has even degree, the point $f(L_{p_i,\bar{p}_i})$ is contained in the conic $\pi_\circ^{-1}(\hat{f}(\pi_\circ(C_3))) \neq C_1, C_2$. In particular, the simple base-point of f^{-1} (which is $f(L_{p_i,\bar{p}_i})$) is not contained in C_1, C_2. By symmetry, the same holds for f. □

Remark 3.2. The group $\mathcal{J}_\circ$ contains standard quintic transformations (Lemma 2.5). Remark that $\sigma_1\colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$ is contained in $\mathcal{J}_\circ$.

The linear map $[x:y:z] \mapsto [z:-y:x]$ exchanges p_1 and p_2 (and $\bar{p}_1$ and $\bar{p}_2$), and the linear map $[x:y:z] \mapsto [-x:y:z]$ exchanges p_1 and $\bar{p}_1$ and fixes p_2. Both are contained in $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ$.

Lemma 3.3. *For any $q \in \mathbb{P}^2(\mathbb{R})$ not collinear with any two of $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ except maybe the pair $(p_2, \bar{p}_2)$, there exists $f \in \mathcal{J}_\circ$ of degree 2 with base-points $p_1, \bar{p}_1, q$.*

In particular: Let $f \in \mathcal{J}_\circ$ of even degree d, the points $p_i, \bar{p}_i$ its base-points of multiplicity $\frac{d}{2}$ and r its simple base-point or the proper point of $\mathbb{P}^2$ to which the simple base-point is infinitely near.

Then there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $p_i, \bar{p}_i, r$.

Proof. Since q is not collinear with $p_1, \bar{p}_1$, there exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that sends $p_1, \bar{p}_1, q$ onto $p_1, \bar{p}_1, [0:0:1]$. Let $t := (\sigma_1\alpha)^\bullet(p_2)$. The quadratic transformation $\sigma_1\alpha$ has base-points $p_1, \bar{p}_1, q$ and sends the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ onto the pencil of conics through $p_1, \bar{p}_1, t, \bar{t}$. By assumption, the point p_2 is not on the lines $L_{q,p_1}, L_{q,\bar{p}_1}$ and thus $t, \bar{t}$ are proper points of $\mathbb{P}^2$ that are not collinear with $p_1, \bar{p}_1$. There exists $\beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that fixes $p_1, \bar{p}_1$ and sends $t, \bar{t}$ onto $p_2, \bar{p}_2$. The quadratic transformation $\beta\sigma_1\alpha$ has base-points $p_1, \bar{p}_1, q$ and sends the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ onto itself, i.e. is contained in $\mathcal{J}_\circ$.

Let $f \in \mathcal{J}_\circ$ of even degree d, $p_i, \bar{p}_i$ its base-points of multiplicity $\frac{d}{2}$ and r its simple base-point or the proper point of $\mathbb{P}^2$ to which the simple base-point is infintely near. By Bézout, $r, p_i, \bar{p}_i$ are not collinear and by Lemma 3.1 the points r, p_i, p_{3-i} and $r, \bar{p}_i, p_{3-i}$ are not collinear. Hence there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $r, p_i, \bar{p}_i$. □

To prove the next lemma (Lemma 3.6), we are forced to introduce another kind of quintic transformation, which is just a degeneration of standard quintic transformations. They will pop up again in Section 5, where we look at relations between quadratic and standard quintic transformations in order to prove that the kernel of the Abelianisation map is equal to the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

Definition 3.4. We define a type of real birational transformation called *special quintic transformation.*

Let $q_1, \bar{q}_1, q_2, \bar{q}_2 \in \mathbb{P}^2$ be two pairs of non-real points of $\mathbb{P}^2$, not on the same line. Denote by $\pi_1 : X_1 \to \mathbb{P}^2$ the blow-up of the four points, and by $E_1, \bar{E}_1 \subset X_1$ the curves contracted onto $q_1, \bar{q}_1$ respectively. Let $q_3 \in E_1$ be a point, and $\bar{q}_3 \in \bar{E}_1$ its conjugate. We assume that there is no conic of $\mathbb{P}^2$ passing through $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$ and let $\pi_2\colon X_2 \to X_1$ be the blow-up of $q_3, \bar{q}_3$.

On X_2 the strict transforms of the two conics $C, \bar{C}$ of $\mathbb{P}^2$ passing through $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3$ and $q_1, \bar{q}_1, q_2, \bar{q}_2, \bar{q}_3$ respectively, are non-real conjugate disjoint (-1) curves. The contraction of these two curves gives a birational morphism $\eta_2\colon X_2 \to Y_1$, contracting $C, \bar{C}$ onto two points $r_3, \bar{r}_3$. On Y_1 we find two pairs of non-real (-1) curves, all four curves being disjoint. These are the strict transforms of the exceptional curves associated to $q_1, \bar{q}_1$, and of the conics passing through $q_1, \bar{q}_1, q_2, q_3, \bar{q}_3$ and $q_1, \bar{q}_1, \bar{q}_2, q_3, \bar{q}_3$ respectively. The contraction of these curves gives a birational morphism $\eta_1 : Y_1 \to \mathbb{P}^2$ and the images of the four curves are points $r_1, \bar{r}_1, r_2, \bar{r}_2$ respectively. The real birational map $\psi = \eta_1\eta_2(\pi_1\pi_2)^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is of degree 5 and called *special quintic transformation.*

Remark 3.5. Let θ be a special quintic transformation and keep the notation of its definition. With similar argument as for the standard quintic transformations (Lemma 2.3) one shows that $q_1, \ldots, \bar{q}_3$

are the base-points of θ, and are of multiplicity 2. Furthermore, θ sends the pencil of conics through $q_1, \bar{q}_1, q_2, \bar{q}_2$ onto the pencil of conics through $r_1, \bar{r}_1, r_2, \bar{r}_2$ and $\theta \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$.

Lemma 3.6. *The group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements.*

Proof. Let $f \in \mathcal{J}_\circ$. We use induction on the degree d of f. We can assume that $d > 2$.

• If d is even, it has a (real) simple base-point. Denote by r the simple base-point of f or, if the simple base-points is not a proper point of $\mathbb{P}^2$, the proper point of $\mathbb{P}^2$ to which the simple base-point is infinitely near to. Let $p_i, \bar{p}_i$, $i \in \{1, 2\}$ be the points of multiplicity $\frac{d}{2}$ (Lemma 3.1). By Lemma 3.3 there exists a quadratic transformation $\tau \in \mathcal{J}_\circ$ with base-points $p_i, \bar{p}_i, r$. The map $f\tau^{-1} \in \mathcal{J}_\circ$ is of degree $\leq d - 1$.

• Suppose that d is odd and has a real base-point q. By Lemma 3.1, the points q, p_1, p_2 are of multiplicity $2, \frac{d-1}{2}, \frac{d-1}{2}$ respectively. We can assume that q is a proper point of $\mathbb{P}^2$ (since no real point is infinitely near $p_1, \dots, \bar{p}_2$). By Bézout, q is not collinear with p_i, p_j, $i, j \in \{1, 2\}$, and so there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $q, p_1, \bar{p}_1$ (Lemma 3.3). The map $f\tau^{-1} \in \mathcal{J}_\circ$ is of degree $d - 1$.

• Suppose that d is odd and has no real base-points. If it has a double point q different from $p_1, \dots, \bar{p}_2$ which is a proper point of $\mathbb{P}^2$ then $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ are not on the same conic (Lemma 3.1). In particular, there exists a standard quintic transformation $\theta \in \mathcal{J}_\circ$ with those points its base-points (Definition 2.2, Lemma 2.5). The map $f\theta^{-1} \in \mathcal{J}_\circ$ is of degree $d - 4$.

If it has no double points that are proper points of $\mathbb{P}^2$, there exists a double point q infinitely near one of the p_i's. By Bézout, $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ are not contained on one conic, hence there exists a special quintic transformation $\theta \in \mathcal{J}_\circ$ with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ (Definition 3.4). The map $f\theta^{-1} \in \mathcal{J}_\circ$ is of degree $d - 4$. By [BM2012, Lemma 3.7] and Remark 3.2, θ is the composition of standard quintic and linear transformations contained in $\mathcal{J}_\circ$. □

Recall that for each element $f \in \mathcal{J}_\circ$ there exists $\hat{f} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$ such that $\hat{f} \circ \pi_\circ = \pi_\circ \circ f$ (Definition 2.1). This induces a group homomorphism $\mathcal{J}_\circ \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$ given by $f \mapsto \hat{f}$ (see Definition 2.1). The next Lemma states that this action corresponds to a real scaling and that every scaling can be realised by a quadratic transformation. The cubic and standard quintic transformations scale by ± 1.

By $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0, \infty])$ we denote the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ that fixes the real interval $[0, \infty]$ in $\mathbb{P}^1(\mathbb{R})$.

Lemma 3.7. *The action of $\mathcal{J}_\circ$ on $\mathbb{P}^1$ gives rise to a surjective homomorphism*

$$\mathcal{J}_\circ \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0, \infty]) \simeq \mathbb{R}_{>0} \ltimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{R}_{>0} \subset \mathrm{PGL}_2(\mathbb{R})$ is given by diagonal maps $[x : y] \mapsto [ax : by]$, $a, b \in \mathbb{R}_{>0}$ and $\mathbb{Z}/2\mathbb{Z}$ is generated by $[x : y] \mapsto [y : x]$.

Moreover, any element of $(\mathbb{R}_{>0})^*$ is the image of a quadratic element of $\mathcal{J}_\circ$ and $\mathbb{Z}/2\mathbb{Z}$ is the image of a linear element.

Furthermore:

• *The cubic transformations are sent onto $(1, 0)$ if they contract $L_{p_i,q}$ onto p_i or $\bar{p}_i$, $i = 1, 2$, where q is the double point, and onto $(1, 1)$ otherwise.*

• *The standard quintic transformations are sent onto $(1, 0)$ or $(1, 1)$.*

Proof. There are exactly three real reducible conics passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$, namely

$$C_1 := L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}, \quad C_2 := L_{p_1,\bar{p}_2} \cup L_{\bar{p}_1,p_2}, \quad C_3 := L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2},$$

and their images by $\pi_\circ \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ are

$$\pi_\circ(C_1) = [0 : 1], \quad \pi_\circ(C_2) = [1 : 0], \quad \pi_\circ(C_3) = [1 : 1].$$

Let $f \in \mathcal{J}_\circ$ and $\hat{f}$ the induced automorphism on $\mathbb{P}^1$. By Lemma 3.1, f preserves or exchanges C_1, C_2, which yields that $\hat{f}$ is of the form $\hat{f} \colon [u : v] \mapsto [au : bv]$ or $\hat{f} \colon [u : v] \mapsto [av : bu]$, $a, b \in \mathbb{R}^*$, where $[a : b] = \hat{f}(\pi_\circ(C_3)) = \pi_\circ(f(C_3))$. This yields a homomorphism

$$\psi \colon \mathcal{J}_\circ \to \mathbb{R}^* \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Lets show that the image of ψ is $\mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z}$ and that any element of $\mathbb{R}_{>0}$ is the image of a quadratic transformation.

By Lemma 3.6, the group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements. The map $\gamma\colon [x:y:z]\mapsto[-x:y:z]$ induces $\hat{\gamma}\colon [u:v]\mapsto[v:u]$, i.e. $\psi(\gamma)=(0,1)$. The linear transformations send lines onto lines, and hence are sent by ψ onto $(1,0)$ or $(1,1)$. The standard quintic transformations preserve the set $\{C_1,C_2,C_3\}$ and are hence sent onto $(1,0)$ or $(1,1)$. Let $\tau\in\mathcal{J}_\circ$ be a quadratic transformation. It has base-points $p_i,\bar{p}_i,q$, for some $i\in\{1,2\}$, and sends $p_{3-i},\bar{p}_{3-i}$ onto proper points of $\mathbb{P}^2$. In particular, q is not collinear with any two of $p_1,\bar{p}_1,p_2,\bar{p}_2$ except maybe $p_{3-i},\bar{p}_{3-i}$. It follows that $q\in\mathbb{P}^2(\mathbb{R})\setminus\{[1:0:1],[1:0:-1]\}$. On the other hand, take $q=[a:b:1]\in\mathbb{P}^2(\mathbb{R})\setminus\{[1:0:1],[1:0:-1]\}$. Then q is not collinear with any two of $p_1,\bar{p}_1,p_2,\bar{p}_2$, except maybe $p_2,\bar{p}_2$. By Lemma 3.3 there exists a quadratic transformation $\tau\in\mathcal{J}_\circ$ with base-points $q,p_1,\bar{p}_1$.

We have $\pi_\circ(\tau^{-1}(C_3))=\pi_\circ(q)=[b^2+(a+1)^2:b^2+(a-1)^2]$, which is not equal to $[0:1],[1:0]$. In particular, $\psi(\tau^{-1})\in(\mathbb{R}_{>0})^*\rtimes\mathbb{Z}/2\mathbb{Z}$, and it follows that $\psi(\mathcal{J}_\circ)\subset(\mathbb{R}_{>0})^*\rtimes\mathbb{Z}/2\mathbb{Z}$.

Note that $\mathrm{pr}_1(\psi(\tau))=\pi_\circ(q)$, so the image by $(\mathrm{pr}_1\circ\psi)$ of the set of quadratic elements of $\mathcal{J}_\circ$ is equal to the image by $\pi_\circ$ of the set $\mathbb{P}^2(\mathbb{R})\setminus\{[1:0:1],[1:0:-1]\}$.

Claim: $\pi_\circ(\ \mathbb{P}^2(\mathbb{R})\setminus\{[1:0:1],[1:0:-1]\}\ )=\{[a:1]\in\mathbb{P}^1(\mathbb{R})\mid a>0\}\simeq\mathbb{R}_{>0}$: The set of points where $\pi_\circ\colon[x:y:z]\dashrightarrow[y^2+(x+z)^2:y^2+(x-z)^2]$ is not defined is $\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$, hence $\pi_\circ$ is defined on $\mathbb{P}^2(\mathbb{R})$ and continuous on it. Thus $\pi_\circ(\mathbb{P}^2(\mathbb{R}))$ is a connected subset of $\{[a:1]\in\mathbb{P}^1(\mathbb{R})\mid a\geq 0\}\cup\{[1:0]\}\subset\mathbb{P}^1(\mathbb{R})$. The claim now follows with $\pi_\circ([1:0:1])=[1:0]$ and $\pi_\circ([1:0:-1])=[0:1]$.

In conclusion, every element of $\mathbb{R}_{>0}$ is the image of a quadratic element of $\mathcal{J}_\circ$, and ψ has image $\mathbb{R}_{>0}\rtimes\mathbb{Z}/2\mathbb{Z}$.

To complete the proof of the lemma, remark that cubic transformations preserve C_3 and they preserve C_1,C_2 if they contract $L_{p_i,q}$ onto p_i or $\bar{p}_i$, $i=1,2$, where q is the double point. □

3.2. **The quotient.** Using Lemma 3.7, we now construct a surjective group homomorphism $\mathcal{J}_\circ\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$.

There are two constructions of the quotient - one geometrical and the other using the spinor norm on $\mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))$. We first give the geometrical construction and then the one via the spinor norm.

Definition 3.8. Let $f\in\mathcal{J}_\circ$. For any *non-real* base-point q of f, we have $\pi_\circ(C_q)=[a+ib:1]$ and $\pi_\circ(C_{\bar{q}})=[a-ib:1]$ for some $a,b\in\mathbb{R}$, $b\neq0$ (see Definition 2.12 for the definition of C_q). We define

$$\nu(C_q):=\frac{a}{\mid b\mid}\ \in\mathbb{R}.$$

Note that $\nu(C_q)=\nu(C_{\bar{q}})$. Moreover, $\nu(C_{q'})=\nu(C_q)$ if and only if $\pi_\circ(C_q)=\lambda\pi_\circ(C_{q'})$ or $\pi_\circ(C_q)=\lambda\pi_\circ(C_{\bar{q}'})$ for some $\lambda\in\mathbb{R}^*$.

Definition 3.9. We define $e_\delta\in\oplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ to be the "standard vector" given by

$$(e_\delta)_\varepsilon=\begin{cases}1, & \delta=\varepsilon\\ 0, & \text{else}\end{cases}$$

Definition 3.10. Let $f\in\mathcal{J}_\circ$ and $S(f)$ be the set of non-real conjugate *pairs* of base-points of f different from $p_1,\dots,\bar{p}_2$. We define

$$\varphi_\circ\colon\mathcal{J}_\circ\longrightarrow\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z},\qquad f\longmapsto\sum_{(q,\bar{q})\in S(f)}e_{\nu(C_q)}$$

which is a well defined map according to Definition 3.8.

Remark 3.11. The following remarks directly follow from the definition of $\varphi_\circ$.

(1) If $S(f)=\emptyset$, then $\varphi_0(f)=0$.
(2) For every $f\in\mathcal{J}_\circ$ of degree ≤4 the set $S(f)$ is empty (follows from its characteristic; Lemma 3.1), hence in particular $\varphi_0(f)=0$.
(3) Let $\theta\in\mathcal{J}_\circ$ be a standard quintic transformation. Then $|S(f)|=1$ and $\varphi_\circ(\theta)$ is a "standard vector".
(4) It follows from the definition of standard quintic transformations (Definition 2.2) that for every $\delta\in\mathbb{R}$ there exists a standard quintic transformation $\theta\in\mathcal{J}_\circ$ such that $\varphi_\circ(\theta)=e_\delta$.

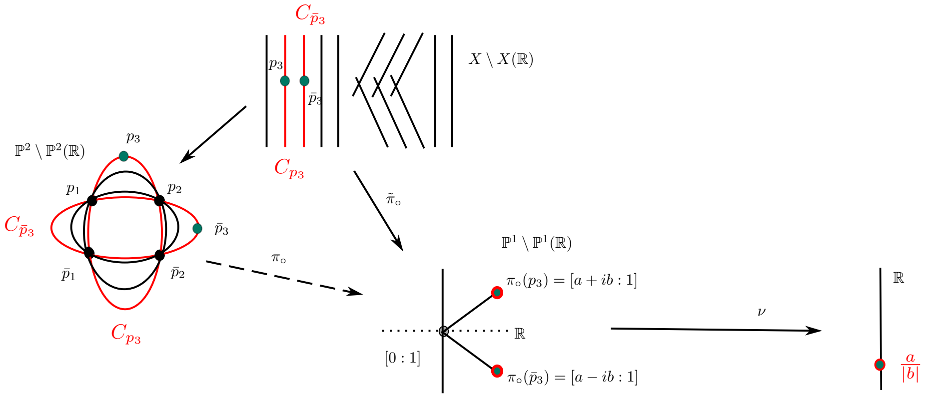

FIGURE 1. The map ν (Definition 3.8)

(5) Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations and $S(\theta_i) = \{(q_i, \bar{q}_i)\}$, $i = 1, 2$. If $C_{q_1} = C_{q_2}$ (or $C_{q_1} = C_{\bar{q}_2}$), then $\varphi_\circ(\theta_1) = \varphi_\circ(\theta_2)$.
(6) Let $\theta \in \mathcal{J}_\circ$ be a standard quintic transformation. Let $S(\theta) = \{(q_1, \bar{q}_1)\}$ and $S(\theta^{-1}) = \{(q_2, \bar{q}_2)\}$. Since θ induces Id or $[x : y] \mapsto [y : x]$ on $\mathbb{P}^1$ (Lemma 3.7), it follows that $\nu(C_{q_1}) = \nu(C_{q_2})$ and in particular $\varphi_\circ(\theta) = \varphi_\circ(\theta^{-1})$.
(7) Let $f \in \mathcal{J}_\circ$ and C be any non-real conic passing through $p_1, \ldots, \bar{p}_2$. The automorphism $\hat{f}$ on $\mathbb{P}^1$ induced by f is a scaling by a positive real number (Lemma 3.7), thus $\nu \circ \hat{f} = \nu$. In particular,

$$e_{\nu(f(C))} = e_{\nu(\hat{f}(C))} = e_{\nu(C)}.$$

Let us finally prove that $\varphi_\circ$ is a homomorphism of groups.

Lemma 3.12. *The map $\varphi_\circ \colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is a surjective group homomorphism and its kernel contains all elements of degree ≤ 4.*

Proof. It suffices to show that $\varphi_\circ$ is a group homomorphism: the surjectivity and the assertion on the kernel then follow from Remark 3.11 (2) and (4).

Let $f, g \in \mathcal{J}_\circ$. We want to show that $\varphi_\circ(fg) = \varphi_\circ(f) + \varphi_\circ(g)$. The group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements (Lemma 3.6), so we can assume that f is a linear, quadratic or standard quintic element of $\mathcal{J}_\circ$. In particular, $S(f)$ is empty if f is linear or quadratic (Remark 3.11 (2)), and $|S(f)| = 1$ if f is a standard quintic transformation.

Suppose that $S(f) \cap S(g^{-1}) = \emptyset$, then $S(fg) = S(g) \cup (g^{-1})^\bullet(S(f))$ [AC2002, Corollary 4.1.14]. If $S(f) = \emptyset$, we have $\varphi_\circ(f) = 0$ (Remark 3.11 (1)), $S(fg) = S(g)$, and in particular $\varphi_\circ(fg) = \varphi_\circ(f) + \varphi_\circ(g)$. If $S(f) \neq \emptyset$, then $S(f) = \{(q, \bar{q})\}$. By Remark 3.11 (7), we have

$$e_{\nu(C_{(g^{-1})^\bullet(q)})} = e_{\nu(g^{-1}(C_q))} = e_{\nu(C_q)}$$

In particular,

$$\begin{aligned}\varphi_\circ(fg) &= \sum_{(p,\bar{p}) \in S(fg)} e_{\nu(C_p)} = e_{\nu(C_{(g^{-1})^\bullet(q)})} + \sum_{(p,\bar{p}) \in S(g)} e_{\nu(C_p)} \\ &= e_{\nu(C_q)} + \sum_{(p,\bar{p}) \in S(g)} e_{\nu(C_p)} = \varphi_\circ(f) + \varphi_\circ(g)\end{aligned}$$

Suppose that $\emptyset \neq S(f) \subset S(g^{-1})$. Then f is a standard quintic transformation. In order to make the argument a bit more simple, lets prove that $\varphi_\circ(g^{-1}f^{-1}) = \varphi_\circ(g^{-1}) + \varphi_\circ(f^{-1})$, which will yield the claim (since $\varphi_\circ(h) = \varphi_\circ(h^{-1})$ by Remark 3.11 (6)). Let $S(f) = \{(q, \bar{q})\}$, $S(f^{-1}) = \{(q', \bar{q}')\}$.

We claim that $S((fg)^{-1}) = f^\bullet\left(S(g^{-1}) \setminus \{(q, \bar{q})\}\right)$. Indeed, the multiplicity of $(fg)^{-1}$ in q' is equal to the intersection of the strict transform of C_q with the strict transform of the linear system of g^{-1} in the blow-up of $q, \bar{q}, p_1, \bar{p}_1, p_2, \bar{p}_2$ in $\mathbb{P}^2$. Since C_q contains exactly one base-point of g^{-1} (Lemma 3.1), which is q, the intersection is precisely

$$\begin{aligned} m_{(fg)^{-1}}(q') &= 2\deg(g^{-1}) - 2m_{g^{-1}}(p_1) - 2m_{g^{-1}}(p_2) - \sum_{r\in C_q} m_{g^{-1}}(r) \\ &= 2\deg(g) - 2(\deg(g)-1) - m_{g^{-1}}(q) = 0 \end{aligned}$$

On the other hand, f does not touch the base-points of g^{-1} different from $q,\bar{q},p_1,\bar{p}_1,p_2,\bar{p}_2$. It follows that $S(g^{-1}f^{-1}) = f^\bullet\big(\, S(g^{-1})\setminus\{(q,\bar{q})\}\big)$ [AC2002, Corollary 4.1.14]. In particular, we have by Remark 3.11 (6), (7)

$$\begin{aligned} \varphi_\circ(g^{-1}f^{-1}) &= \sum_{(p,\bar{p})\in S(g^{-1}f^{-1})} e_{\nu(C_p)} = \sum_{(p,\bar{p})\in f^\bullet(\, S(g^{-1})\setminus\{(q,\bar{q})\})} e_{\nu_\circ(C_p)} \\ &\overset{(7)}{=} \sum_{(p,\bar{p})\in S(g^{-1})\setminus\{(q,\bar{q})\}} e_{\nu(C_p)} = \varphi_\circ(g^{-1}) - e_{\nu(C_q)} \\ &= \varphi_\circ(g^{-1}) - \varphi_\circ(f) \overset{(6)}{=} \varphi_\circ(g^{-1}) + \varphi_\circ(f^{-1}) \end{aligned}$$

□

3.3. **Construction of quotient using the spinor norm.** The quotient $\varphi\colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is in fact given by the spinor norm, as explained in the following.

Blowing up the four base-points $p_1,\bar{p}_1,p_2,\bar{p}_2$ of the rational map $\pi_\circ\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ and contracting the strict transform of $L_{p_1,\bar{p}_1}$ (or $L_{p_2,\bar{p}_2}$) yields a del Pezzo surface X_6 of degree 6. The fibration $\pi_\circ$ becomes a morphism $\pi'_\circ\colon X_6 \to \mathbb{P}^1$, which is a conic bundle with two singular fibres, both having only one real point. The group $\mathcal{J}_\circ$ is the group of birational maps of X_6 preserving this conic bundle structure.

The contraction the two (-1)-sections on X_6 is a morphism

$$X_6 \to S = \{wz = x^2+y^2\} \subset \mathbb{P}^3,$$

onto the quadric in $\mathbb{P}^3$ whose real part is diffeomorphic to the sphere. We can choose the images of the sections to be the points $[0:1:i:0],[0:1:-i:0]\in\mathbb{P}^3$ and obtain that $X_6 = \{([w:x:y:z],[u:v])\in\mathbb{P}^3\times\mathbb{P}^1 \mid uz = vw, wz = x^2+y^2\}$ and

$$\begin{tikzcd} X_6 \arrow[r] \arrow[d, "\pi'_\circ"] & S \arrow[d, dashed] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} ([w:x:y:z],[u:v]) \arrow[r] \arrow[d] & {[w:x:y:z]} \arrow[d, dashed] \\ {[u:v]} \arrow[r, equal] & {[w:z]} \end{tikzcd}$$

The generic fibre of $\pi'_\circ$ is the conic C in $\mathbb{P}^2_{\mathbb{R}(t)}$ given by $x^2+y^2-tz^2=0$. By Lemma 3.7, the projection $\pi'_\circ$ induces an exact sequence

$$1 \to \mathrm{Aut}_{\mathbb{R}(t)}(C) \longrightarrow \mathcal{J}_\circ \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1,[0,\infty]) \simeq \mathbb{R}_{>0}\rtimes\mathbb{Z}/2\mathbb{Z} \to 1,$$

which in fact is split by

$$\begin{aligned} (\lambda,0) &\mapsto \Big([w:x:y:z]\mapsto[\lambda w:\sqrt{\lambda}x:\sqrt{\lambda}y:z]\Big) \\ (0,1) &\mapsto ([w:x:y:z]\mapsto[z:x:y:w]). \end{aligned}$$

Furthermore, $\mathrm{Aut}_{\mathbb{R}(t)}(C)$ is isomorphic to the subgroup of $\mathrm{PGL}_3(\mathbb{R}(t))$ preserving the quadratic form $x^2+y^2-tz^2$, and is therefore isomorphic to $\mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))$.

Consider the spinor norm

$$0\to\mathbb{Z}/2\mathbb{Z}\to\mathrm{Spin}(x^2+y^2-tz^2,\mathbb{R}(t))\to\mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))\xrightarrow{\theta}\mathbb{R}(t)^*/(\mathbb{R}(t)^*)^2.$$

For a reflection f at a vector $v=(a(t),b(t),c(t))$, the spinor norm is the the length of v squared, i.e. $\theta(f)=a(t)^2+b(t)^2-tc(t)^2$. As squares are moded out, we may assume that $a(t),b(t),c(t)\in\mathbb{R}[t]$. An element $g\in\mathbb{R}[t]$ is a square if and only if every root of g appears with even multiplicity. Thus we can identify $\mathbb{R}(t)^*/(\mathbb{R}(t)^*)^2$ with polynomials in $\mathbb{R}[t]$ having only simple roots, i.e. with $\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_{\mathbb{H}}\mathbb{Z}/2\mathbb{Z}$, where $\overline{\mathbb{H}} \subset \mathbb{C}$ is the closed upper half plane and the first factor is the sign of the polynomial. A non-real root $a \pm ib$ is the root of $(t-a)^2+b^2$. In particular, the spinor norm induces a surjective homomorphism

$$\bar{\theta}\colon \mathrm{SO}(x^2+y^2-tz^2, \mathbb{R}(t)) \to \bigoplus_{\mathbb{H}} \mathbb{Z}/2\mathbb{Z}.$$

Let's look at it geometrically. Extending the scalars to $\mathbb{C}(t)$, the isomorphism $\mathrm{Aut}_{\mathbb{R}(t)}(C) \simeq \mathrm{SO}(x^2+y^2-tz^2, \mathbb{R}(t))$ extends to

$$\mathrm{Aut}_{\mathbb{R}(t)}(C) \subset \mathrm{Aut}_{\mathbb{C}(t)}(C) \simeq \mathrm{PGL}_2(\mathbb{C}[t]) \simeq \mathrm{SO}(x^2+y^2-tz^2, \mathbb{C}(t)) \simeq \mathrm{SO}(tx^2-yz, \mathbb{C}(t)),$$

where the isomorphism $\alpha\colon \mathrm{PGL}_2(\mathbb{C}[t]) \simeq \mathrm{SO}(tx^2-yz, \mathbb{C}(t))$ is given by the embedding $\mathbb{P}^1_{\mathbb{C}(t)} \hookrightarrow \mathbb{P}^2_{\mathbb{C}(t)}$, $[u:v] \mapsto [uv : tu^2 : v^2]$. The group $\mathrm{PGL}_2(\mathbb{C}[t])$ is generated by its involutions, all of which are conjugate to matrices of the form

$$P := \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}, \quad p \in \mathbb{C}[t].$$

The image of P via α is

$$\alpha(P) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -tp \\ 0 & -1/tp & 0 \end{pmatrix},$$

which is a reflexion at its eigenvector $(0, -tp, 1)$ of eigenvalue 1. In particular, $\theta(P) = tp$ and so $\bar{\theta}(P) = p = \det(P) \in \mathbb{C}(t)^*/(\mathbb{C}(t)^*)^2$. The isomorphism $\alpha\colon \mathrm{PGL}_2(\mathbb{C}[t]) \simeq \mathrm{SO}(x^2+y^2-tz^2, \mathbb{C}(t))$ is induced by a birational map $X_6 \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ that contracts one component in each singular fibre. The zeros of $\bar{\theta}(P)$ correspond to the fibres contracted by $f_P\colon \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $(x,y) \mapsto (p/x, y)$ "an odd number of times". So, for $f \in \mathrm{Aut}_{\mathbb{R}(t)}(C)$ the spinor norm $\bar{\theta}(f)$ corresponds to the non-real conics contracted by f "an odd number of times". Lemma 3.7 implies that $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0,\infty])$ acts on the image of these conics in $\mathbb{P}^1$ by real positive scaling. Observe that the quotient of $\mathbb{H}$ by $\mathbb{R}_{>0}$ is bijective to $\mathbb{R}$ via the map ν given in Definition 3.8 (see also Figure 1). We obtain a group homomorphism

$$\mathcal{J}_\circ \simeq \mathrm{SO}(x^2+y^2-tz^2, \mathbb{R}(t)) \rtimes \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0,\infty]) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z},$$

$$f \mapsto \begin{cases} \bar{\theta}(f), & f \in \mathrm{SO}(x^2+y^2-tz^2, \mathbb{R}(t)) \\ 0, & f \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0,\infty]) \end{cases}$$

which is exactly the quotient $\varphi_\circ$.

4. A quotient of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

Let $\varphi_0\colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ be the map given in Definition 3.8. By Proposition 2.9, the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to $\mathcal{G}$ (see Defintion 2.7), which, according to Remark 2.8, is the quotient of the free product $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ$ by the normal subgroup generated by all the relations given by the pairwise intersections of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ and the relations (1), (2), (3) of Definition 2.7. Define the map

$$\Phi\colon\ \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad f \mapsto \begin{cases} \varphi_\circ(f), & f \in \mathcal{J}_\circ \\ 0, & f \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \end{cases}$$

It is a surjective homomorphism of groups because $\varphi_\circ$ is a surjective homomorphism of groups (Lemma 3.12).

We shall now show that there exists a homomorphism φ such that the diagram

$$\begin{tikzcd} \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \arrow[dl, dotted, "\exists\, \varphi"] \\ \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

is commutative, where π is the quotient map. For this, it suffices to show that $\ker(\pi) \subset \ker(\Phi)$. We will first show that the relations given by the pairwise intersections of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ are contained in $\ker(\Phi)$ and then it is left to prove that relations (1), (2), (3) are contained $\ker(\Phi)$.

Lemma 4.1.

(1) *Let* $f_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $f_2 \in \mathcal{J}_\circ$ *such that* $\pi(f_1) = \pi(f_2)$*. Then* $\Phi(f_1) = \Phi(f_2) = 0$*.* (2) *Let* $f_1 \in \mathcal{J}_*$, $f_2 \in \mathcal{J}_\circ$ *such that* $\pi(f_1) = \pi(f_2)$*. Then* $\Phi(f_1) = \Phi(f_2) = 0$*.*
In particular, Φ *induces a homomorphism from the generalised amalgamated product of* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ *along all pairwise intersections onto* $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$*.*

Proof. (1) We have $\pi(f_1) = \pi(f_2) \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ \subset \mathcal{J}_\circ$. In particular, $\varphi_\circ(\pi(f_i)) = 0$, $i = 1, 2$ (Remark 3.11, (2)), and so $\Phi(f_1) = \Phi(f_2) = 0$ by definition of Φ.

(2) Lets first figure out what exactly $\mathcal{J}_* \cap \mathcal{J}_\circ$ consists of. First of all, it is not empty because the quadratic involution

$$\tau \colon [x : y : z] \dashrightarrow [y^2 + z^2 : xy : xz]$$

is contained in it. Let $f \in \mathcal{J}_* \cap \mathcal{J}_\circ$ be of degree d. By Lemma 3.1, its characteristic is $(d; \frac{d-1}{2}^4, 2^{\frac{d-1}{2}})$ or $(d; \frac{d}{2}^2, \frac{d-2}{2}^2, 2^{\frac{d-2}{2}}, 1)$. Since $f \in \mathcal{J}_*$, it has characteristic $(d; d-1, 1^{2d-2})$. If follows that $d \in \{1, 2, 3\}$.

Linear and quadratic elements of $\mathcal{J}_\circ$ are sent by $\varphi_\circ$ onto 0 (Remark 3.11 (2)). Elements of $\mathcal{J}_\circ$ of degree 3 decompose into quadratic elements of $\mathcal{J}_\circ$ and are hence sent onto zero by $\varphi_\circ$ as well. In particular, $\Phi(f_1) = \Phi(f_2) = 0$.

Since $\Phi(\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)) = \Phi(\mathcal{J}_*) = 0$, (1) and (2) imply that Φ induces a homomorphism from the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ onto $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. □

Lemma 4.2. *Let* $\theta \in \mathcal{J}_\circ$ *be a standard quintic with* $S(\theta) = \{(q, \bar{q})\}$, $S(\theta^{-1}) = \{(q', \bar{q}')\}$*.*

Let $\alpha_q, \alpha_{q'} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *that fix* p_1 *and send* q *(resp.* q'*) onto* p_2*. Then* $\theta' := \alpha_{q'}\theta(\alpha_q)^{-1} \in \mathcal{J}_\circ$ *is a standard quintic transformation and*

$$\Phi(\theta) = \Phi(\alpha_{q'}\theta(\alpha_q)^{-1}) = \Phi(\theta')$$

Note that the statement still holds if we write $\bar{p}_2$ *instead of* p_2*.*

Proof. Remark that

$$S(\theta') = \{\ (\alpha_q(p_2),\ \alpha_q(\bar{p}_2))\ \}.$$

Hence we need to show that

$$\Phi(\theta') = \varphi_\circ(\theta') = e_{\nu(C_{\alpha_q(p_2)})} = e_{\nu(C_q)} = \varphi_\circ(\theta) = \Phi(\theta).$$

To do this, it suffices to show that $\pi_\circ(C_{\alpha_q(p_2)}) = \lambda\pi_\circ(C_q)$ or $\pi_\circ(C_{\alpha_q(p_2)}) = \lambda\pi_\circ(C_{\bar{q}})$ for some $\lambda \in \mathbb{R}^*$. For this, we need to understand the map α_q. So, we study the non algebraic mapping

$$\psi \colon \mathbb{P}^2(\mathbb{C}) \setminus \{z = 0\} \longrightarrow \mathbb{P}^2(\mathbb{C}) \setminus \{z = 0\}, \quad q \mapsto \alpha_q(p_2)$$

which we can describe, via the parametrisation

$$\iota \colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{C}), \quad (u, v, x, y) \mapsto [u + iv : x + iy : 1],$$

by the real birational involution

$$\hat{\psi} \colon \mathbb{R}^4 \dashrightarrow \mathbb{R}^4, \quad (u, v, x, y) \mapsto \left(\frac{ud - vx}{v^2 + y^2}, \frac{-v}{v^2 + y^2}, \frac{uv + xy}{v^2 + y^2}, \frac{y}{v^2 + y^2} \right).$$

The domain of $\hat{\psi}$ is $\mathbb{R}^4 \setminus \{v = y = 0\} = \iota^{-1}\left(\mathbb{P}^2(\mathbb{C}) \setminus (\{z = 0\} \cup \mathbb{P}^2(\mathbb{R}))\right)$. To understand $\psi(C_q \setminus \{z = 0\})$, we use the parametrisation

$$\begin{aligned} \mathrm{par} \colon \mathbb{C} &\longrightarrow C_q \setminus \{z = 0\}, \\ t &\mapsto \left[\frac{(t-1)(t+1)(\lambda + \mu)}{\lambda t + \mu t + \lambda - \mu} : \frac{i(\lambda t^2 + \mu t^2 + 2\lambda t - 2\mu t + \lambda + \mu)}{\lambda t + \mu t + \lambda - \mu} : 1 \right], \end{aligned}$$

which is the inverse of the projection of C_q centred at p_1. This yields the commutative diagram

$$\begin{tikzcd}
& \iota^{-1}(C_q \setminus \{z = 0\}) \arrow[r, "\hat{\psi}"] \arrow[d, "\iota"] & \hat{\psi}(\iota^{-1}(C_q \setminus \{z = 0\})) \arrow[d, "\iota"] & \\
\mathbb{C} \arrow[r, "\mathrm{par}"] \arrow[rrr, bend right] & C_q \setminus \{z = 0\} \arrow[r, "\psi"] & \psi(C_q \setminus \{z = 0\}) \arrow[r, "\pi_\circ"] & \mathbb{P}^1
\end{tikzcd}$$

The map $(\pi_\circ \circ \psi \circ \mathrm{par})$ is given by

$$x+iy \mapsto \left[\frac{-\rho Q_q(x,y)}{4(\nu^2+\rho^2)} + i\frac{-\nu Q_q(x,y)}{4(\rho^2+\nu^2)} : 1\right]$$

where $\rho,\nu \in \mathbb{R}$ are the real coordinates of $\pi_\circ(C_q)$, i.e. $\pi_\circ(C_q) = [\rho+i\nu : q]$, and $Q_q(x,y) \in \mathbb{R}[x,y]$ is of degree 2. This shows that the points $\pi_\circ(C_q)$ and $\pi_\circ(C_{(\alpha_q)^{-1}(p_2)})$ are equal up to multiplication by $\frac{-Q_q(x,y)}{4(\nu^2+\rho^2)}$, which yields the claim. □

Recall the definition of the homomorphism

$$\Phi\colon\ \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad f \mapsto \begin{cases} \varphi_\circ(f), & f \in \mathcal{J}_\circ \\ 0, & f \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \end{cases}$$

Proposition 4.3. *The homomorphism Φ induces a surjective homomorphism of groups*

$$\varphi\colon\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$$

which is given as follows:

Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and write $f = f_n \cdots f_1$, where $f_1,\dots,f_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$. Then $\varphi(\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_*)) = 0$ and

$$\varphi(f) = \sum_{i=1}^{n} \Phi(f_i) = \sum_{f_i \in \mathcal{J}_\circ} \varphi_\circ(f_i)$$

Its kernel $\ker(\varphi)$ contains all elements of degree ≤ 4.

Proof. Let $\pi\colon\ \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \to \mathcal{G} \simeq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be the quotient map (Remark 2.8). We want to show that there exists a homomorphism $\varphi\colon\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ such that the diagram

$$\begin{tikzcd} \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \arrow[dl, dotted, "\exists\ \varphi"] \\ \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

is commutative. For this, it suffices to show that $\ker(\pi) \subset \ker(\Phi)$. By Lemma 4.1, Φ induces a homomorphism from the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ along all intersections onto $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. So, by Remark 2.8 it suffices to show that Φ sends the relations (1), (2), (3) in Definition 2.7 onto zero.

Linear, quadratic and cubic transformations in $\mathcal{J}_\circ$ and the group $\mathcal{J}_*$ are sent onto zero by φ (definition of Φ and Remark 3.11 (2)), hence relations (2) and (3) are contained in $\ker(\Phi)$. So, we just have to bother with relation (1):

Lets $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations, $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that

$$\theta_2 = \alpha_2 \theta_1 \alpha_1$$

If α_1, α_2 are contained in $\mathcal{J}_\circ$, then $\Phi(\alpha_2\theta_1\alpha_1(\theta_2)^{-1}) = \varphi_\circ(\alpha_2\theta_1\alpha_2(\theta_2)^{-1}) = \varphi_\circ(\mathrm{Id}) = 0$.

So, lets assume that $\alpha_1, \alpha_2 \notin \mathcal{J}_\circ$ and define $\mathfrak{q} := (q, \bar{q})$ for $q \in \mathbb{P}^2$. Denote $S(\theta_1) = \{\mathfrak{p}_3\}$, $S((\theta_1)^{-1}) = \{\mathfrak{p}_4\}$. There exist $i, j \in \{1,2,3\}$ such that $(\alpha_1)^{-1}(\mathfrak{p}_i) = \mathfrak{p}_1$, $(\alpha_1)^{-1}(\mathfrak{p}_j) = \mathfrak{p}_2$. Since $\alpha_1 \notin \mathcal{J}_\circ$, we have $3 \in \{i,j\}$. By Remark 3.2 there exist $\beta, \gamma \in \mathcal{J}_\circ \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $(\alpha_1^{-1}\beta^{-1})(p_1) = p_1$ and $(\alpha_2\gamma)(p_1) = p_1$. We obtain that $(\beta\alpha_1)^{-1}(p_3) \in \mathfrak{p}_2$ and $(\alpha_2\gamma)(p_4) \in \mathfrak{p}_2$. It follows from Lemma 4.2 that

$$\Phi(\theta_2) = \Phi\left((\alpha_2\gamma)(\gamma^{-1}\theta_1\beta^{-1})(\beta\alpha_1)\right) = \Phi(\theta_1),$$

i.e. Φ sends relation (1) onto zero. The surjectivity of φ follows from the surjectivity of φ_0 (Lemma 3.12).

If $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is of degree 2 or 3 there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\beta f \alpha \in \mathcal{J}_*$. Hence $\varphi(f) = 0$. If $\deg(f) = 4$, f is a composition of quadratic maps, hence $\varphi(f) = 0$. □

Let X be a real variety. We denote by $X(\mathbb{R})$ its set of real points of and by $\mathrm{Aut}(X(\mathbb{R}))\subset\mathrm{Bir}(X)$ the subgroup of transformations defined at each point of $X(\mathbb{R})$. It is also called the group of *birational diffeomorphisms* of X, and is, in general, strictly larger than the group of automorphisms $\mathrm{Aut}_{\mathbb{R}}(X)$ of X defined over $\mathbb{R}$. The group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the standard quintic transformations [RV2005, BM2012]. Until now no similar result has been found for $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$.

Corollary 4.4. *There exist surjective group homomorphisms*

$$\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z},\qquad \mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}.$$

Proof. We identify $\mathbb{A}^2(\mathbb{R})$ with $\mathbb{P}^2(\mathbb{R})\setminus L_{p_1,\bar{p}_1}$. All quintic transformations are contained in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ (Lemma 2.3) and preserve $C_3:=L_{p_1,\bar{p}_1}\cup L_{p_2,\bar{p}_2}$. For any standard quintic transformation θ there exists a permutation α of $p_1,\ldots,\bar{p}_2$ such that $\alpha\theta$ preserves $L_{p_i,\bar{p}_i}$, $i=1,2$, i.e. is contained in $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$. Therefore, the restriction of φ onto $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ is surjective. □

Let $\mathcal{Q}_{3,1}\subset\mathbb{P}^2$ be the variety given by the equation $w^2=x^2+y^2+z^2$. Its real part $\mathcal{Q}_{3,1}(\mathbb{R})$ is the 2-sphere $\mathbb{S}^2$.

Lemma 4.5. *There exists a surjective group homomorphism*

$$\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}.$$

Proof. By [KM2009, Theorem 1] (see also [BM2012, Theorem 1.4]), the group $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})=\mathbb{P}\mathrm{O}(3,1)$ and the family of standard cubic transformations (see [BM2012, Example 5.1] for definition). Consider the stereographic projection

$$\mathrm{p}\colon\mathcal{Q}_{3,1}\dashrightarrow\mathbb{P}^2,\quad [w:x:y:z]\mapsto[w-z:x:y]$$

It is a real birational transformation obtained by first blowing-up the point $[1:0:0:1]$ and then blowing down the singular hyperplane section $w=z$ onto the points $p_2,\bar{p}_2$. The inverse p^{-1} is an isomorphism around $p_1,\bar{p}_1$ and p sends a general hyperplane section onto a general conic passing through $p_2,\bar{p}_2$.

To prove the Lemma, it suffices to show that every standard quintic transformation in $\mathcal{J}_\circ$ that contracts the conic passing through all its base-points except p_2 onto p_2 is conjugate via p to a standard cubic transformation in $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. Let $\theta\in\mathcal{J}_\circ$ be a standard quintic transformation with base-points $p_1,\bar{p}_1$, $p_2,\bar{p}_2$, $p_3,\bar{p}_3$, and assume that the conic passing through $p_1,\bar{p}_1,\bar{p}_2,p_3,\bar{p}_3$ is contracted by θ^{-1} onto p_2. Then $\mathrm{p}\,\theta\,\mathrm{p}^{-1}\in\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. Let $C\subset\mathbb{P}^2$ be a general conic passing through $p_2,\bar{p}_2$. Then $\theta^{-1}(C)$ is a curve of degree 6 with multiplicty 3 in $p_2,\bar{p}_2$ and multiplicity 2 in $p_1,\bar{p}_1,p_3,\bar{p}_3$. Therefore, $(\mathrm{p}^{-1}\,\theta^{-1})(C)\subset\mathcal{Q}_{3,1}$ is a curve of self-intersection 18 passing through $\mathrm{p}^{-1}(p_1),\mathrm{p}^{-1}(\bar{p}_1)$, $\mathrm{p}^{-1}(p_3),\mathrm{p}^{-1}(\bar{p}_3)$ with multiplicity 2. It follows that $(\mathrm{p}^{-1}\,\theta\,\mathrm{p})^{-1}$ sends a general hyperplane section onto a cubic section having multiplicity 2 at these four points. [BM2012, Lemma 5.4 (3)] implies that $\mathrm{p}^{-1}\,\theta\,\mathrm{p}\in\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ is a standard cubic transformation. □

Corollary 4.4 and Lemma 4.5 imply Corollary 1.5.

Corollary 4.6 (Corollary 1.6)**.** *For any real birational map $\psi\colon\mathbb{F}_0\dashrightarrow\mathbb{P}^2$, the group $\psi\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi^{-1}$ is a subgroup of* $\ker(\varphi)$.

Proof. By [BM2012, Theorem 1.4], the group $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)\simeq\mathrm{PGL}(2,\mathbb{R})^2\rtimes\mathbb{Z}/\mathbb{Z}$ and the involution

$$\tau\colon([u_0:u_1],[v_0:v_1])\mapsto([u_0,u_1],[u_0v_0+u_1v_1:u_1v_0-u_0v_1]).$$

Consider the real birational map

$$\psi\colon\mathbb{P}^2\dashrightarrow\mathbb{F}_0,\quad [x:y:z]\mapsto([x:z],[y:z]),$$

with inverse $\psi^{-1}\colon([u_0:u_1],[v_0,v_1])\mapsto[u_0v_1:u_1v_0:u_1v_1]$.

A quick calculation shows that the conjugate by ψ of these generators of $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ are of degree at most 3. Proposition 4.3 implies that they are contained in $\ker(\varphi)$. In particular, $\psi^{-1}\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi\subset\ker(\varphi)$. Since $\ker(\varphi)$ is a normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the same statement holds for any other real birational map $\mathbb{P}^2\dashrightarrow\mathbb{F}_0$. □

Corollary 4.7 (Corollary 1.7)**.** *For any $n \in \mathbb{N}$ there is a normal subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *of index* 2^n *containing all elements of degree* ≤ 4.
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Proof. Let $pr_{\delta_1,\dots,\delta_n}\colon \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^n$ be the projection onto the $\delta_1,\dots,\delta_n$-th factors. Then $pr_{\delta_1,\dots,\delta_n} \circ \varphi$ has kernel of index 2^n containing $\ker(\varphi)$ and thus all elements of degree ≤ 4. By Corollary 1.5, the same argument works for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. □

Lemma 2.5 and Theorem 2.4 imply that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \sigma_1, \sigma_0$ and all standard quintic transformations in $\mathcal{J}_\circ$. This generating set is not far from being minimal:

Corollary 4.8 (Theorem 1.1 and Corllary 1.4)**.** *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is not generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *and a countable family of elements.*

The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$, *replacing* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *for the latter two by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{A}^2)$, $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ *respectively.*

Proof. If $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ was generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and a countable family $\{f_n\}_{n\in\mathbb{N}}$ of elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ then by Proposition 4.3, the countable family would yield a countable generating set of $\oplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$, which is impossible.

The same argument works for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ - for the latter two we replace $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ respectively by $\mathbb{P}\mathrm{O}(3,1)$ and by the subgroup of affine automorphisms of $\mathbb{A}^2$, which corresponds to $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$). □

Corollary 4.9 (Corollary 1.8)**.** *The normal subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *generated by any countable set of elements of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is a proper subgroup of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ *and* $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Proof. Let $S \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be a countable set of elements. Its image $\pi(S) \subset \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is a countable set and hence a proper subset of $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. Since π is surjective (Proposition 4.3), the preimage $\pi^{-1}(\pi(S)) \subsetneq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a proper subset. The group $\oplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ is Abelian, so the set $\pi^{-1}(\pi(S))$ contains the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by S, which in particular is a proper subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. □

Remark 4.10. The group homomorphism $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ does not have any sections: If it had a section, then for any $k \in \mathbb{N}$ the group $(\mathbb{Z}/2\mathbb{Z})^k$ would embed into $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is not possible by [Bea2007].

Remark 4.11. Over $\mathbb{C}$, the group $\mathcal{J}_\circ$ is conjugate to $\mathcal{J}_*$ (f.e. by any quadratic transformation having base-points $p_1, \bar{p}_1, p_2$ and sending $\bar{p}_2$ onto $[1:0:0]$). This is not true over $\mathbb{R}$: By Proposition 4.3, one is contained in $\ker(\varphi)$ and the other is not.

Remark 4.12.
(1) No proper normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of finite index is closed with respect to the Zariski or the Euclidean topology because $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is connected with respect to either topology [Bla2010].
(2) The group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ does not contain any proper normal subgroups of countable index: Assume that $\{\mathrm{Id}\} \neq N$ is a normal subgroup of countable index. The image of $\mathrm{PGL}_3(\mathbb{C})$ in the quotient is countable, hence $\mathrm{PGL}_3(\mathbb{C}) \cap N$ is non-trivial. Since $\mathrm{PGL}_3(\mathbb{C})$ is a simple group, we have $\mathrm{PGL}_3(\mathbb{C}) \subset N$. Since the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{C})$ is $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2], we get that $N = \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.

Lemma 4.13. *The normal subgroup of* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ *generated by any non-trivial element of of degree* ≤ 4 *is equal to* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.

Proof. The claim is stated in [Giz1994, Remark on Lemma 2, p. 42] for degree ≤ 7 but only a partial proof is given, which works for all transformations preserving a pencil of lines [Giz1994, Lemma 2].

(deg 1:) For degree 1, it is the fact that the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{C})$ is equal to $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2, Case 1 of proof].

(deg 2, 3:) Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be of degree 2 or 3. There exists a proper base-point q (resp. q') of f (resp. f^{-1}) such that f sends the pencil of lines through q onto the pencil of lines through q'. Pick $\alpha \in \mathrm{PGL}_3(\mathbb{C})$ that exchanges q, q' and such that $f\alpha^{-1}f\alpha \neq \mathrm{Id}$. Then $\mathrm{Id} \neq f\alpha^{-1}f\alpha$ is contained in the normal subgroup generated by f and preserves the pencil of lines through q'. Hence the normal subgroup generated by f is $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2].

(deg 4:) If the transformation has a triple base-point, we prove the claim with a similar argument as above. For transformations without triple points, despite the idea of proof in [Giz1994, Remark on lemma 2], we only succeeded to show the claim with a lot of effort and a rather long case by case study depending on the configuration of the base-points. □

5. The kernel of the quotient

In this section, we prove that the kernel of $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, which will turn out to be the commutator subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. It implies that the quotient $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is in fact the Abelianisation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

For this, we will again use the presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generators and relations given in Proposition 2.9. We will see that $\ker(\varphi)$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$ and $\ker(\varphi_0)$, and then it suffices to prove that $\mathcal{J}_*$ and $\ker(\varphi_\circ)$ are contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

The key idea is to show that if two standard quintic transformations θ_1, θ_2 are sent by $\varphi_\circ$ onto the same image, then θ_2 can be obtained by composing θ_1 with a suitable amount of quadratic elements, which will imply that $\theta_1(\theta_2)^{-1}$ is contained the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. For this to be useful, we need to be able to put the quintic elements next to each other when decomposing an element of $\mathcal{J}_\circ$ into quadratic and standard quintic elements. To do this we need to sidle off to involve special quintic transformations (see Definition 3.4), which is why they pop up again in this section.

More precisely, Lemma 5.6 shows that if two standard or special quintic transformations in $\mathcal{J}_\circ$ have the same image via $\varphi_\circ$, we can obtain one from the other by composing with a suitable amount of quadratic transformations in $\mathcal{J}_\circ$. We then show that every quadratic transformation in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ (Lemma 5.8). Lemma 5.9 shows that $\mathcal{J}_*$ is contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. All of this will yield that that the kernel of φ is indeed the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ (Proposition 5.13)

Definition 5.1. We denote by $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

5.1. **Geometry between cubic and quintic transformations.** One idea in the proof that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ is to see that if two standard quintic transformations are sent onto the same standard vector in $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, then one is obtained from the other by composing from the right and the left with suitable cubic maps, which in turn can be written as composition of quadratic maps. For this, we first have to dig into the geometry of cubic maps.

Remark 5.2. Let $f \in \mathcal{J}_\circ$ of degree 3 and $r \in \mathbb{P}^2(\mathbb{R})$ its double point. The points $p_1, \ldots, \bar{p}_2$ are base-points of f of multiplicity 1 (Lemma 3.1). Note that for $i \in \{1,2\}$, the map f contracts the line passing through r, p_i onto one of $p_1, \bar{p}_1, p_2, \bar{p}_2$ and that by Bézout the (real) double point is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$.

Lemma 5.3. *For every $r \in \mathbb{P}^2(\mathbb{R})$ not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$ there exists $f \in \mathcal{J}_\circ$ of degree 3 with base-points $r, p_1, \bar{p}_1, p_2, \bar{p}_2$ (with double point r).*

Proof. Since r is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, there exists $\tau_1 \in \mathcal{J}_\circ$ quadratic with base-points $r, p_1, \bar{p}_1$ (Lemma 3.3). The base-points of its inverse are $s, p_i, \bar{p}_i$ for some $s \in \mathbb{P}^2(\mathbb{R})$ and $i \in \{1,2\}$. We can assume that $i = 1$ by exchanging p_1, p_2 if necessary (Remark 3.2). Since $r, p_2, \bar{p}_2$ are not collinear, also $s, p_2, \bar{p}_2$ are not collinear because τ_1 sends the lines through r onto the lines through s and preserves $\{p_2, \bar{p}_2\}$. Moreover, s is not collinear with p_1, p_2 because $(\tau_1^{-1})_\bullet(p_2) \in \{p_2, \bar{p}_2\}$ is a proper point of $\mathbb{P}^2$. Hence there exists $\tau_2 \in \mathcal{J}_\circ$ of degree 2 with base-point $s, p_2, \bar{p}_2$ (Lemma 3.3). The map $\tau_2\tau_1 \in \mathcal{J}_\circ$ is of degree 3 with base-points $r, p_1, \bar{p}_1, p_2, \bar{p}_2$. □

Lemma 5.4. *Let $q \in \mathbb{P}^2(\mathbb{C}) \setminus \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ be a non-real point such that $C_q = \pi_\circ^{-1}(\pi_\circ(q))$ is irreducible. Then there exists a real point $r \in \mathbb{P}^2(\mathbb{C})$ and $f \in \mathcal{J}_\circ$ of degree 2 or 3 with r among its base-points such that*

(1) $f(C_q) = C_q$

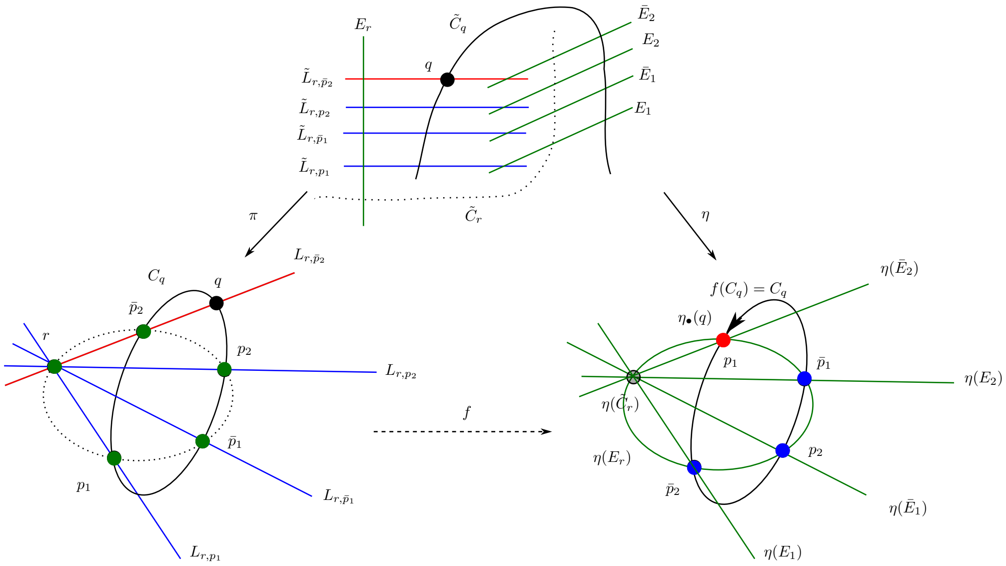

FIGURE 2. The cubic transformation of Lemma 5.4

(2) *$f_\bullet(q)$ is infinitely near p_1 corresponding to the tangent direction of $f(C_q)$*
(3) *either* $\deg(f) = 3$ *and C_r is irreducible or* $\deg(f) = 2$ *and C_r is singular.*
(4) $q \in L_{r,\bar{p}_2}$.

Proof. Let L be the line passing through $q, \bar{p}_2$. Since C_q is irreducible, q is not collinear with any of $p_1, \bar{p}_1, p_2, \bar{p}_2$. It follows that $L \neq \bar{L}$, and so L and $\bar{L}$ intersect in exactly one point r, which is a real point.

If r is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, then Lemma 5.3 states that there exists $f \in \mathcal{J}_\circ$ of degree 3 with singular point r. The line L is contracted onto p_i or $\bar{p}_i$, $i \in \{1, 2\}$. By composing with elements of $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cap \mathcal{J}_\circ$, we can assume that L is contracted onto p_1 and that f preserves the conic $L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}$, and thus induces the identity map on $\mathbb{P}^1$ (Lemma 3.7), and therefore preserves C_q. It follows that $f^\bullet(q)$ is infinitely near p_1 and corresponds to the tangent direction of $f(C_q) = C_q$.

If r is collinear with two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, it is collinear with $p_1, \bar{p}_1$ and not collinear with any other two. Lemma 3.3 implies that there exists $f \in \mathcal{J}_\circ$ of degree 2 with base-points $r, p_2, \bar{p}_2$, and we can choose f such that the line L (through $q, \bar{p}_2, r$) is contracted onto p_1 (then $f(\{p_1, \bar{p}_1\}) = \{p_2, \bar{p}_2\}$) and such that $f(p_1) = p_2$. Then $f_\bullet(q)$ is infinitely near p_1. We claim that $f(C_q) = C_q$: Call $\hat{f}$ the automorphism of $\mathbb{P}^1$ induced by f. We calculate $\hat{f}^{-1}$ (cf. proof of Lemma 3.7). Since $f(L_{p_1,p_2}) = L_{p_1,p_2}$, we see that $\hat{f}^{-1} \colon [u : v] \mapsto [(r_1^2 + (r_0 + r_2)^2)u : (r_1^2 + (r_0 - r_2)^2)v]$, where $r = [r_0 : r_1 : r_2]$. Since $r \in L_{p_1,\bar{p}_1}$, we have $r_2 = 0$ and so $\hat{f}^{-1} = \mathrm{Id}$. In particular, $f(C_q) = C_q$. □

Remark 5.5. Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be special quintic transformations with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$. If $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$, then $q_1 = q_2$ or $q_1 = \bar{q}_2$ respectively. In particular, there exist $\alpha_1, \alpha_2 \in \mathcal{J}_\circ \cap \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\theta_2 = \alpha_2\theta_1\alpha_1$.

Lemma 5.6. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$, $i = 1, 2$. Assume that $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$.*

Then there exist $\tau_1, \ldots, \tau_8 \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \tau_8 \cdots \tau_5\theta_1\tau_4 \cdots \tau_1$.

Proof. By exchanging the names of $q_2, \bar{q}_2$, we can assume that $C_{q_1} = C_{q_2}$. It suffices to show that there exist $g_1, \ldots, g_4 \in \mathcal{J}_\circ$ of degree ≤ 3 such that $\theta_2 = g_4g_3\theta_1g_2g_2$, since every element of $\mathcal{J}_\circ$ of degree 3 can be written as a product of two qudratic elements of $\mathcal{J}_\circ$. We give an explicit construction of the g_i's.

According to Lemma 5.4 there exist for $i = 1, 2$ a real point r_i and $f_i \in \mathcal{J}_\circ$ of degree $d_i \in \{2, 3\}$ with base-point r_i such that f_i preserves C_{q_i} and $t_i := (f_i)_\bullet(q_i)$ is infinitely near p_1 corresponding to the tangent direction of C_{q_i} and that $q_i \in L_{r_i,\bar{p}_2} =: L$. Since C_{r_i} is real, r_i is not on a conic contracted by θ_i, and so $s_i := (\theta_i)_\bullet(r_i) = \theta_i(r_i)$ is a proper point of $\mathbb{P}^2$.

If C_{r_i} is irreducible (and hence $d_i = 3$), then r_i is not collinear with any two of $p_1, \ldots, \bar{p}_2$, and so s_i is not collinear with any two of $p_1, \ldots, \bar{p}_2$ either. Therefore, there exists $h_i \in \mathcal{J}_\circ$ of degree 3 with singular base-point s_i (Lemma 5.3). If C_{r_i} is singular (and hence $d_i = 2$), then $r_i \in L_{p_1,\bar{p}_1}$, and so $s_i \in \theta_i(L_{p_1,\bar{p}_1}) = L_{p_j,\bar{p}_j}$ for some $j \in \{1,2\}$. Therefore, there exists $h_i \in \mathcal{J}_\circ$ of degree 2 with base-points $s_i, p_{3-j}, \bar{p}_{3-j}$ (Lemma 3.3).

By composing h_i with elements in $\mathcal{J}_\circ \cap \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$, we can assume that h_i sends the line $\theta_i(L_{r_i,\bar{p}_2})$ onto p_1 (Remark 5.2). Then $h_i\theta_i(f_i)^{-1} \in \mathcal{J}_\circ$ is of degree 5. Its base-points are $p_1, \bar{p}_1, p_2, \bar{p}_2, (f_i)_\bullet(q_i), (f_i)_\bullet(\bar{q}_i)$, where the latter ones are infinitely near $p_1, \bar{p}_1$ corresponding to the tangent direction of $C_{q_i}, C_{\bar{q}_i}$. By Remark 5.5, $h_1\theta_1(f_1)^{-1}$ and $h_2\theta_2(f_2)^{-1}$ have exactly the same base-points, hence $h_1\theta_1(f_1)^{-1} = \beta h_2\theta_2(f_2)^{-1}\alpha$ for some $\alpha, \beta \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cap \mathcal{J}_\circ$. In particular, $\theta_2 = (h_2)^{-1}\beta^{-1}h_1\theta_1(f_1)^{-1}\alpha^{-1}f_2$. The claim follows with $g_1 = \alpha^{-1}f_2,\ g_2 = (f_1)^{-1},\ g_3 = \beta^{-1}h_1,\ g_4 = (h_2)^{-1}$. □

Lemma 5.7. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be a standard and a special quintic transformation respecitvely with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$. Assume that $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$.*

Then there exists $\tau_1, \ldots, \tau_4 \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \tau_4\tau_3\theta_1\tau_2\tau_1$.

Proof. By exchanging the names of $q_1, \bar{q}_1, q_2, \bar{q}_2$, we can assume that $C_{q_1} = C_{q_2}$ and that q_2 is infinitely near p_i, $i \in \{1,2\}$. By Lemma 5.4 there exists $f \in \mathcal{J}_\circ$ of degree $d \in \{2,3\}$ such that $f(C_{q_1}) = C_{q_1} = C_{q_2}$ and $f_\bullet(q_1)$ is infinitely near p_i. Let r be the real base-point of f. Since r is real, it is not on a conic contracted by θ_1, and so $(\theta_1)_\bullet(r) = \theta_1(r)$ is a proper point of $\mathbb{P}^2$.

If C_r is irreducible (i.e. $d = 3$), the conic $\theta_1(C_r) = C_{\theta(r)}$ is irreducible as well. By Lemma 5.3 there exists $g \in \mathcal{J}_\circ$ of degree 3 with double point $\theta_1(r)$. If C_r is singular (i.e. $d = 2$), the conic $\theta_1(C_r) = C_{\theta(r)}$ is singular as well. By Lemma 3.3 there exists $g \in \mathcal{J}_\circ$ of degree 2 with $\theta(r)$ among its base-points.

The map $g\theta_1 f^{-1}$ is of degree 5 with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, f_\bullet(q_1), f_\bullet(q_1)$, where the latter two are infinitely near $p_i, \bar{p}_i$ corresponding to the tangent directions $C_{q_1} = C_{q_2}, C_{\bar{q}_2}$. Hence there exists $\alpha \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cap \mathcal{J}_\circ$ such that $\alpha g\theta_1 f^{-1} = \theta_2$. The claim follows from the fact that we can write f, g as composition of at most two quadratic transformations in $\mathcal{J}_\circ$. □

5.2. **The normal subgroup generated by** $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$**.** Lemma 5.6 implies that if two standard or special quintic transformations θ_1, θ_2 contract the same conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$, then θ_2 is obtained from θ_1 by composing with suitable quadratic transformations. So, one step of proving that $\ker(\varphi) = \langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$ is to see that all quadratic transformations are contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$.

Lemma 5.8. *Any quadratic map in $\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ is contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$.*

Proof. Let $\tau \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ be of degree 2. Pick two base-points q_1, q_2 of τ that are either a pair of non-real conjugate points or two real base-points, such that either both are proper points of $\mathbb{P}^2$ or q_1 is a proper point of $\mathbb{P}^2$ and q_2 is in the first neighbourhood of q_1. Let t_1, t_2 be base-points of τ^{-1} such that τ sends the pencil of conics through q_1, q_2 onto the pencil of conics through t_1, t_2. Pick a general point $r \in \mathbb{P}^2$ and let $s := \tau(r)$. There exists $\alpha \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ that sends q_1, q_2 onto t_1, t_2 and exchanges r, s. The map $\tilde{\tau} := \tau\alpha$ is of degree 2, fixes s, and t_1, t_2 are base-points of $\tilde{\tau}$ and $\tilde{\tau}^{-1}$.

Since r is general, also s is general, and there exists $\theta \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ of degree 2 with base-points t_1, t_2, s. Observe that the map $\theta\tilde{\tau}\theta^{-1}$ is linear. In particular, τ is contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$. □

Recall that $\mathcal{J}_*$ is contained in $\ker(\varphi)$. Using Lemma 5.8, we now prove that $\mathcal{J}_*$ is contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$:

Lemma 5.9. *The group $\mathcal{J}_*$ is generated by its quadratic and linear elements. In particular, $\mathcal{J}_* \subset \langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$.*

Proof. Let $f \in \mathcal{J}_*$. We do induction on the degree $d = \deg(f)$ of f. If f is linear or quadratic, there is nothing to do. So, we can assume that $d \geq 3$.

Case 1: Assume that there exist two simple base-points q_1, q_2 of f that are proper points of $\mathbb{P}^2$ and either non-real conjugate points or both real points. The points $[1:0:0], q_1, q_2$ are not collinear by Bézout, hence there exists a quadratic map $\tau \in \mathcal{J}_*$ with base-points $[1:0:0], q_1, q_2$. The map $f\tau^{-1} \in \mathcal{J}_*$ is of degree $d-1$.

<u>Case 2:</u> Assume that f has exactly one simple (real) base-point q that is a proper point of $\mathbb{P}^2$. Let r be a general real point in $\mathbb{P}^2$. There exists $\tau_1 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], q, r$ and the map $f(\tau_1)^{-1} \in \mathcal{J}_*$ is of degree d. If t is a base-point of f in the first neighbourhood of $[1:0:0]$ or q, then $(\tau_1)^\bullet(t)$ is a base-point of $f(\tau_1)^{-1}$ that is a proper point of $\mathbb{P}^2$. Thus $f(\tau_1)^{-1}$ has at least two simple base-points that are proper points of $\mathbb{P}^2$ and either non-real conjuagte points (if t is non-real) or both real (if t is real). We proceed as above.

<u>Case 3:</u> Assume that f has no simple proper base-points at all, i.e. any simple base-point is infinitely near $[1:0:0]$.

• If there are at least two base-points q_1, q_2 in the first neighbourhood of $[1:0:0]$, let $r, s \in \mathbb{P}^2$ be general points. There exists $\tau_1 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], r, s$. Call $[1:0:0], r', s'$ the base-points of $(\tau_1)^{-1}$. The map $f(\tau_1)^{-1}$ is of degree $d+1$. We may assume that q_1, q_2 are both real or a pair of non-real conjugate points. Then $(\tau_1)_\bullet(q_1), (\tau_1)_\bullet(q_2)$ are proper points of $\mathbb{P}^2$ and base-points of $f(\tau_1)^{-1}$. Since $[1:0:0], \tau_1(q_1), \tau_1(q_2)$ are not collinear, there exists $\tau_2 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], \tau_1(q_1), \tau_1(q_2)$. The map $f(\tau_1)^{-1}(\tau_2)^{-1}$ is of degree d. We claim that the image by $(\tau_2)_\bullet(\tau_1)^\bullet$ of all base-points of f different from q_1, q_2 in the first neighbourhood of $[1:0:0]$ or of q_1, q_2 are base-points of $f(\tau_1)_{-1}(\tau_2)^{-1}$ that are proper points of $\mathbb{P}^2$ or are in the 1^{st} neighbourhood of $[1:0:0]$: Indeed, let t be a base-point of f in the 1^{st} neighbourhood of $[1:0:0]$ or q_1. Then $(\tau_1)_\bullet(t)$ is either a proper point of $\mathbb{P}^2$ on the line $L_{r',s'}$ or is infinitely near $\tau_1(q_1)$. By Bézout, $[1:0:0], \tau_1(q_1), (\tau_1)_\bullet(t)$ are not collinear. It follows that $(\tau_2)_\bullet((\tau_1)_\bullet(t))$ is either in the 1^{st} neighbourhood of $[1:0:0]$ (if t is in the 1^{st} neighbourhood of $[1:0:0]$ or q_1 and proximate to $[1:0:0]$) or a proper point of $\mathbb{P}^2$ (if t is in the 1^{st} neighbourhood of q_1 but not proximate to $[1:0:0]$). The situation is visualised in the following picture:

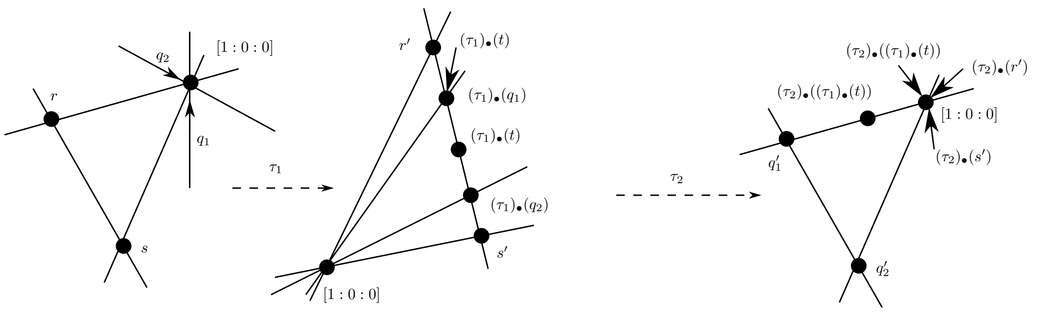

FIGURE: The quadratic maps τ_1, τ_2, and the possibilities for the point $(\tau_2)_\bullet((\tau_1)_\bullet(t))$.

Since not all base-points of f are proximate to $[1:0:0]$, we can repeat all of this until we obtain an element of $\mathcal{J}_*$ of degree d with simple proper base-points. We continue as in Case 1 or Case 2.

• If there is exactly one base-point q of f in the first neighbourhood of $[1:0:0]$, then in particular, q is a real point. Let $r \in \mathbb{P}^2$ be a general real point. There exists $\tau \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], q, r$. The map $f\tau^{-1} \in \mathcal{J}_*$ is of degree d and the image by $\tau_\bullet$ of any base-point in the first neighbourhood of q is a base-points of $f\tau^{-1}$ in the first neighbourhood of $[1:0:0]$. We repeat this step until we reach one of the above cases or until we obtain a linear map. □

5.3. **The kernel is equal to** $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. It now remains to actually prove that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. Take an element of $\ker(\varphi)$. It is the composition of linear, quadratic and standard and special quintic elements (Lemma 3.6). The next three lemmata show that we can choose the order of the linear, quadratic and standard and special quintic elements so that the ones belonging to the same coset are just one after another. These lemmata will be the remaining ingredients to prove that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$

Lemma 5.10. *Let $\tau, \theta \in \mathcal{J}_\circ$ be a quadratic and a standard (or special) quintic transformation respectively. Then there exist $\tilde{\tau}_1, \tilde{\tau}_2 \in \mathcal{J}_\circ$ of degree 2 and $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{J}_\circ$ standard (or special quintic) transformations such that $\tau\theta = \tilde{\theta}_1\tilde{\tau}_1$ and $\theta\tau = \tilde{\tau}_2\tilde{\theta}_2$, i.e. we can "permute" τ, θ.*

Proof. The map τ^{-1} has base-points $p_i, \bar{p}_i, r$, for some $r \in \mathbb{P}^2(\mathbb{R})$, $i \in \{1,2\}$. Since r is not on a conic contracted by θ, the point $\theta_\bullet(r) = \theta(r)$ is a proper point of $\mathbb{P}^2$ that is a base-point of $(\theta\tau)^{-1}$. Let p_{j_i} be the image by θ of the contracted conic not passing through p_i. The map $\theta\tau$ is of degree 6 and $p_{j_i}, \bar{p}_{j_i}$ are base-points of $(\theta\tau)^{-1}$ of multiplicity 3. By Lemma 3.3 there exists $\tilde{\tau} \in \mathcal{J}_\circ$ of degree 2 with base-points $\theta(r), p_{j_i}, \bar{p}_{j_i}$. The map $\tilde{\theta} := \tilde{\tau}\theta\tau \in \mathcal{J}_\circ$ is a standard (or special) quintic transformation. We put $\tilde{\tau}_2 := \tilde{\tau}^{-1}$, $\tilde{\theta}_2 := \tilde{\theta}$. A similar construction yields $\tilde{\theta}_1, \tilde{\tau}_1$. $\square$

Lemma 5.11. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard or special quintic transformations (both can be either) such that $\varphi_0(\theta_1) \neq \varphi_0(\theta_2)$. Then there exist $\theta_3, \theta_4 \in \mathcal{J}_\circ$ standard or special quintic transformations, such that*

$$\theta_2\theta_1 = \theta_4\theta_3, \quad \varphi_0(\theta_1) = \varphi_0(\theta_4), \quad \varphi_0(\theta_2) = \varphi_0(\theta_3)$$

i.e. we can "permute" θ_1, θ_2.

Proof. Let $S(\theta_1) = \{(p_3, \bar{p}_3)\}$ and $S(\theta_2) = \{(p_4, \bar{p}_4)\}$. By definition of φ_0 the assumption $\varphi_0(\theta_1) \neq \varphi_0(\theta_2)$ implies $p_4 \notin C_{p_3} \cup C_{\bar{p}_3}$.

The point $p_5 := ((\theta_1)^{-1})_\bullet(p_4)$ is either a proper point of $\mathbb{P}^2$ or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$. Because $p_4, \bar{p}_4, p_1, \bar{p}_1, p_2, \bar{p}_2$ are not on one conic, the points $p_5, \bar{p}_5, p_1, \ldots, \bar{p}_2$ are not on one conic. So, there exists a standard or special quintic transformation $\theta_3 \in \mathcal{J}_\circ$ with base-points $p_1, \ldots, \bar{p}_2, p_5, \bar{p}_5$. The map $\theta_4 := \theta_2\theta_1(\theta_3)^{-1} \in \mathcal{J}_\circ$ is a standard or special quintic transformation. In fact, its inverse has base-points $p_1, \ldots, \bar{p}_2, (\theta_2)_\bullet(p_3), (\theta_2)_\bullet(\bar{p}_3)$. We have by construction $\theta_2\theta_1 = \theta_4\theta_3$. The equalities $\varphi_\circ(\theta_1) = \varphi_\circ(\theta_4)$ and $\varphi_\circ(\theta_2) = \varphi_\circ(\theta_3)$ follow from the construction and Remark 3.11 (7). $\square$

Lemma 5.12. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard or special quintic transformations (both can be either) such that $\varphi_0(\theta_1) = \varphi_0(\theta_2)$. Then $\theta_1(\theta_2)^{-1} \in \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.*

Proof. Let $S(\theta_1) = \{(p_3, \bar{p}_3)\}$ and $S(\theta_2) = \{(p_4, \bar{p}_4)\}$. The assumption $\varphi_0(\theta_1) = \varphi_0(\theta_2)$ implies that there exists some $\lambda \in \mathbb{R}_{>0}$ such that $\pi_\circ(C_{p_3}) = \lambda\pi_\circ(C_{p_4})$ or $\pi_\circ(C_{p_3}) = \lambda\pi_\circ(C_{\bar{p}_4})$ in $\mathbb{P}^1$. By Lemma 3.7 there exist $\tau_1 \in \mathcal{J}_\circ$ of degree 2 such that $\pi_\circ(\tau_1(C_{p_3})) = \pi_\circ(C_{p_4})$ (resp. $\pi_\circ(C_{\bar{p}_4})$), i.e. $\tau_1(C_{p_3}) = C_{p_4}$ (resp. $C_{\bar{p}_4}$). Let r be the real base-points of τ. Since C_r is a real conic, it is not contracted by θ_1 and hence $(\theta_1)_\bullet(r) = \theta_1(r)$ is a proper point of $\mathbb{P}^2$ and a base-point of $(\theta_1\tau_1)^{-1}$. Let p_{j_i} be the image by θ_1 of the contracted conic not passing through p_i. The map $\theta_1\tau_1$ is of degree 6 and $p_{j_i}, \bar{p}_{j_i}$ are base-points of $(\theta_1\tau_1)^{-1}$ of multiplicity 3. By Lemma 3.3 there exists $\tau_2 \in \mathcal{J}_\circ$ of degree 2 with base-points $\theta(r), p_{j_i}, \bar{p}_{j_i}$. The map $\tau_2\theta_1\tau_1 \in \mathcal{J}_\circ$ is a standard or special quintic transformation contracting the conics $C_{p_4}, C_{\bar{p}_4}$. Hence, by Lemma 5.6, Remark 5.5 and Lemma 5.7, there exist $\nu_1, \ldots, \nu_{2m} \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \nu_{2m}\cdots\nu_{m+1}(\tau_2\theta_1\tau_n^{-1})\nu_m\cdots\nu_1$. Then

$$\theta_1(\theta_2)^{-1} = \big(\theta_1(\nu_m\cdots\nu_1)^{-1}(\tau_1)\theta_1^{-1}\big)\big(\tau_2)^{-1}(\nu_{2m}\cdots\nu_{m+1})^{-1}.$$

By Lemma 5.8, all quadratic elements of $\mathcal{J}_\circ$ belong to $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$, so $\theta_1(\theta_2)^{-1}$ is contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. $\square$

Proposition 5.13. *Let $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ be the surjective group homomorphism defined in Theorem 4.3. Then*

$$\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$$

Proof. By definition of φ (see Proposition 4.3), $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ is contained in $\ker(\varphi)$, hence $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle \subset \ker(\varphi)$. Lets prove the other inclusion. Consider the commutative diagram from Proposition 4.3:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \arrow[dl, "\varphi"] \\
\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} &
\end{tikzcd}$$

It follows that $\ker(\varphi) = \pi(\ker(\Phi))$, which is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_\circ$ and $\ker(\varphi_\circ)$. Moreover, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathcal{J}_*$ are contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ (Lemma 5.9), thus it suffices to prove that $\ker(\varphi_0)$ is contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.

By Lemma 3.6, every $f \in \ker(\varphi_\circ)$ is the composition of linear, quadratic and standard quintic elements of $\mathcal{J}_\circ$. Note that a quadratic or quintic element composed with a linear element is still a quadratic or standard quintic element respectively, so we can assume that f decomposes into quadratic and standard quintic elements. For every $\delta \in \mathbb{R}$ the number of standard quintic elements in the decomposition of f with image e_δ is even. According to Lemma 5.10 and Lemma 5.11, we can write

f as a composition of quadratic, and standard and special quintic transformations, such that for each $\delta \in \mathbb{R}$, all the standard and special quintic transformations with image e_δ are next to each other. In particular, for any δ the number of standard and special quintic transformations next to each other that are sent onto e_δ is even. It follows from Lemma 5.12, Lemma 5.8 and $\varphi_0(\theta) = \varphi_0(\theta^{-1})$ (Remark 3.11 (6)) that $f \in \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. □

Corollary 5.14. *We have*

$$\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle = \ker(\varphi) = \big[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\big]$$

Proof. The first equality is Proposition 5.13. The normal subgroup $\big[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\big]$ contains non-trivial linear elements, and since $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ is a simple group, $\big[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\big]$ contains $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and therefore also $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. Thus, the Abelianisation homomorphism factors through φ. As φ is a homomorphism onto an Abelian group it implies that φ *is* the Abelianisation homomorphism. □

Corollary 5.15 (Corollary 1.3)**.** *The sequence of iterated commutated subgroups of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is stationary. More specifically: Let* $H := [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$*. Then* $[H, H] = H$*.*

Proof. Since $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset H$, the group $[H, H]$ contains non-trivial elements of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. But $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ is simple, hence $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset [H, H]$. By Corollary 5.14, we have

$$H = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle \subset [H, H].$$

□

Theorem 5.16 (Theorem 1.2)**.** *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is not perfect: its Abelianisation is isomorphic to*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)] \simeq \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the commutator subgroup of $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ *is the normal subgroup generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$*, and contains all elements of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *of degree* ≤ 4*.*

Proof. Follows from Proposition 4.3, Proposition 5.13 and Corollary 5.14. □

Remark 5.17. The kernel of φ is the normal subgroup N generated by all squares in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$: On one hand, for any group G, its commutator subgroup $[G, G]$ is contained in the normal subgroup of G generated by all squares. On the other hand, since $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is Abelian and all its elements are of order 2, the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by the squares is contained in $\ker(\varphi)$. The claim now follows from $\ker(\varphi) = [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ (Corollary 5.14).

Remark 5.18. Endowed with the Zariski topology or the Euclidean topology (see [BF2013]), the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ does not contain any non-trivial proper closed normal subgroups and $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ is dense in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ [BZ2015]. In particular, the quotient topology on $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is the trivial topology.

6. Presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generating sets and relations

This section is devoted to the rather technical proof of Proposition 2.9. We remind of the notation $p_1 := [1 : i : 0], p_2 := [0 : 1 : i]$.

Recall that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ (Corollary 2.6).
Consider F_S, the free group generated by the set

$$S = \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ.$$

There is a natural word map $w\colon S \to F_S$, sending an element to its corresponding word.

Remark 6.1. Let $\mathcal{G}$ as in Definition 2.7. There exists a natural surjective homomorphism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By abuse of notation, we also denote by

$$w\colon \mathrm{Aut}_{\mathbb{R}} \cup \mathcal{J}_* \cup \mathcal{J}_\circ \to \mathcal{G}$$

the composition of $S \to F_S$ with the canonical projection $F_S \to \mathcal{G}$.

Remark 6.2. In the proof that $\mathcal{G} \simeq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (Proposition 2.9) the relations given in the definition of $\mathcal{G}$ (list in Definition 2.7) mostly turn up in the form of the following examples:

(1) Let $\theta \in \mathcal{J}_\circ$ be a standard quintic transformation (see Definition 2.2). Call its base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, p_3, \bar{p}_3$, and the base-points of its inverse $p_1, \bar{p}_1, p_2, \bar{p}_2, p_4, \bar{p}_4$ where p_3, p_4 are non-real proper points of $\mathbb{P}^2$. There exist $i, j \in \{1,2\}$ such that θ sends the pencil of conics passing through $p_i, \bar{p}_i, p_3, \bar{p}_3$ onto the pencil of conics passing through $p_j, \bar{p}_j, p_4, \bar{p}_4$. Let $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that α_1 sends the set $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ onto $\{p_i, \bar{p}_i, p_3, \bar{p}_3\}$, and α_2 sends the set $\{p_j, \bar{p}_j, p_4, \bar{p}_4\}$ onto the set $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$, $i \in \{1,2\}$. Then $\alpha_2\theta\alpha_1 \in \mathcal{J}_\circ$ is a standard quintic transformation. The relation $w(\alpha_2)w(\theta)w(\alpha_1) = w(\alpha_2\theta\alpha_1)$ holds in $\mathcal{G}$ (Definition 2.7 (1)).

(2) Let $\tau \in \mathcal{J}_\circ$ be of degree 2 or 3. Let r be the real base-point of τ and s the real base-point of τ^{-1}. Observe that τ sends the pencil of lines through r onto the pencil of lines through s. There exist $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $(\alpha_1)^{-1}(r) = [1:0:0] = \alpha_2(s)$. Then $\alpha_2\tau\alpha_1$ is an element of $\mathcal{J}_*$ and the relation $w(\alpha_2)w(\tau)w(\alpha_1) = w(\alpha_2\tau\alpha_1)$ holds in $\mathcal{G}$ (Definition 2.7 (2)) .

(3) Let $\tau_1, \tau_2 \in \mathcal{J}_\circ$ of degree 2 with base-points $p_i, \bar{p}_i, r$ and $p_j, \bar{p}_j, s$ respectively, and $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha(p_i) = p_j$ and $\alpha(r) = s$. Then $\tau_2\alpha(\tau_1)^{-1}$ is linear. The relation $w(\tau_2)w(\alpha)w((\tau_1)^{-1}) = w(\tau_2\alpha(\tau_1)^{-1})$ holds in $\mathcal{G}$ ((Definition 2.7 (2)).

(4) Let $\tau_1, \tau_2 \in \mathcal{J}_*$ be of degree 2 with base-points $p := [1:0:0], r_1, r_2$ and p, s_1, s_2 respectively, and $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ with $\alpha(r_i) = s_i$ but $\alpha(p) \neq p$ (i.e. $\alpha \notin \mathcal{J}_*$). Suppose that the base-points of $(\tau_1)^{-1}, (\tau_2)^{-1}$ are p, r_1', r_2' and p, s_1', s_2' respectively. Then $\tau_3 := \tau_2\alpha(\tau_1)^{-1}$ is quadratic with base-points $r_1', r_2', \tau_1(\alpha^{-1}(p))$ and its inverse has base-points $s_1', s_2', \tau_2(\alpha(p))$. There exist $\beta_1, \beta_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tilde{\tau}_3 \in \mathcal{J}_*$ of degree 2 such that $\tau_3 = \beta_2\tilde{\tau}_3\beta_3$. The relation $w(\beta_2)w(\tilde{\tau}_3)w(\beta_2) = w(\tau_2)w(\alpha)w(\tau_1)$ holds in $\mathcal{G}$ (Definition 2.7 (3)).

Remark 6.3. Suppose $\theta_1, \theta_2 \in \mathcal{J}_\circ$ are special quintic transformations (see Definition 3.4). If there exist $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_2 = \alpha_2\theta_1\alpha_1$ then α_1, α_2 permute $p_1, \bar{p}_1, p_2, \bar{p}_2$ and are thus contained in $\mathcal{J}_\circ$. So, the relation

$$w(\theta_2) = w(\alpha_2)w(\theta_1)w(\alpha_2) \quad \text{if} \quad \theta_2 = \alpha_2\theta_1\alpha_1 \text{ in } \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$$

is true in $\mathcal{G}$ and even in the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ $\mathcal{J}_*, \mathcal{J}_\circ$ along all the pairwise intersections.

Lookout 6.4. We are going to look at the following three situations: Let $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $f, h \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ belonging to $\mathcal{J}_*$ or being standard quintic transformations.

Suppose that Λ is a real linear system of degree $D := \deg(\Lambda)$ and that

$$\deg(h^{-1}(\Lambda)) \leq D, \quad \deg(fg(\Lambda)) < D$$

We want to find $\theta_1, \ldots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that $w(f)w(g)w(h) = w(\theta_n)\cdots w(\theta_1)$

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & {} \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda)
\end{tikzcd}$$

and such that the successive images of $h^{-1}(\Lambda)$ have degree $< D$ or such that the degree certain elements $\theta_i \in \mathcal{J}_*$ drop (Lemma 6.7, Lemma 6.10, Lemma 6.11).

This will then be the key ingredient to prove that $\mathcal{G}$ is isomorphic to $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (Proposition 6.12).

Lemma 6.5. *Let $f \in \mathcal{J}_* \cup \mathcal{J}_\circ$ be non-linear and Λ be a real linear system of degree $\deg(\Lambda) = D$. Suppose that*

$$\deg(f(\Lambda)) \leq D \quad (\textit{resp. } \deg(f(\Lambda)) < D).$$

(1) *If $f \in \mathcal{J}_*$, there exist two real or a pair of non-real conjugate base-points q_1, q_2 of f such that*

$$m_\Lambda([1:0:0]) + m_\Lambda(q_1) + m_\Lambda(q_2) \geq D \quad (\textit{resp. } > D)$$

(2) *Suppose that $f \in \mathcal{J}_\circ$. Then there exists a base-point $q \notin \{p_1, \ldots, \bar{p}_2\}$ of f of multiplicity 2 such that*

$$(2.1) \quad m_\Lambda(p_1) + m_\Lambda(p_2) + m_\Lambda(q) \geq D \quad (\textit{resp. } > D)$$

or f has a simple base-point r and there exists $i \in \{1,2\}$ such that

$$(2.2) \qquad 2m_\Lambda(p_i) + m_\Lambda(r) \geq D \quad (\textit{resp. } >).$$

Moreover, if inequality (2.1) *does not hold, we can replace* $\geq$ *with* $>$ *in* (2.2) *if* $\deg(f) > 2$.

Proof. Define $d := \deg(f)$ to be the degree of f.

(1) Suppose that $f \in \mathcal{J}_*$. Its characteristic is $(d; d-1, 1^{2d-2})$. Let $r_1, \dots, r_{2d-2}$ be its simple base-points. Since non-real base-points come in pairs, f has an even number N of real base-points. Call $m_i := m_\Lambda(r_i)$ the multiplicity of Λ in r_i and $m_0 = m_\Lambda([1:0:0])$ the one in $[1:0:0]$. We order the base-points such that either r_{2i-1}, r_{2i} are real or $r_{2i} = \bar{r}_{2i-1}$ for $i = 1, \dots, d-1$. Then

$$\begin{aligned} D \geq \deg(f(\Lambda)) =& dD - (d-1)m_0 - \sum_{i=1}^{d-1}(m_{2i-1} + m_{2i}) \\ =& D + \sum_{i=1}^{d-1}(D - m_0 - m_{2i-1} - m_{2i}) \end{aligned}$$

Hence there exists i_0 such that $D \leq m_0 - m_{2i_0-1} - m_{2i_0}$. The claim for ">" follows analogously.

(2) Suppose that $f \in \mathcal{J}_\circ$. By Lemma 3.1, its characteristic is $(d; \frac{d-1}{2}^4, 2^{\frac{d-1}{2}})$ or $(d; \frac{d}{2}^2, \frac{d-2}{2}^2, 2^{\frac{d-2}{2}}, 1)$.

Assume that f has no simple base-point. Call $r_1, \dots, r_{(d-1)/2}$ its base-points of multiplicity 2. Let $m_i := m_\Lambda(p_i)$ be the multiplicity of Λ in p_i, $i = 1, 2$ and $a_i := m_\Lambda(r_i)$ the one in r_i. Then

$$\begin{aligned} D \geq \deg(f(\Lambda)) &= dD - 2m_1 \cdot \frac{d-1}{2} - 2m_2 \cdot \frac{d-1}{2} - 2\sum_{i=1}^{(d-1)/2} a_i \\ &= D + 2\sum_{i=1}^{(d-1)/2}(D - m_1 - m_2 - a_i) \end{aligned}$$

which implies that there exists i_0 such that $0 \geq D - m_1 - m_2 - a_{i_0}$. The claim for ">" follows analogously.

Assume that f has a simple base-point r. Let $r_1, \dots, r_{(d-2)/2}$ be its base-points of multiplicity 2, $a_i := m_\Lambda(r_i)$ the multiplicity of Λ in r_i, and $m_i := m_\Lambda(p_i)$ the one in p_i. Then

$$\begin{aligned} D \geq \deg(f(\Lambda)) &= dD - 2m_j \cdot \frac{d}{2} - 2m_k \cdot \frac{d-2}{2} - (2\sum_{i=1}^{(d-2)/2} a_i) - m_\Lambda(r) \\ &= D + (D - 2m_j - m_\Lambda(r)) + 2\sum_{i=1}^{(d-2)/2}(D - m_j - m_k - a_i) \end{aligned}$$

where $\{j, k\} = \{1, 2\}$. The inequality implies there exist i_0 such that $0 \geq D - m_j - m_k - a_{i_0}$ or that $0 \geq D - 2m_j - m_\Lambda(r)$. The claim for ">" follows analougously.

Suppose that $0 < D - m_j - m_k - a_i$ for all $i = 1, \dots, \frac{d-2}{2}$, i.e. , $1 \leq D - m_j - m_k - a_i$ for all $i = 1, \dots, \frac{d-2}{2}$. We obtain from the calculations above that

$$\begin{aligned} 0 \geq\ & (D - 2m_j - m_\Lambda(r)) + 2\sum_{i=1}^{(d-2)/2}(D - m_j - m_k - a_i) \\ \geq\ & (D - 2m_j - m_\Lambda(r)) + (d-2) \end{aligned}$$

Assume that $d > 2$, i.e. since d is even here, $d \geq 4$. The inequality above implies

$$-2 \geq -(d-2) \geq D - 2m_j - m_\Lambda(r)$$

and so

$$2m_j + m_\Lambda(r) \geq D + 2 > D.$$

$\square$

Notation 6.6. For a pair of non-real points $q, \bar{q} \in \mathbb{P}^2$ or infinitely near, we denote by $\mathfrak{q}$ the set $\{q, \bar{q}\}$.

Lemma 6.7. *Let* $f, h \in \mathcal{J}_\circ$ *be standard or special quintic transformations,* $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *and* Λ *be a real linear system of degree* D. *Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D \quad \textit{and} \quad \deg(fg(\Lambda)) < D.$$

Then there exists $\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\theta_2, \dots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_\circ$ such that

(1) *$w(f)w(g)w(h) = w(\theta_n) \cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following diagram corresponds to a relation in $\mathcal{G}$:*

$$\begin{tikzcd} & \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\ h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & {} \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda) \end{tikzcd}$$

(2) $\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D$ *for $i = 2, \dots, n$.*

Proof. The maps h^{-1} and f have base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, p_3, \bar{p}_3$ and $p_1, \bar{p}_1, p_2, \bar{p}_2, p_4, \bar{p}_4$ respectively, for some non-real points p_3, p_4 that are in $\mathbb{P}^2$ or infinitely near one of $p_1, \dots, \bar{p}_2$. Denote by $m(q) := m_\Lambda(q)$ the multiplicity of Λ at q. According to Lemma 6.5 we have

$$(\mathrm{Ineq}^0) \qquad m(p_1) + m(p_2) + m(p_3) \geq D, \quad m_{g(\Lambda)}(p_1) + m_{g(\Lambda)}(p_2) + m_{g(\Lambda)}(p_4) > D$$

We choose $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3$ with $\{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ such that $m(r_1) \geq m(r_2) \geq m(r_3)$ and such that if r_i is infinitely near r_j, then $j < i$. Similarly, we choose $\mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6$ with $\{\mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\} = g^{-1}(\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_4\})$. In particular, r_1, r_4 are proper points of $\mathbb{P}^2$.

The two inequalities (Ineq^0) translate to

$$(\mathrm{Ineq}^1) \qquad m(r_1) + m(r_2) + m(r_3) \geq D, \quad m(r_4) + m(r_5) + m(r_6) > D$$

We now look at four cases, depending of the number of common base-points of fg and h^{-1}.

<u>Case 0:</u> If h^{-1} and fg have six common base-points, then $\alpha := fgh$ is linear and $w(g)w(h)w(\alpha^{-1}) = w(f^{-1})$ (Definition 2.7 (1)).

<u>Case 1</u>: *Suppose that h^{-1} and fg have exactly four common base-points.* There exists $\alpha_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that α_1 sends the common base-points onto $p_1, \dots, \bar{p}_2$ if all the common points are proper points of $\mathbb{P}^2$, and onto $p_i, \bar{p}_i, p_3, \bar{p}_3$ if $p_3, \bar{p}_3$ are infinitely near $p_i, \bar{p}_i$ (cf. Remark 6.2). There exist $\alpha_2, \alpha_3 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\tilde{f} := \alpha_3 fg(\alpha_1)^{-1} \in \mathcal{J}_\circ$ and $\tilde{h} := \alpha_1 h \alpha_2 \in \mathcal{J}_\circ$ (see Lemma 2.5). The commutative diagram

$$\begin{tikzcd} & & \Lambda \arrow[d, "\alpha_1"] \arrow[drr, dashed, "fg"] & & \\ h^{-1}(\Lambda) \arrow[urr, dashed, "h"] \arrow[r, "\alpha_2"'] & \alpha_2 h^{-1}(\Lambda) \arrow[r, dashed, "\tilde{h}"'] \arrow[rr, dashed, bend right, "\theta_2"'] & \alpha_1(\Lambda) \arrow[r, dashed, "\tilde{f}"'] & \alpha_3 fg(\Lambda) & fg(\Lambda) \arrow[l, "\alpha_3"] \end{tikzcd}$$

is generated by relations in $\mathcal{G}$ (Definition 2.7 (1), Remark 6.2, Remark 6.3). Write $\theta_2 := \tilde{f}\tilde{h} \in \mathcal{J}_\circ$. The claim now follows with $\theta_1 := \alpha_1, \theta_2,\ \theta_3 := (\alpha_3)^{-1}$.

<u>Case 2:</u> *Suppose that the set $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ consists of 6 points $r_{i_1}, \bar{r}_{i_1}, \dots, r_{i_3}, \bar{r}_{i_3}$.*

If at least four of them are proper points of $\mathbb{P}^2$, inequality (Ineq^1) yields

$$2m(r_{i_1}) + 2m(r_{i_2}) + 2m(r_{i_3}) > D,$$

which implies that the six points $r_{i_1}, \bar{r}_{i_1}, \dots, r_{i_3}, \bar{r}_{i_3}$ are not contained in one conic. By this and by the chosen ordering of the points, there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$, $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that those six points are the base-points of $\theta\alpha$. By construction, we have

$$\deg(\theta\alpha(\Lambda)) = 5D - 4m(r_{i_1}) - 4m(r_{i_2}) - 4m(r_{i_3}) < D,$$

and $h^{-1}, \theta\alpha$ and $\theta\alpha, fg$ each have four common base-points. We apply Case 1 to h, α, θ and to $\theta^{-1}, g\alpha^{-1}, f$.

If only two of the six points are proper points of $\mathbb{P}^2$, then the chosen ordering yields $\mathfrak{q} = \mathfrak{r}_1 = \mathfrak{r}_4$ and the points in $\mathfrak{r}_2 \cup \mathfrak{r}_5$ are infinitely near points. Since h, f are standard or special quintic transformations, it follows that r_3, r_6 are both proper points of $\mathbb{P}^2$. We choose $i \in \{3,6\}$, $j \in \{2,5\}$ with $m(r_i) = \max\{m(r_3), m(r_6)\}$ and $m(r_j) = \max\{m(r_2), m(r_5)\}$. We have

$$2m(r_1) + 2m(r_j) + 2m(r_i) \geq 2m(r_4) + 2m(r_5) + 2m(r_6) > D.$$

Thus the six points in $\mathfrak{r}_1 \cup \mathfrak{r}_i \cup \mathfrak{r}_j$ are not contained on one conic and there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$, $\alpha \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that the base-points of $\theta\alpha$ are $\mathfrak{r}_1 \cup \mathfrak{r}_i \cup \mathfrak{r}_j$. Again, the maps $h^{-1}, \theta\alpha$ and $\theta\alpha, fg$ have four common base-points, $\deg(\theta\alpha(\Lambda)) < D$ and we apply Case 1 to h, α, θ and to $\theta^{-1}, g\alpha^{-1}, f$.

Case 3: *Suppose that $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ consists of eight points.* Then $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ and $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ each consist of six points. We have by inequality Ineq[1] and by the chosen ordering that

$$2m(r_1) + 2m(r_2) + 2m(r_4) > 2D, \qquad 2m(r_1) + 2m(r_4) + 2m(r_5) > 2D,$$

so the points in each set $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ and $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ are not on one conic. Moreover, at least four points in each set are proper points of $\mathbb{P}^2$ ($r_1, r_4 \in \mathbb{P}^2$). Therefore, there exist standard or special quintic transformations $\theta_1, \theta_2 \in \mathcal{J}_\circ$, $\alpha_1, \alpha_2 \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\theta_1\alpha_1$ (resp. $\theta_2\alpha_2$) has base-points $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ (resp. $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$). Then $\deg(\theta_i\alpha_i(\Lambda)) < D$ and we can apply Case 1 to $h, \alpha_1^{-1}, \theta_1$ and to $(\theta_1)^{-1}, \alpha_2(\alpha_1)^{-1}, \theta_2$ and to $(\theta_2)^{-1}, g(\alpha_2)^{-1}, f$. ☐

Remark 6.8. Let $f \in \mathcal{J}_*$, and q_1, q_2 two simple base-points of f. Then by Bézout, the points $[1:0:0], q_1, q_2$ are not collinear. (This means that they do not belong, as proper points of $\mathbb{P}^2$ or infinitely near points, to the same line.)

Notation 6.9. In the following diagrams, the points in the brackets are the base-points of the corresponding birational map (arrow). A dashed arrow indicates a birational map, and a drawn out arrow a linear tranformation.

Lemma 6.10. *Let $f, h \in \mathcal{J}_*$, $g \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ and Λ be a real linear system of degree D. Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D, \quad \deg(fg(\Lambda)) < D$$

Then there exist $\theta_1, \ldots, \theta_n \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

1. *$w(f)w(g)w(h) = w(\theta_n)\cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following commutative diagram corresponds to a relation in $\mathcal{G}$:*

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & {} \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda)
\end{tikzcd}$$

2. *$\theta_1 \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$, $\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D$ for $i = 2, \ldots, n$*
3. *or $\theta_1 \in \mathcal{J}_*$, $\theta_2 \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$, $\deg(\theta_1) = \deg(h) - 1$ and*

$$\deg(\theta_1(\Lambda)) = \deg(\theta_2\theta_1(\Lambda)) \leq D$$
$$deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D, \quad i = 3, \ldots, n.$$

Proof. If $g \in \mathcal{J}_*$ then $w(f)w(g)w(h) = w(fgh)$ in $\mathcal{J}_*$. So, lets assume that $g \notin \mathcal{J}_*$. Let $p := [1:0:0], q := g^{-1}([1:0:0])$. Let $m(q)$ be the multiplicity of Λ in q. By Lemma 6.5 there exists r_1, r_2 base-points of h^{-1} and s_1, s_2 base-points of fg such that

$$(\bigstar) \qquad m(p) + m(r_1) + m(r_2) \geq D, \quad m(q) + m(s_1) + m(s_2) > D$$

and either r_1, r_2 (resp. s_1, s_2) are both real or a pair of non-real conjugate points. We can assume that $m(r_1) \geq m(r_2)$, $m(s_1) \geq m(s_2)$ and that r_1 (resp. s_1) is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p (resp. q) and that r_2 (resp. s_2) is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p (resp. q) or r_1 (resp. s_1).

Note that if $\deg(h^{-1}(\Lambda)) < D$, then by Lemma 6.5 ">" holds in all inequalities. We split the remain of the proof into three Situations, depending on whether or not there exist $\tau_1, \tau_2 \in \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ with base-point p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively.

- Situation 1 - *Assume that there exist $\tau_1, \tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively, and that $\tau_1, \tau_2 g$ have common base-points.*

Observe that $\tau_1 h,\ f(\tau_2)^{-1} \in \mathcal{J}_*$ and $\deg(\tau_1 h) = \deg(h) - 1$, and by inequality ($\bigstar$) that

$$\deg(\tau_1(\Lambda)) = 2D - m(p) - m(r_1) - m(r_2) \leq D,$$
$$deg(\tau_2 g(\Lambda)) = 2D - m(q) - m(s_1) - m(s_2) < D$$

We are going to look at three cases, depending on the common base-points of τ_1, τ_2.

• If τ_1 and $\tau_2 g$ have three common base-points, the map $\tau_2 g(\tau_1)^{-1}$ is linear. The commutative diagram

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] \arrow[d, dashed, "\tau_1"] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed] & \tau_1(\Lambda) \arrow[r] & \tau_2 g(\Lambda) \arrow[r, dashed] & {}
\end{tikzcd}$$

is generated by relations in $\mathcal{G}$ (Definition 2.7 (2)), and the claim follows with $\theta_1 := \mathrm{Id},\ \theta_2 ;= \tau_1 h,\ \theta_3 := \tau_2 g(\tau_1)^{-1},\ \theta_4 := f(\tau_2)^{-1}$.

• If τ_1 and $\tau_2 g$ have exactly two common base-points, the map $\tau_2 g(\tau_1)^{-1}$ is of degree 2 and there exists $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tau_3 \in \mathcal{J}_*$ such that $\tau_2 g(\tau_1)^{-1} = \alpha_2 \tau_3 \alpha_1$.

The situation is summarised in the following commutative diagram

$$\begin{tikzcd}
& \Lambda \arrow[rr, "g"] \arrow[d, dashed, "{[p,r_1,r_2]}"', "\tau_1"] & & g(\Lambda) \arrow[dr, dashed, "{[g(q),g(s_1),g(s_2)]}"', "\tau_2"] \arrow[drr, "f"] & & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau_1(\Lambda) \arrow[r, "\alpha_1"] & \alpha_1\tau_1(\Lambda) \arrow[r, dashed, "\tau_3"] & \tau_3\alpha_1\tau_1(\Lambda) \arrow[r, "\alpha_2"] & \tau_2(\Lambda) \arrow[r, dashed] & fg(\Lambda)
\end{tikzcd}$$

Observe that is generated by relations in $\mathcal{G}$ (Definition 2.7 (3)) and that

$$\deg(\alpha_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D, \quad \deg(\tau_3\alpha_1\tau_1(\Lambda)) = \deg(\tau_2(\Lambda)) < D.$$

The claim follows with $\theta_1 := \tau_1 h$, $\theta_2 := \alpha_1$, $\theta_3 := \tau_3$, $\theta_4 := \alpha_2$, $\theta_5 := f(\tau_2)^{-1}$.

• If τ_1 and $\tau_2 g$ have exactly one common base-point, then $\tau_2 g(\tau_1)^{-1}$ is of degree 3 and there exists $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tau_3 \in \mathcal{J}_*$ of degree 3 such that $\tau_2 g(\tau_1)^{-1} = \alpha_2\tau_3\alpha_1$, which corresponds to a relation in $\mathcal{G}$ (Definition 2.7 (3)). The situation can be visualised with the diagram of the previous case, and here too, $\deg(\alpha_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D$, $\deg(\tau_3\alpha_1\tau_1(\Lambda)) = \deg(\tau_2(\Lambda)) < D$. The claim follows, as above, with $\theta_1 := \tau_1 h$, $\theta_2 := \alpha_1$, $\theta_3 := \tau_3$, $\theta_4 := \alpha_2$, $\theta_5 := f(\tau_2)^{-1}$.

- Situation 2 - *As in Situation 1, we assume that there exist $\tau_1, \tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively. Opposed to Situation 1, we now assume that $\tau_1, \tau_2 g$ have no common base-points.*

We put $\theta_1 := \tau_1 h, \theta_n := f(\tau_2)^{-1}$ and construct $\theta_2, \ldots, \theta_{n-1}$ as follows in the below three cases, which depend on the r_i's and s_i's begin real point or non-real points:

• If r_1, r_2, s_1, s_2 are real points, let $\{a_1, a_2, a_3\} = \{p, r_1, r_2\}$ and $\{b_1, b_2, b_3\} = \{g(q), g(s_1), g(s_2)\}$ such that $m(a_i) \leq m(a_{i+1})$ and $m(b_i) \leq m(b_{i+1})$, $i = 1, 2, 3$, and if a_i (resp. b_i) is infinitely near a_j (resp. b_j) then $j > i$. From inequalities ($\bigstar$), we obtain

$$m(a_1) + m(a_2) + m(b_1) > D, \quad m(a_1) + m(b_1) + m(b_2) > D.$$

By them and the chosen ordering, there exists $\tau_3, \tau_4 \in \mathcal{J}_*$ of degree 2, $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\tau_3\alpha_1,\ \tau_4\alpha_2$ have base-points a_1, a_2, b_1 and a_1, b_1, b_2 respectively. The situation is summarised in the following commutative diagram

$$\begin{tikzcd}
& & \Lambda \arrow[r] \arrow[dl, dashed, "{[a_1,a_2,a_3]}", "\tau_1"'] \arrow[d, dashed, "{[a_1,a_2,b_1]}", "\tau_3\alpha_1"'] & g(\Lambda) \arrow[d, dashed, "{[a_1,b_1,b_2]}"', "\tau_4\alpha_2"] \arrow[dr, dashed, "{[b_1,b_2,b_3]}", "\tau_2"'] \arrow[drr, dashed, bend left, "f"] & & \\
h^{-1}(\Lambda) \arrow[urr, dashed, bend left, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau_1(\Lambda) & \tau_3\alpha_1(\Lambda) & \tau_4\alpha_2 g(\Lambda) & \tau_2 g(\Lambda) \arrow[r, dashed] & fg(\Lambda)
\end{tikzcd}$$

By construction of τ_3, τ_4, we have

$$\deg(\tau_3\alpha_1(\Lambda)) = 2D - m(a_1) - m(a_2) - m(b_1) < D,$$
$$\deg(\tau_4\alpha_2 g(\Lambda)) = 2D - m(a_1) - m(b_1) - m(b_2) < D$$

The maps $\tau_1, \tau_3\alpha_1$, the maps $\tau_3\alpha_1, \tau_4\alpha_2$ and the maps $\tau_4\alpha_2, \tau_2$ each have two common base-points, and we proceed with each pair as in Situation 1 to obtain $\theta_2, \dots, \theta_{n-1}$.

- Assume that $r_2 = \bar{r}_1$ and s_1, s_2 are real points. If $m(q) \geq m(p)$, then

$$m(q) + 2m(r_1) > D$$

hence $q, r_1, \bar{r}_1$ are not collinear and there exists $\tau_3 \in \mathcal{J}_*$ of degree 2 with base-points $g(q), g(r_1), g(r_2)$.

If $m(q) < m(p)$, then

$$m(p) + m(q) + m(s_1) > m(q) + m(s_1) + m(s_2) > D$$

hence there exists $\tau_4 \in \mathcal{J}_*$ of degree 2 with base-points p, q, s_1. Note that $\tau_2(\tau_3)^{-1}, \tau_4(\tau_1)^{-1} \in \mathcal{J}_*$. The situation is summarised in the following commutative diagrams.

$$\begin{tikzcd}
\Lambda \arrow[rr, "g"] \arrow[d, dashed, "\tau_1"] & & g(\Lambda) \arrow[d, dashed, "{[g(q),g(r_1),g(r_2)]}"', "\tau_3"] \arrow[dr, dashed, "{[g(q),g(s_1),g(s_2)]}", "\tau_2"'] & \\
\tau_1(\Lambda) & & \tau_3(\Lambda) \arrow[r, dashed] & \tau_2 g(\Lambda),
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Lambda \arrow[r, "g"] \arrow[dl, dashed, "{[p,r_1,r_2]}"', "\tau_1"] \arrow[d, dashed, "{[p,q,s_1]}", "\tau_4"'] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \\
\tau_1(\Lambda) \arrow[r, dashed] & \tau_4(\Lambda) & \tau_2 g(\Lambda)
\end{tikzcd}$$

By construction of τ_3, τ_4, we have

$$\deg(\tau_3 g(\Lambda)) < D, \quad \deg(\tau_4(\Lambda)) < D$$

The maps $\tau_1, \tau_3 g$, the maps $\tau_4, \tau_2 g$ are of degree 2 with one common base-point and we obtain $\theta_2, \dots, \theta_{n-1}$ as in Situation 1.

- If $r_2 = \bar{r}_1$ and $s_2 = \bar{s}_1$, then $r_1, \bar{r}_2, s_1, \bar{s}_1$ are proper points of $\mathbb{P}^2$. Moreover, no three collinear: Else, all four would be on one line and so $2m(r_1) + 2m(s_1) \leq D$. But then the inequality (obtained from inequalities ($\bigstar$))

$$\text{(Ineq}^2\text{)} \qquad (m(p) + 2m(r_1)) + (m(q) + 2m(s_1)) > 2D$$

would imply $m(p)+m(q) > D$, which is impossible by Bézout. Since no three are collinear, there exists $\alpha, \beta, \gamma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha(r_1) = p_1, \alpha(s_1) = p_2$ and $\tilde{\tau}_1 := \beta\tau_1\alpha^{-1} \in \mathcal{J}_\circ$, $\tilde{\tau}_2 := \gamma\tau_2 g\alpha^{-1} \in \mathcal{J}_\circ$ (see Remark 6.2). These correspond to relations in $\mathcal{G}$ (Definition 2.7 (2)).

$$\begin{tikzcd}
 & \Lambda \arrow[r, "g"] \arrow[dl, "\tau_1"'] \arrow[d, "\alpha"] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \\
\tau_1(\Lambda) \arrow[dr, "\beta"] & \alpha(\Lambda) \arrow[d, dashed, "\tilde{\tau}_1"] \arrow[dr, dashed, "\tilde{\tau}_2"] & \tau_2(\Lambda) \arrow[d, "\gamma"] \\
 & \beta\tau_1(\Lambda) \arrow[r, dashed] & \gamma\tau_2 g(\Lambda)
\end{tikzcd}$$

Note that $\tilde{\tau}_2(\tilde{\tau}_1)^{-1} \in \mathcal{J}_\circ$ and we get from the inequalities at the very beginning of the proof that

$$\deg(\beta_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D, \quad \deg(\gamma\tau_2 g(\Lambda)) = \deg(\tau_2 g(\Lambda)) < D.$$

The claim follows with $\theta_2 := \beta, \theta_3 := \tilde{\tau}_2(\tilde{\tau}_1)^{-1}, \theta_3 = \theta_{n-1} = \gamma^{-1}$.

- Situation 3 - *Assume that there exists no $\tau_1 \in \mathcal{J}_*$ or no $\tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively.*

We essentially look at two cases, depending on who of τ_1, τ_2 exists:

- Assume that neither τ_1 nor τ_2 exists. Since p, r_1, r_2 (resp. q, s_1, s_2) are not collinear by Lemma 6.8, it follows that r_1, r_2 are both proximate to p and s_1, s_2 are both proximate to q [AC2002, §2]. Then $m(p) \geq m(r_1) + m(r_2)$, and from Inequalities ($\bigstar$) we obtain $2m(p) \geq m(p) + m(r_1) + m(r_2) \geq D$. Similarly we get $2m(q) > D$. But then $m(p) \geq \frac{D}{2}$ and $m(q) > \frac{D}{2}$, which is impossible by Bézout. So, this case does not appear.

• Assume that τ_1 exists, but τ_2 does not. As above, it follows that s_1, s_2 are both proximate to q and hence $m(q) > \frac{D}{2}$. In particular, by Bézout,

$$m(q) > m(s_1), m(s_2), m(p), m(r_1), m(r_2).$$

Furthermore, $\tau_1 h \in \mathcal{J}_*$ and (from Inequalities ($\bigstar$))

$$\deg(\tau_1 h) = \deg(h) - 1, \quad \deg(\tau_1(\Lambda)) = 2D - m(p) - m(r_1) - m(r_2) \leq D.$$

We define $\theta_1 := \tau_1 h$ and construct $\theta_2, \ldots, \theta_n$.

If r_1, r_2 are real, let $\{t_1, t_2, t_3\} = \{p, r_1, r_2\}$ such that $m(t_i) \geq m(t_{i+1})$ and such that if t_i is infinitely near t_j then $i > j$. By the chosen ordering, we have

$$m(t_1) + m(t_2) + m(q) \geq \frac{2D}{3} + \frac{D}{2} > D.$$

Moreover, t_1, t_2 are proper points of $\mathbb{P}^2$ or t_2 is in the first neighbourhood of t_1, hence there exist $\tau_3 \in \mathcal{J}_*$ with base-points $[1:0:0] = g(q), g(t_1), g(t_2)$.

If $r_2 = \bar{r}_1$, then $r_1, \bar{r}_2$ are proper points of $\mathbb{P}^2$ (they are base-points of τ_1). We have from inequalities ($\bigstar$) and $m(q) > m(p)$ and that

$$m(q) + 2m(r_1) > m(p) + 2m(r_1) \geq D.$$

Thus there exists $\tau_4 \in \mathcal{J}_*$ with base-points $[1:0:0] = g(q), g(r_1), g(\bar{r}_2)$.

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] \arrow[d, dashed, "{[p,r_1,r_2]}", "\tau_1"'] & g(\Lambda) \arrow[d, dashed, "{[g(q),g(t_1),g(t_2)]/}"', "{\tau_3/}"] \arrow[d, dashed, "{[g(q),g(r_1),g(\bar{r}_2)]}"', "\tau_4"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1", "\deg(h)-1"'] & \tau_1(\Lambda) & \tau_3 g(\Lambda)/\tau_4 g(\Lambda) \arrow[r, dashed, "\theta_7"] & fg(\Lambda)
\end{tikzcd}$$

The maps $f(\tau_3)^{-1}$ and $f(\tau_4)^{-1}$ are contained in $\mathcal{J}_*$ and

$$\deg(\tau_3 g(\Lambda)) = 2D - m(q) - m(t_1) - m(t_2) < D,$$
$$\deg(\tau_4 g(\Lambda)) = 2D - m(q) - 2m(r_1) < D$$

Define $\theta_7 := f(\tau_3)^{-1}$ (resp. $= f(\tau_4)^{-1}$). We obtain $\theta_2, \ldots, \theta_6$ by applying Situation 1 to $\tau_1, \tau_3 g$ (resp. $\tau_1, \tau_4 g$).

• The case where τ_1 does not exist and τ_2 exists is treated similarly. □

Lemma 6.11. *Let $f \in \mathcal{J}_\circ$ be a standard or special quintic transformation, $h \in \mathcal{J}_*$, $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and Λ be a real linear system of degree D. Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D, \quad \deg(fg(\Lambda)) < D$$

Then there exist $\theta_1, \ldots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

(1) *$w(f)w(g)w(h) = w(\theta_n) \cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following commutative diagram corresponds to a relation in $\mathcal{G}$:*

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & \cdots \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda)
\end{tikzcd}$$

(2) *$\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D$ for $i = 2, \ldots, n$*
or $\theta_1 \in \mathcal{J}_*$, $\theta_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_1) = \deg(h) - 1$ and

$$\deg(\theta_1(\Lambda)) = \deg(\theta_2\theta_1(\Lambda)) \leq D,$$
$$\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D, \quad i = 3, \ldots, n.$$

(3) *If $h \in \mathcal{J}_\circ$ is a standard or special quintic transformation and $f \in \mathcal{J}_*$, the same statements holds with*

$$\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \ \deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D, \ i = 2, \ldots, n$$

If $\deg(h^{-1}(\Lambda)) < D$, then "$<$" holds everywhere.

Proof. We only look at the situation, where $f \in \mathcal{J}_\circ$, $h \in \mathcal{J}_*$, since for $f \in \mathcal{J}_*$, $h \in \mathcal{J}_\circ$ the proof works similarly.

Let $p := [1:0:0]$, and define $m(q) := m_\Lambda(q)$ to be the multiplicity of Λ at q. Call $p_1, \dots, \bar{p}_2, p_3, \bar{p}_3$ the base-points of f. By Lemma 6.5 we have

$$m(p_1) + m(p_2) + m(p_3) > D \tag{Ineq^3}$$

By Lemma 6.5 there exist two real or two non-real conjugate base-points r_1, r_2 of h, such that

$$m(p) + m(r_1) + m(r_2) \geq D \tag{Ineq^4}$$

Note that if $\deg(h^{-1}(\Lambda)) < D$, then ">" holds everywhere (Lemma 6.5) and we will have "<" everywhere.

We order r_1, r_2 such that $m(r_1) \geq m(r_2)$ and such that r_1 is a proper point of $\mathbb{P}^2$ or infinitely near p and r_2 is a proper base-point of $\mathbb{P}^2$ or infinitely near p or r_1. Let $\mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_3 = g^{-1}(\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_3)$ such that $m(s_1) \geq m(s_2) \geq m(s_3)$ and if s_i is infinitely near s_j, then $i > j$. In particular, s_1 is a proper point of $\mathbb{P}^2$. We now look at two cases, depending on whether r_1, r_2 are real or not. Inequality (Ineq^3) translates to

$$m(s_1) + m(s_2) + m(s_3) > D \tag{Ineq^5}$$

We look at two cases, depending on whether r_1, r_2 are real or not.

<u>Case 1:</u> Suppose that r_1, r_2 are real points. Let $t \in \{p, r_1, r_2\} \cap \mathbb{P}^2$ such that $m(t) = \max\{m(p), m(r_1), m(r_2)\}$. Then

$$m(t) + 2m(s_1) > D$$

There exists $\tau \in \mathcal{J}_*$ of degree 2, $\alpha \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\theta\alpha$ has base-points $t, s_1, \bar{s}_1$. There exists $\beta_1, \beta_2, \beta_3 \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\tilde{\tau} := \beta_1 g(\tau\alpha)^{-1}\beta_1$, $\tilde{f} := \beta_3 f(\beta_2)^{-1} \in \mathcal{J}_\circ$ (see Remark 6.2). The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
& \Lambda \arrow[rr, "g"] \arrow[d, dashed, "{[q,s_1,\bar{s}_1]}" ', "\tau\alpha"] & & g(\Lambda) \arrow[d, "\beta_2"] \arrow[rrd, dashed, bend left, "f"] & & \\
h^{-1}(\Lambda) \arrow[ru, dashed, "h"] & \tau\alpha(\Lambda) & (\beta_1)^{-1}\tau\alpha(\Lambda) \arrow[l, "\beta_1"] \arrow[r, dashed, "\tilde{\tau}"] \arrow[rr, dashed, bend right] & {} \arrow[r, dashed, "\tilde{f}"] & \beta_3 f g(\Lambda) & f g(\Lambda) \arrow[l, "\beta_3" ']
\end{tikzcd}$$

It is generated by relations in $\mathcal{G}$ (Definition 2.7 (2)). Moreover,

$$\deg((\beta_1)^{-1}\tau\alpha(\Lambda)) = \deg(\tau\alpha(\Lambda)) = 2D - m(q) - 2m(s_1) < D$$

The claim now follows from applying Lemma 6.10 to h, α, τ.

<u>Case 2:</u> Assume that $r_2 = \bar{r}_1$. If $r_1, \bar{r}_1 \in \mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_2$, then in particular, $r_1, \bar{r}_1$ are proper points of $\mathbb{P}^2$, and by Remark 6.8 the points $p, r_1, \bar{r}_1$ are not collinear. So, there exists $\tau \in \mathcal{J}_*$ of degree 2 with base-points $p, r_1, \bar{r}_1$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{J}_\circ$ such that $\tilde{\tau} := \alpha_2\tau\alpha_1 \in \mathcal{J}_\circ$, $\tilde{f} := \alpha_3 f g \alpha_1 \in \mathcal{J}_\circ$. The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
& h(\Lambda) \arrow[rr, "g"] \arrow[dd, dashed, "\tau"] & & g(\Lambda) \arrow[rd, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ru, dashed, "h"] \arrow[rd, dashed, "\deg(h)-1" '] & & (\alpha_1)^{-1}(\Lambda) \arrow[lu, "\alpha_1" '] \arrow[d, dashed, "\tilde{\tau}"] \arrow[rd, dashed, "\tilde{f}"] & & f g(\Lambda) \arrow[ld, "\alpha_3"] \\
& \tau(\Lambda) \arrow[r, "\alpha_2"] & \tilde{\tau}(\alpha_1)^{-1}(\Lambda) \arrow[r, dashed] & \alpha_3 f g(\Lambda) &
\end{tikzcd}$$

It is generated by relations in $\mathcal{G}$ (Definition 2.7 (2)). Note that $\deg(\tau h) = \deg(h) - 1$ and

$$\deg(\tilde{\tau}(\alpha_1)^{-1}(\Lambda)) = \deg(\tau(\Lambda)) \leq D, \quad \deg(\alpha_3 f g h(\Lambda)) = \deg(f g(\Lambda)) < D$$

The claim follows with $\theta_1 := \tau h, \theta_2 := \alpha_2, \theta_3 := \tilde{f}\tilde{\tau}, \theta_4 := (\alpha_3)^{-1}$.

So, lets assume that $r_1, \bar{r}_1 \notin \mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_2$.

• If $m(p) < m(r_1)$, then in particular $r_1, \bar{r}_1$ are proper points of $\mathbb{P}^2$ and there exists $\tau \in \mathcal{J}_*$ with base-points $p, r_1, \bar{r}_1$. Remark that

$$\deg(\tau(\Lambda)) \leq D, \quad \deg(\tau h) = \deg(h) - 1.$$

Furthermore, from inequality (Ineq[5]) and the order of the s_i's we derive the inequality $2m(r_1) + 2m(s_1) + 2m(s_2) > 2D$. Since moreover r_1, s_1 are proper points of $\mathbb{P}^2$, there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$, $\alpha \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\theta\alpha$ has base-points $g(\mathfrak{r}_1 \cup \mathfrak{s}_1 \cup \mathfrak{s}_2)$. Consider the following diagram

$$\begin{tikzcd}
 & \Lambda \arrow[r, "g"] \arrow[d, dashed, "\tau"] & g(\Lambda) \arrow[d, dashed, "\theta\alpha"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau(\Lambda) & \theta\alpha g(\Lambda) & fg(\Lambda)
\end{tikzcd}$$

Note that by construction of θ, we have

$$\deg(\theta\alpha g(\Lambda)) = 5D - 4m(r_1) - 4m(s_1) - 4m(s_2) < D$$

The maps $\tau, \alpha g, \theta$ are in the situation of the Case 1, and θ, α, f satisfy the assumptions of Lemma 6.7, and the claim follows from them.

• If $m(p) \geq m(r_1)$, then $m(p) + 2m(s_1) > D$ and so there exists $\tau \in \mathcal{J}_*$ with base-points $p, s_1, \bar{s}_1$. We proceed as in Case 1 (where r_1, r_2 are real but the map we construct is of the same kind). □

Proposition 6.12. ([Proposition 2.9]) *Let $f_1, \ldots, f_m \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that*

$$f_m \cdots f_1 = \mathrm{Id} \quad \textit{in } \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2).$$

Then

$$w(f_m) \cdots w(f_1) = 1 \quad \textit{in } \mathcal{G}.$$

In particular, the natural surjective homomorphism $\mathcal{G} \to \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ is an isomorphism.

Proof. Let Λ_0 be the linear system of lines in $\mathbb{P}^2$, and define

$$\Lambda_i := (f_i \cdots f_1)(\Lambda_0)$$

It is the linear system of the map $(f_i \cdots f_1)^{-1}$ and of degree $d_i := \deg(f_i \cdots f_1)$. Define

$$D := \max\{d_i \mid i = 1, \ldots, m\}, \; n := \max\{i \mid d_i = D\}, \; k := \sum_{i=1}^{n} (\deg(f_i) - 1)$$

We use induction on the lexicographically ordered pair (D, k).

If $D = 1$, then $f_1, \ldots, f_m$ are linear maps, and thus $w(f_m) \cdots w(f_1) = 1$ holds in $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ (and hence in $\mathcal{G}$). So, lets assume that $D > 1$. Note that by construction $\deg(f_{n+1}) \geq 2$. We may assume that f_n is a linear map - else we can insert Id after f_n, i.e. $w(f_m) \cdots w(f_1) = w(f_m) \cdots w(f_{n+1}) w(\mathrm{Id}) w(f_n) \cdots w(f_1)$, which does not change (D, k).

We now construct maps $\theta_1, \ldots, \theta_N \in \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

$$w(f_{n+1}) w(f_n) w(f_{n-1}) = w(\theta_N) \cdots w(\theta_1)$$

and such that the pair $(\tilde{D}, \tilde{k})$ associated to $f_m \cdots f_{n+1} \theta_N \cdots \theta_1 f_{n-2} \cdots f_1$ is strictly smaller than (D, k).

If $f_{n-1}, f_{n+1} \in \mathcal{J}_*$, we apply Lemma 6.10 to f_{n-1}, f_n, f_{n+1} to decrease (D, k).

If $f_{n-1} \in \mathcal{J}_\circ$ or $f_{n+1} \in \mathcal{J}_\circ$, we have to look at three cases, depending on to which group they belong to. We will only do one case as the other two are done similarly.

Suppose that $f_{n-1} \in \mathcal{J}_\circ$ and $f_{n+1} \in \mathcal{J}_*$. By Lemma 6.5, there exists a base-point q of $(f_{n-1})^{-1}$ of multiplicity 2 such that $m(p_1) + m(p_2) + m(q) \geq D$, or there exists $i \in \{1, 2\}$ such that $2m(p_i) + m(r) \geq D$, where r is the simple base-point of $(f_{n-1})^{-1}$. We can assume that q is either a proper point of $\mathbb{P}^2$ or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$.

• If $m(p_1) + m(p_2) + m(q) \geq D$ for some *non-real* base-point q of $(f_{n-1})^{-1}$ of multiplicity 2, then $p_1, \dots, \bar{p}_2, q, \bar{q}$ are not one one conic (Lemma 3.1). So, there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$ with base-points $p_1, \dots, \bar{p}_2, q, \bar{q}$. Then $\theta f_{n-1} \in \mathcal{J}_\circ$ and

$$(*) \qquad \deg(\theta f_{n-1}) = \deg(f_{n-1}) - 4 < \deg(f_{n-1}), \quad \deg(\theta(\Lambda_{n-1})) \leq D.$$

Applying Lemma 6.11 to $\theta^{-1}, f_n, f_{n+1}$ decreases (D, k).

• If $m(p_1) + m(p_2) + m(q) \geq D$ for some *real* base-point q of f of multiplicity 2, then q is a proper point of $\mathbb{P}^2$. If $\deg(f_{n-1})$ is odd, then by Bézout, q is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, and there exists $\theta_1 \in \mathcal{J}_\circ$ of degree 3 with base-points $q, p_1, \dots, \bar{p}_2$ (Lemma 5.3). If $\deg(f_{n-1})$ is even, let p_i be a base-point of multiplicity $\frac{\deg(f_{n-1})}{2}$. By Bézout, q is not collinear with any two of $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ except maybe $p_{3-i}, \bar{p}_{3-i}$. It follows from Lemma 3.3 that there exists $\theta_2 \in \mathcal{J}_\circ$ of degree 2 with base-points $q, p_i, \bar{p}_i$. Note that for $i = 1, 2$, $\theta_i f_{n+1} \in \mathcal{J}_\circ$ and

$$(**) \qquad \deg(\theta_i f_{n-1}) = \deg(f_{n-1}) - 2 < \deg(f_{n-1}), \quad \deg(\theta(\Lambda_{n-1})) \leq D$$

There exist $\tilde{\theta}_i \in \mathcal{J}_*$ and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_i = \alpha_2 \tilde{\theta}_i \alpha_1$. By Definition 2.7 (2), $w(\theta_i) = w(\alpha_2) w(\tilde{\theta}_i) w(\alpha_1)$ and we can apply Lemma 6.10 to $\tilde{\theta}^{-1}$, $f_n(\alpha_1)^{-1}, f_{n+1}$, which decreases (D, k).

• Suppose that there is no base-point q of multiplicity 2 such that $m(p_1)+m(p_2)+m(q) \geq D$, which means by Lemma 6.5 that

(1) d is even,
(2) $m(r) + 2m(p_i) \geq D$, $i \in \{1, 2\}$,
(3) $2m(p_i) + m(r) > D$ if $\deg(f_{n-1}) > 2$.

If $\deg(f_{n-1}) = 2$, there exist $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \tau \in \mathcal{J}_*$ such that $f_{n-1} = \beta\tau\alpha \in \mathcal{J}_*$. Aplplying Lemma 6.10 to $\tau, f_n\alpha^{-1}, f_{n+1}$ decreases (D, k).

If $\deg(f_{n-1}) > 2$, the point r may not be a proper point of $\mathbb{P}^2$. We denote by s the proper point of $\mathbb{P}^2$ to which r is infinitely near to, if r is not a proper point of $\mathbb{P}^2$, and $s = r$ if r is a proper point of $\mathbb{P}^2$. The above list still holds if we write s instead of r. In particular, $p_i, \bar{p}_i, s$ are not collinear and so there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $s, p_i, \bar{p}_i$ (Lemma 3.3). Then $\tau f_{n-1} \in \mathcal{J}_\circ$ and

$$\deg(\tau(\Lambda_{n-1})) = 2D - m(r) - 2m(p_i) < D.$$

The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
 & \Lambda_{n-1} \arrow[r, "f_n"] \arrow[d, dashed, "\tau"] & \Lambda_n \arrow[dr, dashed, "f_{n+1}"] & \\
\Lambda_{n-1} \arrow[ur, dashed, "f_{n-1}"] \arrow[r, dashed] & \tau(\Lambda_{n-1}) & & {}
\end{tikzcd}$$

There exist $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tilde{\tau} \in \mathcal{J}_*$ of degree 2 such that $\tau = \beta\tilde{\tau}\alpha$. Applying Lemma 6.10 to $\tau, f_n\alpha^{-1}, f_{n+1}$ decreases (D, k). □

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V Punctual transformations

In this chapter, we encounter a specific type of transformation contained in $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$, $n \geq 2$, the *punctual transformations* (see Definition V.1.6). Any birational transformation of $\mathbb{P}^2$ is punctual, and for $n \geq 3$, the punctual transformations of $\mathbb{P}^n$ are geometrically similar to plane Cremona transformations and there exist easy formulae for the degree and multiplicities of compositions with the standard Cremona involution, just like for $n = 2$. The standard Cremona involution σ_n of $\mathbb{P}^n$ is the most prominent example of a punctual transformation.

In [Kan1897], S. Kantor studies birational transformations of $\mathbb{P}^3$ not having any curves of the first species, that is, curves which are the image of a surface in $\mathbb{P}^3$, and claims that any such transformation is contained in the group $G_3(\mathrm{k})$ generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^3)$ and σ_3. The statement is false, a counterexample was given in [BlaHed2014, Proposition 8.1].

Punctual transformations are a specific family of transformations without curves of the first species, and not all transformations having no curve of the first species are punctual [BlaHed2014, Proposition 8.1]. The notion of punctual transformations had first been mentioned in [DolOrt1988, p.93], where they study pseudo-isomorphisms of varieties. They suggest that the following statement is true.

Suggestion ([DolOrt1988, p. 93])**.** *The set of punctual transformations is contained in $G_n(\mathrm{k})$, the group generated by σ_n and $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$.*

However, they mention that, although they believe the statement to be true, they could not find a proof.

In [BlaHed2014, Example 8.3], Blanc and Hedén prove that the set of punctual transformations is not a group. Their counterexample to Kantor's claim is not a punctual transformation, and the Suggestion remains unproven.

In this chapter, we list some general properties of punctual transformations, which might help to prove or disprove the Suggestion. Candidates of punctual transformations not contained in $G_n(\mathrm{k})$ are punctual stellar transformations; stellar transformations were studied in [Pan1999, Pan2000], where they are used to prove that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is not generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ and countably many transformations if $n \geq 3$.

Outline of the chapter: We first revisit some prelimenaries, where we define a family of curves associated to a birational transformation of $\mathbb{P}^n$, $n \geq 3$, and define the punctual transformations and state some basic properties. Then, we compute the formulae for degree and multiplicities of compositions of punctual transformations, and use them to list further properties. In Chapter V.4, we recall the stellar transformations and prove that if a stellar transformation is punctual then it projects onto a punctual transformation.

V.1 Preliminaries revisited

To any $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$, we associate a linear system Λ_f of hypersurfaces, each of its elements passing through the base-locus of f. To use intersection theory in higher dimension, we will further associate to f a family of curves.

Definition V.1.1 (Multiplicity of a curve)**.** Let $c \subset \mathbb{P}^n$ be a curve and $p \in \mathcal{B}(\mathbb{P}^n)$. Let $\eta\colon X \to \mathbb{P}^n$ be a sequence of blow-ups of points such that p corresponds to a proper point of X and denote by $\tilde{c}^\eta \subset X$ the strict transform of c. Let $\eta'\colon Y \to \mathbb{P}^n$ be the blow-up of p and $E_p \subset Y$ its exceptional divisor. We define

$$m_p(c) := \tilde{c}^{\eta\eta'} \cdot E_p.$$

Definition V.1.2 (Total transform of a curve)**.** Let $c \subset \mathbb{P}^n$ be a curve and $\eta\colon X \to \mathbb{P}^n$ the blow-up of $p_1, \dots, p_k \in \mathcal{B}(\mathbb{P}^n)$. Denote by $\tilde{E}_i \subset X$ the strict transform of the exceptional divisor of p_i and $e_i \subset \tilde{E}_i$ a general curve. We define the 1-cycle

$$\bar{c}^\eta := \tilde{c}^\eta + \sum_{i=1}^{k} m_{p_i}(c) e_i \quad \in N_1(X)$$

and call it the *total transform* of c in X.

Lemma V.1.3. *Let $c \subset \mathbb{P}^n$ be a curve and $\eta\colon X \to \mathbb{P}^n$ the blow-up of $p_1, \dots, p_k \in \mathcal{B}(\mathbb{P}^n)$. For any general line $l \subset \mathbb{P}^n$, the 1-cycles $\deg(c)\tilde{l}^\eta$ and $\bar{c}^\eta$ are numerically equivalent.*

Proof. We just have check that their intersections with the generators of $\mathrm{Pic}(X)$ are the same. Denote by $E_i \subset X$ the total transform of the exceptional divisors of p_i and pick a general hyperplane $H \subset \mathbb{P}^n$. Then

$$\mathrm{Pic}(X) = \overline{H}^\eta \oplus E_1\mathbb{Z} \oplus \cdots \oplus E_k\mathbb{Z}.$$

From Lemma I.1.5, we get that $\overline{H}^\eta e_i = 0$, $E_i e_i = -1$ and $E_i e_j = 0$ for $i, j = 1, \dots, k$ and $i \neq j$. Since l and H are general, we obtain

$$\overline{H}^\eta(\deg(c)\tilde{l}^\eta) = \deg(c), \quad E_i\tilde{l}^\eta = 0, \quad i = 1, \dots, k,$$

and, by definition of the multiplicity $m_{p_i}(c)$ and the projection formula,

$$\begin{aligned} \overline{H}^\eta \bar{c}^\eta &= \overline{H}^\eta(\tilde{c}^\eta + \sum_{j=1}^{k} m_{p_j}(c) e_j) = \deg(c) \\ E_i \bar{c}^\eta &= E_i \tilde{c}^\eta + \sum_{j=1}^{k} m_{p_j}(c) E_i e_j = m_{p_i}(c) - m_{p_i}(c) = 0, \quad i = 1, \dots, k. \end{aligned}$$

□

Let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$, and let $\mathcal{L}_f \subset \mathbb{G}(1, n)$ be the open subset of all lines in $\mathbb{P}^n$ not passing though $\mathrm{Base}(f)$ and not contained in any hypersurface contracted by f. Then for any $l \in \mathcal{L}_f$, the curve $f(l)$ is of degree $\deg(f)$ [Pan1999, Proposition 1.1]. If we write

$$f\colon [x_0 : \cdots : x_n] \dashmapsto [f_0 : \cdots : f_n],$$

where $f_0, \dots, f_n \in \mathrm{k}[x_0, \dots, x_n]$ are homogenous of equal degree and without common factors, and parametrise a general line $l \in \mathcal{L}_f$ by a linear map $l\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$, then $f(l)$ is parametrised by

$$[u : v] \dashrightarrow [f_0(l([u : v])) : \dots : f_n(l([u : v]))].$$

Therefore, the line $f(l)$ passes through the base-locus of f^{-1}.

Definition V.1.4 (System of curves associated to a transformation)**.** To $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$, we associate the set

$$\mathcal{C}_f := \cup_{l \in \mathcal{L}_{f^{-1}}} f^{-1}(l)$$

of all pre-images of the lines in $\mathcal{L}_{f^{-1}}$, and call it the *system of curves* of f. We define $\deg(\mathcal{C}_f) := \deg(f^{-1})$. Further, we call $\mathrm{Ind}(\mathcal{C}_f) = \cap_{c \in \mathcal{C}_f} c$ the *indeterminacy points* of $\mathcal{C}_f$.

Note that $\mathcal{C}_f$ is parametrised by an open subset of $\mathbb{G}(1, n)$ and hence carries the structure of an algebraic variety.

Example V.1.5. Lets figure out the system of curves $\mathcal{C}_{\sigma_n}$ of σ_n, $n \geq 3$. Let $l \subset \mathbb{P}^n$ be a general line parametrised by

$$l\colon [u : v] \mapsto [a_0 u + b_0 v : \dots : a_n u + b_n v]$$

Its pre-image $\sigma_n(l)$ is the rational curve of degree n parametrised by

$$c\colon [u : v] \dashrightarrow \left[\prod_{i \neq 0} (a_i u + b_i v) : \dots : \prod_{i \neq n} (a_i u + b_i v) \right].$$

Then $c([b_i : -a_i])$ is the ith coordinate point p_i of $\mathbb{P}^n$, and so any curve in $\mathcal{C}_f$ passes through all coordinate points of $\mathbb{P}^n$. Blowing up p_0, we obtain that the strict transform of $\sigma_n(l)$ is parametrised by

$$[u : v] \mapsto \left(\left[\prod_{i \neq 0} (a_i u + b_i v) : \dots : \prod_{i \neq n} (a_i u + b_i v) \right], \left[\prod_{i \neq 0,1} (a_i u + b_i v) : \dots : \prod_{i \neq 0,n} (a_i u + b_i v) \right] \right).$$

The strict transform of $\sigma_n(l)$ intersects the exceptional divisor in exactly one point, namely the image of $[b_0 : -a_0]$. Changing the parametrisation, we can choose $a_0 \neq 0$ and $b_0 = 0$. Now, we see that the differential of c at $[0 : 1]$ has full rank, hence $m_{p_0}(\sigma_n(l)) = 1$. By symmetry, we obtain $m_{p_i}(\sigma_n(l)) = 1$ for all $i = 0, \dots, n$.

The base-locus of σ_n is the union of the varieties C_{ij} given by $x_i = x_j = 0$ for $i \neq j$, and the intersection of $\sigma_n(l)$ and C_{ij} are the coordinate points contained in C_{ij}.

Definition V.1.6 (Pseudo-isomorphism, punctual transformation)**.**

1. A birational map $g\colon X \dashrightarrow Y$ between smooth projective varieties is called *pseudo-isomorphism* (or *isomorphism in codimension* 1) if there exist dense open subsets $U_g \subset X, U_{g^{-1}} \subset Y$ with $\mathrm{codim}_X(X \setminus U_g) \geq 2$, $\mathrm{codim}_Y(Y \setminus U_{g^{-1}}) \geq 2$ such that $g|_{U_g}\colon U_g \xrightarrow{\simeq} U_{g^{-1}}$ is an isomorphism.

2. We call an element $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ *punctual* if there exist sequences of blow-ups of

points $\pi_i\colon X_i \to \mathbb{P}^n$, $i = 1, 2$ and a pseudo-isomorphism $\hat{f}$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \mathbb{P}^n \arrow[r, dashed, "f"] & \mathbb{P}^n \end{tikzcd}$$

is commutative. We choose π_1, π_2 to be minimal and call the points blown-up by π_1 the *central base-points* of f.

Definition V.1.7 (Strict transforms)**.** For a pseudo-isomorphism $g\colon X \dashrightarrow Y$ and a hypersurface $D \subset Y$, we call the hypersurface $\widetilde{D}^g := \overline{g^{-1}(D \cap U_{g^{-1}})}$ the *strict transform* of D via g.

For a curve $c \subset Y$ not contained in $Y \setminus U_{g^{-1}}$, we call the curve $\tilde{c}^g := \overline{g^{-1}(c \cap U_{g^{-1}})}$ the *strict tranform* of c via g.

Remark V.1.8. For $n \geq 1$, any linear element of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is punctual.

Any element of $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is punctual and $\deg(f) = \deg(f^{-1})$ (see for instance [AC2002, §1.3, 2.1.12]).

For any $n \geq 2$, the standard Cremona involution $\sigma_n \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is punctual [BlaHed2014, Proposition 3.1].

For any $n \geq 3$, the group $G_n(\mathrm{k})$ generated by σ_n and $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ contains non-punctual elements [BlaHed2014, Example 8.3].

Remark V.1.9. Let $g\colon Y_1 \dashrightarrow Y_2$ be a pseudo-isomorphism between two smooth projective varieties of dimension n. Then the pullback $g^*(K_{Y_2})$ of the canonical divisor K_{Y_2} via g is equivalent to the canonical divisor K_{Y_1} of Y_1.

Let $g\colon X \dashrightarrow Y$ be a pseudo-isomorphism, and let $D \subset Y$ be a hypersurface and $c \subset U_{g^{-1}} \subset Y$. Then g^{-1} is an isomorphism around c and therefore preserves the intersection intersection with c. More concretely,

$$\widetilde{D}^f \cdot \tilde{c}^f = D \cdot c, \quad K_Y \cdot c = K_X \cdot \tilde{c}^f$$

Lemma V.1.10. *For $n \geq 2$ and $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ punctual, the base-points of $\mathcal{C}_f$ are exactly the central base-points of f.*

Moreover, for each central base-point $p \in \mathcal{B}(\mathbb{P}^n)$, there exists a positive integer $m \in \mathbb{Z}$ such that a general element of $\mathcal{C}_f$ has multiplicity m in p. We define $m_p(\mathcal{C}_f) := m$.

Furthermore, for any central base-point $p \in \mathcal{B}(\mathbb{P}^n)$ of f that has no infinitely near central base-point, there exists a rational hypersurface $S_p \subset \mathbb{P}^n$ of degree $m_p(\mathcal{C}_f)$ that is contracted onto p by f^{-1}.

Proof. For $n = 2$, this follows from the existence of resolution of birational transformations between smooth projective surfaces (see Chapter I.2). Let $n \geq 3$ and $\pi_i\colon X_i \to \mathbb{P}^n$, $i = 1, 2$, be sequences of blow-ups of points and $\hat{f}\colon X_1 \dashrightarrow X_2$ be a pseudo-isomorphism such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \mathbb{P}^n \arrow[r, dashed, "f"] & \mathbb{P}^n \end{tikzcd}$$

is commutative. The base-locus of $\mathcal{C}_f$ consists of finitely many points because two distinct curves in $\mathbb{P}^n$ can meet at most in finitely many points. Since the family of lines in $\mathbb{P}^n$ has empty base-locus, $\hat{f}$ is an pseudo-isomorphism by assumption and a general element of $\mathcal{C}_f$ does not meet the image of $X_1 \setminus U_{\hat{f}}$, every point in the base-locus of $\mathcal{C}_f$ is blown-up by π_1, i.e. is a central base-point of f.

Let $p \in \mathcal{B}(\mathbb{P}^n)$ be a central base-point of f such that f has no infinitely near base-points infinitely near p, and let $E_p \subset X_1$ be its exceptional divisor. A general element of Λ_f passes through p, and as there are no central base-points of f infinitely near p, the strict transform of a general element of Λ_f intersects E_p. Then, since $\hat{f}$ is an pseudo-isomorphism, $S_p := \widetilde{E_p}^{\hat{f}^{-1}} \subset X_2$ is a hypersurface that intersects the strict transform of all general hyperplanes in $\mathbb{P}^n$. The linear system of hyperplanes in $\mathbb{P}^n$ has empty base-locus, hence S_p is not contracted by π_2 (recall that π_2 contracts divisors onto points only). In particular, $\hat{f}|_{E_p} \colon E_p \dashrightarrow S_p$ is birational. Let $l \subset \mathbb{P}^n$ be a general line. Then $\hat{f}^{-1}$ is an isomorphism around $\tilde{l}^{\pi_2}$, and Lemma I.1.5 and Remark V.1.9 imply that

$$0 \neq S_p \cdot l = \widetilde{S_p}^{\pi_2} \cdot \tilde{l}^{\pi_2} \overset{V.1.9}{=} E_p \cdot \tilde{l}^{\pi_2 \hat{f}} = E_p \cdot (\deg(f^{-1})\tilde{l}^{\pi_1} - \sum m_q(\tilde{l}^{\pi_2 \hat{f}})e_q) \overset{I.1.5}{=} m_p(\tilde{l}^{\pi_2 \hat{f}})$$

where we sum over all central base-points of f and $e_q \subset E_q$ is a general line in the strict transform of the exceptional divisor of q. Figure V.1 visualises the calculation. Then $c := \pi_1(\tilde{l}^{\pi_2 \hat{f}}) = f^{-1}(l) \in \mathcal{C}_f$ and $m_p(c) \neq 0$. A general element of $\mathcal{C}_f$ is the pre-image of a line satisfying the assumptions on l, hence $m_p(c) \neq 0$ for a general $c \in \mathcal{C}_f$. In other words, p is contained in a general elements of $\mathcal{C}_f$. Further, it shows that all the central base-points to which p is infinitely near are contained in the base-locus of $\mathcal{C}_f$, and therefore, all central base-points of f are contained in $\mathrm{Base}(\mathcal{C}_f)$. Moreover, a general element of $\mathcal{C}_f$ has multiplicity $m_p(c)$ in p. Therefore, for all central base-points q of f, a general element of $\mathcal{C}_f$ has multiplicity $m_q(c)$ in q.

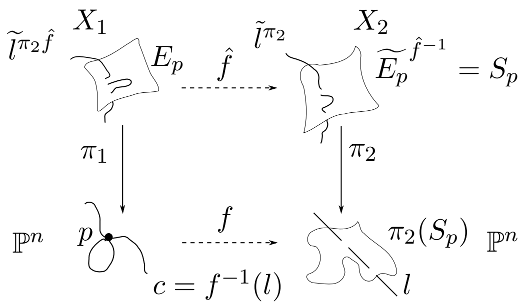

Figure V.1: Calculation of the multiplicity $m_p(\mathcal{C}_f)$.

□

Lemma V.1.11. *Let $n \geq 2$ and $f \in \mathrm{Bir}_k(\mathbb{P}^n)$ be punctual. Let $V \subset \mathbb{P}^n$ be of codimension ≥ 2. Then a general element $c \in \mathcal{C}_f$ intersects V only in the central base-points of f contained in V.*

Proof. For $i = 1, 2$, let $\pi_i \colon X_i \to \mathbb{P}^n$ be blow-up of the central base-points of f and f^{-1} respectively and $\hat{f} \colon X_1 \dashrightarrow X_2$ the induced pseudo-isomorphism of f, i.e. the following diagram is commutative

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \mathbb{P}^n \arrow[r, dashed] & \mathbb{P}^n \end{tikzcd}$$

Observe that the claim is equivalent to $\tilde{c}^{\pi_1}$ and $\widetilde{V}^{\pi_1}$ not intersecting. There exist open dense subsets $U_{\hat{f}} \subset X_1$ and $U_{\hat{f}^{-1}} \subset X_2$ such that $\hat{f}|_{U_{\hat{f}}} : U_{\hat{f}} \xrightarrow{\simeq} U_{\hat{f}^{-1}}$ is an isomorphism. For a general line $l \subset \mathbb{P}^n$, the strict transform $\tilde{l}^{\pi_2} = \bar{l}^{\pi_1} \subset X_2$ is contained in $U_{\hat{f}^{-1}}$, and so a general element $c \in \mathcal{C}_f$ is contained in $U_{\hat{f}}$. If $\widetilde{V}^{\pi_1} \subset X_1 \setminus U_{\hat{f}}$, then we are done. Suppose that $\widetilde{V}^{\pi_1} \cap U_{\hat{f}} \neq \emptyset$. Since $\tilde{c}^{\pi_1} \subset U_{\hat{f}}$, $\hat{f}$ preserves the intersection of $\tilde{c}^{\pi_1}$ with varieties. As $\tilde{c}^{\pi_1}$ is sent by $\hat{f}$ onto a general line, which does not intersect the codimension≥ 2 set $\hat{f}(\widetilde{V}^{\pi_1} \cap U_{\hat{f}})$, the curve $\tilde{c}^{\pi_1}$ does not intersect $\widetilde{V}^{\pi_1} \cap U_{\hat{f}}$. Since $\tilde{c}^{\pi_1} \subset U_{\hat{f}}$, the claim follows. □

V.2 Composition revisited

For $n \geq 3$, there is no general fomula known that computes the degree of compositions, let alone multiplicities of the linear system. For punctual transformations we can find formulae using the intersection form of hyperplanes and curves. For $n = 2$, they are very classical and can for instance be found in [AC2002, §4] (see Lemma I.3.4).

Lemma V.2.1 (Composition). *For $n \geq 2$, let $D \subset \mathbb{P}^n$ be a hypersurface, $g \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be punctual and $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be any transformation. Then*

$$\deg(g(D)) = \deg(g^{-1})\deg(D) - \sum_{p \in \mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_g) m_p(D),$$

where $m_p(D)$ is the multiplicity of D in p, and

$$\deg(fg) = \deg(f)\deg(g) - \sum_{p \in \mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_{g^{-1}}) m_p(\Lambda_f). \tag{V.1}$$

By Lemma V.1.10, $m_p(\mathcal{C}_g) = 0$ if p is not a central base-point of g, thus the sums in the above lemma are finite. Further, for $n = 2$, the formulae translate to the classical ones given in Lemma I.3.4.

Proof. By definition of punctual transformation, there exist sequences π_1, π_2 of blow-ups of points and a pseudo-isomorphism $\hat{g}$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\hat{g}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \mathbb{P}^n \arrow[r, dashed, "g"] & \mathbb{P}^n \end{tikzcd}$$

is commutative. Let $l \subset \mathbb{P}^n$ be a general line. The degree of $g(D)$ is equal to the intersection $g(D) \cdot l$. The blow-up π_2 and the pseudo-isomorphism $\hat{g}^{-1}$ are isomorphisms around

$\tilde{l}^{\pi_2}$, and so Lemma I.1.5 and Remark V.1.9 imply that

$$\begin{aligned}
\deg(g(D)) = g(D)\cdot l &= \overline{g(D)}^{\pi_2}\cdot \overline{l}^{\pi_2} = \widetilde{g(D)}^{\pi_2}\cdot \tilde{l}^{\pi_2} \\
&\overset{V.1.9}{=} \widetilde{g(D)}^{\pi_2\hat{g}}\cdot \tilde{l}^{\hat{g}\pi_2} \\
&= \Big(\deg(D)\overline{H}^{\pi_1} - \sum m_p(D)E_p\Big)\cdot\Big(\deg(g^{-1})\overline{l}^{\pi_1} - \sum m_p(\mathcal{C}_g)e_p\Big) \\
&\overset{I.1.5}{=} \deg(g^{-1})\deg(D) - \sum m_p(D)m_p(\mathcal{C}_g)
\end{aligned}$$

where we sum over all points blown-up by π_1, $H\subset\mathbb{P}^n$ is a general hyperplane, E_p the total transform of the exceptional divisor of a point p and $e_p\subset E_p$ a general line in the strict transform of the exceptional divisor of p. Since $m_p(\mathcal{C}_g)=0$ if p is not a central base-point of f (Lemma V.1.10), we can sum over all points of $\mathcal{B}(\mathbb{P}^n)$ in the last line.

The degree of fg is equal to the degree of $(fg)^{-1}(H)$, where H is a general hyperplane in $\mathbb{P}^n$. Putting $D=f^{-1}(H)$ in the above formula, we obtain

$$\begin{aligned}
\deg(fg) = \deg(g^{-1}(f^{-1}(H))) &= \deg(g)\deg(f^{-1}(H)) - \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_{g^{-1}})m_p(f^{-1}(H)) \\
&= \deg(g)\deg(f) - \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_{g^{-1}})m_p(\Lambda_f).
\end{aligned}$$

□

Lemma V.2.2 (Noether equations). *Let $n\geq 2$ and let $f\in\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be punctual transformation. Then the intersection of general two elements of $\mathcal{C}_f$ and Λ_f have exactly one free intersection point and*

$$\deg(f)\deg(f^{-1})-1 = \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\Lambda_f)m_p(\mathcal{C}_f), \tag{V.2}$$

$$(n+1)(\deg(f^{-1})-1) = \sum_{p\in\mathcal{B}(\mathbb{P}^n)} (n-1)m_p(\mathcal{C}_f). \tag{V.3}$$

If $n=2$, then $\deg(f)=\deg(f^{-1})$ and $\Lambda_f=\mathcal{C}_f$, and the equations translate to the classical Noether equations (see Lemma I.3.3).

Proof. The first equation follows from equation (V.1) in Lemma V.2.1 with $g=f^{-1}$.

Let $H,l\subset\mathbb{P}^n$ respectively be a general hyperplane and line and consider the commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\
\mathbb{P}^n \arrow[r, dashed, "f"] & \mathbb{P}^n
\end{tikzcd}$$

where π_1,π_2 are the blow-ups the central base-points of f,f^{-1} respectively, and $\hat{f}$ a pseudo-isomorphism. Both H and l do not contain any of the central base-points of f^{-1}, and $\hat{f}^{-1}$ is an isomorphism in an open neighbourhood of $\tilde{l}^{\pi_2}$. Using Remark V.1.9 and Lemma I.1.5, we obtain

$$1 = H\cdot l = \widetilde{H}^{\pi_2\hat{f}}\cdot\tilde{l}^{\pi_2\hat{f}} = \widetilde{f^{-1}(H)}^{\pi_1}\cdot\widetilde{f^{-1}(l)}^{\pi_1},$$

which says that any two general elements of Λ_f and $\mathcal{C}_f$ have exactly one free intersection point. It is also another way to obtain the first equation. Remark V.1.9 and Lemma I.1.5 yield

$$\begin{aligned}
-(n+1) = K_{\mathbb{P}^n}\cdot l =&(\pi_2)^*(K_{\mathbb{P}^n})\cdot \bar{l}^{\pi_2} = (K_{X_2} + \sum (n-1)E_p)\cdot \widetilde{l}^{\pi_2} = K_{X_2}\cdot \widetilde{l}^{\pi_2} \\
&= \hat{f}^*(K_{X_2})\cdot \widetilde{l}^{\hat{f}\pi_2} \overset{V.1.9}{=} K_{X_1}\cdot \widetilde{l}^{\hat{f}\pi_2} \\
&= \Big(-(n+1)\overline{H}^{\pi_1} + \sum (n-1)E_p\Big)\cdot\Big(\deg(f^{-1})\bar{l}^{\pi_1} - \sum m_p(\mathcal{C}_f)e_p\Big) \\
&\overset{I.1.5}{=} -(n+1)\deg(f^{-1}) + \sum (n-1)m_p(\mathcal{C}_f)
\end{aligned}$$

where we sum over the points blown-up by π_1. Since $m_p(\mathcal{C}_f)=0$ if p is not a central base-point of f by Lemma V.1.10, we can sum over all points of $\mathcal{B}(\mathbb{P}^n)$ in the last equation. □

The following consequence is a motivation for the name "punctual".

Corollary V.2.3 ([Kan1897, Theorem LIV]). *Let $f\in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be punctual and $S\subset\mathbb{P}^n$ a hypersurface of degree $\deg(S)=\deg(f)$ passing through all central base-points p of f with multiplicity $m_p(\Lambda_f)$. Then $S\in\Lambda_f$. In other words:*

"The linear system of a punctual transformation is determined by its central base-points."

Proof. By Lemmata V.2.1 and V.2.2, the hypersurface $f(S)$ has degree

$$\begin{aligned}
\deg(f(S)) &\overset{V.2.1}{=} \deg(f^{-1})\deg(S) - \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_f)m_p(S) \\
&= \deg(f^{-1})\deg(f) - \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_f)m_p(\Lambda_f) \overset{V.2.2}{=} 1.
\end{aligned}$$

Thus $f(S)$ is a hyperplane, which means that S belongs to Λ_f. □

Remark V.2.4. Example V.1.5 shows that $m_p(\mathcal{C}_f)=1$ if p is a coordinate point of $\mathbb{P}^n$. If we put $f=\sigma_n$ or $g=\sigma_n$ in Lemma V.2.1, we obtain the formulae

$$\deg(f\sigma_n) = n\deg(f) - \sum_{\mathrm{cpb}(\sigma_n)} m_p(\Lambda_f)$$

and

$$\deg(\sigma_n g) = n\deg(g) - \sum_{\mathrm{cpb}(\sigma_n)} (n-1)m_p(\mathcal{C}_{g^{-1}}),$$

where $\mathrm{cpb}(\sigma_n)$ is the set of central base-points of σ_n.

We also want to be able to compute the multiplicities of $f\sigma_n$ for some punctual transformation f. For this, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\hat{\sigma}_n"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\
\mathbb{P}^n \arrow[r, dashed, "\sigma_n"] & \mathbb{P}^n
\end{tikzcd}$$

where π is the blow-up of all coordinate points of $\mathbb{P}^n$ and $\hat{\sigma}_n$ the induced pseudo-isomorphism. Let $H,l\subset\mathbb{P}^n$ be a general hyperplane and line respectively and denote by $E_0,\dots,E_n\subset$

X_1 the exceptional divisors of coordinate points of $\mathbb{P}^n$. Further, denote by $e_i \subset E_i$ a general line in E_i. They yield a basis of the Picard group and the group of curves in X modulo numerical equivalence,

$$\operatorname{Pic}(X) = \overline{H}^{\pi}\mathbb{Z} \oplus E_0\mathbb{Z} \oplus \cdots E_n\mathbb{Z}, \quad N_1(X) = \bar{l}^{\pi}\mathbb{Z} \oplus e_0\mathbb{Z} \oplus \cdots \oplus e_n\mathbb{Z}.$$

The pseudo-isomorphism $\hat{\sigma}_n$ induces a linear involution $\operatorname{Pic}(X) \to \operatorname{Pic}(X)$ by sending divisors onto their strict transforms via $\hat{\sigma}_n$. It also induces a linear involution $(\hat{\sigma}_n)^* \colon N_1(X) \to N_1(X)$ as follows: For $i = 0, \ldots, n$, let H_i be the hyperplane given by $x_i = 0$. The line l is a general line, hence $\bar{l}^{\pi} = \tilde{l}^{\pi}$ is contained in the open set $U_{\hat{\sigma}_n}$ where $\hat{\sigma}_n$ is an isomorphism. We define $(\hat{\sigma}_n)^*(\bar{l}^{\pi}) = \widetilde{l}^{\pi\sigma_n}$, which is precisely the strict transform via π of the curve $\sigma_n(l)$. The map σ_n sends H_i onto the ith coordinate point. The restriction of $\hat{\sigma}_n$ onto $\widetilde{H_i}^{\pi}$ induces the standard Cremona involution

$$\sigma_{n-1} \colon \mathbb{P}^{n-1} \simeq \widetilde{H_i}^{\pi} \xrightarrow{\hat{\sigma}_n|_{E_i}} E_i \simeq \mathbb{P}^{n-1}.$$

As $e_i \subset E_i$ is a general line, we define the image of e_i, just like the image of $\tilde{l}^{\pi}$ above, to be the curve $(\hat{\sigma}_n)^*(e_i) := \sigma_{n-1}(e_i) \subset \widetilde{H_i}^{\pi}$.

These linear involutions which can be used to compute multiplicities when composing with σ_n.

Lemma V.2.5. *Let $n \geq 2$. The $\mathbb{Z}$-module isomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X)$ induced by $\hat{\sigma}_n$ with respect to the basis $(\overline{H}^{\pi}, E_0, \ldots, E_n)$ is given by the involution*

$$M_{\sigma_n,\mathrm{Pic}} := \begin{bmatrix} n & 1 & 1 & 1 & \cdots & 1 \\ -(n-1) & 0 & -1 & -1 & \cdots & -1 \\ -(n-1) & -1 & 0 & -1 & \cdots & -1 \\ -(n-1) & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -1 \\ -(n-1) & -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \in \mathrm{GL}_{n+2}(\mathbb{Z})$$

and $(\hat{\sigma}_n)^* \colon N_1(X) \to N_1(X)$ with respect to the basis $(\bar{l}^{\pi}, e_0, \ldots, e_n)$ is given by the involution

$$M_{\sigma_n,N_1} := \begin{bmatrix} n & (n-1) & (n-1) & (n-1) & \cdots & (n-1) \\ -1 & 0 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \in \mathrm{GL}_{n+2}(\mathbb{Z})$$

Furthermore, for any $c \subset U_{\hat{\sigma}_n} \subset X$, we have $(\hat{\sigma}_n)^*(c) \equiv \widetilde{c}^{\,\hat{\sigma}_n}$.

For $n = 2$, they are presented in [AC2002, §2.4].

Proof. A general hyperplane of $\mathbb{P}^n$ is sent by σ_n onto a surface passing through the $n+1$ coordinate points with multiplicity $(n-1)$. The hyperplane through n of these $n+1$ points is sent onto the exceptional divisor of the $(n+1)$th point [BlaHed2014, Proposition 3.1].

A general line is sent by σ_n onto a curve of degree n passing through the $n+1$ coordinate points with multiplicity 1. By Lemma V.1.3, its class in $N_1(X)$ is the element $n\bar{l}^{\pi} - \sum_{i=0}^{n} e_i$, which is precisely $(\hat{\sigma}_n)^*(\bar{l}^{\pi})$. As explained above, $\hat{\sigma}_n(e_i)$ is a curve in H_i of degree $n-1$ passing through all coordinate points contained in H_i. By Lemma V.1.3, its class in $N_1(X)$ is $(n-1)\bar{l}^{\pi} - \sum_{j \neq i} e_i$, which is precisely $(\hat{\sigma}_n)^*(e_i)$.

For the last claim, we have to show that $\hat{\sigma}_n$ acts linearly one the curves in $U_{\hat{\sigma}_n}$. As $\hat{\sigma}_n(c \cup c') = \hat{\sigma}_n(c) \cup \hat{\sigma}_n(c')$ for any two curves $c, c' \subset U_{\hat{\sigma}_n}$, we just have to show that it sends numerically trivial curves onto numerically trivial curves. Remark V.1.9 explains that $\hat{\sigma}_n$ preserves the intersection of curves in $U_{\hat{\sigma}_n}$ and divisors, and this yields the claim. □

Lemma V.2.6. *For $n \geq 2$, let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be a punctual transformation, $p_0, \dots, p_n$ the central base-points of σ_n and $p_{n+1}, \dots, p_m$ the central base-points of f different from $p_0, \dots, p_n$. Define*

$$\varepsilon := \sum_{i=0}^{n} m_{p_i}(\Lambda_f), \qquad \delta := \sum_{i=0}^{n} m_{p_i}(\mathcal{C}_f).$$

If $f\sigma_n$ is punctual, then

$$\begin{aligned} m_{p_i}(\Lambda_{f\sigma_n}) &= (n-1)\deg(f) - \varepsilon + m_{p_i}(\Lambda_f), && i = 0, \dots, n \\ m_{(\sigma_n)_\bullet(p_i)}(\Lambda_{f\sigma_n}) &= m_{p_i}(\Lambda_f), && i \geq n+1 \end{aligned}$$

and

$$\begin{aligned} m_{p_i}(\mathcal{C}_{f\sigma_n}) &= \deg(f^{-1}) - \delta + m_{p_i}(\mathcal{C}_f), && i = 0, \dots, n \\ m_{(\sigma_n)_\bullet(p_i)}(\mathcal{C}_{f\sigma_n}) &= m_{p_i}(\mathcal{C}_f), && i \geq n+1 \end{aligned}$$

Note that for $n = 2$, the equations translate to the ones given in Lemma II.4.2.

Proof. Since for $n = 2$ the claim is stated in Lemma II.4.2, we can assume that $n \geq 3$. Both σ_n and f are punctual transformations. Denoting by $\pi\colon X \to \mathbb{P}^n$ be the blow-up of $p_0, \dots, p_n$ and for $i = 1, 2$ by $\eta_i\colon Y_i \to \mathbb{P}^n$ the blow-up of the central base-points of f, f^{-1} respectively, and by $\hat{\sigma}_n$ and $\hat{f}$ repsectively the lifts of σ_n and f onto the blow-ups. We can view the base-points of η_1 not blown up by π as points of X. We denote by $\eta_1'\colon Z \to X$ the blow-up of these points. Similarly, we denote by $\pi'\colon Z \to Y_1$ the blow-up of the base-points of π not blown up by η. The situation is summarised in the following commutative diagram.

$$\begin{tikzcd} & & Z \arrow[dl, "\eta_1'"'] \arrow[dr, "\pi'"] & & \\ X \arrow[r, dashed, "\hat{\sigma}_n"'] \arrow[d, "\pi"'] & X \arrow[dr, "\pi"] & & Y_1 \arrow[r, dashed, "\hat{f}"] \arrow[dl, "\eta_1"'] & Y_2 \arrow[d, "\eta_2"] \\ \mathbb{P}^n \arrow[rr, dashed, "\sigma_n"] & & \mathbb{P}^n \arrow[rr, "f"] & & \mathbb{P}^n \end{tikzcd}$$

Let $H, l \subset \mathbb{P}^n$ respectively be a general hyperplane and line. Denote by $E_0, \dots, E_m \subset Z$ be the total transforms of the exceptional divisors of $p_0, \dots, p_m$ under the map $\pi\eta_1'$ and by $e_i \subset E_i$ the strict transform of a general line in the the exceptional divisor of p_i. Then

$$\begin{aligned} \mathrm{Pic}(Z) &= \overline{H}^{\eta_1'\pi}\mathbb{Z} \oplus E_0\mathbb{Z} \oplus \cdots \oplus E_m\mathbb{Z}, \\ N_1(Z) &= \overline{l}^{\eta_1'\pi}\mathbb{Z} \oplus e_0\mathbb{Z} \oplus \cdots \oplus e_m\mathbb{Z}. \\ \mathrm{Pic}(X) &= \overline{H}^{\pi}\mathbb{Z} \oplus (\eta_1')_*(E_0)\mathbb{Z} \oplus \cdots \oplus (\eta_1')_*(E_n)\mathbb{Z}, \\ N_1(X) &= \overline{l}^{\pi}\mathbb{Z} \oplus (\eta_1')_*(e_0)\mathbb{Z} \oplus \cdots \oplus (\eta_1')_*(e_n)\mathbb{Z} \end{aligned}$$

We look at the birational map $\hat{\sigma}_n\eta_1'\colon Z \dashrightarrow X$. The isomorphism $M_{\sigma_n,\mathrm{Pic}}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ extends to a linear map $\mathrm{Pic}(Z) \to \mathrm{Pic}(X)$, which, written with respect to the basis above, is given by the matrix

$$M'_{\sigma_n,\mathrm{Pic}} := \begin{bmatrix} M_{\sigma_n,\mathrm{Pic}} & 0 \end{bmatrix} \in \mathrm{M}_{n+2,m+2}(\mathbb{Z}).$$

The sequence of blow-up of points η'_1 induces a linear projection $(\eta'_1)_*\colon N_1(Z) \to N_1(X)$ by sending $\tilde{l}^{\eta'_1\pi}$ onto $\tilde{l}^{\pi}$ and e_i onto $(\eta'_1)_*(e_i)$. Then $(\hat{\sigma}_n)^*$ extends to a linear map $(\sigma_n)^* \circ (\eta_1)_*\colon N_1(Z) \to N_1(X)$. Written with respect to the basis above, is given by the matrix

$$M'_{\sigma_n,N_1} := \begin{bmatrix} M_{\sigma_n,N_1} & 0 \end{bmatrix} \in \mathrm{M}_{n+2,m+2}(\mathbb{Z}).$$

By Lemma V.2.5, we obtain

$$\begin{aligned} &M'_{\sigma_n,\mathrm{Pic}}\,{}^t\left(\deg(f), -m_{p_0}(\Lambda_f), \dots, -m_{p_m}(\Lambda_f)\right) \\ &= {}^t\left(\deg(f\sigma_n),\ -\left((n-1)\deg(f) - \sum_{i=1}^{n} m_{p_i}(\Lambda_f)\right), \dots, -\left((n-1)\deg(f) - \sum_{i=0}^{n-1} m_{p_i}(\Lambda_f)\right), m_{p_{n+1}}(\Lambda_f), \dots, m_{p_m}(\Lambda_f)\right), \end{aligned}$$

which the class of the divisor $\widetilde{\sigma_n(D)}^{\pi}$ for a general element $D \in \Lambda_f$.

For $i \neq j$, denote by $C_{ij} \subset \mathbb{P}^n$ the codimension 2 subvariety given by $x_i = x_j = 0$. By [BlaHed2014, Proposition 3.1], the union of the $\widetilde{C_{ij}}^{\pi}$ is the base-locus of $\hat{\sigma}_n$, and $\hat{\sigma}_n$ is an isomorphism outside of it. Let $c \in \mathcal{C}_f$ be a general element. By Lemma V.1.11, c only intersects C_{ij} in the central base-points of f. We claim that any of these must be coordinate points, i.e. points blown up by π. Suppose that there exists a central base-point p of f that is contained in C_{ij}. By Lemma V.1.10 there exists a surface $S \subset \mathbb{P}^n$ which is contracted by f^{-1} onto p. Then $(f\sigma_n)$ contracts S onto C_{ij}, which is a contradiction to $f\sigma_n$ being punctual. It follows that c only intersects C_{ij} in points blown up by π, hence $\tilde{c}^{\pi} \subset U_{\hat{\sigma}_n}$. By Lemma V.2.5, we get $\hat{\sigma}_n(\tilde{c}^{\pi}) = (\hat{\sigma}_n)^*(\tilde{c}^{\pi}) = (\hat{\sigma}_n)^*(\eta_1)_*(\tilde{c}^{\pi\eta'_1})$ and obtain

$$\begin{aligned} &M'_{\sigma_n,N_1}\,{}^t\left(\deg(f^{-1}), -m_{p_0}(\mathcal{C}_f), \dots, -m_{p_m}(\mathcal{C}_f)\right) \\ &= {}^t\left(\deg((f\sigma_3)^{-1}),\ -\left(\deg(f^{-1}) - \sum_{i=1}^{n} m_{p_i}(\mathcal{C}_f)\right), \dots, -\left(\deg(f^{-1}) - \sum_{i=0}^{n-1} m_{p_i}(\mathcal{C}_f)\right), m_{p_{n+1}}(\mathcal{C}_f), \dots, m_{p_m}(\mathcal{C}_f)\right), \end{aligned}$$

which is the class of a general element in $\mathcal{C}_{f\sigma_n}$. □

V.3 Properties of punctual transformations

The following lemma shows that if we can decompose a punctual transformation f into linear maps and σ_n such that all successive compositions are punctual, then f has rather nice properties:

Lemma V.3.1. *For $n \geq 3$, let $\alpha_0, \dots, \alpha_m \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ such that for every $i = 1, \dots m$, the transformation $f_i := \alpha_i\sigma_n\alpha_{i-1}\cdots\alpha_1\sigma_n\alpha_0$ is punctual. Then*

1. *$m_p(\Lambda_{f_m}) = (n-1)m_p(\mathcal{C}_{f_m})$ for all central base-points $p \in \mathcal{B}(\mathbb{P}^n)$ of f,*
2. *$\deg(f_m) = \deg(f_m^{-1})$.*

For $n = 2$, the first part of the lemma is redundant, while the second part is classical ([Hud1927, §I.1.3]).

Proof. Let $n \geq 3$. For $i = 1, \dots, m$, we define $g_i := \alpha_m\sigma_n\alpha_{m-1}\cdots\alpha_{m-i}$ and use induction on m and the g_i. Note that the punctual transformation $g_1 = \alpha_m\sigma_n\alpha_{m-1}$ has the desired properties, which yields the claim for $m = 1$, and that $f_m = g_m = g_{m-1}\sigma_n\alpha_0$. The induction hypothesis is

$$m_p(\Lambda_{g_{m-1}}) = (n-1)m_p(\mathcal{C}_{g_{m-1}}), \quad \deg(g_{m-1}) = \deg(g_{m-1}^{-1})$$

for all central base-points p of g_{m-1}. It follows from Lemma V.2.6 that
• If $q = (\sigma_n)_\bullet(p_i),\ i \geq n+1$, then

$$\begin{aligned} m_q(\Lambda_{f_m}) = m_q(\Lambda_{g_{m-1}\sigma_n}) &\overset{V.2.6}{=} m_{p_i}(\Lambda_{g_{m-1}}) \overset{\text{ind.}}{=} (n-1)m_{p_i}(\mathcal{C}_{g_{m-1}}) \\ &\overset{V.2.6}{=} (n-1)m_{p_i}(\mathcal{C}_{g_{m-1}\sigma_n}) = (n-1)m_q(\mathcal{C}_{f_m}). \end{aligned}$$

• If $q = p_i,\ 0 = 1,\dots,n$, then

$$\begin{aligned} m_q(\Lambda_{f_m}) =& m_q(\Lambda_{g_{m-1}\sigma_n}) \overset{V.2.6}{=} (n-1)\deg(g_{m-1}) - \sum_{j\neq i} m_{p_j}(\Lambda_{g_{m-1}}) \\ &\overset{\text{ind.}}{=} (n-1)\left(\deg(g_{m-1}) - \sum_{j\neq i} m_{p_j}(\mathcal{C}_{g_{m-1}})\right) \\ &\overset{\text{ind.}}{=} (n-1)\left(\deg(g_{m-1}^{-1}) - \sum_{j\neq i} m_{p_j}(\mathcal{C}_{g_{m-1}})\right) \\ &\overset{V.2.6}{=} (n-1)m_q(\mathcal{C}_{g_{m-1}\sigma_n}) = (n-1)m_q(\mathcal{C}_{f_m}) \end{aligned}$$

With Remark V.2.4 it follows that

$$\begin{aligned} \deg(f_m) = \deg(g_{m-1}\sigma_n) =& n\deg(g_{m-1}) - \sum_{p\in\mathrm{cpb}(\sigma_n)} m_p(\Lambda_{g_{m-1}}) \\ &\overset{\text{ind.}}{=} n\deg(g_{m-1}^{-1}) - \sum_{p\in\mathrm{cpb}(\sigma_n)} (n-1)m_p(\mathcal{C}_{g_{m-1}}) \\ &= \deg(\sigma_n g_{m-1}^{-1}) = \deg(f_m^{-1}) \end{aligned}$$

where $\mathrm{cpb}(\sigma_n)$ is the set of central base-points of σ_n. □

The following lemmata aim at showing that the non-linear punctual transformation of smallest degree with only proper base-points is in fact the standard Cremona involution.

Remark V.3.2. The generalised Noether-inequalities from Lemma V.2.2 imply that:

• If $n \geq 3$ is even, then $\gcd(n-1, n+1) = 1$ and hence $d = N(n-1)+1$ for some $N \in \mathbb{N}$.

• If $n \geq 3$ is odd, then $\gcd(n-1, n+1) = 2$ and hence $d = N\frac{n-1}{2}+1$ for some $N \in \mathbb{N}$. Note that for $n = 3$, this does not say anything at all.

Lemma V.3.3. *For $n \geq 2$, there are no non-linear punctual transformations of degree $\leq n-1$ in* $\mathrm{Bir}_k(\mathbb{P}^n)$.

Proof. For $n = 2$, the lemma is trivial. Let $n \geq 3$, $f \in \mathrm{Bir}_k(\mathbb{P}^n)$ be punctual and non-linear, and $\deg(f^{-1}) = d$. It suffices to show that $d \geq n$. Suppose that $d = n-k$ for some

$1 \leq k \leq n-2$. Equation (V.2) in Lemma V.2.2 implies

$$(n-1)\sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_f) \overset{V.2.2}{=} (n+1)(d-1) = (n+1)(n-k-1)$$
$$= (n-1)(n-k-1) + 2(n-k-1)$$

As $n-1$ is a factor of $(n-1)(n-k-1)$, it is also a factor or $2(n-k-1)$, which is impossible for $n \geq 4$.

If $n = 3$, then $d = 2$ and equation (V.3) in Lemma V.2.2 implies that

$$2 = \sum_{p\in\mathcal{B}(\mathbb{P}^n)} m_p(\mathcal{C}_f).$$

Thus f has exactly two base-points p_1, p_2, both of multiplicitiy $m_{p_1}(\mathcal{C}_f) = m_{p_2}(\mathcal{C}_f) = 1$. Then equation (V.2) in Lemma V.2.2 states

$$2\deg(f) - 1 = m_{p_1}(\Lambda_f) + m_{p_2}(\Lambda_f).$$

Bézout theorem implies that $m_{p_1}(\Lambda_f) \leq \deg(f) - 1$, which yields

$$2\deg(f) - 1 = m_{p_1}(\Lambda_f) + m_{p_2}(\Lambda_f) \leq \deg(f) - 1 + m_{p_2}(\Lambda_f),$$

which implies $\deg(f) \leq m_{p_2}(\Lambda_f)$. Impossible.

□

Lemma V.3.4. *For $n \geq 2$, any non-linear punctual tranformation in $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ has at least $n+1$ central base-points.*

Proof. Suppose that $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is has degree $d := \deg(f^{-1}) > 1$ and has at most n central base-points, say $p_0, \dots, p_m \in \mathcal{B}(\mathbb{P}^n)$, $m \leq n$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ that contains $p_0, \dots, p_m$. Pick a curve $c \in \mathcal{C}_f$. Then

$$(n-1)d = (n-1)c \cdot H \geq (n-1)\sum_{i=0}^{m} m_{p_i}(\mathcal{C}_f) \overset{V.2.2}{=} (n+1)(d-1),$$

which implies that $2d \leq n+1$ and $\deg(f) \leq \frac{n+1}{2} < n$ for $n \geq 2$. That is a contradiction to Lemma V.3.3, which states that $\deg(f) \geq n$. □

Lemma V.3.5. *Let $n \geq 2$. If a punctual transformation $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ satisfies $\deg(f^{-1}) = n$, then f has exactly $n+1$ central base-points in $\mathcal{B}(\mathbb{P}^n)$ and*

$$m_p(\mathcal{C}_f) = 1 \quad \textit{for all central base-points } p \in \mathcal{B}(\mathbb{P}^n).$$

If furthermore all central base-points are proper points of $\mathbb{P}^n$, then there exist $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ such that $f = \beta\sigma_n\alpha$.

Proof. Let $p_0, \dots, p_m \in \mathcal{B}(\mathbb{P}^n)$ be the central base-points of f and write $\deg(f) = D$ and $\deg(f^{-1}) = d = n$. Recall the equations in Lemma V.2.2:

$$(V.2.2, V.2)\ dD - 1 = \sum_{i=0}^{m} m_{p_i}(\Lambda_f) m_{p_i}(\mathcal{C}_f), \quad (V.2.2, V.3)\ (n+1)(d-1) = \sum_{i=0}^{m} m_{p_i}(\mathcal{C}_f)(n-1).$$

We plug the assumption $d = n$ into equation (V.2.2, V.3) and get

$$(n+1)(n-1) = \sum_{i=0}^{m} m_{p_i}(\mathcal{C}_f)(n-1),$$

which implies

$$\sum_{i=0}^{m} m_{p_i}(\mathcal{C}_f) = n+1. \tag{V.4}$$

As $m_{p_i}(\mathcal{C}_f) \geq 1$ for $i = 0, \dots, m$, the above equation implies $m \leq n$. By Lemma V.3.4, f has at least $n+1$ central base-points, therefore $m = n$. Now, equation (V.4) implies that $m_{p_0}(\mathcal{C}_f) = m_{p_1}(\mathcal{C}_f) = \cdots = m_{p_n}(\mathcal{C}_f) = 1$ and that $p_0, \dots, p_n$ are not contained in one hyperplane (cf. proof of Lemma V.3.4).

Suppose that $p_0, \dots, p_n \in \mathbb{P}^n$. Then equation (V.2.2, V.2) translates to

$$Dn - 1 = \sum_{i=0}^{n} m_{p_i}(\Lambda_f) m_{p_i}(\mathcal{C}_f) = \sum_{i=0}^{n} m_{p_i}(\Lambda_f). \tag{V.5}$$

As $p_0, \dots, p_n$ are not contained in one hyperplane, there exists $\alpha \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ that sends $p_0, \dots, p_n$ onto the $n+1$ coordinate points. The formula for composing with σ_n given in Remark V.2.4 and equation (V.5) imply

$$\deg(f\alpha^{-1}\sigma_n) \overset{V.2.4}{=} nD - \sum_{i=0}^{n} m_{p_i}(\Lambda_f) \overset{(V.5)}{=} 1.$$

In other words, $f\alpha^{-1}\sigma_n \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$. □

Lemma V.3.6. *Let $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ be a punctual transformation with exactly $n+1$ central base-points. Then $\deg(f^{-1}) = n$.*

In particular, if all its central base-points are proper points of $\mathbb{P}^n$, then there exists $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ such that $f = \beta\sigma_n\alpha$.

Proof. Let $p_0, \dots, p_n \in \mathcal{B}(\mathbb{P}^n)$ the central base-points of f and order them such that $m_{p_0}(\mathcal{C}_f) \leq m_{p_1}(\mathcal{C}_f) \leq \cdots \leq m_{p_n}(\mathcal{C}_f)$ and that there is no base-points infinitely near to p_0. Let $d := \deg(f^{-1})$. Then equation V.3 yields

$$(n+1)(d-1) = \sum_{i=0}^{n} m_p(\mathcal{C}_f)(n-1) \geq (n+1)(n-1)m_{p_0}(\mathcal{C}_f)$$

which implies

$$d - 1 \geq (n-1)m_{p_0}(\mathcal{C}_f).$$

Further, the n points $p_1, \dots, p_n$ are contained in a hyperplane H. For any $c \in \Lambda_f$, we get by Lemma V.1.10 that

$$d = c \cdot H \geq \sum_{i=1}^{n} m_{p_i}(\mathcal{C}_f) = \frac{n+1}{n-1}(d-1) - m_{p_0}(\mathcal{C}_f)$$

which implies

$$(n-1)m_{p_0}(\mathcal{C}_f) \geq 2d - (n+1).$$

Together with the above inequality, we obtain

$$d - 1 \geq (n-1)m_{p_0}(\mathcal{C}_f) \geq 2d - (n+1)$$

which yields

$$n \geq d.$$

Lemma V.3.3 implies that $\deg(f^{-1}) = d = n$. Lemma V.3.5 implies that if $p_0, \ldots, p_n \in \mathbb{P}^n$, then $f = \beta\sigma_n\alpha$ for some $\alpha, \beta \in \operatorname{Aut}_{\mathrm{k}}(\mathbb{P}^n)$. □

V.4 Punctual stellar transformations

In this chapter, we attack the question, whether all punctual transformations can be decomposed into linear transformations and σ_n. Although we do not succeed to prove it or give a counterexample, we take a close look at transformations that are not obviously compositions of σ_n and automorphisms.

We now recall the family of stellar transformations in $\operatorname{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ as presented in [Pan1999, Pan2000]; for any field k, $n \geq 3$ and for any curve $\Gamma \subset \mathbb{P}^2$, the family contains an element that contracts a non-rational hypersurface birational to $\Gamma \times \mathbb{P}^{n-2}$, which proves that $\operatorname{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ cannot be generated by $\operatorname{Aut}_{\mathrm{k}}(\mathbb{P}^n)$ and a countable set of elements [Pan1999, Théorème 1].

They are perfect candidates to explore properties of punctual maps far away from the prejudice induced by σ_n and in the end of this chapter, we will attempt to determine when they are punctual.

Definition V.4.1. Let $t_1, \ldots, t_n \in \mathrm{k}[x_1, \ldots, x_n]$ be homogenous polynomials of equal degree $e := \deg(t_i)$ without common factors. They define a rational map

$$t\colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}, \quad [x_1 : \cdots : x_n] \mapstochar\dashrightarrow [t_1 : \cdots : t_n].$$

Pick an integer $d > e$ and homogenous polynomials $g_d, g_{d-1}, h_{d-e}, h_{d-e-1} \in \mathrm{k}[x_1, \ldots, x_n]$ of degree as indexed and define

$$g := g_{d-1}x_0 + g_d, \qquad h := h_{d-e-1}x_0 + h_{d-e},$$

which are homogenous polynomials of degree $\deg(g) = d$ and $\deg(h) = d - e > 0$. This yields a rational map $T_{g,h,t}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$,

$$T_{g,h,t}\colon [x_0 : \cdots : x_n] \mapstochar\dashrightarrow [\frac{g}{h} : t_1 : \cdots : t_n] = [\frac{g_{d-1}x_0 + g_d}{h_{d-e-1}x_0 + h_{d-e}} : t_1 : \cdots : t_n],$$

which is called *stellar*.

Remark V.4.2. The name *stellar* is motivated by the fact that a rational transformation $T_{g,h,t}$ respects the projection $\mathrm{pr}_O\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ centered at $O := [1 : 0 : \cdots : 0]$, i.e. the following diagram is commutative

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed, "T_{g,h,t}"] \arrow[d, dashed, "\mathrm{pr}"'] & \mathbb{P}^n \arrow[d, dashed, "\mathrm{pr}"] \\ \mathbb{P}^{n-1} \arrow[r, dashed, "t"] & \mathbb{P}^{n-1} \end{tikzcd}$$

In other words, if $T_{g,h,t}$ is birational, it sends the bunch of lines through $[1:0:\cdots:0]$ onto the bunch of lines through $[1:0:\cdots:0]$.

Lemma V.4.3 ([Pan1999, Lemm 2]). *Suppose that*

$$\det \begin{pmatrix} g_{d-1} & g_d \\ h_{d-e-1} & h_{d-e} \end{pmatrix} = g_{d-1}h_{d-e} - g_d h_{d-e-1} \neq 0.$$

Then $T_{g,h,t}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *is birational if and only if* $t\colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ *is.*

The proof of Lemma V.4.3 in [Pan1999] is given for algebraically closed fields of characteristic zero but works over any field.

Definition V.4.4. We denote the set of stellar birational transformations in by $\mathbf{St}_{\mathrm{k}}(\mathbb{P}^n) \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$.

Lemma V.4.5 ([Pan2000, Proposition 2.1]). *The set* $\mathbf{St}_{\mathrm{k}}(\mathbb{P}^n) \subset \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ *of birational stellar transformation is a group and is isomorphic to*

$$\mathbf{St}_{\mathrm{k}}(\mathbb{P}^n) \simeq \mathrm{PGL}_2(\mathrm{k}(y_1,\ldots,y_{n-1})) \rtimes \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^{n-1}).$$

The proof of Lemma V.4.5 in [Pan2000] is given for algebraically closed fields of characteristic zero but works over any field.

Definition V.4.6. For homogeneous, rational functions $f_0,\ldots,f_n \in \mathrm{k}(x_0,\ldots,x_n)$ of equal degree, we define the *Jacobian*

$$\mathrm{Jac}(f_0,\ldots,f_n) = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=0}^{n} \in \mathrm{k}(x_0,\ldots,x_n).$$

For a rational transformation

$$f\colon [x_0:\cdots:x_n] \mapstochar\dashrightarrow [f_0(x_0,\ldots,x_n):\cdots:f_n(x_0,\ldots,x_n)]$$

with $f_0,\ldots,f_n \in \mathrm{k}[x_0,\ldots,x_n]$ without common factor, we define the *Jacobian* of f

$$\mathrm{Jac}(f) = \mathrm{Jac}(f_0,\ldots,f_n) \in \mathrm{k}[x_0,\ldots,x_n],$$

which is a homogenous polynomial of degree $\deg(\mathrm{Jac}(f)) = (n+1)(\deg(f)-1)$.

Remark V.4.7. For $n \geq 1$, the zero set of the Jacobian $\mathrm{Jac}(f)$ of an element $f \in \mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^n)$ is the union of hypersurfaces of $\mathbb{P}^n$ contracted by f.

Let $f \in \mathbf{St}(\mathbb{P}^n)$ and $p \in \mathcal{B}(\mathbb{P}^n)$ be a central base-point that has no infinitely near central base-points. By Lemma V.1.10, there exists a hypersurface $S_p \subset \mathbb{P}^n$ that is contracted by f^{-1} onto p. It follows that S_p is the zero set of some irreducible factor of $\mathrm{Jac}(f)$.

Lemma V.4.8 ([BlaHed2014, Lemma 2.3]). *Let* k *be a field of characteristic zero, let* $h \in \mathrm{k}[x_0,\ldots,x_n]$ *be a homogenous polynomial of degree* $d \in \mathbb{N}$, *and let* $t_0,\ldots,t_n \in \mathrm{k}(x_0,\ldots,x_n)$ *homogenous rational functions of degree* $e \in \mathbb{Z}\setminus\{0\}$. *Then*

$$\mathrm{Jac}(ht_0,\ldots,ht_n) = (1 + {}^{d}/_{e})h^{n+1}\mathrm{Jac}(t_0,\ldots,t_n).$$

Lemma V.4.9. *Let* $\mathrm{char}(\mathrm{k})$ *and let* $T_{g,h,t}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *be a stellar map. Then*

$$\mathrm{Jac}(T_{g,h,t}) = \left(1 + \frac{\deg(h)}{e}\right) h^{n-1}\left(g_{d-1}h_{d-e} - g_d h_{d-e-1}\right)\mathrm{Jac}(t)$$

Proof. Write

$$T_{g,h,t}\colon [x_0 : \cdots : x_n] \mapsto [\frac{g}{h} : t_1 : \cdots : t_n].$$

Since $t_1, \ldots, t_n \in \mathrm{k}[x_1, \ldots, x_n]$ are homogenous of equal degree without common factor, we get

$$\mathrm{Jac}(\frac{g}{h}, t_1, \ldots, t_n) = \begin{pmatrix} \partial \frac{g}{h}/\partial x_0 & \partial \frac{g}{h}/\partial x_j \\ 0 & \partial t_i/\partial t_j \end{pmatrix} = \left(\frac{\partial \frac{g}{h}}{\partial x_0}\right)\mathrm{Jac}(t_1, \ldots, t_n) = \left(\frac{\partial \frac{g}{h}}{\partial x_0}\right)\mathrm{Jac}(t).$$

Then Lemma V.4.8 yields, with $e = \deg(t_i)$,

$$\begin{aligned}
\mathrm{Jac}(T_{g,h,t}) &= \mathrm{Jac}(g, ht_1, \ldots, ht_n) \\
&\overset{V.4.8}{=} \left(1 + \frac{\deg(h)}{e}\right) h^{n+1}\mathrm{Jac}(g/h, t_1, \ldots, t_n) \\
&= \left(1 + \frac{\deg(h)}{e}\right) h^{n+1}\left(\frac{\partial \frac{g}{h}}{\partial x_0}\right)\mathrm{Jac}(t) \\
&= \left(1 + \frac{\deg(h)}{e}\right) h^{n+1}\left(\frac{g_{d-1}h_{d-e} - g_d h_{d-e-1}}{h^2}\right)\mathrm{Jac}(t)
\end{aligned}$$

□

Notation V.4.10. For any homogeneous polynomial $f \in \mathrm{k}[x_0, \ldots, x_n] \setminus \mathrm{k}^*$, we denote by $S_f \subset \mathbb{P}^n$ the hypersurface given by the equation $f = 0$.

Remark V.4.11. It follows from Remark V.4.7 and Lemma V.4.9 that $T_{g,h,t}$ contracts the hypersurfaces S_h, $S_{\mathrm{Jac}(t)}$ and $S_{g_{d-1}h_{d-e} - g_d h_{d-e-1}}$ and no other hypersurfaces.

Lemma V.4.12. *Let* $T_{g,h,t} \in \mathbf{St}(\mathbb{P}^n)$ *and* $S \subset \mathbb{P}^{n-1}$ *an irreducible hypersurface. Suppose that* t *contracts* S *onto a point.*

If $T_{g,h,t}$ *contracts* $\mathrm{pr}_O^{-1}(S)$ *onto a point, then* $g_{d-1}h_{d-e} - g_d h_{d-e-1}$ *vanishes on* S *and* $\mathrm{pr}_O^{-1}(S)$*.*

Proof. We may suppose that t contracts S onto $p := [1 : 0 : \cdots : 0] \in \mathbb{P}^{n-1}$ because composing with an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$ that fixes O and corresponds via pr_O to an element of $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^{n-1})$ moving the point $p \in \mathbb{P}^{n-1}$ does not change the claim. Writing

$$T_{g,h,t}\colon [x_0 : \cdots : x_n] \dashrightarrow [\frac{g_{d-1}x_0 + g_d}{h_{d-e-1}x_0 + h_{d-e}} : t_1 : \cdots : t_n],$$

we see that any point of $T_{g,h,t}(\mathrm{pr}_O^{-1}(S))$ is of the form

$$[\frac{g_{d-1}(y_1, \ldots, y_n)x_0 + g_d(y_1, \ldots, y_n)}{h_{d-e-1}(y_1, \ldots, y_n)x_0 + h_{d-e}(y_1, \ldots, y_n)} : t_1(y_1, \ldots, y_n) : 0 : \cdots : 0]$$

for some $[y_1 : \cdots : y_n] \in S$ and t_1 does not vanish on S.

If $T_{g,h,t}(\mathrm{pr}_O^{-1}(S))$ is a point, then the matrix

$$\begin{pmatrix} g_{d-1} & g_d \\ ah_{d-e-1} & ah_{d-e} \end{pmatrix}$$

has rank ≤ 1 on S and hence $g_{d-1}h_{d-e} - g_d h_{d-e-1}$ vanishes on S.

□

Lemma V.4.13. *Let $T_{g,h,t} \in \mathbf{St}_{\mathrm{k}}(\mathbb{P}^n)$ and suppose that h is irreducible and $h_{d-e-1} \neq 0$. Let $\pi_O \colon X_O \to \mathbb{P}^n$ be the blow-up of O and $E_O \subset X_O$ be its exceptional divisor. Then $T_{g,h,t}$ lifts to a birational map*

$$S_h \dashrightarrow E_O, \quad [x_0 : y_1 : \ldots, y_n] \mapsto (O, [y_1 : \cdots : y_n]).$$

Proof. Since $T_{g,h,t}$ sends S_h onto O, we get the rational map above. It is in fact birational because $h = h_{d-e-1}x_0 + h_{d-e}$ is irreducible and $h_{d-e-1} \neq 0$; we can recover x_0 by writing $x_0 = \frac{h_{d-e}}{h_{d-e-1}}$. □

Proposition V.4.14. *Let k be of characteristic zero, $T_{g,h,t} \in \mathbf{St}_{\mathrm{k}}(\mathbb{P}^n)$ and let $T_{g',h',t^{-1}} = T_{g,h,t}^{-1}$. Suppose that $T_{g,h,t}$ is punctual, that h does not vanish on any component of $S_{\mathrm{Jac}(t)}$ and that h' does not vanish on any component of $S_{\mathrm{Jac}(t^{-1})}$. Then*

- *t is punctual,*
- *the factors of $\mathrm{Jac}(t)$ are factors of $g_{d-1}h_{d-e} - g_d h_{d-e-1}$,*
- *T contracts a component of $S_{g_{d-1}h_{d-e}-g_d h_{d-e-1}}$ onto O if and only if h_{d-e} or all t_i are factors of $g_{d-1}h_{d-e} - g_d h_{d-e-1}$.*
- *the projection of the points in $\mathcal{B}(\mathbb{P}^n)$ different from O onto which $S_{\mathrm{Jac}(t)}$ is contracted are the central base-points of t.*

Proof. If $T_{g,h,t} =: T$ is punctual, then there exist sequences of blow-ups of point $\pi_i \colon X_i \to \mathbb{P}^n$, $i = 1, 2$ and a pseudo-isomorphism $\hat{T} \colon X_1 \dashrightarrow X_2$ such that the following diagram is commutative

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\hat{T}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ \mathbb{P}^n \arrow[r, dashed, "T"] & \mathbb{P}^n \end{tikzcd}$$

By Remark V.4.11, T only contracts the hypersurfaces S_h, $S_{d-1h_{d-e}-g_d h_{d-e-1}}$ and $S_{\mathrm{Jac}(t)}$, and it contracts them onto points because it is punctual. More concretely, $\pi_2\hat{T}$ contracts the strict transforms of these surfaces onto points in $\mathcal{B}(\mathbb{P}^n)$. Lemma V.4.12 yields the second claim. Note that T contracts S_h onto the point O. We write

$$T_{g,h,t} \colon [x_0 : \cdots : x_n] \mapsto [\frac{g_{d-1}x_0 + g_d}{h_{d-e-1}x_0 + h_{d-e}} : t_1 : \cdots : t_n].$$

It contracts the components of $S_{g_{d-1}h_{d-e}-g_d h_{d-e-1}}$ onto points of the form

$$[\frac{g_d(y_1, \ldots, y_n)}{h_{d-e}(y_1, \ldots, y_n)} : t_1(y_1, \ldots, y_n) : \cdots : t_n(y_1, \ldots, y_n)]$$

for some $[0 : y_1 : \cdots : y_n] \in S_{g_{d-1}h_{d-e}-g_dh_{d-e-1}}$. Such a point is different from O if and only if h_{d-e} and not all t_j do not vanish on the corresponding component of $S_{g_{d-1}h_{d-e}-g_dh_{d-e-1}}$. This yields the third claim.

By Remark V.4.7, t contracts the hypersurface $R_{\mathrm{Jac}(t)} \subset \mathbb{P}^{n-1}$ given by $\mathrm{Jac}(t) = 0$ onto a variety of dimension$\leq n-2$ whose points are of the form

$$[t_1(y_1, \ldots, y_n) : \cdots : t_n(y_1, \ldots, y_n)]$$

for some $[y_1 : \cdots : y_n] \in R_{\mathrm{Jac}(t)}$. Hence not all t_i vanish on the components of $R_{\mathrm{Jac}(t)}$ and therefore also not on the components of $S_{\mathrm{Jac}(t)} = \mathrm{pr}_O^{-1}(R_{\mathrm{Jac}(t)})$. Then T contracts $S_{\mathrm{Jac}(t)}$ onto points of the form

$$[\frac{g_{d-1}(y_1, \ldots, y_n)x_0 + g_d(y_1, \ldots, y_n)}{h_{d-e-1}(y_1, \ldots, y_n)x_0 + h_{d-e}(y_1, \ldots, y_n)} : t_1(y_1, \ldots, y_n) : \cdots : t_n(y_1, \ldots, y_n)]$$

for some $[x_0 : y_1 : \cdots : y_n] \in S_{\mathrm{Jac}(t)}$. By assumption, h does not vanish on any component of $S_{\mathrm{Jac}(t)}$, hence these points are different from O.

Call $q_1, \ldots, q_m \in \mathcal{B}(\mathbb{P}^n)$ the images by $\pi_2\hat{T}$ of the strict transform of the components of $S_{\mathrm{Jac}(t)}$. The situation is similar for T^{-1} and we call $p_1, \ldots, p_k \in \mathcal{B}(\mathbb{P}^n)$ the images of the strict transforms of the components of $S_{\mathrm{Jac}(t^{-1})}$ by $\pi_1\hat{T}^{-1}$. Since all these points are different from O, the projection $(\mathrm{pr}_O)_\bullet$ is defined around them and we call $r_i, s_i \in \mathcal{B}(\mathbb{P}^{n-1})$ respectively the image of p_i, q_i by $(\mathrm{pr}_O)_\bullet$. Denote by $\eta_1 \colon Y_1 \to \mathbb{P}^{n-1}$ the blow-up of the r_i and by $\eta_2 \colon Y_2 \to \mathbb{P}^{n-1}$ the blow-up of the s_i. Then t lifts to a birational map $\hat{t} \colon Y_1 \dashrightarrow Y_2$ and pr_O lifts to projections $\mathrm{pr}_i \colon X_i \dashrightarrow Y_i$, $i = 1, 2$, that respectively project the strict transform $\tilde{E}_{p_i} \subset X_1$ of the exceptional divisor of p_i onto $\tilde{E}_{r_i} \subset Y_1$, the strict transform of the exceptional divisor of r_i, and similarly $\tilde{E}_{q_i} \subset X_2$ onto $\tilde{E}_{s_i} \subset Y_2$. The situation is summarised in the following commutative diagram.

$$\begin{tikzcd}
& X_1 \arrow[rr, dashed, "\hat{T}"] \arrow[dl, dashed, "\mathrm{pr}_1"'] \arrow[dd, "\pi_1"] & & X_2 \arrow[dl, dashed, "\mathrm{pr}_2"'] \arrow[dd, "\pi_2"] \\
Y_1 \arrow[rr, dashed, "\hat{t}"] \arrow[dd, "\eta_1"'] & & Y_2 \arrow[dd, "\eta_2"] & \\
& \mathbb{P}^n \arrow[rr, dashed, "T"] \arrow[dl, dashed, "\mathrm{pr}_O"] & & \mathbb{P}^n \arrow[dl, dashed, "\mathrm{pr}_O"] \\
\mathbb{P}^{n-1} \arrow[rr, dashed, "t"] & & \mathbb{P}^{n-1} &
\end{tikzcd}$$

We claim that $\hat{t}$ is a pseudo-isomorphism. Because of the symmetry of the contstruction, we only need to check that $\hat{t}$ does not contract any hypersurfaces of Y_1.

By Remark V.4.7, any irreducible hypersurface $R \subset Y_1$ contracted by $\hat{t}$ is a component of $\overline{R_{\mathrm{Jac}(t)}}^{\eta_1}$ and hence is either a component of $\widetilde{R_{\mathrm{Jac}(t)}}^{\eta_1}$ or one of the $\tilde{E}_{r_i}$, and is contracted to a variety $W \subset Y_2$ of dimension $\dim(W) \leq n-3$. Then $\hat{T}$ sends the hypersurface $\mathrm{pr}_1^{-1}(R)$ onto a variety contained in $\mathrm{pr}_2^{-1}(W)$, which is of dimension $\dim(\mathrm{pr}_2^{-1}(W)) \leq n-2$. That is a contradiction to $\hat{T}$ being a pseudo-isomorphism. This yields the first and last claim. $\square$

Lemma V.4.15. *Let $n \geq 3$ and k of characteristic zero. If $t = \sigma_{n-1}$ and $T_{g,h,t}$ a punctual stellar transformation satisfying the assumptions of Proposition V.4.14, then $T = \beta\sigma_n\alpha$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$.*

Proof. It follows from Proposition V.4.14, that all central base-points of T project to central base-points of σ_{n-1}. Furthermore, T contracts S_h onto O, which is therefore a central base-points as well. Hence T has exactly $n+1$ central base-points, and Lemma V.3.6 implies that $T = \beta\sigma_n\alpha$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^n)$. □

It will be quite interesting to explore examples of punctual stellar transformations and to see when they are compositions of σ_n and linear maps.

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