Local and nonlocal problems regarding the Q-curvature and the Adams-Moser-Trudinger inequalities

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Chapter 1

Introduction

In this dissertation we study higher order (local and non-local) partial differential equations (PDEs). These equations naturally arise in the study of differential geometry, calculus of variations, functional analysis and various mathematical inequalities. In general they are of critical type which makes them more interesting from the point of view of PDEs.

We mainly focus on the prescribed Q-curvature equation

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \Omega \subseteq \mathbb{R}^n \tag{1.1}$$

where n is a positive integer and Ω is a domain in $\mathbb{R}^n$. We also study the Adams-Moser-Trudinger equation

$$(-\Delta)^{\frac{n}{2}}u = \lambda ue^{bu^2} \quad \text{in } \Omega \subseteq \mathbb{R}^n, \tag{1.2}$$

which appears in the study of critical points to a Adams-Moser-Trudinger functional. Below we describe them briefly.

1.1 Prescribing Q-curvature problem

Let (M^2, g) be a 2 dimensional smooth Riemannian manifold and let $g_u := e^{2u}g$ be a conformal metric where u is a smooth function on M^2. Then the Laplace-Beltrami operator transform according to the rule (conformal covariance)

$$\Delta_g = e^{2u}\Delta_{g_u}. \tag{1.3}$$

A famous and old problem in differential geometry is the following: Given a smooth function K on (M^2, g), does there exist a conformal metric g_u such that K is the Gaussian curvature of g_u?

This problem is equivalent to solving

$$-\Delta_g u + K_g = Ke^{2u} \quad \text{in } M^2, \tag{1.4}$$

where K_g is the Gaussian curvature of g.

The above equation (1.4) relates the Gaussian curvatures of the metrics g and $g_u = e^{2u}g$. Moreover, if M^2 is closed then integrating (1.4) on M^2 we obtain

$$\int_{M^2} K_g dv_g = \int_{M^2} K dv_{g_u}, \quad g_u := e^{2u}g, \tag{1.5}$$

and hence the total Gaussian curvature is invariant under conformal transformations. In fact, the total Gaussian curvature is exactly $2\pi\chi(M^2)$, which is the well-known Gauss-Bonnet theorem, where $\chi(M^2)$ is the Euler characteristic of M^2.

On a $2m$ dimensional Riemannian manifold (M^{2m}, g), higher order curvatures Q_g^{2m} and higher order operators P_g^{2m} were introduced in [8, 29]. The curvature Q_g^{2m} and the operator P_g^{2m} are known as Q-curvature and Paneitz operator (or GJMS operator) respectively. An interesting fact about the operator P_g^{2m} is that it is conformally covariant, that is (analogous to (1.3))

$$P_g^{2m} = e^{2mu} P_{g_u}^{2m}, \quad g_u := e^{2u}g.$$

The curvature function $Q_{g_u}^{2m}$ (Q-curvature of g_u) satisfies

$$P_g^{2m} u + Q_g^{2m} = Q_{g_u}^{2m} e^{2mu}, \tag{1.6}$$

which is a higher dimensional analog of (1.4). When M^{2m} is closed, integrating (1.6) and using that the operator P_g^{2m} is in divergence form, one obtains

$$\int_{M^{2m}} Q_g^{2m} dv_g = \int_{M^{2m}} Q_{g_u}^{2m} dv_{g_u}.$$

That means the total Q-curvature is invariant under conformal transformations.

In dimension 4 an explicit expression of Q_g^4 and P_g^4 was obtained by Branson-Orsted [10], Paneitz [62]:

$$Q_g^4 := \frac{1}{6}(R_g^2 - 3|Ric_g|^2 - \Delta_g R_g),$$

$$P_g^4 := (-\Delta_g)^2 - div(\frac{2}{3}R_g g - 2Ric_g)d,$$

where R_g and Ric_g are the scaler and Ricci curvatures of g respectively, and d is the differential.

Although an explicit formulas for the Q-curvature and Paneitz operator is not known in general manifold, we know them on the Euclidean space $\mathbb{R}^n$ and on the round sphere S^n. For instance we have $P_g^n = (-\Delta)^{\frac{n}{2}}$, $Q_g^n \equiv 0$ on $\mathbb{R}^n$ (with $g = |dx|^2$), and on (S^n, g) (g is the round metric)

$$P_g^n = \begin{cases} \prod_{k=0}^{\frac{n-2}{2}} (-\Delta_g + k(n-k-1)) & \text{if } n \text{ is even} \\ \left(-\Delta_g + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (-\Delta_g + k(n-k-1)) & \text{if } n \text{ is odd,} \end{cases}$$

and $Q_g^n \equiv (n-1)!$.

We consider Eq. (1.6) when the manifold M^{2m} is the Euclidean space $\mathbb{R}^{2m}$. Then (1.6) reduces to

$$(-\Delta)^m u = Qe^{2mu} \quad \text{in } \mathbb{R}^{2m}. \tag{1.7}$$

The above equation (1.7) with $m = 1$ ($Q \equiv const > 0$) is the well-known Liouville equation. It was shown by Liouville [47] that any solution u to (1.7) with $m = 1$ and $Q \equiv 1$ can be given by

$$u(\xi, \zeta) = \log\left(\frac{2|f'(z)|}{1+|f(z)|^2}\right), \quad z = \xi + i\zeta \in \mathbb{C},$$

for some meromorphic function f on $\mathbb{C}$ such that $|f'| \neq 0$ at all regular points. For example, the classical solution (of the Liouville equation) namely (see [13])

$$u_1(x) = \log\left(\frac{2}{1+|x|^2}\right), \quad x \in \mathbb{R}^2,$$

can be obtained from

$$f(z) = z, \quad f(z) = \frac{z+1}{z-1} \text{ and } f(z) = \frac{e^{i\theta}}{z}, \quad \theta \in \mathbb{R}.$$

Another explicit solution (depending on one variable only)

$$u_2(x, y) = \log\left(\frac{1}{\cosh x}\right), \quad (x, y) \in \mathbb{R}^2,$$

can be obtained from

$$f(z) = e^z, \quad f(z) = \tan(-\frac{iz}{2}) \text{ and } f(z) = \tanh(z).$$

Observe that the conformal metric $g_1 := e^{2u_1}|dx|^2$ has finite volume, that is

$$\int_{\mathbb{R}^2} dv_{g_1} = \int_{\mathbb{R}^2} e^{2u_1} dx < \infty,$$

whereas $g_2 := e^{2u_2}|dx|^2$ has infinite volume. An interesting point is that the finite volume condition characterizes the solution u_1. More precisely, if u is a solution to (1.7) (in dimension 2, $Q \equiv 1$) such that the metric $g_u := e^{2u}|dx|^2$ has finite volume, then up to a translation and dilation we have $u = u_1$.

The geometric meaning of the equation (1.7) also leads us to find solutions of the form u_1. Indeed, any smooth solution of (1.7) corresponds to a conformal metric $g_u := e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ such that Q-curvature of g_u is Q. Since the round metric of the sphere S^{2m} has the constant Q-curvature $(2m-1)!$, pulling back the round metric on $\mathbb{R}^{2m}$ via the stereographic projection, one can obtain a solution

$$u(x) = \log\left(\frac{2}{1+|x|^2}\right), \quad x \in \mathbb{R}^{2m},$$

to (1.7) with $Q \equiv (2m-1)!$. In fact, by translation and dilation, one has a family of solutions, namely

$$u_{\lambda, x_0}(x) = \log\left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right), \quad \lambda > 0,\ x_0 \in \mathbb{R}^{2m}.$$

These solutions are known by spherical solution.

Now we consider odd dimensional analogous equation to (1.7), namely

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n. \tag{1.8}$$

We also assume that the total Q-curvature of the conformal metric $g_u := e^{2u}|dx|^2$ is finite, that is

$$\int_{\mathbb{R}^n} Qe^{nu}dx =: \kappa \in (-\infty, \infty). \tag{1.9}$$

A simple application of maximum principle shows that (see [21, 52]) there exists no solution to (1.8)-(1.9) for $n = 1, 2$ and $Q \equiv -1$.

In the following chapters (Chapters 2,3,4 and 5) we address the following three questions:

(i) What are the solutions to (1.8)-(1.9)?

(ii) How do they behave at infinity?

(iii) What are the possible values of κ?

1.2 Adams-Moser-Trudinger type inequalities

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. The Sobolev embedding theorem states that the space $W_0^{k,p}(\Omega)$ continuously embeds into $L^q(\Omega)$ for all $1 \leq q \leq \frac{np}{n-kp}$ if $kp < n$ and into $C^{m,\alpha}(\Omega)$ if $kp > n$ where m is an integer such that $k-m-\alpha = \frac{n}{p}$ and $\alpha \in (0, 1)$. However, it is not true that $W_0^{k,p}(\Omega) \subset L^\infty(\Omega)$ for $kp = n$. In the borderline case, as shown by Trudinger [76], $W_0^{1,n}(\Omega)$ embeds into an Orlicz space and in fact

$$\sup_{u\in W_0^{1,n}(\Omega),\, \|\nabla u\|_{L^n(\Omega)}\leq 1} \int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}dx < \infty,$$

for some $\alpha > 0$. This leads to a natural question: is there a function $F : \mathbb{R} \to [0, \infty)$ with "optimal growth" such that

$$\sup_{u\in W_0^{1,n}(\Omega),\, \|\nabla u\|_{L^n(\Omega)}\leq 1} \int_\Omega F(u)dx < \infty? \tag{1.10}$$

In a famous work Moser [61] found an optimal function $F(t) = F_n(t) := e^{\alpha_n t^{\frac{n}{n-1}}}$ which satisfies (1.10), where $\alpha_n := n|S^{n-1}|^{\frac{1}{n-1}}$. In fact, if F satisfies

$$F(t) = f(t)F_n(t), \quad \lim_{t\to\infty} f(t) = \infty,$$

then the supremum in (1.10) is infinite. Adams [2] generalized this result for higher order derivatives. More precisely, if k is a positive integer less than n, then

$$\sup_{u\in C_c^k(\Omega),\, \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)}\leq 1} \int_\Omega F(u)dx < \infty, \tag{1.11}$$

for

$$F(t) = F_{k,n}(t) := e^{\alpha_{k,n} t^{\frac{n}{n-k}}},$$

where

$$\alpha_{k,n} = \frac{n}{|S^{n-1}|} \begin{cases} \left[\frac{\pi^{\frac{n}{2}} 2^k \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)} \right]^{\frac{n}{n-k}}, & \text{if } k \text{ is odd,} \\ \left[\frac{\pi^{\frac{n}{2}} 2^k \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)} \right]^{\frac{n}{n-k}}, & \text{if } k \text{ is even,} \end{cases}$$

and

$$\nabla^k := \begin{cases} \nabla \Delta^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \\ \Delta^{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

Moreover, the supremum in (1.11) is infinite if F satisfies

$$F(t) = f(t) F_{k,n}(t), \quad \lim_{t \to \infty} f(t) = \infty.$$

Notice that if F is monotone increasing then the supremum in (1.11) is equivalent to

$$\sup_{u \in W_0^{k,\frac{n}{k}}(\Omega),\, \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)} = 1} \int_\Omega F(u) dx. \tag{1.12}$$

In the particular case $n = 2k$ and $F(t) = e^{bt^2}$ $(b > 0)$, if the supremum in (1.12) is attained by some $u \in W_0^{\frac{n}{2},2}(\Omega)$, then u satisfies

$$(-\Delta)^{\frac{n}{2}} u = \lambda u e^{bu^2} \quad \text{in } \Omega, \tag{1.13}$$

for some $\lambda > 0$.

We study (in Chapter 7) the sharpness of some fractional Adams-Moser-Trudinger type inequalities. As an application, for every $\lambda \in (0, \lambda_1)$, we prove the existence of solutions to (1.13) with Dirichlet boundary condition, where λ_1 is the first eigenvalue of $(-\Delta)^{\frac{n}{2}}$ on Ω.

1.3 Structure of the chapters

In Chapter 2 we classify all solutions to (1.8)-(1.9) with $Q \equiv const > 0$ in terms of their behavior at infinity for every $n \geq 3$ odd. Then we develop some criteria to characterize the spherical solutions. This result is very crucial in studying blow-up analysis.

In Chapter 3 we prove the existence of solutions with prescribed volume (equivalently, total Q-curvature) and asymptotic behavior to (1.8)-(1.9) with $Q \equiv const \neq 0$ in even dimension $n \geq 4$. In the negative case we can prescribe any $\kappa \in (-\infty, 0)$, but in the positive case only in $(0, \Lambda_1)$ (Λ_1 is a dimensional constant). This will be done by a Schauder fixed point argument, and blow-up analysis.

The main difference between the positive and negative cases is that if (u_k) is a sequence of solutions to

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } B_R \subset \mathbb{R}^{2m}, \quad \int_{B_R} e^{2mu_k} dx \leq C, \quad \int_{B_R} |\Delta u_k| dx \leq C,$$

for some $Q_k \to Q_\infty$ in $C^0_{loc}(B_R)$, such that

$$\max_{x \in \bar{B}_{\frac{R}{2}}} u_k(x) = u(x_k) \to \infty, \quad x_k \to x_\infty,$$

then $Q_\infty(x_\infty) > 0$ and for any $\varepsilon > 0$ (small)

$$\lim_{k\to\infty} \int_{B_\varepsilon(x_\infty)} Q_k e^{2mu_k} dx \geq \Lambda_1.$$

Thus one rules out a possible blow-up easily if $Q_k \leq 0$. On the other hand, if $Q_k > 0$, to rule out a possible blow-up we require that

$$\limsup_{k\to\infty} \int_{B_R} Q_k e^{2mu_k} dx < \Lambda_1.$$

In Chapter 4 we extend the results of Chapter 3 to odd dimension using a variational approach. We find the solutions as critical points of some energy functional. Again, we need to assume that $\kappa \in (0, \Lambda_1)$ in the case when Q is a positive constant and κ could be anything in $(-\infty, 0)$ in the negative case. This restriction $\kappa < \Lambda_1$ (in the positive case) plays an important role in showing that the functional is coercive.

In Chapter 5 we address the following problem: what are the possible values of κ (as defined in (1.9))? We prove that for every $n \geq 5$ and for every $\kappa \in (0, \infty)$ there exists a solution to (1.8)-(1.9) with $Q \equiv const > 0$ (in fact Q can be a non-constant function). Our approach is again based on a Schauder fixed point argument, however, this time we need a delicate blow-up analysis as we are allowing $\kappa \geq \Lambda_1$. The main idea is that one can recover compactness (for a sequence of radial solutions) on a bounded domain by prescribing "boundary value at infinity", that is, by prescribing asymptotic behavior. More precisely, if (ψ_k) is a sequence of radial solutions to

$$\psi_k(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) e^{-|y|^4} e^{n\psi_k(y)} dy + \frac{1}{2n} |\Delta\psi_k(0)| (|x|^2 - |x|^4) + c_k,$$

where $c_k \in \mathbb{R}$ is a normalization constant such that

$$\int_{\mathbb{R}^n} e^{-|y|^4} e^{n\psi_k(y)} dy = \kappa,$$

then the sequence (ψ_k) is precompact.

Our approach explains why $n = 5$ is the first dimension in which large total Q-curvature (equivalently, large volume) appears. Moreover, when Q is non constant and decays fast enough at infinity, this approach works also in lower dimension $n = 3, 4$.

In Chapter 6 we study (1.7) on a domain in $\mathbb{R}^{2m}$. We construct blowing-up sequences of solutions (u_k) to the prescribed Q-curvature problem

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega, \quad \int_\Omega e^{2mu_k} dx \le C, \quad \|Q_k\|_{L^\infty(\Omega)} \le C, \tag{1.14}$$

where $\Omega \subset \mathbb{R}^{2m}$ is an open domain and $m \ge 2$. For a given $\varphi \in C^\infty(\Omega)$ satisfying

$$\Delta^m \varphi = 0 \quad \text{in } \Omega, \quad \varphi \le 0, \quad \varphi \not\equiv 0, \quad S_\varphi := \{x \in \Omega : \varphi(x) = 0\} \neq \emptyset,$$

and given Q_k (uniformly bounded) we prove the existence of solutions (u_k) to (1.14) such that $u_k \to \infty$ locally uniformly on S_φ and $u_k \to -\infty$ locally uniformly on $\Omega \setminus S_\varphi$. In addition to this, (under certain conditions on Q_k) we can also prescribe the total Q-curvature of the metric $e^{2mu_k}|dx|^2$ in $(0, \frac{\Lambda_1}{2})$.

In Chapter 7 we study (1.13) for every odd integer $n \ge 1$. We prove the existence of solution for $\lambda \in (0, \lambda_1)$ by minimizing a suitable energy functional.

Since any non-trivial (weak) solution to (1.13) belongs to the Nehari manifold S, that is

$$S := \left\{ u \in \tilde{H}^{\frac{n}{2},2}(\Omega) \setminus \{0\} : \|u\|^2 = \lambda \int_\Omega u^2 e^{bu^2} dx \right\},$$

we look for a minimizer of the energy functional J on S, where

$$\tilde{H}^{\frac{n}{2},2}(\Omega) := \left\{ u \in L^2(\Omega) : u \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega, \, (-\Delta)^{\frac{n}{4}} u \in L^2(\mathbb{R}^n) \right\}.$$

The main difficulty in the variational approach is the lack of compactness, more precisely the global Plais-Smale condition does not hold. However, the Palais-Smale condition still holds on $(-\infty, c_0)$ for some $c_0 > 0$. Therefore, in order to recover the compactness, it is sufficient to show that for a minimizing sequence $(u_k) \subset S$ one has $\lim_{k\to\infty} J(u_k) < c_0$. It turns out that the constant c_0 is related with the best constant in Adams' inequality (it also depends on b), and in fact, a sharp Adams type inequality in a fractional settings yields that $\lim_{k\to\infty} J(u_k) < c_0$.

The content of the Chapters 2, 3, 4, 5, 6 and 7 corresponds to the papers [32], [37], [33], [34], [36] and [35] respectively.

Chapter 2

Classification of solutions to a fractional Liouville equation in $\mathbb{R}^n$

In this chapter we study the nonlocal equation

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{nu}dx < \infty,$$

which arises in the conformal geometry. Inspired by the previous work of Lin and Martinazzi in even dimension and Jin-Maalaoui-Martinazzi-Xiong in dimension three we classify all solutions to the above equation in terms of their behavior at infinity.

2.1 Introduction to the problem and the main theorems

We consider the equation

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n. \tag{2.1}$$

Here we assume that

$$V := \int_{\mathbb{R}^n} e^{nu}dx < \infty, \tag{2.2}$$

and we shall see both the left and right-hand side of (2.1) as tempered distributions. In order to define the left-hand side of (2.1) as a tempered distribution, one possibility is to follow the approach of [40], i.e. we see the operator $(-\Delta)^{\frac{n}{2}}$ as $(-\Delta)^{\frac{n}{2}} := (-\Delta)^{\frac{1}{2}} \circ (-\Delta)^{\frac{n-1}{2}}$ for $n \geq 1$ odd integer with the convention that $(-\Delta)^0$ is the identity, where $(-\Delta)^{\frac{1}{2}}$ is defined as follows. First for $s > 0$ consider the space

$$L_s(\mathbb{R}^n) := \left\{ v \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}dx < \infty \right\}. \tag{2.3}$$

Then for $v \in L_s(\mathbb{R}^n)$ we define $(-\Delta)^s v$ as the tempered distribution defined by

$$\langle (-\Delta)^s v, \varphi \rangle := \int_{\mathbb{R}^n} v(-\Delta)^s \varphi dx \qquad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.4}$$

where

$$\mathcal{S}(\mathbb{R}^n) := \left\{ u \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x|^N |D^\alpha u(x)| < \infty \text{ for all } N \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^n \right\}$$

is the Schwartz space, and

$$\widehat{(-\Delta)^s \varphi}(\xi) = |\xi|^{2s}\hat{\varphi}(\xi), \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Here the normalized Fourier transform is defined by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \quad f \in L^1(\mathbb{R}^n).$$

Notice that the integral in (2.4) converges thanks to Proposition 2.2.1 below.

Then a possible definition of the equation

$$(-\Delta)^{\frac{n}{2}} u = f \quad \text{in } \mathbb{R}^n \tag{2.5}$$

is the following:

Definition 2.1.1. Given $f \in \mathcal{S}'(\mathbb{R}^n)$, we say that u is a solution of (2.5) if

$$u \in W^{n-1,1}_{loc}(\mathbb{R}^n), \quad \Delta^{\frac{n-1}{2}} u \in L_{\frac{1}{2}}(\mathbb{R}^n),$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} u(x) (-\Delta)^{\frac{1}{2}} \varphi(x) dx = \langle f, \varphi \rangle, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.6}$$

While Definition 2.1.1 is general enough for our purposes, requiring a priori that a solution to (2.1) belongs to $W^{n-1,1}_{loc}(\mathbb{R}^n)$ might sound unnecessarily restrictive. In fact it is possible to relax Definition 2.1.1 as follows.

Definition 2.1.2. Given $f \in \mathcal{S}'(\mathbb{R}^n)$, a function $u \in L_{\frac{n}{2}}(\mathbb{R}^n)$ is a solution of (2.5) if

$$\int_{\mathbb{R}^n} u(x) (-\Delta)^{\frac{n}{2}} \varphi(x) dx = \langle f, \varphi \rangle, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.7}$$

Notice again that the integral in (2.6) and (2.7) are converging by Proposition 2.2.1 below.

As we shall see, a function u solving (2.1)-(2.2) in the sense of Definition 2.1.2 also solves (2.1) in the sense of Definition 2.1.1, and conversely, see Proposition 2.2.6 below. Therefore, from now on a solution of (2.1)-(2.2) will be intended in the sense of Definition 2.1.1. In fact it turns out that such solutions enjoy even more regularity:

Theorem 2.1.1. *Let u be a solution of (2.1)-(2.2) (in the sense of Definition 2.1.1 or 2.1.2). Then u is smooth.*

Geometrically any solution u of (2.1)-(2.2) corresponds to a conformal metric $g_u := e^{2u}|dx|^2$ on $\mathbb{R}^n$ ($|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) such that the Q-curvature of g_u is constant $(n-1)!$. Moreover the volume and the total Q-curvature of the metric g_u

are $V = \int_{\mathbb{R}^n} e^{nu}dx < \infty$ and $\kappa = \int_{\mathbb{R}^n}(n-1)!e^{nu}dx < \infty$ respectively. When $n = 1$ a geometric interpretation of (2.1) in terms of holomorphic immersion of $\overline{D^2}$ into $\mathbb{C}$ was given in [21, Theorem 1.3]. If u is a solution of (2.1) then for any constant c, $\tilde{u} := u - c$ satisfies

$$(-\Delta)^{\frac{n}{2}}\tilde{u} = (n-1)!e^{nc}e^{n\tilde{u}} \quad \text{in } \mathbb{R}^n.$$

This shows that we could take any arbitrary positive constant instead of $(n-1)!$ in (2.1), but we restrict ourselves to the fixed constant $(n-1)!$ because it is the constant Q-curvature of the round sphere S^n.

Now we shall address the following question: What are the solutions to (2.1) and in particular how do they behave at infinity?

It is well known that the equation (2.1) possesses the following explicit solution

$$u(x) = \log\left(\frac{2}{1+|x|^2}\right),$$

obtained by pulling back the round metric on S^n via the stereographic projection. By translating and rescaling this function u one can produce a class of solutions, namely

$$u_{\lambda,x_0}(x) := \log\left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right),$$

for every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Any such u_{λ,x_0} is called spherical solution. Chen-Li [19] showed that these are the only solutions in dimension two but in higher dimension non-spherical solutions do exist as shown by Chang-Chen [15]. Lin [45] for $n = 4$ and Martinazzi [56] for $n \geq 4$ even classified all solutions of (2.1)-(2.2) and they proved:

Theorem A ([45], [56])**.** *Let $n \geq 4$ be an even integer. If u solves (2.1)-(2.2), then u has the asymptotic behavior*

$$u(x) = -\alpha\log(|x|) - P(x) + C + o(1), \quad o(1) \to 0 \text{ as } |x| \to \infty,$$

where $\alpha = \frac{2V}{|S^{2m}|}$ and P is a polynomial of degree at most $2m-2$ bounded from below. Moreover, P is constant if and only if u is spherical. When $m = 2$ one has $V \in (0, |S^4|]$ and $V = |S^4|$ if and only if u is spherical.

When n is odd things are more complex as the operator $(-\Delta)^{\frac{n}{2}}$ is nonlocal. In a recent work Jin-Maalaoui-Martinazzi-Xiong have proven the following theorem in dimension three:

Theorem B ([40])**.** *Let u be a smooth solution of (2.1)-(2.2) with $n = 3$. Then u has the asymptotic behavior given by*

$$u(x) = -P(x) - \alpha\log|x| + o(\log|x|),$$

where P is a polynomial of degree 0 or 2 bounded from below, $\alpha \in (0, 2]$ and $\alpha = 2$ if and only if degree$(P) = 0$.

The restriction to dimension 3 in Theorem B dramatically simplifies the proof, since in this case one can easily show that $\Delta u < 0$ (Lemma 17 in [40]) and use the classical maximum principle, or Harnack's inequality. This argument is essentially the same as in Lin's previous work [45], both cases resting on the formula

$$-\Delta u(x) = c_n \int_{\mathbb{R}^n} \frac{e^{nu(y)}}{|x-y|^2} dy + a, \quad \text{for some } a \geq 0,$$

which holds for $n = 3, 4$. When $n \geq 4$ the constant a should be replaced by a polynomial of degree $n-4$ for n even and $n-3$ for n odd, whose sign one cannot control. In spite of this Martinazzi [56] was able to handle the even-dimensional case using explicit Green representation formulas and divergence theorems (see in particular Lemmas 12 and 13 in [56]) that are not available for fractional powers of the Laplacian. This makes the generalization of Theorem B to odd dimension $n \geq 5$, namely our following Theorem 2.1.2 particularly challenging.

In order to state it we define

$$v(x) := \frac{(n-1)!}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) e^{nu(y)} dy, \quad \gamma_n := \frac{(n-1)!}{2} |S^n|, \tag{2.8}$$

where u is a smooth solution of (2.1)-(2.2). Then we have

Theorem 2.1.2. *Let $n \geq 3$ be any odd integer and let u be a smooth solution of (2.1)-(2.2). Then*

$$u = v + P,$$

where P is a polynomial of degree at most $n-1$ bounded from above, v is given by (2.8) and it satisfies

$$v(x) = -\alpha \log|x| + o(\log|x|), \quad \text{as } |x| \to \infty,$$

where $\alpha = \frac{2V}{|S^n|}$. Moreover

$$\lim_{|x|\to\infty} D^\beta v(x) = 0 \ \text{ for every multi-index } \beta \in \mathbb{N}^n \text{ with } 0 < |\beta| \leq n-1.$$

As a corollary of Theorem 2.1.2 one can obtain necessary and sufficient conditions under which any solution of (2.1)-(2.2) is spherical, in analogy with [45, 56]. More precisely we have the following theorem.

Theorem 2.1.3. *Let u be a smooth solution of (2.1)-(2.2). Then the following are equivalent:*

(i) *u is a spherical solution.*

(ii) *$deg(P) = 0$, where P is the polynomial given by Theorem 2.1.2.*

(iii) *$u(x) = o(|x|^2)$ as $|x| \to \infty$.*

(iv) *$\lim_{|x|\to\infty} \Delta^j u(x) = 0$ for $j = 1, 2, ..., \frac{n-1}{2}$.*

(v) *$\liminf_{|x|\to\infty} R_{g_u} > -\infty$, where R_{g_u} is the scalar curvature of g_u.*

(vi) *$\pi^* g_u$ can be extended to a Riemannian metric on S^n, where π is the stereographic projection.*

Moreover, if u is not a spherical solution then there exists a j with $1 \leq j \leq \frac{n-1}{2}$ and a constant $c < 0$ such that

$$\lim_{|x|\to\infty} \Delta^j u(x) = c. \tag{2.9}$$

Due to the equivalence of Definitions 2.1.1 and 2.1.2 (Proposition 2.2.6), Theorems 2.1.2 and 2.1.3 have numerous applications. For instance, Theorems 2.1.2 and 2.1.3 have been used in [48] under Definition 2.1.2, whereas in Chapter 5 under Definition 2.1.1. More generally we expect Theorems 2.1.2 and 2.1.3 to play the same central role that Theorem A had in the study of problems of prescribed Q-curvature (see e.g. [24], [53]) or the Adams-Moser-Trudinger embeddings (see e.g. [58]), this time in odd dimension and in the non-local context, the idea being that after a blow-up procedure one naturally ends up with solutions of (2.1)-(2.2), whose classification is then crucial.

In dimension 3 and 4 if u is a smooth solution of (2.1)-(2.2) then $V \in (0, |S^n|]$ (see [45], [40]), but V could be any positive number in dimension $n \geq 5$ (see Chapter 5).

We also mention that using different techniques Da Lio-Martinazzi-Rivière [21] have discussed the case in one dimension, proving that all solutions are spherical (see also [20] for yet a different proof, avoiding the moving-plane technique).

2.2 Equivalence of definitions

Proposition 2.2.1. *For any $s > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$|(-\Delta)^s \varphi(x)| \leq \frac{C}{|x|^{n+2s}},$$

where $(-\Delta)^s \varphi := (-\Delta)^\sigma \circ (-\Delta)^k \varphi$, where $\sigma \in [0,1)$, $k \in \mathbb{N}$ and $s = k + \sigma$.

In order to prove Proposition 2.2.1 let us introduce the spaces

$$\begin{aligned} \mathcal{S}_k(\mathbb{R}^n) :&= \{\varphi \in \mathcal{S}(\mathbb{R}^n) : D^\alpha \hat{\varphi}(0) = 0, \text{ for } |\alpha| \leq k\} \\ &= \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} y^\alpha \varphi(y) dy = 0, \text{ for } |\alpha| \leq k\right\}, \quad k = 0, 1, 2, \dots \\ \mathcal{S}_{-1}(\mathbb{R}^n) :&= \mathcal{S}(\mathbb{R}^n) \end{aligned}$$

Proposition 2.2.1 easily follows from the remark that $\Delta^k \varphi \in \mathcal{S}_{2k-1}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and from Lemma 2.2.2 below.

Lemma 2.2.2. *Let $\varphi \in \mathcal{S}_k(\mathbb{R}^n)$ and $\sigma \in (0,1)$. Then*

$$|(-\Delta)^\sigma \varphi(x)| \leq \frac{C}{|x|^{n+2\sigma+k+1}}, \quad x \in \mathbb{R}^n.$$

Proof. Since $(-\Delta)^\sigma \varphi \in C^\infty(\mathbb{R}^n)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it suffices to prove the lemma for large x. For a fixed $x \in \mathbb{R}^n$ we split $\mathbb{R}^n$ into

$$A_1 := B_{\frac{|x|}{2}} \quad \text{and } A_2 := \mathbb{R}^n \setminus B_{\frac{|x|}{2}}.$$

Then using (2.28) we have

$$|(-\Delta)^\sigma \varphi(x)| \le \frac{1}{2} C_{n,\sigma} \left(I_1 + I_2\right),$$

where

$$I_i := \left| \int_{A_i} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}} dy \right| \quad i = 1, 2.$$

Noticing that on A_1

$$|\varphi(x+y) + \varphi(x-y) - 2\varphi(x)| \le \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} |y|^2,$$

we get

$$I_1 \le \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} \int_{A_1} \frac{dy}{|y|^{n-2+2\sigma}} \le C \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} |x|^{2-2\sigma}.$$

On the other hand

$$\begin{aligned} I_2 &\le 2|\varphi(x)| \int_{A_2} \frac{dy}{|y|^{n+2\sigma}} + 2 \left| \int_{A_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}} dy \right| \\ &\le 2 \left| \int_{A_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}} dy \right| + C|\varphi(x)||x|^{-2\sigma} \\ &=: 2I_3 + C|\varphi(x)||x|^{-2\sigma}. \end{aligned}$$

Changing the variable $y \mapsto x - y$ we have

$$\begin{aligned} I_3 &= \left| \int_{|x-y| > \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| \\ &\le \left| \int_{|x-y| > \frac{|x|}{2}, |y| > \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| + \left| \int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| \\ &\le \left| \int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| + C\|\varphi\|_{L^\infty(A_2)} |x|^{-2\sigma} \\ &=: I_4 + C\|\varphi\|_{L^\infty(A_2)} |x|^{-2\sigma}. \end{aligned}$$

Finally, to bound I_4 we use the fact that $\varphi \in S_k$. Setting $f(x) = \frac{1}{|x|^{n+2\sigma}}$ and using

$$\sum_{|\alpha| \le k} \frac{D^\alpha f(x)}{\alpha!} \int_{\mathbb{R}^n} y^\alpha \varphi(y) dy = 0, \quad x \neq 0,$$

we obtain

$$
\begin{aligned}
&\int_{|y|<\frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}}dy \\
&= \int_{B_{\frac{|x|}{2}}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}}dy - \sum_{|\alpha|\le k} \frac{D^\alpha f(x)}{\alpha!}\int_{B_{\frac{|x|}{2}}} y^\alpha \varphi(y)dy - \sum_{|\alpha|\le k} \frac{D^\alpha f(x)}{\alpha!}\int_{B^c_{\frac{|x|}{2}}} y^\alpha \varphi(y)dy \\
&= \int_{B_{\frac{|x|}{2}}} \varphi(y)\left(f(x-y) - \sum_{|\alpha|\le k} y^\alpha \frac{D^\alpha f(x)}{\alpha!}\right)dy - \sum_{|\alpha|\le k} \frac{D^\alpha f(x)}{\alpha!}\int_{B^c_{\frac{|x|}{2}}} y^\alpha \varphi(y)dy \\
&= \int_{B_{\frac{|x|}{2}}} \varphi(y)\sum_{|\beta|=k+1} y^\beta R_\beta(\xi_y)dy - \sum_{|\alpha|\le k} \frac{D^\alpha f(x)}{\alpha!}\int_{B^c_{\frac{|x|}{2}}} y^\alpha \varphi(y)dy,
\end{aligned}
$$

where $R_\beta(\xi_y)$ satisfies

$$
f(x-y) = \sum_{|\alpha|\le k} y^\alpha \frac{D^\alpha f(x)}{\alpha!} + \sum_{|\beta|=k+1} y^\beta R_\beta(\xi_y), \quad |y| < \frac{|x|}{2},\ \xi_y \in B_{\frac{|x|}{2}}(x),
$$

and

$$
|R_\beta(\xi_y)| \le C \max_{|\alpha|=k+1} \max_{z\in B_{\frac{|x|}{2}}(x)} |D^\alpha f(z)| \le \frac{C}{|x|^{n+2\sigma+k+1}}.
$$

Therefore,

$$
\begin{aligned}
I_4 &\le \sum_{|\beta|=k+1} \int_{|y|<\frac{|x|}{2}} |\varphi(y)||y|^{|\beta|}|R_\beta(\xi_y)|dy + \sum_{|\alpha|\le k} \frac{|D^\alpha f(x)|}{\alpha!}\int_{A_2} |y|^{|\alpha|}|\varphi(y)|dy \\
&\le \frac{C}{|x|^{n+2\sigma+k+1}} \int_{\mathbb{R}^n} |\varphi(y)||y|^{k+1}dy + \|\sqrt{|\varphi|}\|_{L^\infty(A_2)} \sum_{|\alpha|\le k} \frac{|D^\alpha f(x)|}{\alpha!}\int_{\mathbb{R}^n} |y|^{|\alpha|}\sqrt{|\varphi(y)|}dy.
\end{aligned}
$$

We conclude the proof. □

Lemma 2.2.3. *Let $f \in L^1(\mathbb{R}^n)$. We set*

$$
\tilde{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)dy, \quad x \in \mathbb{R}^n. \tag{2.10}
$$

Then

(i) $\tilde{v} \in W^{n-1,1}_{loc}(\mathbb{R}^n)$ *and*

$$
D^\alpha \tilde{v} = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} D^\alpha_x \log\left(\frac{1+|y|}{|x-y|}\right) f(y)dy, \quad 0 \le |\alpha| \le n-1.
$$

(ii) $D^\alpha \tilde{v} \in L_{\frac{1}{2}}(\mathbb{R}^n)$ *for every multi-index* $\alpha \in \mathbb{N}^n$ *with* $0 \le |\alpha| \le n-1$.

(iii) For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$
\int_{\mathbb{R}^n} \tilde{v}(x)(-\Delta)^{\frac{n}{2}}\varphi(x)dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}}\tilde{v}(x)(-\Delta)^{\frac{1}{2}}\varphi(x)dx = \int_{\mathbb{R}^n} \varphi(x)f(x)dx,
$$

that is $\tilde{v}$ solves (2.5) in the sense of Definition 2.1.1 and 2.1.2.

Proof. Proof of (i) is trivial.

To prove (ii) first we consider $0 < |\alpha| \leq n-1$ and we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|D^\alpha \tilde{v}(x)|}{1+|x|^{n+1}} dx \\
&\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|^{n+1})|x-y|^{|\alpha|}} dx \right) dy \\
&= C \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1(y)} \frac{dx}{(1+|x|^{n+1})|x-y|^{|\alpha|}} + \int_{\mathbb{R}^n \setminus B_1(y)} \frac{dx}{(1+|x|^{n+1})|x-y|^{|\alpha|}} \right) dy \\
&\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1(y)} \frac{dx}{|x-y|^{|\alpha|}} + \int_{\mathbb{R}^n \setminus B_1(y)} \frac{dx}{(1+|x|^{n+1})} \right) dy \\
&< \infty.
\end{aligned}$$

The case when $\alpha = 0$ follows from

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|\tilde{v}(x)|}{1+|x|^{n+1}} dx \leq \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+1}} \left(\int_{\mathbb{R}^n} \left| \log \frac{1+|y|}{|x-y|} \right| |f(y)| dy \right) dx \\
&= \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1^c(y)} \frac{\left|\log \frac{1+|y|}{|x-y|}\right|}{1+|x|^{n+1}} dx + \int_{B_1(y)} \frac{\left|\log \frac{1+|y|}{|x-y|}\right|}{1+|x|^{n+1}} dx \right) dy \\
&\leq \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1^c(y)} \frac{\log(2+|x|)}{1+|x|^{n+1}} dx + \int_{B_1(y)} \left(\frac{\log(2+|x|)}{1+|x|^{n+1}} + |\log|x-y|| \, dx \right) \right) dy \\
&= \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{\log(2+|x|)}{1+|x|^{n+1}} dx + \| \log(\cdot) \|_{L^1(B_1)} \right) dy \\
&< \infty,
\end{aligned}$$

where in the first inequality we used

$$\frac{1}{1+|x|} \leq \frac{1+|y|}{|x-y|} \leq 2+|x|, \; 1+|y| \leq 2+|x| \text{ for } |x-y| \geq 1.$$

(iii) follows from integration by parts and Lemma 2.5.1. □

Lemma 2.2.4. *Let u be a solution of (2.5) with $f \in L^1(\mathbb{R}^n)$ in the sense of Definition 2.1.2. Let $\tilde{v}$ be given by (2.10). Then $p := u - \tilde{v}$ is a polynomial of degree at most $n-1$.*

Proof. Let us consider a function $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. We set

$$\varphi := \mathcal{F}^{-1}\left(\frac{\bar{\psi}}{|\xi|^n} \right) \in \mathcal{S}(\mathbb{R}^n), \quad \bar{\psi}(x) := \psi(-x), \, x \in \mathbb{R}^n.$$

Now the growth assumption to u in Definition 2.1.2 implies that u is a tempered distribution and at the same time the function v is also a tempered distribution thanks to Lemma 2.2.3. Therefore $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ and $\hat{p} \in \mathcal{S}'(\mathbb{R}^n)$. Indeed,

$$\langle \hat{p}, \psi \rangle = \int_{\mathbb{R}^n} p \hat{\psi} dx = \int_{\mathbb{R}^n} p(x)(-\Delta)^{\frac{n}{2}} \varphi(x) dx = 0,$$

where the last equality follows from the Definition 2.1.2 and Lemma 2.2.3.

Thus $\hat{p}$ is a tempered distribution with support $\hat{p} \subseteq \{0\}$ which implies that p is a polynomial and combining with $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ we conclude that degree of p is at most $n-1$. $\square$

Lemma 2.2.5. *Let u be a solution of (2.5) with $f \in L^1(\mathbb{R}^n)$ in the sense of Definition 2.1.1 and let $\tilde{v}$ be given by (2.10). If u also satisfies*

$$\int_{B_R} u^+ dx = o(R^{2n}) \quad \text{or} \int_{B_R} u^- dx = o(R^{2n}) \quad \text{as } R \to \infty, \tag{2.11}$$

then $p := u - \tilde{v}$ is a polynomial of degree at most $n-1$.

Proof. We have $\Delta^{\frac{n-1}{2}} p \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and it satisfies

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} p (-\Delta)^{\frac{1}{2}} \varphi dx = 0, \qquad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.12}$$

thanks to Lemma 2.2.3. Moreover, by Schauder's estimate (see e.g. [40, Proposition 22]) for some $\alpha > 0$

$$\|(-\Delta)^{\frac{n-1}{2}} p\|_{C^{0,\alpha}(B_1)} \leq C \|(-\Delta)^{\frac{n-1}{2}} p\|_{L_{\frac{1}{2}}(\mathbb{R}^n)}.$$

Adapting the arguments in [40, Lemma 15] one can get that $(-\Delta)^{\frac{n-1}{2}} p$ is constant in $\mathbb{R}^n$ and hence $(-\Delta)^{\frac{n+1}{2}} p = 0$ in $\mathbb{R}^n$. Noticing that $\tilde{v} \in L_{\frac{n}{2}}(\mathbb{R}^n)$ we conclude the proof by Lemma 2.5.6 below. $\square$

Proposition 2.2.6. *Let $f \in L^1(\mathbb{R}^n)$. Then the following are equivalent:*

(i) u is a solution of (2.5) in the sense of Definition 2.1.2.

(ii) u is a solution of (2.5) in the sense of Definition 2.1.1 and u satisfies (2.11).

In particular, Definition 2.1.1 and Definition 2.1.2 are equivalent for the solutions of (2.1)-(2.2).

Proof. If p is a polynomial of degree at most $n-1$ then $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} p(-\Delta)^{\frac{n}{2}} \varphi dx = \int_{\mathbb{R}^n} p(-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} \varphi dx = C_p \int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} \varphi dx = 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $C_p := (-\Delta)^{\frac{n-1}{2}} p$ is a constant and the second equality follows from integration by parts (which can be justified thanks to Lemma 2.2.2). Now the equivalence of (i) and (ii) follows immediately from Lemmas 2.2.3, 2.2.4 and 2.2.5. To conclude the proposition notice that the condition (2.2) implies

$$\int_{B_R} u^+ dx = \frac{1}{n} \int_{B_R} n u^+ dx \leq \frac{1}{n} \int_{B_R} e^{nu} dx \leq \frac{V}{n}.$$

$\square$

2.3 Regularity results

Proof of Theorem 2.1.1 First we write $(n-1)!e^{nu} = f_1 + f_2$ where $f_1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $f_2 \in L^1(\mathbb{R}^n)$. Let us define the functions

$$u_i(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) f_i(y)dy, \quad x \in \mathbb{R}^n,\ i = 1, 2.$$

Then we have that $u_1 \in C^{n-1}(\mathbb{R}^n)$ and $u_2 \in W^{n-1,1}_{loc}(\mathbb{R}^n)$. Indeed, for $p \in \left(0, \frac{\gamma_n}{\|f_2\|}\right)$ using Jensen's inequality

$$\begin{aligned}
\int_{B_R} e^{np|u_2|}dx &= \int_{B_R} \exp\left(\int_{\mathbb{R}^n} \frac{np\|f_2\|}{\gamma_n} \log\left(\frac{1+|y|}{|x-y|}\right) \frac{f_2(y)}{\|f_2\|}dy\right) dx \\
&\le \int_{B_R}\int_{\mathbb{R}^n} \exp\left(\frac{np\|f_2\|}{\gamma_n} \log\left(\frac{1+|y|}{|x-y|}\right)\right) \frac{|f_2(y)|}{\|f_2\|}dydx \\
&= \frac{1}{\|f_2\|}\int_{\mathbb{R}^n} |f_2(y)| \int_{B_R} \left(\frac{1+|y|}{|x-y|}\right)^{\frac{np\|f_2\|}{\gamma_n}} dxdy \\
&\le C(n, p, \|f_2\|, R),
\end{aligned} \tag{2.13}$$

where $\|\cdot\|$ denotes the $L^1(\mathbb{R}^n)$ norm. Moreover, by Lemma 2.2.3 (with $\tilde{v} = u_i$ and $f = f_i$) we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} u_i (-\Delta)^{\frac{1}{2}} \varphi dx = \int_{\mathbb{R}^n} f_i \varphi dx, \quad \text{for every } \varphi \in \mathcal{S}.$$

We set

$$u_3 := u - u_1 - u_2.$$

We claim that the function u_3 is smooth in $\mathbb{R}^n$ whenever u is a solution of (2.1)-(2.2) in the sense of Definition 2.1.1 or 2.1.2. Then taking (2.13) into account we have $e^{nu} \in L^p_{loc}(\mathbb{R}^n)$ for every $p < \infty$ and hence $f_2 \in L^p_{loc}(\mathbb{R}^n)$. Therefore, for every $x \in B_R$ by Hölder's inequality

$$\begin{aligned}
|u_2(x)| &\le C\int_{|y|<2R} \left|\log\left(\frac{1+|y|}{|x-y|}\right)\right| |f_2(y)|dy + C\int_{|y|\ge 2R} \left|\log\left(\frac{1+|y|}{|x-y|}\right)\right| |f_2(y)|dy \\
&\le C\left(\log(1+2R)\|f_2\|_{L^1(B_{2R})} + \|\log(\cdot)\|_{L^2(B_{3R})}\|f_2\|_{L^2(B_{2R})}\right) \\
&\quad + C\log(3R)\|f_2\|_{L^1(B^c_{2R})},
\end{aligned}$$

and for every $0 < |\alpha| \le n-1$ again by Hölder's inequality

$$\begin{aligned}
|D^\alpha u_2(x)| &\le C\int_{|y|<2R} \frac{1}{|x-y|^{|\alpha|}}|f_2(y)|dy + C\int_{|y|\ge 2R} \frac{1}{|x-y|^{|\alpha|}}|f_2(y)|dy \\
&\le C\||(\cdot)|^{-|\alpha|}\|_{L^p(B_{3R})}\|f_2\|_{L^{p'}(B_{2R})} + CR^{-|\alpha|}\|f_2\|_{L^1(B^c_{2R})},
\end{aligned}$$

where $p \in (1, \frac{n}{n-1})$. Thus $u_2 \in W^{n-1,\infty}_{loc}(\mathbb{R}^n)$ and by Sobolev embeddings we have $u_2 \in C^{n-2}(\mathbb{R}^n)$, which implies that $u = u_1 + u_2 + u_3 \in C^{n-2}(\mathbb{R}^n)$. Now to prove $u \in C^\infty(\mathbb{R}^n)$ we proceed by induction.

Set $\tilde{u} = u_1 + u_2$. Then for $0 < |\alpha| \le n-1$

$$D^\alpha \tilde{u}(x) = \frac{(n-1)!}{\gamma_n} \int_{\mathbb{R}^n} D_x^\alpha \log\left(\frac{1+|y|}{|x-y|}\right) e^{nu(y)} dy =: \int_{\mathbb{R}^n} K_\alpha(x-y) e^{nu(y)} dy, \quad x \in \mathbb{R}^n.$$

Notice that the function K_α is smooth in $\mathbb{R}^n \setminus \{0\}$ and it also satisfies the estimate

$$|D^\beta K_\alpha(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+|\beta|}}, \quad \beta \in \mathbb{N}^n,\, x \in \mathbb{R}^n \setminus \{0\}.$$

We rewrite the function $D^\alpha \tilde{u}(x)$ as

$$\begin{aligned} D^\alpha \tilde{u}(x) &= \int_{\mathbb{R}^n} \eta(x-y) K_\alpha(x-y) e^{nu(y)} dy + \int_{\mathbb{R}^n} (1-\eta(x-y)) K_\alpha(x-y) e^{nu(y)} dy \\ &= \int_{\mathbb{R}^n} \eta(x-y) K_\alpha(x-y) e^{nu(y)} dy + \int_{\mathbb{R}^n} (1-\eta(y)) K_\alpha(y) e^{nu(x-y)} dy, \end{aligned}$$

where $\eta \in C^\infty(\mathbb{R}^n)$ satisfies

$$\eta(x) = \begin{cases} 0 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| \ge 2. \end{cases}$$

If we assume $u \in C^k(\mathbb{R}^n)$ for some integer $k \ge 1$ then observing that $\eta K_\alpha \in C^\infty(\mathbb{R}^n)$, $D^\beta(\eta K_\alpha) \in L^\infty(\mathbb{R}^n)$ and $1-\eta$ is compactly supported, one has for $|\beta| \le k$

$$D^{\alpha+\beta} \tilde{u}(x) = \int_{\mathbb{R}^n} D_x^\beta(\eta(x-y) K_\alpha(x-y)) e^{ny(y)} dy + \int_{\mathbb{R}^n} (1-\eta(y)) K_\alpha(y) D_x^\beta e^{nu(x-y)} dy.$$

Thus $u \in C^{k+n-1}(\mathbb{R}^n)$ thanks to the claim that $u_3 \in C^\infty(\mathbb{R}^n)$, which proves our induction argument.

It remains to show that $u_3 \in C^\infty(\mathbb{R}^n)$ whenever u is a solution of (2.1)-(2.2) in the sense of Definition 2.1.1 or 2.1.2.

In the case of Definition 2.1.2 from Lemma 2.2.4 we have that u_3 is a polynomial of degree at most $n-1$ and hence it is smooth. On the other hand, if we consider Definition 2.1.1 then by Lemma 2.2.3 we get $\Delta^{\frac{n-1}{2}} u_3 \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and it also satisfies (2.12) with $p = u_3$. Therefore, by [71, Proposition 2.22] we have $\Delta^{\frac{n-1}{2}} u_3 \in C^\infty(\mathbb{R}^n)$ which implies that $u_3 \in C^\infty(\mathbb{R}^n)$.

□

2.4 Classification of solutions

2.4.1 A fractional version of a lemma of Brézis and Merle

Theorem 2.4.2 below is a fractional version of a lemma of Brézis and Merle [12, Theorem 1], compare also [21, Theorem 5.1], which we shall later need in the proof of Lemma 2.4.8. Although, in our case Theorem 2.4.2 will be used in a smooth setting, here we shall prove it with more generality because of its independent interest. Before stating the theorem we need the following definition, partially inspired by [1, Section 3.3].

Definition 2.4.1. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Assume $f \in L^1(\Omega)$ and $g_j \in L^1(\partial\Omega)$ for $j = 0, 1, ..., \frac{n-3}{2}$. We say that $w \in L_{\frac{1}{2}}(\mathbb{R}^n)$ is a solution of

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}(-\Delta)^{\frac{1}{2}}w = f & in\, \Omega \\ (-\Delta)^j(-\Delta)^{\frac{1}{2}}w = g_j & on\, \partial\Omega,\ j = 0, 1, ..., \frac{n-3}{2} \end{cases} \tag{2.14}$$

if w satisfies

$$\int_{d(x,\partial\Omega)<2, x\in\Omega^c} \frac{|w(x)|}{\sqrt{\delta(x)}} dx < \infty, \tag{2.15}$$

and there exists a function $W \in L^1(\Omega)$ such that $(-\Delta)^{\frac{1}{2}}w = W$ in Ω, i.e.

$$\int_{\mathbb{R}^n} w(-\Delta)^{\frac{1}{2}}\varphi dx = \int_\Omega W\varphi dx \quad \text{for every } \varphi \in T_1, \tag{2.16}$$

and the function W satisfies

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}W = f & in\, \Omega \\ (-\Delta)^j W = g_j & on\, \partial\Omega,\ j = 0, 1, ..., \frac{n-3}{2}, \end{cases} \tag{2.17}$$

i.e.

$$\int_\Omega W(-\Delta)^{\frac{n-1}{2}}\varphi dx = \int_\Omega f\varphi dx - \sum_{j=0}^{\frac{n-3}{2}} \int_{\partial\Omega} g_j \frac{\partial}{\partial\nu}(-\Delta)^{\frac{n-3}{2}-j}\varphi d\sigma \quad \text{for every } \varphi \in T_2,$$

where the spaces of test functions T_1 and T_2 are defined by

$$T_1 := \left\{ \varphi \in C^\infty(\Omega) \cap C^{\frac{1}{2}}(\mathbb{R}^n) : \begin{cases} (-\Delta)^{\frac{1}{2}}\varphi = \psi & \text{in } \Omega \\ \varphi = 0 & \text{on } \Omega^c \end{cases} \text{for some } \psi \in C_c^\infty(\Omega), \right\},$$

and

$$T_2 := \left\{ \varphi \in C^{n-1}(\overline{\Omega}) : \Delta^j\varphi = 0 \text{ on } \partial\Omega,\ j = 0, 1, \dots, \frac{n-3}{2} \right\}.$$

Notice that the left hand side of (2.16) is well-defined thanks to the assumption (2.15) and Lemma 2.4.4 below.

Lemma 2.4.1 (Maximum Principle). *Let w be a solution of (2.14) with $f, g_j \geq 0$ in the sense of Definition 2.4.1. If $w \geq 0$ on Ω^c then $w \geq 0$ in Ω.*

Proof. First notice that the conditions $f \geq 0$, $g_j \geq 0$ implies that $W \geq 0$ in Ω, where $W \in L^1(\Omega)$ is a solution of (2.17). Now consider a test function $\psi \in C_c^\infty(\Omega)$ such that $\psi \geq 0$ in Ω. Let $\varphi \in T_1$ be the solution of $(-\Delta)^{\frac{1}{2}}\varphi = \psi$ in Ω. Then by classical maximum principle one has $\varphi \geq 0$ in Ω. Since the constant $C_{n,\frac{1}{2}} > 0$ in Proposition 2.5.2 we get

$$(-\Delta)^{\frac{1}{2}}\varphi(x) < 0 \qquad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega},$$

and from (2.16)

$$\int_{\Omega} w\psi dx = \int_{\Omega} w(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\Omega} W\varphi dx - \int_{\Omega^c} w(-\Delta)^{\frac{1}{2}}\varphi dx \geq 0,$$

which completes the proof. □

Theorem 2.4.2. *Let $f \in L^1(B_R)$. Let $u \in L^1(B_R)$ be a solution of (2.14) (in the sense of Definition 2.4.1) with $g_j = 0$ for $j = 0, 1, ..., \frac{n-3}{2}$ and $u = 0$ on B_R^c. Then for any $p \in \left(0, \frac{\gamma_n}{\|f\|_{L^1(B_R)}}\right)$*

$$\int_{B_R} e^{np|u|} dx \leq C(p, R).$$

Proof. We set

$$\overline{W}(x) = \int_{B_R} \Psi(x-y)|f(y)|dy \quad x \in \mathbb{R}^n,$$

where

$$\Psi(x) := \frac{\Gamma(\frac{1}{2})}{n2^{n-2}|B_1|\Gamma(\frac{n}{2})\left(\frac{n-3}{2}\right)!}\frac{1}{|x|},$$

is a fundamental solution of $(-\Delta)^{\frac{n-1}{2}}$ in $\mathbb{R}^n$ (see [26, Section 2.6]). Then $\overline{W} \in L^1(B_R)$ satisfies

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}\overline{W} = |f| & \text{in } B_R \\ (-\Delta)^j \overline{W} \geq 0 & \text{on } \partial B_R,\ j = 0, 1, ..., \frac{n-3}{2}, \end{cases}$$

and by maximum principle $\overline{W} \geq |W|$ in B_R, where $W \in L^1(B_R)$ is a solution of (2.17). Let us define

$$\overline{u}(x) := \Phi * (\overline{W}\chi_{B_R})(x) = \frac{(\frac{n-3}{2})!}{2\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1}}\overline{W}(y)\chi_{B_R}(y)dy, \quad x \in \mathbb{R}^n,$$

where Φ is given in Lemma 2.5.1 below. Noticing

$$\frac{1}{\gamma_n} = |S^{n-1}|\frac{\Gamma(\frac{1}{2})}{n2^{n-2}|B_1|\Gamma(\frac{n}{2})\left(\frac{n-3}{2}\right)!}\frac{(\frac{n-3}{2})!}{2\pi^{\frac{n+1}{2}}},$$

in view of Lemma 2.4.3 below one has

$$|\overline{u}(x)| \leq C + \frac{1}{\gamma_n}\int_{|y|<R}|f(y)||\log|x-y||dy, \quad x \in \mathbb{R}^n,$$

which yields

$$\overline{u} \in L^q_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \setminus B_{R+\delta}), \quad q \in [1, \infty),\ \delta > 0.$$

Moreover, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{B_R}\overline{W}\varphi dx = \int_{\mathbb{R}^n}\overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{B_R}\overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx + \int_{B_R^c}\overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx, \tag{2.18}$$

thanks to Lemma 2.5.1 below.

We claim that (2.18) holds for $\varphi \in T_1$. Then for any $\varphi \in T_1$ with $\varphi \geq 0$

$$\int_{B_R} (\overline{u} \pm u)(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{B_R} \underbrace{(\overline{W} \pm W)}_{\geq 0} \varphi dx - \int_{B_R^c} \overline{u} \underbrace{(-\Delta)^{\frac{1}{2}}\varphi}_{\leq 0} dx \geq 0,$$

and by maximum principle one has $\overline{u} \geq |u|$ in B_R and the theorem follows at once.

To prove the claim we consider a mollifying sequence $\varphi_k := \varphi * \rho_k$, where $\rho_k(x) = k^n \rho(kx)$. Then (see [1, Section A])

$$(-\Delta)^{\frac{1}{2}}\varphi_k(x) = \varphi * (-\Delta)^{\frac{1}{2}}\rho_k(x) \quad x \in \mathbb{R}^n,$$

and

$$(-\Delta)^{\frac{1}{2}}\varphi_k(x) = \rho_k * (-\Delta)^{\frac{1}{2}}\varphi(x), \quad dist(x, \partial B_R) > \frac{1}{k}. \tag{2.19}$$

Then the uniform convergence of φ_k to φ imply

$$\int_{B_R} \overline{W}\varphi_k dx \xrightarrow{k\to\infty} \int_{B_R} \overline{W}\varphi dx.$$

Using the uniform convergence of $(-\Delta)^{\frac{1}{2}}\varphi_k$ to $(-\Delta)^{\frac{1}{2}}\varphi$ on the compact sets in B_R and the fact that $\mathrm{supp}(-\Delta)^{\frac{1}{2}}\varphi|_{B_R} \subseteq B_R$ we get

$$\int_{B_R} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi_k dx \xrightarrow{k\to\infty} \int_{B_R} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx.$$

It remains to verify that

$$\int_{B_R^c} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi_k dx \xrightarrow{k\to\infty} \int_{B_R^c} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx,$$

which follows immediately from

$$(-\Delta)^{\frac{1}{2}}\varphi_k \xrightarrow{k\to\infty} (-\Delta)^{\frac{1}{2}}\varphi \text{ in } L^q(B_{R+1} \setminus B_R), \text{ for some } q > 1, \tag{2.20}$$

and

$$(-\Delta)^{\frac{1}{2}}\varphi_k \xrightarrow{k\to\infty} (-\Delta)^{\frac{1}{2}}\varphi \text{ in } L^1(B_{R+1}^c). \tag{2.21}$$

With the help of Lemma 2.4.4 below and (2.19) one can get (2.21). To conclude (2.20) first notice that $(-\Delta)^{\frac{1}{2}}\varphi_k$ converges to $(-\Delta)^{\frac{1}{2}}\varphi$ point-wise and that $(-\Delta)^{\frac{1}{2}}\varphi \in L^q(B_{R+1} \setminus B_R)$ for any $q \in [1, 2)$ thanks to Lemma 2.4.4 below. By [44, Theorem 1.9 (Missing term in Fatou's lemma)] it is sufficient to show that for some $q > 1$

$$\int_{R<|x|<R+1} |(-\Delta)^{\frac{1}{2}}\varphi_k(x)|^q dx \leq \int_{R<|x|<R+1} |(-\Delta)^{\frac{1}{2}}\varphi(x)|^q dx + o(1),$$

where $o(1) \to 0$ as $k \to \infty$. Now using the estimate (see for instance [1, Section A])

$$|(-\Delta)^{\frac{1}{2}}\rho_k(x)| \leq Ck^{n+1} \quad x \in \mathbb{R}^n,$$

and fixing t and q such that

$$\frac{2n}{2n+1} < t < 1, \quad 1 < q < \min\left\{\frac{1+nt}{t+nt}, \frac{2nt+t+2}{2n+2}\right\},$$

we bound

$$\begin{aligned}
&\int_{R<|x|<R+1} |(-\Delta)^{\frac{1}{2}}\varphi_k(x)|^q dx \\
&= \int_{R<|x|<R+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}}\varphi_k(x)|^q dx + \int_{R+\frac{1}{k}<|x|<R+1} |(-\Delta)^{\frac{1}{2}}\varphi_k(x)|^q dx \\
&= \int_{R<|x|<R+\frac{1}{k}} |\varphi * (-\Delta)^{\frac{1}{2}}\rho_k(x)|^q dx + \int_{R+\frac{1}{k}<|x|<R+1} |\rho_k * (-\Delta)^{\frac{1}{2}}\varphi(x)|^q dx \\
&\le \|\varphi\|_{L^1}^{q-1} \int_{R<|x|<R+\frac{1}{k}} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}}\rho_k(y)|^q |\varphi(x-y)| dy dx \\
&\quad + \int_{R+\frac{1}{k}<|x|<R+1} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}}\varphi(y)|^q \rho_k(x-y) dy dx \\
&= \int_{R<|y|<R+1+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}}\varphi(y)|^q dy \\
&\quad + \|\varphi\|_{L^1}^{q-1} \int_{R<|x|<R+\frac{1}{k}} \int_{|y|>\frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}}\rho_k(y)|^q |\varphi(x-y)| dy dx \\
&\quad + \|\varphi\|_{L^1}^{q-1} \int_{R<|x|<R+\frac{1}{k}} \int_{|x-y|<R, |y|<\frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}}\rho_k(y)|^q |\varphi(x-y)| dy dx \\
&\le \int_{R<|y|<R+1+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}}\varphi(y)|^q + C\|\varphi\|_{L^1}^{q-1} k^{t(q+nq-n)-1} + C\|\varphi\|_{L^1}^{q-1} k^{q(n+1)-nt-\frac{t}{2}-1} \\
&= \int_{R<|y|<R+1} |(-\Delta)^{\frac{1}{2}}\varphi(y)|^q + o(1),
\end{aligned}$$

where in the last inequality we have used (for the second term)

$$\begin{aligned}
\int_{|x|>\frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}}\rho_k(x)|^q dx &= \int_{|x|>\frac{1}{k^t}} \left| C_{1/2} P.V. \int_{\mathbb{R}^n} \frac{\rho_k(x)-\rho_k(y)}{|x-y|^{n+1}} dy \right|^q dx \\
&\le C \int_{|x|>\frac{1}{k^t}} \int_{|y|<1} \frac{\rho(y)^q}{|x-\frac{y}{k}|^{nq+q}} dy dx \\
&\le C \int_{|y|<1} \int_{|x|>\frac{1}{k^t}} \frac{1}{|x|^{nq+q}} dx dy \\
&\le C k^{t(q+nq-n)}.
\end{aligned}$$

□

Lemma 2.4.3. *Let Ω be a domain in $\mathbb{R}^n$. Let p and q be two positive real numbers. Then*

$$\int_\Omega \frac{dy}{|x-y|^{n+p}} \le \frac{|S^{n-1}|}{p} \frac{1}{\delta(x)^p}, \quad \text{if } \delta(x) := \operatorname{dist}(x, \Omega) > 0,$$

and

$$\int_\Omega \frac{dz}{|x-z|^p |y-z|^q} \le \frac{C_{n,p,q}}{|x-y|^{p+q-n}}, \quad \text{if } p+q>n, p<n, q<n,\ x \ne y,$$

where the constant $C_{n,p,q}$ is given by (an explicit formula can be found in [44, Section 5.10])

$$C_{n,p,q} = \int_{\mathbb{R}^n} \frac{dz}{|z|^p |e_1 - z|^q}.$$

In addition if we also assume that the domain Ω is bounded then

$$\int_\Omega \frac{dy}{|x-y|^n} \leq |\Omega| + |S^{n-1}||\log \delta(x)| \ \ \textit{if } \delta(x) > 0,$$

and

$$\int_\Omega \frac{dz}{|x-z|^p |y-z|^q} \leq C + |S^{n-1}| \left|\log(|x-y|)\right|, \ \textit{if } p+q = n, p < n, q < n, \, x \neq y.$$

Proof. Let us denote the set $\{y - x : y \in \Omega\}$ by $\Omega - x$. Using a change of variable $z \mapsto z - x$ and setting $w = y - x$ we have

$$\int_\Omega \frac{dz}{|x-z|^p |y-z|^q} = \int_{\Omega - x} \frac{dz}{|z|^p |w-z|^q} =: I.$$

If $p + q > n$ then changing the variable $z \mapsto |w| z$ one has

$$I = \frac{1}{|w|^{p+q-n}} \int_{\frac{1}{|w|}(\Omega - x)} \frac{dz}{|z|^p |\frac{w}{|w|} - z|^q} \leq \frac{1}{|w|^{p+q-n}} \int_{\mathbb{R}^n} \frac{dz}{|z|^p |\frac{w}{|w|} - z|^q} = \frac{C_{n,p,q}}{|w|^{p+q-n}}.$$

In the case when $p + q = n$, we split the domain $\Omega - x$ into two disjoint domains:

$$\Omega_1 := (\Omega - x) \cap B_1, \quad \Omega_2 = (\Omega - x) \cap B_1^c.$$

Then

$$I = \sum_{i=1}^2 I_i, \quad I_i := \int_{\Omega_i} \frac{dz}{|z|^p |w-z|^q}.$$

Since Ω_2 is bounded and $q < n$, we have

$$I_2 \leq \int_{\Omega_2} \frac{dz}{|w-z|^q} \leq C.$$

Now using

$$\frac{1}{|\frac{w}{|w|} - z|} \leq \frac{1}{|z|} \left(1 + \frac{2}{|z|}\right) \quad \text{for } |z| \geq 2,$$

and

$$(1+x)^q \leq 1 + C_q x \quad \text{for } x \in (0,1),$$

we bound

$$
\begin{aligned}
I_1 &\le \int_{B_1} \frac{dz}{|z|^p|w-z|^q} = \int_{|z|\le \frac{1}{|w|}} \frac{dz}{|z|^p|\frac{w}{|w|}-z|^q} \\
&\le \underbrace{\int_{|z|\le 2} \frac{dz}{|z|^p|\frac{w}{|w|}-z|^q}}_{\le C} + \int_{2<|z|\le \frac{1}{|w|}} \frac{1}{|z|^n}\left(1+\frac{2}{|z|}\right)^q dz \\
&\le \int_{2<|z|\le \frac{1}{|w|}} \frac{1}{|z|^n}\left(1+\frac{C}{|z|}\right) dz \\
&\le C + |S^{n-1}||\log|w||.
\end{aligned}
$$

Finally, we conclude the lemma by showing that for $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$
\int_\Omega \frac{dy}{|x-y|^{n+p}} \le \int_{|z|>\delta(x)} \frac{dy}{|z|^{n+p}} = \frac{|S^{n-1}|}{p}\frac{1}{\delta(x)^p}, \quad p>0,
$$

and

$$
\int_\Omega \frac{dy}{|x-y|^n} \le |\Omega| + \int_{\Omega\cap B_1(x)} \frac{dy}{|x-y|^n} \le |\Omega| + \int_{\delta(x)<|z|<1} \frac{dy}{|z|^n} = |\Omega| + |S^{n-1}||\log\delta(x)|.
$$

□

Lemma 2.4.4. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\varphi \in C^{k,\sigma}(\mathbb{R}^n)$ for some nonnegative integer k and $0 \le \sigma \le 1$ be such that $\varphi = 0$ on $\mathbb{R}^n \setminus \Omega$. Then for $0 < s < 1$ and for $x \in \mathbb{R}^n \setminus \overline{\Omega}$*

$$
|(-\Delta)^s\varphi(x)| \le C \begin{cases} \min\{\max\{1, \delta(x)^{-2s+k+\sigma}\}, \delta(x)^{-n-2s}\} & \text{if } k+\sigma \ne 2s \\ \min\{|\log\delta(x)|, \delta(x)^{-n-2s}\} & \text{if } k+\sigma = 2s, \end{cases}
$$

where $\delta(x) := \operatorname{dist}(x, \Omega)$.

Proof. We claim that

$$
|\varphi(y)| \le C|x-y|^{k+\sigma}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega},\ y \in \Omega,
$$

which can be verified using the Taylor's expansion

$$
\varphi(y) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!}\underbrace{D^\alpha\varphi(x)}_{=0}(y-x)^\alpha + \sum_{|\beta|=k} \frac{|\beta|}{\beta!}(y-x)^\beta \int_0^1 (1-t)^{|\beta|-1} D^\beta\varphi(x+t(y-x))dt,
$$

and

$$
|D^\beta\varphi(x+t(y-x))| = |D^\beta\varphi(x+t(y-x)) - D^\beta\varphi(x)| \le C|t(x-y)|^\sigma \le C|x-y|^\sigma.
$$

Therefore, by Proposition 2.5.2

$$
|(-\Delta)^s\varphi(x)| = \left| C_{n,s}\int_\Omega \frac{\varphi(y)}{|x-y|^{n+2s}}dy \right| \le C\int_\Omega \frac{dy}{|x-y|^{n+2s-k-\sigma}},\ x \in \mathbb{R}^n \setminus \overline{\Omega},
$$

and

$$|(-\Delta)^s\varphi(x)| \leq C\int_\Omega \frac{|\varphi(y)|}{|x-y|^{n+2s}}dy \leq C\int_\Omega \frac{|\varphi(y)|}{\delta(x)^{n+2s}}dy \leq \frac{C}{\delta(x)^{n+2s}}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Now the proof follows at once from Lemma 2.4.3. □

2.4.2 Asymptotic behavior of solutions

First we study the asymptotic behavior of v defined in (2.8).

Lemma 2.4.5. *Let u be a smooth solution of (2.1)-(2.2) and let v be given by (2.8). Then there exists a constant $C > 0$ such that*

$$v(x) \geq -\alpha \log|x| - C, \quad |x| \geq 4.$$

Proof. The proof follows as in the proof of [45, Lemma 2.1]. □

A consequence of the above lemma is the following Proposition, compare Lemmas 2.2.4, 2.2.5.

Proposition 2.4.6. *Let u be a smooth solution of (2.1)-(2.2) in the sense of Definition 2.1.1 or 2.1.2 and let v be defined by (2.8). Then the function*

$$P(x) := u(x) - v(x), \quad x \in \mathbb{R}^n,$$

is a polynomial of degree at most $n-1$ and P is bounded above.

Proof. Since (2.2) implies (2.11), by Lemmas 2.2.4 and 2.2.5 we have that P is a polynomial of degree at most $n-1$. On the other hand, using Lemma 2.4.5 one can get that P is bounded above (the proof is very similar to [56, Lemma 11]). □

Lemma 2.4.7. *Let $n \geq 3$ be an odd integer and let u be a smooth solution of (2.1)-(2.2) and v be given by (2.8). Then*

(i) $v \in C^\infty(\mathbb{R}^n)$ and $D^\alpha v \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| \leq n-1$.

(ii) There exists a constants $C > 0$ such that

$$\int_{\partial B_4(x)} |(-\Delta)^j(-\Delta)^{\frac{1}{2}}v(y)|d\sigma(y) \leq C \text{ for every } x \in \mathbb{R}^n,\ j = 0,1,2,...,\frac{n-3}{2}.$$

(iii) v is a pointwise solution of

$$(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}}v = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n.$$

(iv) v solves (2.14) with $f = (n-1)!e^{nu}$ and $g_j = (-\Delta)^j(-\Delta)^{\frac{1}{2}}v$ for $j = 0,1,2,\ldots,\frac{n-3}{2}$.

Proof. We divide the proof into several steps.

Step 1. From Proposition 2.4.6 we have the smoothness of v and by Lemma 2.2.3 we get $D^\alpha v \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}^n$ with $0 \le |\alpha| \le n-1$.

Step 2. In this step we use (i) to prove (ii). In fact by Lemmas 2.5.3, 2.5.4, below we have

$$\begin{aligned}\int_{\partial B_4(x)} |(-\Delta)^j(-\Delta)^{\frac{1}{2}}v(y)|d\sigma(y) &= \int_{\partial B_4(x)} |(-\Delta)^{\frac{1}{2}}(-\Delta)^j v(z)|d\sigma(z)\\ &\le C\int_{\partial B_4(x)}\int_{\mathbb{R}^n}\frac{e^{nu(y)}}{|y-z|^{2j+1}}dyd\sigma(z)\\ &= C\int_{\mathbb{R}^n} e^{nu(y)}\int_{\partial B_4(x)}\frac{1}{|y-z|^{2j+1}}d\sigma(z)dy\\ &\le C.\end{aligned}$$

Step 3. We claim that for $g \in C^\infty(\mathbb{R}^n) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}g\varphi dx = \int_{\mathbb{R}^n} g(-\Delta)^{\frac{1}{2}}\varphi dx \text{ for every } \varphi \in C_c^\infty(\mathbb{R}^n).$$

To prove the claim we consider a approximating sequence

$$g_k(x) := g(x)\psi(\frac{x}{k}), \quad \psi \in C^\infty(\mathbb{R}^n), \quad \psi(x) = \begin{cases} 1 & \text{if } |x| < 1\\ 0 & \text{if } |x| > 2.\end{cases}$$

Then $g_k \in \mathcal{S}(\mathbb{R}^n)$ and hence

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}g_k\varphi dx = \int_{\mathbb{R}^n} g_k(-\Delta)^{\frac{1}{2}}\varphi dx.$$

Now the claim follows from the locally uniform convergence of $(-\Delta)^{\frac{1}{2}}g_k$ to $(-\Delta)^{\frac{1}{2}}g$ and the $L_{\frac{1}{2}}(\mathbb{R}^n)$ convergence of g_k to g.

Step 4. Using *Step 3* with $g = (-\Delta)^{\frac{n-1}{2}}v$ we have

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}}v\varphi dx = \int_{\mathbb{R}^n}(-\Delta)^{\frac{n-1}{2}}v(-\Delta)^{\frac{1}{2}}\varphi dx = (n-1)!\int_{\mathbb{R}^n} e^{nu}\varphi dx,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, which implies (iii).

To complete (iv) it suffices to show that $W := (-\Delta)^{\frac{1}{2}}v \in C^\infty(\mathbb{R}^n)$ and it satisfies (2.15)-(2.17) with $w = v$.

The smoothness of v implies $W \in C^\infty(\mathbb{R}^n)$ and (2.15). Moreover, using integration by parts (see [1, Proposition 1.2.1]) one can get (2.16).

One must notice that the function u in [1, Proposition 1.2.1] is in $C^{1+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^n)$ but still we can use it since our function $v \in C^\infty(\mathbb{R}^n) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$.

Finally, we prove (2.17) by showing that W is a classical solution of (2.17). Since W is smooth in $\mathbb{R}^n$ clearly it satisfies the boundary conditions. Using *step 3* (with $g = v$)

and Lemma 2.2.3 (with $f = (n-1)!e^{nu}$) we have for every $\varphi \in C_c^\infty(\Omega)$

$$\begin{aligned}\int_\Omega (-\Delta)^{\frac{n-1}{2}} W\varphi dx &= \int_\Omega W(-\Delta)^{\frac{n-1}{2}}\varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} v(-\Delta)^{\frac{n-1}{2}}\varphi dx \\ &= \int_{\mathbb{R}^n} v(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}}\varphi dx = (n-1)!\int_{\mathbb{R}^n} e^{nu}\varphi dx,\end{aligned}$$

that is

$$(-\Delta)^{\frac{n-1}{2}} W = (n-1)!e^{nu} \quad \text{in } \Omega.$$

□

The following lemma is the crucial part in the proof of Theorem 2.1.2.

Lemma 2.4.8. *Let u be a smooth solution of (2.1)-(2.2) and v be given by (2.8). Then for any $\varepsilon > 0$ there exists $R > 0$ such that for $|x| > R$*

$$v(x) \le (-\alpha + \varepsilon)\log|x|.$$

Proof. Step 1. For any $\varepsilon > 0$ there exists a $R > 0$ such that for $|x| \ge R$

$$v(x) \le (-\alpha + \frac{\varepsilon}{2})\log|x| - \frac{(n-1)!}{2}\int_{B_1(x)} \log|x-y|e^{nu(y)}dy. \tag{2.22}$$

The proof of (2.22) is very similar to the proof of [45, Lemma 2.4]. As a consequence of (2.22) using Jensen's inequality we have the following estimate

$$\|v^+\|_{L^p(\mathbb{R}^n)} \le |\alpha - \frac{\varepsilon}{2}|\|\log|\cdot|\|_{L^p(B_1)} + \frac{(n-1)!}{2}\|e^{nu}\|_{L^1(\mathbb{R}^n)}\|\log|\cdot|\|_{L^p(B_1)}, \quad 1 \le p < \infty. \tag{2.23}$$

Step 2. We claim that there exists $p > 1$ and $C > 0$ independent of x_0 such that $\|e^{nu}\|_{L^p(B_1(x_0))} \le C$. Then using Hölder inequality one can bound the second term on the right hand side of (2.22) uniformly in x and that completes the proof of the lemma.

To prove the claim, first notice that it is sufficient to consider $x_0 \in \mathbb{R}^n \setminus B_R$ for any fixed $R > 0$. We choose $R > 0$ large enough such that

$$(n-1)!\|e^{nu}\|_{L^1(B^c_{R-1})} < \frac{\gamma_n}{2}.$$

Let $w \in C^0(\mathbb{R}^n)$ be the solution of

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}(-\Delta)^{\frac{1}{2}} w = (n-1)!e^{nu} & in\, B_4(x_0) \subset \mathbb{R}^n \\ (-\Delta)^j(-\Delta)^{\frac{1}{2}} w = 0 & on\, \partial B_4(x_0), for\, j = 0, 1, ..., \frac{n-3}{2} \\ w = 0 & on\, \mathbb{R}^n \setminus B_4(x_0), \end{cases}$$

in the sense of Definition 2.4.1. Since u is smooth by Schauder's estimates and bootstrap argument we have $W = (-\Delta)^{\frac{1}{2}} w \in C^\infty(\overline{B_4(x_0)})$ which solves (2.14) with $f = (n-1)!e^{nu}$ and $g_j = (-\Delta)^j(-\Delta)^{\frac{1}{2}} v$ for every $j = 0, 1, 2, \ldots, \frac{n-3}{2}$. Then using Green's representation formula (see [14, Theorem 3]) one can get $w \in C^0(\mathbb{R}^n)$ (in fact $w \in C^{\frac{1}{2}}(\mathbb{R}^n)$, see [68]), which is the pointwise continuous unique solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}} w = W & \text{in } B_4(x_0) \\ w = 0 & \text{on } B_4(x_0)^c. \end{cases}$$

Moreover, w satisfies (2.16) thanks to [1, Proposition 3.3.3].

We set $h = v - w$. Then we have that $h \in C^0(\mathbb{R}^n)$, $(-\Delta)^{\frac{1}{2}} h \in C^\infty(\overline{B_4(x_0)})$ and

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} h = 0 & \textit{in } B_4(x_0) \\ (-\Delta)^j (-\Delta)^{\frac{1}{2}} h = (-\Delta)^j (-\Delta)^{\frac{1}{2}} v & \textit{on } \partial B_4(x_0),\ j = 0, 1, ..., \frac{n-3}{2} \\ h = v & \textit{on } \mathbb{R}^n \setminus B_4(x_0), \end{cases} \tag{2.24}$$

thanks to Lemma 2.4.7. Indeed, by Lemma 2.4.9 below there exists a constant $C > 0$ independent of the choice of $x_0 \in \mathbb{R}^n$ such that

$$h(x) \le C \quad \text{for every } x \in B_1(x_0).$$

Hence by Proposition 2.4.6

$$u = v + P \le C + h + w \le C + w,$$

and by Theorem 2.4.2 we have the proof. □

A simple consequence of Lemma 2.4.8 is that

$$\lim_{|x| \to \infty} u(x) = -\infty, \tag{2.25}$$

thanks to Proposition 2.4.6. Using (2.25) one can show that

$$\lim_{|x| \to \infty} D^\beta v(x) = 0 \text{ for every } \beta \in \mathbb{N}^n \text{ with } 0 < |\beta| < n - 1\ .$$

Now the proof of Theorem 2.1.2 follows at once from Lemmas 2.4.5, 2.4.8 and Proposition 2.4.6.

Lemma 2.4.9. *Let $h \in C^0(\mathbb{R}^n)$ be given by (2.24). Then there exists a constant $C > 0$ (independent of x_0) such that*

$$h(x) \le C, \textit{ for every } x \in B_1(x_0).$$

Proof. Let us write $h = h_1 + h_2$ where $h_1, h_2 \in C^0(\mathbb{R}^n)$ be such that

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_1 = (-\Delta)^{\frac{1}{2}} h & \textit{in } B_4(x_0) \\ h_1 = 0 & \textit{on } B_4(x_0)^c, \end{cases}$$

and

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_2 = 0 & \textit{in } B_4(x_0) \\ h_2 = h = v & \textit{on } B_4(x_0)^c. \end{cases}$$

Let $h_3 \in C^0(\mathbb{R}^n)$ be such that

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_3 = 0 & \textit{in } B_4(x_0) \\ h_3 = v^+ & \textit{on } B_4(x_0)^c. \end{cases}$$

Then by maximum principle

$$h_2 \le h_3 \text{ on } \mathbb{R}^n.$$

Without loss of generality we can assume that $x_0 = 0$. Then the Poisson formula gives (see [14, Theorem 1])

$$h_3(x) = \int_{|y|>4} P(x,y)v^+(y)dy, \quad x \in B_4,$$

where

$$P(x,y) = C_n \left(\frac{16-|x|^2}{|y|^2-16}\right)^{\frac{1}{2}} \frac{1}{|x-y|^n}.$$

Now for $x \in B_2$ by Hölder's inequality we get

$$\begin{aligned}
|h_3(x)| &\le C \int_{|y|>4} \left(\frac{1}{|y|^2-16}\right)^{\frac{1}{2}} \frac{1}{|y|^n} v^+(y)dy \\
&\le C \left(\int_{|y|>4} v^+(y)^3 dy\right)^{\frac{1}{3}} \left(\int_{|y|>4} \frac{1}{(|y|^2-16)^{\frac{3}{4}}} \frac{1}{|y|^{\frac{3n}{2}}} dy\right)^{\frac{2}{3}} \\
&\le C\|v^+\|_{L^3(\mathbb{R}^n)} \le C,
\end{aligned}$$

where the last inequality follows from (2.23). By Lemma 2.4.10 below we have

$$h \le C, \text{ for every } x \in B_1(x_0),$$

where C being independent of x_0. □

Lemma 2.4.10. *Let $h \in C^0(\mathbb{R}^n)$ solves (2.24). Let $h_1 \in C^0(\mathbb{R}^n)$ be the solution of*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_1 = (-\Delta)^{\frac{1}{2}} h & \text{in } B_4(x_0) \\ h_1 = 0 & \text{on } B_4(x_0)^c. \end{cases}$$

Then there exists a constant $C = C(n)$ such that

$$\|h_1\|_{L^\infty(B_1(x_0))} \le C.$$

Proof. We assume that $x_0 = 0$. Using Green's representation formula (see [14, Theorem 3]) the solution is given by

$$h_1(x) = \int_{B_4} G_2(x,y)(-\Delta)^{\frac{1}{2}} h(y)dy, \quad x \in B_4,$$

where

$$G_2(x,y) = C_n|x-y|^{1-n} \int_0^{r_0(x,y)} \frac{r^{\frac{1}{2}-1}}{(1+r)^{\frac{n}{2}}} dr, \quad r_0(x,y) = \frac{(16-|x|^2)(16-|y|^2)}{|x-y|^2}.$$

Since

$$\frac{r^{-\frac{1}{2}}}{(1+r)^{\frac{n}{2}}} \in L^1((0,\infty)),$$

we have

$$|G_2(x,y)| \le C|x-y|^{1-n}.$$

For $|z| \le 1$ using (2.24), Lemma 2.4.7 and Lemma 2.5.5 below we bound

$$\begin{aligned}
|h_1(z)| &\le \int_{B_4} |G_2(z,y)||(-\Delta)^{\frac{1}{2}} h(y)|dy \\
&\le \sum_{i=0}^{\frac{n-3}{2}} \int_{B_4} |G_2(z,y)| \int_{\partial B_4} \left|(-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x)\right| \left| \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(y,x)\right)\right| d\sigma(x) dy \\
&\le C \sum_{i=0}^{\frac{n-3}{2}} \int_{B_4} |z-y|^{1-n} \int_{\partial B_4} \left|(-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x)\right| |x-y|^{1+2i-n} d\sigma(x) dy \\
&= C \sum_{i=0}^{\frac{n-3}{2}} \int_{|x|=4} \left|(-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x)\right| \int_{|y|<4} |z-y|^{1-n} |x-y|^{1+2i-n} dy d\sigma(x) \\
&\le C \sum_{i=0}^{\frac{n-3}{2}} \int_{|x|=4} \left|(-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x)\right| d\sigma(x) \\
&\le C.
\end{aligned}$$

□

2.4.3 Characterization of spherical solution

Proof of Theorem 2.1.3 One can verify easily that $(i) \Rightarrow (ii)$-(vi). On the other hand, by Theorem 2.1.2 (ii) -(iv) are equivalent. Moreover, $(ii) \Rightarrow (i)$ thanks to [78, Theorem 4.1]. To show that $(v) \Rightarrow (i)$ and $(vi) \Rightarrow (i)$ one can follow the arguments in [56].

Finally to prove (2.9) we use [56, Theorem 6 and Lemma 3]. Since the polynomial P is bounded from above, $\deg(P)$ must be even and let it be $2k$. Then $\Delta^k P = C_0$ on $\mathbb{R}^n$ and $\Delta^{k+1} P = 0$ on $\mathbb{R}^n$. By [56, Lemma 3] we have

$$\sum_{i=0}^{k} c_i R^{2i} \Delta^i P(0) = \frac{1}{|B_R|} \int_{B_R} P(x)dx \le \sup_{\mathbb{R}^n} P \le C, \quad \text{for every } R > 0,$$

where the constants $c_i's$ are positive and hence $C_0 = \Delta^k P(0) \le 0$. We claim that $C_0 < 0$. Otherwise, by Theorem 2.1.2 and [56, Theorem 6] one gets $\deg(P) \le 2k-2$, which is a contradiction. □

2.5 Some useful lemmas

Lemma 2.5.1 (Fundamental solution)**.** *For $n \ge 3$ odd integer the function*

$$\Phi(x) := \frac{(\frac{n-3}{2})!}{2\pi^{\frac{n+1}{2}}} \frac{1}{|x|^{n-1}} = \frac{1}{\gamma_n} (-\Delta)^{\frac{n-1}{2}} \log \frac{1}{|x|}$$

is a fundamental solution of $(-\Delta)^{\frac{1}{2}}$ *in* $\mathbb{R}^n$ *in the sense that for all* $f \in L^1(\mathbb{R}^n)$ *we have* $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^n)$ *and for all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}}(\Phi * f)\varphi dx := \int_{\mathbb{R}^n} (\Phi * f)(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\mathbb{R}^n} f\varphi dx. \tag{2.26}$$

Proof. To show $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^n)$ we bound

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\Phi * f(x)|}{1+|x|^{n+1}}dx &\le C\int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+1}}\left(\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}}|f(y)|dy\right)dx \\ &= C\int_{\mathbb{R}^n} |f(y)|\left(\int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+1}}\frac{1}{|x-y|^{n-1}}dx\right)dy \\ &\le C\int_{\mathbb{R}^n} |f(y)|\left(\int_{B_1} \frac{dx}{|x|^{n-1}} + \int_{\mathbb{R}^n} \frac{dx}{1+|x|^{n+1}}\right)dy \\ &\le C\|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{2.27}$$

If $f \in C_c^\infty(\mathbb{R}^n)$ then (2.26) is true by [44, Theorem 5.9]. For the general case $f \in L^1(\mathbb{R}^n)$ choose $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$. Then using (2.27) with $f \equiv f_k - f$ one has

$$\int_{\mathbb{R}^n} |\Phi * (f_k - f)||(-\Delta)^{\frac{1}{2}}\varphi|dx \le C\int_{\mathbb{R}^n} \frac{|\Phi * (f_k - f)(x)|}{1+|x|^{n+1}}dx \le C\|f_k - f\|_{L^1(\mathbb{R}^n)} \to 0,$$

that is

$$\int_{\mathbb{R}^n} (\Phi * f_k)(-\Delta)^{\frac{1}{2}}\varphi dx \to \int_{\mathbb{R}^n} (\Phi * f)(-\Delta)^{\frac{1}{2}}\varphi dx.$$

Now the proof follows from

$$\int_{\mathbb{R}^n} (\Phi * f_k)(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\mathbb{R}^n} f_k\varphi dx \to \int_{\mathbb{R}^n} f\varphi dx.$$

□

Combining [71, Proposition 2.4] and [22, Lemma 3.2] we state the following proposition:

Proposition 2.5.2. *Let* Ω *be an open set in* $\mathbb{R}^n$. *Let* $u \in C^{2\sigma+\epsilon}(\Omega) \cap L_\sigma(\mathbb{R}^n)$ *for some* $\sigma \in (0,1)$ *and* $\epsilon > 0$. *Then* $(-\Delta)^\sigma u$ *is continuous in* Ω *and for every* $x \in \Omega$ *we have*

$$\begin{aligned}(-\Delta)^\sigma u(x) &= C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}dy \\ &= -\frac{1}{2}C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2\sigma}}dy, \end{aligned} \tag{2.28}$$

where $C^{2\sigma+\epsilon}(\Omega) := C^{0,2\sigma+\epsilon}(\Omega)$ *for* $2\sigma+\epsilon \le 1$ *and* $C^{2\sigma+\epsilon}(\Omega) = C^{1,2\sigma+\epsilon-1}(\Omega)$ *for* $2\sigma+\epsilon > 1$ *and the constant* $C_{n,\sigma}$ *is given by*

$$C_{n,\sigma} := \left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2\sigma}}dx\right)^{-1}.$$

The advantage of (2.28) is that the integral is not singular at the origin for a C^2 function.

Lemma 2.5.3. *Let ℓ be a nonnegative integer. Let v be a smooth function on $\mathbb{R}^n$ such that $D^\alpha v \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index α with $|\alpha| \leq \ell$. Then*

$$(-\Delta)^{\frac{1}{2}} D^\alpha v(x) = D^\alpha (-\Delta)^{\frac{1}{2}} v(x), \qquad \textit{for every } x \in \mathbb{R}^n,\ |\alpha| \leq \ell.$$

Proof. It suffices to show the case for $|\alpha| = 1$. Let $\varphi \in C_c^\infty(B_2)$ be such that $\varphi = 1$ on B_1 and $0 \leq \varphi \leq 1$. Let us define $v_k(x) := \varphi(\frac{x}{k})v(x)$. Then we have

$$(-\Delta)^{\frac{1}{2}} D^\alpha v_k(x) = D^\alpha (-\Delta)^{\frac{1}{2}} v_k(x). \tag{2.29}$$

We claim that

$$(-\Delta)^{\frac{1}{2}} D^\alpha v_k \xrightarrow{k\to\infty} (-\Delta)^{\frac{1}{2}} D^\alpha v \quad \text{in } C^0_{loc}(\mathbb{R}^n), \quad |\alpha| = 0, 1.$$

To prove our claim first we fix a $R > 0$. Then for $x \in B_R$ and $k \geq R + 1$ we get

$$\begin{aligned}
&\left|(-\Delta)^{\frac{1}{2}} D^\alpha v_k(x) - (-\Delta)^{\frac{1}{2}} D^\alpha v(x)\right| \\
&= C_{n,\frac{1}{2}} \left| P.V. \int_{\mathbb{R}^n} \frac{D^\alpha v_k(x) - D^\alpha v_k(y) - D^\alpha v(x) + D^\alpha v(y)}{|x-y|^{n+1}} dy \right| \\
&\leq C_{n,\frac{1}{2}} \int_{|y|>k} \frac{2|D^\alpha v(y)| + |\alpha| k^{-1} |(D^\alpha \varphi)(\frac{y}{k})||v(y)|}{|x-y|^{n+1}} dy \\
&\xrightarrow{k\to\infty} 0.
\end{aligned}$$

Thus $\{D^\alpha(-\Delta)^{\frac{1}{2}} v_k\} = \{(-\Delta)^{\frac{1}{2}} D^\alpha v_k\}$ and $\{(-\Delta)^{\frac{1}{2}} v_k\}$ are Cauchy sequences in $C^0_{loc}(\mathbb{R}^n)$ and consequently

$$D^\alpha(-\Delta)^{\frac{1}{2}} v_k(x) \xrightarrow{k\to\infty} D^\alpha(-\Delta)^{\frac{1}{2}} v(x),$$

and together with (2.29) complete the proof. □

Lemma 2.5.4. *We set*

$$f_0(x) := \log|x|, \quad f_j(x) := \frac{1}{|x|^j} \ \textit{for } j = 1, 2, \dots, n-1.$$

Then for $0 < \sigma < 1$ we have

$$(-\Delta)^\sigma f_j(x) = \frac{1}{|x|^{j+2\sigma}} (-\Delta)^\sigma f_j(e_1), \qquad \textit{for } |x| > 0 \textit{ and } 0 \leq j \leq n-1.$$

Proof. Since $f_j \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$ using (2.28) we get

$$\begin{aligned}
(-\Delta)^\sigma f_j(x) = (-\Delta)^\sigma f_j(|x|e_1) &= c_n P.V. \int_{\mathbb{R}^n} \frac{f_j(|x|e_1) - f_j(y)}{||x|e_1 - y|^{n+2\sigma}} dy \\
&= \frac{1}{|x|^{j+2\sigma}} c_n P.V. \int_{\mathbb{R}^n} \frac{f_j(e_1) - f_j(y)}{|e_1 - y|^{n+2\sigma}} dy \\
&= \frac{1}{|x|^{j+2\sigma}} (-\Delta)^\sigma f_j(e_1),
\end{aligned}$$

where in the first equality we used that the function $(-\Delta)^\sigma f_j$ is radially symmetric. □

Lemma 2.5.5. *Let $h \in C^{n-1}(\bar{B}_r)$ be such that*

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} h = 0 & \text{in } B_r \\ (-\Delta)^j h = f_j & \text{on } \partial B_r,\ j = 0, 1, ..., \frac{n-3}{2}. \end{cases} \tag{2.30}$$

Then for every $x \in B_r$

$$h(x) = -\sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} f_i(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x,y) \right) d\sigma(y),$$

and

$$|h(x)| \le C \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} |f_i(y)| \frac{1}{|x-y|^{n-1-2i}} d\sigma(y), \tag{2.31}$$

where G is the Green's function corresponding to the problem (2.30).

Proof. Using integration by parts we have

$$\begin{aligned} 0 &= \int_{B_r} G(x,y)(-\Delta)^{\frac{n-1}{2}} h(y) dy \\ &= \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} (-\Delta)^i h(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x,y) \right) d\sigma(y) + \int_{B_r} (-\Delta)^{\frac{n-1}{2}} G(x,y) h(y) dy \\ &= h(x) + \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} f_i(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x,y) \right) d\sigma(y) \end{aligned}$$

To get (2.31) we only need to show that

$$\left| \frac{\partial}{\partial y_i} (-\Delta)^j G(x,y) \right| \le \frac{1}{|x-y|^{2+2j}}, \quad x, y \in B_r,\ 0 \le j \le \frac{n-3}{2}.$$

In order to do that we use the following representation formula of G given by (see e.g. [30])

$$G(x,y) = \underbrace{\int_{B_r} \cdots \int_{B_r}}_{\frac{n-3}{2} \text{ times}} G_1(x, z_1) G_1(z_1, z_2) \dots G_1(z_{\frac{n-3}{2}}, y) dz_1 dz_2 \dots dz_{\frac{n-3}{2}}, \quad x, y \in B_r,$$

where

$$G_1(x,y) = \frac{1}{n(n-2)|B_1|} \left(\frac{1}{|x-y|^{n-2}} - \frac{r^{n-2}}{||x|(y - \frac{r^2 x}{|x|^2})|^{n-2}} \right) \quad x, y \in B_r,$$

is the Green's function for Laplacian on B_r. Then for $0 \le j \le \frac{n-3}{2}$

$$(-\Delta)^j G(x,y) = \underbrace{\int_{B_r} \cdots \int_{B_r}}_{\frac{n-3-2j}{2} \text{ times}} G_1(x, z_1) G_1(z_1, z_2) \dots G_1(z_{\frac{n-3-2j}{2}}, y) dz_1 dz_2 \dots dz_{\frac{n-3-2j}{2}},$$

and

$$\frac{\partial}{\partial y_i}(-\Delta)^j G(x,y) = \underbrace{\int_{B_r}\cdots\int_{B_r}}_{\frac{n-3-2j}{2}\text{ times}} G_1(x,z_1)G_1(z_1,z_2)\ldots\frac{\partial}{\partial y_i}G_1(z_{\frac{n-3-2j}{2}},y)dz_1\ldots dz_{\frac{n-3-2j}{2}}.$$

A repeated use of Lemma 2.4.3 and the estimate

$$0 < G_1(x,y) \le \frac{C}{|x-y|^{n-2}} \text{ and } \left|\frac{\partial}{\partial x_i}G_1(x,y)\right| \le \frac{C}{|x-y|^{n-1}} \quad x,y\in B_r,$$

gives

$$\left|\frac{\partial}{\partial y_i}(-\Delta)^j G(x,y)\right| \le C\int_{B_r}\frac{1}{|x-z|^{3+2j}}\frac{1}{|y-z|^{n-1}}dz \le C\frac{1}{|x-y|^{2+2j}}, \quad 0\le j\le\frac{n-3}{2}.$$

□

The following lemma is a variant of [56, Theorem 6].

Lemma 2.5.6. *Let $v\in L_{\frac{n}{2}}(\mathbb{R}^n)$ and let $h=u-v$ be $\frac{n+1}{2}$-harmonic in $\mathbb{R}^n$, that is*

$$\Delta^{\frac{n+1}{2}}h = 0, \quad \textit{in } \mathbb{R}^n.$$

If u satisfies (2.11) then h is a polynomial of degree at most $n-1$.

Proof. First notice that the condition $v\in L_{\frac{n}{2}}(\mathbb{R}^n)$ implies that

$$\int_{B_R}|v|dx = o(R^{2n}) \quad \text{as } R\to\infty.$$

For a fixed $x\in\mathbb{R}^n$ by [56, Proposition 4] we have for $\alpha\in\mathbb{N}^n$ with $|\alpha|=n$

$$|D^\alpha h(x)| \le \frac{C}{R^{2n}}\int_{B_R(x)}|h(y)|dy \le \frac{C}{R^{2n}}\int_{B_{2R}}|h(y)|dy, \quad \text{as } R\to\infty.$$

Now using (2.11)

$$\int_{B_R}h^+dx \le \int_{B_R}(u^++|v|)dx = o(R^{2n}) \quad \text{or} \quad \int_{B_R}h^-dx \le \int_{B_R}(u^-+|v|)dx = o(R^{2n}).$$

On the other hand, Pizzetti's formula (see e.g. [56, Lemma 3]) implies that

$$\int_{B_R}hdx = O(R^{2n-1}), \quad \text{as } R\to\infty.$$

Therefore,

$$\begin{aligned}|D^\alpha h(x)| &\le \frac{C}{R^{2n}}\min\left\{\int_{B_{2R}}(2h^+-h)dy, \int_{B_{2R}}(2h^-+h)dy\right\}\\ &= \frac{1}{R^{2n}}\left(O(R^{2n-1})+o(R^{2n})\right)\\ &\xrightarrow{R\to\infty} 0,\end{aligned}$$

and hence h is a polynomial of degree at most $n-1$. $\square$

Chapter 3

Conformal metrics on $\mathbb{R}^{2m}$ with prescribed volume and asymptotic behavior

We study the solutions $u \in C^\infty(\mathbb{R}^{2m})$ of the problem

$$(-\Delta)^m u = \bar{Q}e^{2mu}, \text{ where } \bar{Q} = \pm(2m-1)!, \quad V := \int_{\mathbb{R}^{2m}} e^{2mu}dx < \infty,$$

particularly when $m > 1$. Extending previous works of Chang-Chen, and Wei-Ye, we show that both the value V and the asymptotic behavior of $u(x)$ as $|x| \to \infty$ can be simultaneously prescribed, under certain restrictions. When $\bar{Q} = (2m-1)!$ we need to assume $V < |S^{2m}|$, but surprisingly for $\bar{Q} = -(2m-1)!$ the volume V can be chosen arbitrarily.

3.1 Introduction and statement of the main theorems

We consider the equation

$$(-\Delta)^m u = (2m-1)!e^{2mu} \text{ in } \mathbb{R}^{2m}, \tag{3.1}$$

where $u \in C^\infty(\mathbb{R}^{2m})$ and satisfies

$$V := \int_{\mathbb{R}^{2m}} e^{2mu}dx < \infty. \tag{3.2}$$

The assumption that $u \in C^\infty(\mathbb{R}^{2m})$ is not restrictive, since any weak solution $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ of (3.1) with right-hand side in $L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ is smooth, see e.g. [56, Corollary 8] (see also Theorem 2.1.1).

We recall that for $m \geq 2$ Chang-Chen [15] showed the existence of (non-spherical) solutions to (3.1)-(3.2) for every $V \in (0, |S^{2m}|)$. From the classifiction result (Theorem A) these solutions have the asymptotic behavior given by

$$u(x) = -\alpha \log(|x|) - P(x) + C + o(1), \quad o(1) \to 0 \text{ as } |x| \to \infty, \tag{3.3}$$

where $\alpha = \frac{2V}{|S^{2m}|}$ and P is a polynomial of degree at most $2m-2$ bounded from below.

Wei-Ye [77] complemented the result of Lin by showing, among other things:

Theorem C ([77]). *For any $V \in (0, |S^4|)$ and $P(x) = \sum_{j=1}^4 a_j x_j^2$ with $a_j > 0$, for $m = 2$ Problem (3.1)-(3.2) has a solution with asymptotic expansion (3.3) for some $C \in \mathbb{R}$.*

The first result which we prove here is an extension of the result of Wei-Ye to the case $m > 2$. We will prove the existence of solutions to (3.1)-(3.2) having the asymptotic behavior (3.3) where P will be any given polynomial of degree at most $2m-2$ satisfying

$$\lim_{|x|\to\infty} x \cdot \nabla P(x) = \infty, \tag{3.4}$$

while $\alpha > 0$ is determined by $V \in (0, |S^{2m}|)$. More precisely, define

$$\mathcal{P}_m := \left\{ P \text{ polynomial in } \mathbb{R}^{2m} : \deg P \leq 2m-2, \text{ (3.4) holds} \right\}.$$

It is worth noticing that (3.4) is equivalent to the apparently stronger condition

$$\liminf_{|x|\to\infty} \frac{P(x)}{|x|^a} > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} \frac{x \cdot \nabla P(x)}{|x|^a} > 0, \quad \text{for some } a > 0. \tag{3.5}$$

Indeed (3.4) implies the second inequality of (3.5) by a subtle result of E. Gorin (see [28, Theorem 3.1]), and the second inequality in (3.5) implies the first one, since one can write

$$P(x) = \int_0^{|x|} \frac{d}{dr} P\left(r \frac{x}{|x|} \right) dr + P(0).$$

However, the first condition in (3.5), that is

$$\lim_{|x|\to\infty} P(x) = \infty, \tag{3.6}$$

does not imply (3.4) when $\deg P \geq 4$, see Proposition 3.5.4. A simple example of polynomial belonging to $\mathcal{P}_m$ is

$$P(x) = \sum_{j=1}^{2m} a_j x_j^{2i_j} + p(x),$$

where $a_j > 0$, $i_j \in \{1, 2, \ldots, m-1\}$ for $1 \leq j \leq 2m$, and p is a polynomial of degree at most $2\min\{i_j\} - 1$, but in general $\mathcal{P}_m$ contains polynomials whose higher degree monomials do not split in such a simple way.

Theorem 3.1.1. *For any integer $m \geq 2$, given $P \in \mathcal{P}_m$ and $V \in (0, |S^{2m}|)$, there exists a solution of (3.1)-(3.2) having the asymptotic behavior (3.3) with $\alpha = \frac{2V}{|S^{2m}|}$.*

The restriction $V < |S^{2m}|$ in Theorem 3.1.1 is necessary when $m = 2$ because of the result of Lin (Theorem A). However, for $m \geq 3$ there are solutions to (3.1)-(3.2) with V arbitrarily large (see Chapter 5). The crucial step in which we need V to be smaller than $|S^{2m}|$ is Theorem 3.4.2 below, a compactness result which follows form the blow-up analysis of sequences of prescribed Q-curvature in open domains of $\mathbb{R}^{2m}$ (Theorem 3.4.1) proven by Martinazzi, and inspired by previous works of Brézis-Merle [12] and Robert

[65]. This compactness is used to prove the a priori bounds necessary to run the fixed point argument of [77], which we closely follow.

From the work of Brézis-Merle we also borrow a simple but fundamental critical estimate, whose generalization is Lemma 3.5.2, which is used in Lemma 3.3.6.

As we shall now show, things go differently when the prescribed Q-curvature is negative. Consider the equation

$$(-\Delta)^m u = -(2m-1)!e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{3.7}$$

whose solutions give rise to metrics $g_u = e^{2u}|dx|^2$ of Q-curvature $-(2m-1)!$ in $\mathbb{R}^{2m}$. One can easily verify that under the assumption (3.2) Eq. (3.7) has no solutions when $m = 1$, see e.g. [52, Proposition 6]. On the other hand, when $m \geq 2$ we have:

Theorem D ([52])**.** *For every $m \geq 2$ there is some $V > 0$ such that Problem (3.7)-(3.2) has a radially symmetric solution. Every solution to (3.7)-(3.2) (a priori not necessarily radially symmetric) has the asymptotic behavior given by (3.3) where $\alpha = -\frac{2V}{|S^{2m}|}$ and P is a non-constant polynomial of degree at most $2m-2$ bounded from below.*

Notice that, contrary to Chang-Chen's result [15], the existence part of Theorem D does not allow to prescribe V. Moreover its proof is based on an ODE argument which only produces radially symmetric solutions. It is then natural to address the following question: For which values of V and which polynomials P does Problem (3.7)-(3.2) have a solution with asymptotic behavior (3.3) (with $\alpha = -\frac{2V}{|S^{2m}|}$)? In analogy with Theorem 3.1.1 we will show:

Theorem 3.1.2. *For any integer $m \geq 2$, given $P \in \mathcal{P}_m$ and $V > 0$, there exists a solution of (3.7)-(3.2) having the asymptotic behavior (3.3) for $\alpha = -\frac{2V}{|S^{2m}|}$.*

The remarkable fact which allows for large values of V in Theorem 3.1.2 (but not in Theorem 3.1.1) is that, as shown in [55], when the Q-curvature is negative, compactness is obtained even for large volumes, compare Theorems 3.4.1 and 3.4.2 below. This in turn depends on Theorem D above, and in particular on the fact that the polynomial in the expansion (3.3) of a solution to (3.7)-(3.2) is necessarily non-constant.

About the assumption that $P \in \mathcal{P}_m$ in Theorems 3.1.1 and 3.1.2, we do not claim nor believe that it is optimal, but it is technically convenient in the crucial Lemma 3.3.5 below, where it is needed in (3.21). Since a solution to (3.1)-(3.2) or (3.7)-(3.2) must satisfy (3.3) for $\alpha = \pm\frac{2V}{|S^{2m}|}$, a necessary condition on P and V is

$$\int_{\mathbb{R}^{2m}\setminus B_1} e^{-2m(P(x)+\alpha\log|x|)}dx < \infty, \tag{3.8}$$

but it is unknown whether this condition is also sufficient to guarantee the existence of a solution to (3.1)-(3.2) or (3.7)-(3.2) with asymptotic expansion (3.3), at least in the negative case, or for $V < |S^{2m}|$ in the positive case.

3.2 Strategy of the proof

Fix $u_0 \in C^\infty(\mathbb{R}^{2m})$ such that $u_0(x) = \log|x|$ for $|x| \geq 1$. Integration by parts yields

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m u_0 dx = -\gamma_{2m},$$

where γ_{2m} is defined by

$$(-\Delta)^m \log\frac{1}{|x|} = \gamma_{2m}\delta_0 \text{ in } \mathbb{R}^{2m}, \text{ i.e. } \gamma_{2m} = \frac{(2m-1)!}{2}|S^{2m}|. \tag{3.9}$$

Let V, $\alpha = \pm\frac{2V}{|S^{2m}|}$ and $P \in \mathcal{P}_m$ be given as in Theorem 3.1.1 or 3.1.2. We would like to find a solution to (3.1) or (3.7) of the form

$$u = -\alpha u_0 - P + v + C, \tag{3.10}$$

for a suitable choice of $C \in \mathbb{R}$ and of a smooth function $v(x) = o(1)$ as $|x| \to \infty$. Define

$$K = \frac{\alpha\gamma_{2m}}{V}e^{-2mP-2m\alpha u_0} = \operatorname{sign}(\alpha)(2m-1)!e^{-2mP-2m\alpha u_0}, \tag{3.11}$$

and notice that (3.4) implies

$$|K(x)| \leq C_1 e^{-C_2|x|^a} \tag{3.12}$$

for some $C_1, C_2 > 0$.

Now if we assume (3.2), then the constant C in (3.10) is determined by the function v. Indeed (3.2) implies

$$V = \int_{\mathbb{R}^{2m}} e^{2mu}dx = \frac{e^{2mC}}{(2m-1)!}\int_{\mathbb{R}^{2m}} |K|e^{2mv}dx,$$

hence $C = c_v$, where

$$c_v := -\frac{1}{2m}\log\left(\frac{1}{(2m-1)!V}\int_{\mathbb{R}^{2m}}|K|e^{2mv}dx\right) = -\frac{1}{2m}\log\left(\frac{1}{\alpha\gamma_{2m}}\int_{\mathbb{R}^{2m}} Ke^{2mv}dx\right). \tag{3.13}$$

An easy computation shows that u given by (3.10) satisfies

$$(-\Delta)^m u = \operatorname{sign}(\alpha)(2m-1)!e^{2mu}$$

and (3.2) if and only if $C = c_v$ and

$$(-\Delta)^m v = Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0. \tag{3.14}$$

Then we will use a fixed point method in the spirit of [77] to find a solution v to (3.14) in the Banach space

$$C_0(\mathbb{R}^{2m}) := \left\{f \in C^0(\mathbb{R}^{2m}) : \lim_{|x|\to\infty} f(x) = 0\right\}, \quad \|f\|_{C_0} := \sup_{\mathbb{R}^{2m}}|f|,$$

and of course v will also be smooth by elliptic estimates. In order to run the fixed-point argument we use the following weighted Sobolev spaces.

Definition 3.2.1. For $k \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $p \geq 1$ we set $M^p_{k,\delta}(\mathbb{R}^{2m})$ to be the completion of $C^\infty_c(\mathbb{R}^{2m})$ in the norm

$$\|f\|_{M^p_{k,\delta}} := \sum_{|\beta|\leq k} \|(1+|x|^2)^{\frac{(\delta+|\beta|)}{2}} D^\beta f\|_{L^p(\mathbb{R}^{2m})}.$$

We also set $L^p_\delta(\mathbb{R}^{2m}) := M^p_{0,\delta}(\mathbb{R}^{2m})$. Finally we set

$$\Gamma^p_\delta(\mathbb{R}^{2m}) := \left\{ f \in L^p_{2m+\delta}(\mathbb{R}^{2m}) : \int_{\mathbb{R}^{2m}} f dx = 0 \right\},$$

whenever $\delta p > -2m$, so that $L^p_{2m+\delta}(\mathbb{R}^{2m}) \subset L^1(\mathbb{R}^{2m})$ and the above integral is well defined.

Lemma 3.2.1. *Fix $p \geq 1$ and $\delta > -\frac{2m}{p}$. For $v \in C_0(\mathbb{R}^{2m})$ and c_v as in (3.13) we have*

$$S(v) := Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0 \in \Gamma^p_\delta(\mathbb{R}^{2m}),$$

and the map $S : C_0(\mathbb{R}^{2m}) \to \Gamma^p_\delta(\mathbb{R}^{2m})$ is continuous.

Proof. This follows easily from (3.12) and dominated convergence. □

Lemma 3.2.2 (Theorem 5 in [60])**.** *For $1 < p < \infty$ and $\delta \in \left(-\frac{2m}{p}, -\frac{2m}{p}+1\right)$, the operator $(-\Delta)^m$ is an isomorphism from $M^p_{2m,\delta}(\mathbb{R}^{2m})$ to $\Gamma^p_\delta(\mathbb{R}^{2m})$.*

We postpond the proof of the following lemma until the end of this chapter.

Lemma 3.2.3. *For $\delta > -\frac{2m}{p}$, $p \geq 1$, the embedding $E : M^p_{2m,\delta}(\mathbb{R}^{2m}) \hookrightarrow C_0(\mathbb{R}^{2m})$ is compact.*

Fix $p \in (1, \infty)$ and $\delta \in \left(-\frac{2m}{p}, -\frac{2m}{p}+1\right)$. Then by Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3, one can define a compact map

$$T := E \circ ((-\Delta)^m)^{-1} \circ S : C_0(\mathbb{R}^{2m}) \to C_0(\mathbb{R}^{2m}) \tag{3.15}$$

given by $Tv = \bar{v}$ where $\bar{v}$ is the only solution to

$$(-\Delta)^m \bar{v} = Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0,$$

and compactness follows from the continuity of S and $((-\Delta)^m)^{-1}$ and the compactness of E.

If v is a fixed point of T, then it solves (3.14) and $u = v + c_v - P - \alpha u_0$ is a solution of (3.1) or (3.7) (depending on the sign of K in (3.11)) and (3.2), with asymptotic expansion (3.3). Then in order to prove Theorems 3.1.1 and 3.1.2 it remains to prove that T has a fixed point, and we shall do that using the following fixed-point theorem.

Lemma 3.2.4 (Theorem 11.3 in [27])**.** *Let T be a compact mapping of a Banach space X into itself, and suppose that there exists a constant M such that*

$$\|x\|_X < M$$

for all $x \in X$ and $t \in (0, 1]$ satisfying $tTx = x$. Then T has a fixed point.

In order to apply Lemma 3.2.4 to the operator T defined in (3.15) we will prove in Section 3.3 the following a priori bound, which completes the proof of Theorems 3.1.1 and 3.1.2.

Proposition 3.2.5. *For any $0 < t \le 1$ and $v \in C_0(\mathbb{R}^{2m})$ with $tTv = v$ we have*

$$\|v\|_{C_0(\mathbb{R}^{2m})} \le M, \tag{3.16}$$

where M is independent of v and t.

3.3 A priori estimates and proof of Proposition 3.2.5

Throughout this section let $t \in (0,1]$ and $v \in C_0(\mathbb{R}^{2m})$ be fixed and satisfy $tTv = v$, that is

$$(-\Delta)^m v = t(Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0),$$

where c_v is as in (3.13). Also define

$$\bar{w} := v + c_v + \frac{\log t}{2m}. \tag{3.17}$$

Lemma 3.3.1. *We have*

$$v(x) = -\frac{t}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log(|x-y|) K(y) e^{2m(v(y)+c_v)} dy + t\alpha u_0(x). \tag{3.18}$$

Proof. Let $\tilde{v}(x)$ be defined as the right-hand side of (3.18). Then for $|x| \ge 1$, using (3.13) we write

$$\tilde{v}(x) = \frac{t}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} K(y) e^{2m(v(y)+c_v)} (\log|x| - \log|x-y|) dy$$

We first show that

$$\lim_{|x|\to\infty} \tilde{v}(x) = 0. \tag{3.19}$$

Let $R > 1$ be fixed. Then for $|x| > 2R$, we split

$$\tilde{v}(x) = \sum_{i=1}^{5} I_i, \quad I_i := \frac{t}{\gamma_{2m}} \int_{A_i} K(y) e^{2m(v(y)+c_v)} \log\left(\frac{|x|}{|x-y|}\right) dy,$$

where

$$\begin{aligned}
A_1 &:= B_R(0) \\
A_2 &:= B_1(x) \\
A_3 &:= B_{|x|/2}(x) \setminus B_1(x) \\
A_4 &:= (B_{2|x|}(x) \setminus B_{|x|/2}(x)) \setminus B_R(0) \\
A_5 &:= \mathbb{R}^{2m} \setminus B_{2|x|}(x),
\end{aligned}$$

and we will show that $I_i \to 0$ as $|x| \to \infty$ for $1 \le i \le 5$.

For $i = 1$, since $\lim_{|x|\to\infty} \log\left(\frac{|x|}{|x-y|}\right) = 0$ uniformly with respect to $y \in B_R(0)$, from the dominated convergence theorem we get

$$|I_1| \le C \int_{B_R(0)} |K(y)| \left|\log\left(\frac{|x|}{|x-y|}\right)\right| dy \to 0 \quad \text{as } |x| \to \infty.$$

From (3.12) we also have

$$\begin{aligned}|I_2| &\le C \int_{B_1(x)} |K(y)| \left(\log|x| + |\log|x-y||\right) dy \\ &\le C\|K\|_{L^\infty(B_1(x))} \left(\log|x| + \|\log|\cdot|\|_{L^1(B_1(0))}\right) \\ &\to 0, \quad \text{as } |x| \to \infty.\end{aligned}$$

Since (3.12) yields $K\log(|\cdot|) \in L^1(\mathbb{R}^{2m})$, we infer with the dominated convergence theorem

$$\begin{aligned}|I_3| &\le C \int_{\{1\le|x-y|<|x|/2\}} |K(y)| \left(\log|x| + \log(|x|/2)\right) dy \\ &\le C \int_{\{1\le|x-y|<|x|/2\}} |K(y)| \left(\log|2y| + \log(|y|)\right) dy \\ &\to 0, \quad \text{as } |x| \to \infty.\end{aligned}$$

Using that $\frac{1}{2} < \frac{|x|}{|x-y|} < 2$ on A_4 and that $K \in L^1(\mathbb{R}^{2m})$ we find that for every $\varepsilon > 0$ it is possible to choose R so large that

$$|I_4| \le C \int_{A_4} |K(y)| \left|\log\left(\frac{|x|}{|x-y|}\right)\right| dy \le C \int_{A_4} |K| dy \le C \int_{\mathbb{R}^{2m}\setminus B_R(0)} |K| dy \le \varepsilon.$$

Finally, again using that $K\log(|\cdot|) \in L^1(\mathbb{R}^{2m})$ with the dominated convergence theorem we get

$$\begin{aligned}|I_5| &\le C \int_{\{|x-y|>2|x|\}} |K(y)|(\log|x| + \log|x-y|) dy \\ &\le C \int_{\{|x-y|>2|x|\}} |K(y)|(\log|y| + \log|2y|) dy \\ &\to 0, \quad \text{as } |x| \to \infty.\end{aligned}$$

Since ε can be chosen arbitrarily small, (3.19) is proven. Since $v \in C_0(\mathbb{R}^{2m})$, and $\Delta^m \tilde{v} = \Delta^m v$, the difference $w := v - \tilde{v}$ satisfies

$$\Delta^m w = 0 \quad \text{in } \mathbb{R}^{2m}, \qquad \lim_{|x|\to\infty} w(x) = 0.$$

Then by the Liouville theorem for polyharmonic functions (see e.g. Theorem 5 in [56]) w is a polynomial, and since it vanishes at infinity, it must be identically zero, i.e. $v \equiv \tilde{v}$. □

Using Lemma 3.3.1 one can prove the following decay estimate for the derivatives of v at infinity.

Lemma 3.3.2. *For $1 \le \ell \le 2m-1$ we have*

$$\lim_{|x|\to\infty} |x|^\ell \nabla^\ell v(x) = \lim_{|x|\to\infty} |x|^\ell \nabla^\ell \bar{w}(x) = 0.$$

Proof. Notice that $\nabla v = \nabla \bar{w}$, so it is enough to work with v.

Using (3.18) for $|x| > 1$ one can compute

$$\nabla^\ell v(x) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} K(y) e^{2m\bar{w}(y)} \Big(\nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big)\, dy.$$

Fix $\varepsilon > 0$ and $R_1 > 1$ such that

$$\int_{\mathbb{R}^{2m}\setminus B_{R_1}} |K| e^{2m\bar{w}} dy < \varepsilon.$$

For $|x| > 2R_1$, we split $\mathbb{R}^{2m}$ into three disjoint domains:

$$A_1 := B_{R_1}(0), \quad A_2 := B_{|x|/2}(x), \quad A_3 := \mathbb{R}^{2m} \setminus (A_1 \cup A_2).$$

Then

$$|x|^\ell \nabla^\ell v(x) = \frac{1}{\gamma_{2m}} \sum_{i=1}^3 I_i, \quad I_i := |x|^\ell \int_{A_i} K(y) e^{2m\bar{w}(y)} \Big(\nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big)\, dy.$$

Since R_1 is fixed, for $|x|$ large enough we have by the mean-value theorem

$$\Big| \nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big| \le |y| \sup_{B_{|y|}(x)} \Big| \nabla^{\ell+1} \log(|z|) \Big| \le \frac{C}{|x|^{\ell+1}} \quad \text{for } y \in A_1,$$

hence with (3.13) we get

$$|I_1| \le \frac{C}{|x|} \int_{A_1} |K| e^{2m\bar{w}} dy \le \frac{C}{|x|} |\alpha| \gamma_{2m} \to 0, \quad \text{as } |x| \to \infty.$$

Since K goes to zero rapidly at infinity, $\bar{w}$ is bounded, and $|x-y| \le |x|/2$ on A_2, we have

$$\begin{aligned} |I_2| &\le C \|K\|_{L^\infty(A_2)} \|e^{2m\bar{w}}\|_{L^\infty} |x|^\ell \int_{A_2} \left(\frac{1}{|x|^\ell} + \frac{1}{|x-y|^\ell} \right) dy \\ &\le C \|K\|_{L^\infty(A_2)} \|e^{2m\bar{w}}\|_{L^\infty} |x|^{2m} \\ &\to 0, \qquad \text{as } |x| \to \infty. \end{aligned}$$

On A_3 we have $|x-y| \ge |x|/2$, which implies $\frac{|x|^\ell}{|x-y|^\ell} \le 2^\ell$. Hence

$$|I_3| \le C(1+2^\ell) \int_{A_3} |K| e^{2m\bar{w}} dy < C\varepsilon.$$

Since ε is arbitrarily small, the proof is complete. □

Lemma 3.3.3. *The function $\bar{w}$ given by (3.17) is locally uniformly upper bounded, i.e. for every $R > 0$ there exists $C = C(R)$ such that $\bar{w} \le C$ in B_R.*

Proof. Since u_0 is a fixed function and locally bounded, it is enough to prove that $w := \bar{w} - t\alpha u_0$ is locally uniformly upper bounded. Now

$$(-\Delta)^m w = tKe^{2m(v+c_v)} = Qe^{2mw}, \quad Q := Ke^{2mt\alpha u_0}.$$

We bound

$$\int_{B_R} e^{2mw} dx = t\int_{B_R} e^{2m(v+c_v)-2mt\alpha u_0} dx \leq C(R)\int_{B_R} |K|e^{2m(v+c_v)} dx \leq C(R)|\alpha|\gamma_{2m},$$

where we used (3.13) and that $|K|$ is positive and continuous.

In addition in the case when $Q > 0$ we have

$$\int_{B_R} Qe^{2mw} dx \leq \int_{B_R} Ke^{2m(v+c_v)} dx < \alpha\gamma_{2m} < (2m-1)!|S^{2m}|.$$

Moreover, Lemma 3.3.1 gives

$$\Delta w(x) = -\frac{t}{\gamma_{2m}}\int_{\mathbb{R}^{2m}} \frac{2m-2}{|x-y|^2} K(y)e^{2m(v(y)+c_v)} dy$$

and with Fubini's theorem we get

$$\begin{aligned}\int_{B_R} |\Delta w(x)| dx &= \frac{t}{\gamma_{2m}}(2m-2)\int_{\mathbb{R}^{2m}} |K(y)|e^{2m(v(y)+c_v)} \left(\int_{B_R} \frac{dx}{|x-y|^2}\right) dy \\ &\leq C\int_{\mathbb{R}^{2m}} |K(y)|e^{2m(v(y)+c_v)} \left(\int_{B_R(y)} \frac{dx}{|x-y|^2}\right) dy \\ &\leq CR^{2m-2}.\end{aligned}$$

Therefore Theorem 3.4.2 implies that there exists $C = C(R) > 0$ (independent of w) such that

$$\sup_{B_{R/2}} w \leq C.$$

□

A consequence of the local uniform upper bounds of $\bar{w}$ is the following local uniform bound for the derivatives of v:

Lemma 3.3.4. *For every $R > 0$ there exists a constant $C = C(R) > 0$ independent of v and t such that for $1 \leq \ell \leq 2m-1$ we have*

$$\sup_{B_R} |\nabla^\ell v| \leq C.$$

Proof. Let $x \in B_R$. Then from (3.18) and Lemma 3.3.3, we have

$$\begin{aligned}|\nabla^\ell(v - t\alpha u_0)| &\leq C\int_{\mathbb{R}^{2m}} |K(y)|e^{2m\bar{w}(y)} \frac{1}{|x-y|^\ell} dy \\ &\leq C\|K\|_{L^\infty}\|e^{2m\bar{w}}\|_{L^\infty(B_{2R})}\int_{B_{2R}} \frac{1}{|x-y|^\ell} dy + \frac{C}{R^\ell}\int_{\mathbb{R}^{2m}\setminus B_{2R}} |K|e^{2m\bar{w}} dy \\ &\leq C(R),\end{aligned}$$

where the last integral is bounded using (3.13). Since u_0 is smooth, α is fixed and $t \in (0,1]$, the lemma follows. □

Now to prove uniform upper bounds for $\bar{w}$ outside a fixed compact set, first we will need the following result, which relies on a Pohozaev-type identity.

Lemma 3.3.5. *For given $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon) > 0$ only depending on K (and not on v or t) such that*

$$\int_{\mathbb{R}^{2m}\setminus B_{R_0}} |K| e^{2m\bar{w}} dx < \varepsilon.$$

Proof. Taking $R \to \infty$ in Lemma 3.5.1 and noticing that the first term on the right-hand side of (3.27) vanishes thanks to (3.12) and last two terms vanish thanks to Lemma 3.3.2, we find

$$\int_{\mathbb{R}^{2m}} (x\cdot \nabla K) e^{2m\bar{w}} dx + 2m \int_{\mathbb{R}^{2m}} K e^{2m\bar{w}} dx - 2mt\alpha \int_{B_1} (x\cdot \nabla v)(-\Delta)^m u_0 dx = 0. \quad (3.20)$$

Thanks to (3.5) we can find $C_1 > 0$ and $R_1 \geq 1$ such that

$$x\cdot \nabla |K(x)| = -2m\,(x\cdot \nabla P(x) + \alpha)\,|K(x)| \leq -\frac{1}{C_1}|x|^a |K(x)| \quad \text{for } |x| \geq R_1. \quad (3.21)$$

Then for some $R \geq R_1$ to be fixed later we bound

$$\begin{aligned}\frac{1}{C_1}R^a \int_{\mathbb{R}^{2m}\setminus B_R} |K| e^{2m\bar{w}} dx &\leq \frac{1}{C_1}\int_{\mathbb{R}^{2m}\setminus B_R} |x|^a |K(x)| e^{2m\bar{w}} dx \\ &\leq -\int_{\mathbb{R}^{2m}\setminus B_R} x\cdot \nabla |K(x)| e^{2m\bar{w}} dx \\ &= 2m \int_{\mathbb{R}^{2m}} |K| e^{2m\bar{w}} dx + \int_{B_R} (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx \\ &\quad - 2mt|\alpha| \int_{B_1} (x\cdot \nabla v(x))(-\Delta)^m u_0 dx \\ &=: (I) + (II) + (III),\end{aligned} \quad (3.22)$$

where in the equality on the third line we used (3.20). Now using (3.13) and (3.17), we compute $(I) = 2mt|\alpha|\gamma_{2m}$, and using Lemma 3.3.4 we bound

$$\begin{aligned}(I) + (II) + (III) &\leq C_1 + \int_{B_R} (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx \\ &\leq C_1 + \int_{\Omega} (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx\end{aligned}$$

where

$$\Omega := \left\{x \in \mathbb{R}^{2m} : x\cdot \nabla P(x) + \alpha < 0\right\}.$$

From (3.21) we infer that $\Omega \subset B_{R_1}$. Then with Lemma 3.3.3 we find

$$(I) + (II) + (III) \leq C_1 + \sup_{x\in B_{R_1}} (|x\cdot \nabla K(x)|) \int_{B_{R_1}} e^{2m\bar{w}} dx \leq C_2 = C_2(R_1),$$

where C_2 does not depend on t or v. To complete the proof it suffices to take $R_0 = R$ so large that

$$\frac{R^a}{C_1} \geq \frac{C_2}{\varepsilon}.$$

□

To prove uniform upper bound of $\bar{w}$ on the complement of a compact set, we use the Kelvin transform. For $R > 1$ define

$$\xi_R(x) := \bar{w}\left(\frac{Rx}{|x|^2}\right), \quad 0 < |x| \leq 1. \tag{3.23}$$

Lemma 3.3.6. *There exists $\varepsilon > 0$ sufficiently small such that if $R_0 = R_0(\varepsilon) > 1$ is as in Lemma 3.3.5, then $\xi(x) := \xi_{R_0}(x)$ is uniformly upper bounded on B_1, i.e. $\bar{w}$ is uniformly upper bounded in $\mathbb{R}^{2m} \setminus B_{R_0}$.*

Proof. Using (3.30) for $n = 2m$ and $k = m$ and recalling that

$$(-\Delta)^m \bar{w} = Ke^{2m\bar{w}} \quad \text{in } \mathbb{R}^{2m} \setminus B_1,$$

we have

$$\begin{aligned}
(-\Delta)^m \xi(x) &= \frac{R_0^{2m}}{|x|^{4m}}((-\Delta)^m \bar{w})\left(\frac{R_0 x}{|x|^2}\right) \\
&= \left(\frac{R_0}{|x|^2}\right)^{2m} K\left(\frac{R_0 x}{|x|^2}\right) e^{2m\xi(x)} \\
&=: f(x).
\end{aligned}$$

Then with the change of variable $y = \frac{R_0 x}{|x|^2}$ and Lemma 3.3.5 we obtain for $R_0 = R_0(\varepsilon)$ large enough (and $\varepsilon > 0$ to be fixed later)

$$\int_{B_1} f(x)dx < \varepsilon.$$

We write $\bar{\xi} := \xi_1 + \xi_2$, where

$$\begin{cases} (-\Delta)^m \xi_1 = f & \text{in } B_1 \\ (-\Delta)^k \xi_1 = 0 & \text{on } \partial B_1 \text{ for } k = 0, 1, 2, .., m-1 \end{cases}$$

and

$$\begin{cases} (-\Delta)^m \xi_2 = 0 & \text{in } B_1 \\ (-\Delta)^k \xi_2 = (-\Delta)^k \xi & \text{on } \partial B_1 \text{ for } k = 1, 2, .., m-1 \\ \xi_2 = \xi^+ := \max\{\xi, 0\} & \text{on } \partial B_1. \end{cases}$$

Iteratively using the *maximum principle* it is easy to see that

$$\xi \leq \bar{\xi} \text{ in } B_1. \tag{3.24}$$

Now fix $\varepsilon > 0$ small enough (and consequently $R_0 = R_0(\varepsilon) > 0$ large enough) so that by Lemma 3.5.2 below, there exists $p > 1$ such that $e^{2m\xi_1}$ is bounded in $L^p(B_1)$. As usual this bound, as well as ε, R_0 are independent of t and v.

Since $|\Delta^k \xi_2|$ is uniformly bounded on ∂B_1 for $k = 0, 1, 2, ..., m-1$ by Lemma 3.3.4 and $\bar{w}^+$ is uniformly bounded on ∂B_{R_0} by Lemma 3.3.3, so that ξ^+ is uniformly bounded on ∂B_1, by the maximum principle we get uniform bounds of ξ_2 in B_1. Hence, noticing that

$$\frac{R_0^{2m}}{|x|^{4m}} K\left(\frac{R_0 x}{|x|^2}\right) \leq C \quad \text{for } x \in B_1$$

by (3.12), and using (3.24), we can bound

$$\begin{aligned}
\|f\|_{L^p(B_1)} &\leq C\|e^{2m\xi}\|_{L^p(B_1)} \\
&\leq C\|e^{2m\bar{\xi}}\|_{L^p(B_1)} \\
&\leq C\|e^{2m\xi_1}\|_{L^p(B_1)}\|e^{2m\xi_2}\|_{L^\infty(B_1)} \\
&\leq C.
\end{aligned}$$

Consequently by elliptic estimates and Sobolev embedding there exists a constant $C > 0$ (independent of v and t) such that

$$\|\xi_1\|_{L^\infty(B_1)} \leq C'\|\xi_1\|_{W^{2m,p}(B_1)} \leq C,$$

and therefore

$$\xi \leq \bar{\xi} \leq |\xi_1| + |\xi_2| \leq C \quad \text{in } B_1,$$

with C not depending on v and t. □

By Lemma 3.2.2 and (3.12), we have

$$\begin{aligned}
\frac{1}{C}\|v\|_{M^p_{2m,\delta}} &\leq \|(-\Delta)^m v\|_{L^p_{2m+\delta}} \\
&= \|Ke^{2m\bar{w}} + t\alpha(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \\
&\leq \|K\|_{L^p_{2m+\delta}}\|e^{2m\bar{w}}\|_{L^\infty} + \alpha\|(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \\
&\leq C\|e^{2m\bar{w}}\|_{L^\infty} + C,
\end{aligned}$$

with C independent of t and v, and together with Lemma 3.3.3 and Lemma 3.3.6 we obtain

$$\|v\|_{M^p_{2m,\delta}} \leq C,$$

where C is independent of v and t. Now Proposition 3.2.5 follows at once from the continuity of the embedding $M^p_{2m,\delta}(\mathbb{R}^{2m}) \hookrightarrow C_0(\mathbb{R}^{2m})$ (see Lemma 3.2.3).

Remark. An alternative way of getting uniform bounds on $\|v\|_{C_0}$ is to get uniform upper bounds of $\bar{w}$ and use them in (3.18).

3.4 Local uniform upper bounds for the equation $(-\Delta)^m u = Ke^{2mu}$

Here we state a slightly simplified version of Theorem 1 from [55] which we will use to prove the uniform upper bound of Theorem 3.4.2 below. This theorem was originally proved by Robert [65] in dimension 4 and under the assumption $V_k > 0$, and is a delicate counterpart to the blow-up analysis initiated by Brézis-Merle [12] in dimension 2. The crucial fact which we shall use is that in order to lose compactness V_0 must be positive somewhere *and* $\|V_k e^{2mu_k}\|_{L^1}$ must approach or go above $\Lambda_1 := (2m-1)!|S^{2m}|$.

Theorem 3.4.1 ([55]). *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a connected set. Let $(u_k) \subset C^{2m}_{\mathrm{loc}}(\Omega)$ be such that*

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in} \quad \Omega$$

where $V_k \to V_0$ in $C^0_{\mathrm{loc}}(\Omega)$ and, for some $C_1, C_2 > 0$,

$$\int_\Omega e^{2mu_k} dx \leq C_1, \qquad \int_\Omega |\Delta u_k| dx \leq C_2.$$

Then one of the following is true:

(i) up to a subsequence $u_k \to u_0$ in $C^{2m-1}_{\mathrm{loc}}(\Omega)$ for some $u_0 \in C^{2m}(\Omega)$, or

(ii) there is a finite (possibly empty) set $S = \{x^{(1)}, ..., x^{(I)}\} \subset \Omega$ such that $V_0(x^{(i)}) > 0$ for $1 \leq i \leq I$, and up to a subsequence $u_k \to -\infty$ locally uniformly in $\Omega \setminus S$, and

$$V_k e^{2mu_k} dx \rightharpoonup \sum_{i=1}^{I} \alpha_i \delta_{x^{(i)}}$$

in the sense of measures in Ω, where

$$\alpha_i = L_i \Lambda_1 \text{ for some } L_i \in \mathbb{N} \setminus \{0\}, \quad \Lambda_1 := (2m-1)!|S^{2m}|.$$

In particular, in case (ii) for any open set $\Omega_0 \Subset \Omega$ with $S \subset \Omega_0$ we have

$$\int_{\Omega_0} V_k e^{2mu_k} \to L\Lambda_1 \text{ for some } L \in \mathbb{N}, \text{ and } L = 0 \Leftrightarrow S = \emptyset. \tag{3.25}$$

Theorem 3.4.2. *Let $u \in C^{2m}(B_R)$ solve*

$$(-\Delta)^m u = K e^{2mu} \quad \text{in } B_R$$

for a function $K \in C^0(B_R)$ and assume that for given $C_1, C_2 > 0$ one has

(a) $\int_{B_R} e^{2mu} dx \leq C_1$,

(b) $\int_{B_R} |\Delta u| dx \leq C_2$,

(c_1) either $\int_{B_R} K e^{2mu} dx \leq \Lambda$ for some $\Lambda < (2m-1)!|S^{2m}|$, or

(c_2) $K \leq 0$ in B_R.

Then

$$\sup_{B_{R/2}} u \leq C$$

where C only depends on R, C_1, C_2, Λ (in case (c_1) holds and not (c_2)) and K.

Proof. Assume that there is a sequence of functions $u_n \in C^{2m}(B_R)$ and a sequence of points $x_n \in B_{R/2}$ such that u_n satisfies the conditions (a), (b), and (c_1) or (c_2), and assume that

$$\lim_{n\to\infty} u_n(x_n) = \infty. \tag{3.26}$$

Then we can apply Theorem 3.4.1 with $V_k = K$ for every k, and because of (3.26), we clearly are in case (ii) of the theorem. Assume that $S \neq \emptyset$. Then $K > 0$ on S, hence condition (c_2) does not hold. On the other hand condition (c_1) contradicts (3.25). Then $S = \emptyset$, hence $u_k \to -\infty$ uniformly in $B_{R/2}$, contradicting (3.26). □

3.5 Some useful lemmas

Lemma 3.5.1 (Pohozaev-type identity). *Consider $K \in C^1(\overline{B_R})$ for some $R > 1$, and let $u_0 \in C^{2m}(\mathbb{R}^{2m})$ be such that $\operatorname{supp}(\Delta^m u_0) \subseteq \overline{B_1}$. Let $\bar{w} \in C^{2m}(\overline{B_R})$ be a solution of*

$$(-\Delta)^m \bar{w} = K e^{2m\bar{w}} + t\alpha(-\Delta)^m u_0.$$

Then we have

$$\begin{aligned}\int_{B_R} (x\cdot \nabla K) e^{2m\bar{w}} dx + 2m \int_{B_R} K e^{2m\bar{w}} dx - 2mt\alpha \int_{B_1} (x\cdot \nabla \bar{w})(-\Delta)^m u_0 dx \\ = R\int_{\partial B_R} K e^{2m\bar{w}} d\sigma - mR \int_{\partial B_R} |\Delta^{\frac{m}{2}} \bar{w}|^2 d\sigma - 2m \int_{\partial B_R} f d\sigma,\end{aligned} \tag{3.27}$$

where,

$$f(x) := \sum_{j=0}^{m-1} (-1)^{m+j} \frac{x}{R} \cdot \left(\Delta^{j/2}(x\cdot \nabla \bar{w}) \Delta^{(2m-1-j)/2} \bar{w}\right) \quad \text{on } \partial B_R,$$

and for k odd $\Delta^{k/2} := \nabla \Delta^{(k-1)/2}$.

Proof. Integrating by parts we find

$$\begin{aligned}2m \int_{B_R} (1 + x\cdot \nabla \bar{w}) K e^{2m\bar{w}} dx &= \int_{B_R} K \operatorname{div}(x e^{2m\bar{w}}) dx \\ &= -\int_{B_R} (x\cdot \nabla K) e^{2m\bar{w}} dx + R\int_{\partial B_R} K e^{2m\bar{w}} d\sigma.\end{aligned}$$

Now

$$\int_{B_R} (x\cdot \nabla \bar{w}) K e^{2m\bar{w}} dx = \int_{B_R} (x\cdot \nabla \bar{w})(-\Delta)^m \bar{w} dx - t\alpha \int_{B_1} (x\cdot \nabla \bar{w})(-\Delta)^m u_0 dx, \tag{3.28}$$

and integrating by parts m times the first term on the right-hand side of (3.28) we find

$$\int_{B_R} (x\cdot \nabla \bar{w})(-\Delta)^m \bar{w} dx = \int_{B_R} \Delta^{\frac{m}{2}}(x\cdot \nabla \bar{w}) \Delta^{\frac{m}{2}} \bar{w} dx + \int_{\partial B_R} f d\sigma =: I \tag{3.29}$$

Using

$$\Delta^{\frac{m}{2}}(x\cdot \nabla \bar{w}) \Delta^{\frac{m}{2}} \bar{w} = \frac{1}{2} \operatorname{div}(x |\Delta^{\frac{m}{2}} \bar{w}|^2)$$

(see e.g. [57, Lemma 14] for a simple proof) and using the divergence theorem we obtain

$$I = \frac{1}{2}\int_{\partial B_R} R|\Delta^{\frac{m}{2}}\bar{w}|^2 d\sigma + \int_{\partial B_R} f d\sigma,$$

and putting together the above equations we conclude. □

The proof of the following lemma can be found in [56, Theorem 7]. It extends to arbitrary dimension Theorem 1 of [12].

Lemma 3.5.2. *Let $f \in L^1(B_R)$ and let v solve*

$$\begin{cases} (-\Delta)^m v = f & \text{in } B_R \subset \mathbb{R}^{2m}, \\ \Delta^k v = 0 & \text{on } \partial B_R \text{ for } k = 0, 1, \dots, m-1. \end{cases}$$

Then, for any $p \in \left(0, \frac{\gamma_{2m}}{\|f\|_{L^1(B_R)}}\right)$, we have $e^{2mp|v|} \in L^1(B_R)$ and

$$\int_{B_R} e^{2mp|v|} dx \le C(p) R^{2m},$$

where γ_{2m} is definde by (3.9).

Lemma 3.5.3. *Given $u \in C^\infty(\mathbb{R}^n)$, define $\tilde{u}(x) := u\big(\frac{x}{|x|^2}\big)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Then for any $k \in \mathbb{N}$ we have*

$$\Delta^k \left(\frac{1}{|x|^{n-2k}}\tilde{u}(x)\right) = \frac{1}{|x|^{n+2k}}(\Delta^k u)\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{3.30}$$

Proof. We shall prove the lemma by induction on $k \in \mathbb{N}$. Notice that for $k = 0$ (3.30) is trivial.

For a smooth function f and $g(x) := |x|^2$, we have the formula

$$\Delta^{k+1}(fg) = g\Delta^{k+1} f + 2(k+1)(n+2k)\Delta^k f + 4(k+1)x\cdot\nabla(\Delta^k f),$$

which can be easily proven by induction on $k \in \mathbb{N}$. Choosing

$$f(x) = \frac{\tilde{u}(x)}{|x|^{n-2k}}$$

and assuming that (3.30) is true for a given $k \in \mathbb{N}$, we compute

$$\begin{aligned}
&\Delta^{k+1}\left(\frac{\tilde{u}(x)}{|x|^{n-2(k+1)}}\right) \\
&= \Delta^{k+1}(fg) \\
&= g\Delta(\Delta^k f) + 2(k+1)(n+2k)\Delta^k f + 4(k+1)x\cdot\nabla(\Delta^k f) \\
&= |x|^2\Delta\left(\frac{1}{|x|^{n+2k}}(\Delta^k u)\left(\frac{x}{|x|^2}\right)\right) + 2(k+1)(n+2k)\frac{1}{|x|^{n+2k}}(\Delta^k u)\left(\frac{x}{|x|^2}\right) \\
&\quad + 4(k+1)x\cdot\nabla\left(\frac{1}{|x|^{n+2k}}(\Delta^k u)\left(\frac{x}{|x|^2}\right)\right) \\
&= \frac{1}{|x|^{n+2(k+1)}}(\Delta^{k+1} u)\left(\frac{x}{|x|^2}\right),
\end{aligned}$$

hence completing the induction. □

Proposition 3.5.4. *For $n \geq 2$ there exists a polynomial P of degree 4 in $\mathbb{R}^n$ satisfying (3.6) but not (3.4).*

Proof. In $\mathbb{R}^2$ consider $P(x) = P(x_1, x_2) = x_1^2 + x_2^4 - \beta x_1 x_2^2$, with $0 < \beta < 2$. Then

$$P(x) \geq x_1^2 + x_2^4 - \beta\left(\frac{x_1^2}{2} + \frac{x_2^4}{2}\right) = \left(1 - \frac{\beta}{2}\right)(x_1^2 + x_2^4),$$

so that P satisfies (3.6). Moreover

$$x \cdot \nabla P(x) = 2x_1^2 + 4x_2^4 - 3\beta x_1 x_2^2.$$

Choosing $x = (ax_2^2, x_2)$ we obtain

$$(ax_2^2, x_2) \cdot \nabla P(ax_2^2, x_2) = x_2^4(2a^2 - 3\beta a + 4).$$

Then, since for $\beta > \sqrt{2}\frac{4}{3}$ the polynomial $2a^2 - 3\beta a + 4$ has positive discriminant, fixing $\beta \in \left(\sqrt{2}\frac{4}{3}, 2\right)$ and a such that $2a^2 - 3\beta a + 4 < 0$ we see that

$$\liminf_{|x|\to\infty} x \cdot \nabla P(x) \leq \lim_{|x_2|\to\infty} (ax_2^2, x_2) \cdot \nabla P(ax_2^2, x_2) = -\infty.$$

This proves the proposition for $n = 2$. For $n > 2$ it suffices to consider

$$\tilde{P}(x_1, x_2, \ldots, x_n) = P(x_1, x_2) + \sum_{j=3}^{n} x_j^2,$$

where P is as before. □

We end this chapter by giving a proof of Lemma 3.2.3.

Proof of Lemma 3.2.3 For any $R \geq 1$ set

$$A_R := \{x \in \mathbb{R}^{2m} : R < |x| < 2R\}, \quad A := A_1 = \{x \in \mathbb{R}^{2m} : 1 < |x| < 2\}.$$

Given $f \in W^{2m,p}(A_R)$, define

$$\tilde{f}(x) := f(Rx), \quad \text{for } x \in A.$$

For $|\beta| \leq 2m$, we have

$$\begin{aligned}\int_A |D^\beta \tilde{f}(x)|^p dx &= R^{p|\beta|} \int_A |(D^\beta f)(Rx)|^p dx \\ &= R^{p|\beta| - 2m} \int_{A_R} |D^\beta f(x)|^p dx.\end{aligned}$$

From the embedding $W^{2m,p}(A) \hookrightarrow C^0(A)$ there exists a constant $S > 0$, such that

$$\|u\|_{C^0(A)} \leq S \|u\|_{W^{2m,p}(A)}, \quad \text{for all } u \in W^{2m,p}(A).$$

Hence

$$\begin{aligned}\|f\|_{C^0(A_R)} &= \|\tilde{f}\|_{C^0(A)} \\ &\leq S\|\tilde{f}\|_{W^{2m,p}(A)} \\ &= S\sum_{|\beta|\leq 2m} \|D^\beta \tilde{f}\|_{L^p(A)} \\ &= S\sum_{|\beta|\leq 2m} R^{|\beta|-2m/p}\|D^\beta f\|_{L^p(A_R)} \\ &\leq CS\sum_{|\beta|\leq 2m} R^{-2m/p-\delta}\|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta f\|_{L^p(A_R)} \\ &\leq CSR^{-\gamma}\|f\|_{M^p_{2m,\delta}}, \quad \gamma = 2m/p+\delta > 0.\end{aligned} \tag{3.31}$$

Since $R \geq 1$ is arbitrary, and on B_2 we have

$$\|f\|_{C^0(B_2)} \leq S'\|f\|_{W^{2m,p}(B_2)} \leq CS'\|f\|_{M^p_{2m,\delta}}, \tag{3.32}$$

we conclude that $M^p_{2m,\delta}(\mathbb{R}^{2m}) \subset C_0(\mathbb{R}^{2m})$, and actually

$$\sup_{n\in\mathbb{N}} \|f_n\|_{M^p_{2m,\delta}} < \infty \quad \Rightarrow \quad \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \|f_n\|_{C^0(A_R)} = 0. \tag{3.33}$$

By (3.31) and (3.32), on any compact set $\Omega \Subset \mathbb{R}^{2m}$ the sequence $\|f_n\|_{W^{2m,p}(\Omega)}$ is bounded and from the compact embedding $W^{2m,p}(\Omega) \hookrightarrow C^0(\Omega)$, we can extract a subsequence converging in $C^0(\Omega)$. Then up to choosing $\Omega = B_n$ and extracting a diagonal subsequence we have $f_n \to f$ locally uniformly for a continuous function f, and actually $f \in C_0(\mathbb{R}^{2m})$ and the convergence is globally uniform thanks to (3.33). □

Chapter 4

Existence of solutions to a fractional Liouville equation in $\mathbb{R}^n$

In this chapter we study the existence of solutions to the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad V := \int_{\mathbb{R}^n} e^{nu}dx < \infty,$$

where $Q = (n-1)!$ or $Q = -(n-1)!$. We show that to a certain extent the asymptotic behavior of u and the constant V can be prescribed simultaneously. Furthermore if $Q = -(n-1)!$ then V can be chosen to be any positive number. This is in contrast to the case $n = 3$, $Q = 2$, where Jin-Maalaoui-Martinazzi-Xiong showed that necessarily $V \leq |S^3|$, and to the case $n = 4$, $Q = 6$, where Lin showed that $V \leq |S^4|$.

4.1 Introduction and the main results

We consider the equation

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n, \tag{4.1}$$

where $n \geq 1$ and

$$V := \int_{\mathbb{R}^n} e^{nu}dx < \infty. \tag{4.2}$$

For the definition of the nonlocal operator $(-\Delta)^{\frac{n}{2}}$ we refer to Chapter 2.

We recall that (Theorem C and Theorem 3.1.1) in even dimension $n \geq 4$, for a given $V \in (0, |S^n|)$ and a given polynomial P such that $degree(P) \leq n-2$ and

$$x \cdot \nabla P(x) \to \infty \quad \text{as } |x| \to \infty, \tag{4.3}$$

there exists a solution u to (4.1)-(4.2) having the asymptotic behavior

$$u(x) = -P(x) - \alpha \log|x| + C + o(1), \tag{4.4}$$

where $\alpha := \frac{2V}{|S^n|}$, and $o(1) \to 0$ as $|x| \to \infty$.

When n is odd things are more complex as the operator $(-\Delta)^{\frac{n}{2}}$ is nonlocal. In a recent work Jin-Maalaoui-Martinazzi-Xiong [40] have proven:

Theorem E ([40]). *For every $V \in (0, |S^3|)$ there exists at least one smooth solution to (4.1)-(4.2) with $n = 3$.*

Extending the results of [15, 40, 77] and Theorem 3.1.1 to arbitrary odd dimension $n \geq 3$ we prove the following theorem about the existence of solutions to (4.1)-(4.2) with prescribed asymptotic behavior:

Theorem 4.1.1. *Let $n \geq 3$ be an odd integer. For any given $V \in (0, |S^n|)$ and any given polynomial P of degree at most $n-1$ such that*

$$P(x) \to \infty \quad \text{as } |x| \to \infty, \tag{4.5}$$

there exists $u \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ solution to (4.1)-(4.2) having the asymptotic behavior given in (4.4) with $\alpha = \frac{2V}{|S^n|}$.

Notice that, contrary to the result of Theorem E, in Theorem 4.1.1 we can now prescribe both the asymptotic behaviour and the volume, similar to Theorem 3.1.1, but in fact in more generality, since the condition (4.3) has been replaced by the weaker condition (4.5). Actually with minor modifications one can prove that the condition (4.5) also suffices in even dimension. On the other hand we do not expect this assumption to be optimal, compare to Theorem A and Theorem 2.1.2.

We remark that the condition $0 < V < |S^n|$ is necessary for the existence of non-spherical solution to (4.1)-(4.2) in dimension 3 and 4 as shown in [40] and [45] respectively, but in higher dimension solutions do exist for every $V > 0$ (see Chapter 5).

The condition $n \geq 3$ in Theorem 4.1.1 is necessary, since for $n = 1$ any solution of (4.1)-(4.2) is spherical, see Da Lio-Martinazzi-Rivière [21].

Now we shall discuss the case when the Q-curvature is negative. We consider the equation

$$(-\Delta)^{\frac{n}{2}} u = -(n-1)! e^{nu} \quad \text{in } \mathbb{R}^n. \tag{4.6}$$

In even dimension $n \geq 4$ for any $V > 0$ and any given polynomial P of degree at most $n-2$ satisfying (4.3), the existence of solutions to (4.6)-(4.2) having the asymptotic behavior given by (4.4) with $\alpha = -\frac{2V}{|S^n|}$ has been shown in Theorem 3.1.2. As in the positive case, we shall extend this existence result to arbitrary odd dimension $n \geq 3$, again replacing condition (4.3) with the weaker condition (4.5).

Theorem 4.1.2. *Let $n \geq 3$ be an odd integer. For any given $V > 0$ and any given polynomial P of degree at most $n-1$ satisfying (4.5) there exists $u \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ solution to (4.6)-(4.2) having the asymptotic behavior given in (4.4) with $\alpha = -\frac{2V}{|S^n|}$.*

4.2 Existence results

The proof of Theorems 4.1.1 and 4.1.2 rest on the following theorem:

Theorem 4.2.1. *Let* $w_0(x) = \log \frac{2}{1+|x|^2}$ *and let* $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$ *be the stereographic projection and* $N = (0, \ldots, 0, 1) \in S^n$ *be the North pole. Take any number* $\alpha \in (-\infty, 0) \cup (0, 2)$ *and consider two functions* $K, \varphi \in C^\infty(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n} \varphi dx = \gamma_n := \frac{(n-1)!}{2} |S^n|, \tag{4.7}$$

$\alpha K > 0$ *everywhere in* $\mathbb{R}^n$ *and whenever* $\alpha < 0$ *then* $|K| > \delta e^{-\delta |x|^p}$ *for some* $\delta > 0$, $0 < p < n$. *If both of* Ke^{-nw_0} *and* φe^{-nw_0} *can be extended as* C^{2n+1} *function on* S^n *via the stereographic projection* π *then the problem*

$$(-\Delta)^{\frac{n}{2}} w = Ke^{n(w+c_w)} - \alpha\varphi \quad \text{in } \mathbb{R}^n, \quad c_w := -\frac{1}{n} \log \left(\frac{1}{\alpha\gamma_n} \int_{\mathbb{R}^n} Ke^{nw} dx \right), \tag{4.8}$$

has at least one solution $w \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ *so that* $\lim_{|x| \to \infty} w(x) \in \mathbb{R}$.

Now the proof of Theorem 4.1.1 and Theorem 4.1.2 follows at once by taking

$$u := -P + \alpha u_0 + w + c_w,$$

where $u_0 \in C^\infty(\mathbb{R}^n)$ is given by Lemma 4.2.2 with $k = 2n + 3$, w is the solution in Theorem 4.2.1 with $\varphi = (-\Delta)^{\frac{n}{2}} u_0$ which satisfies (4.7) thanks to Lemma 4.2.3, and $K := \operatorname{sign}(\alpha)(n-1)! e^{-nP + n\alpha u_0}$. Notice that Ke^{-nw_0} can be extended smoothly on S^n via the stereographic projection π where as φe^{-nw_0} can be extended as a C^{2n+1} function.

Lemma 4.2.2. *For every positive integer* k *there exists* $u_0 \in C^\infty(\mathbb{R}^n)$ *such that*

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } |x| \geq 1 \text{ and } \quad |D^\alpha (-\Delta)^{\frac{n}{2}} u_0(x)| \leq \frac{C}{|x|^{2n+k+|\alpha|}} \quad \text{for } x \neq 0, \tag{4.9}$$

for any multi-index $\alpha \in \mathbb{N}^n$.

Proof. Inductively we define

$$v_j(x) = \int_0^{x_1} v_{j-1}((t, \bar{x})) dt, \quad \text{for } x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}, \, j = 1, 2, \ldots, k,$$

where

$$v_0(x) = \log \frac{1}{|x|}.$$

Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that

$$\chi(x) = \begin{cases} 0 & for\, |x| \leq \frac{1}{2} \\ 1 & for\, |x| \geq 1. \end{cases}$$

We claim that $u_0 := \frac{\partial^k}{\partial x_1^k}(\chi v_k)$ satisfies (4.9). It is easy to see that $u_0(x) = \log \frac{1}{|x|}$ for $|x| \geq 1$. By Lemma 2.5.1 $\frac{1}{\gamma_n}(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial x_1^k} v_k$ is a fundamental solution of $(-\Delta)^{\frac{1}{2}}$ on $\mathbb{R}^n$ and hence for $x \neq 0$, $(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial x_1^k} v_k(x) = 0$. For $|x| > 2$ using integration by parts we compute

$$\begin{aligned}(-\Delta)^{\frac{n}{2}}u_0(x) &= (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}}\frac{\partial^k}{\partial x_1^k}(\chi v_k - v_k)(x)\\ &= C_n\int_{|y|<1}\frac{(-\Delta)^{\frac{n-1}{2}}\frac{\partial^k}{\partial y_1^k}(\chi v_k - v_k)(y)}{|x-y|^{n+1}}dy\\ &= C_n\int_{|y|<1}(\chi v_k - v_k)(y)\frac{\partial^k}{\partial y_1^k}(-\Delta)^{\frac{n-1}{2}}\left(\frac{1}{|x-y|^{n+1}}\right)dy,\end{aligned}$$

and

$$D^\alpha(-\Delta)^{\frac{n}{2}}u_0(x) = C_n\int_{|y|<1}(\chi(y)v_k(y) - v_k(y))\,D_x^\alpha\frac{\partial^k}{\partial y_1^k}(-\Delta)^{\frac{n-1}{2}}\left(\frac{1}{|x-y|^{n+1}}\right)dy.$$

Hence

$$|D^\alpha(-\Delta)^{\frac{n}{2}}u_0(x)| \le C\frac{\|v_k\|_{L^1(B_1)}}{|x|^{2n+k+|\alpha|}}.$$

□

Lemma 4.2.3. *Let $u_0 \in C^\infty(\mathbb{R}^n)$ be as given by Lemma 4.2.2 for a given $k \in \mathbb{N}$. Then u_0 satisfies (4.7), that is,*

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{n}{2}}u_0(x)dx = \gamma_n.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le 1\\ 0 & \text{if } |x| \ge 2.\end{cases}$$

We set $\eta_k(x) = \eta(\frac{x}{k})$. Then noticing that $(-\Delta)^{\frac{n}{2}}u_0 \in L^1(\mathbb{R}^n)$ one has

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^{\frac{n}{2}}u_0(x)dx &= \lim_{k\to\infty}\int_{\mathbb{R}^n}(-\Delta)^{\frac{n}{2}}u_0(x)\eta_k(x)dx\\ &= \lim_{k\to\infty}\int_{B_1}\left(u_0(x) - \log\frac{1}{|x|}\right)(-\Delta)^{\frac{n}{2}}\eta_k(x)dx + \gamma_n\\ &= \gamma_n,\end{aligned}$$

where in the second equality we used the fact that $\frac{1}{\gamma_n}\log\frac{1}{|x|}$ is a fundamental solution of $(-\Delta)^{\frac{n}{2}}$ and the third equality follows from the locally uniform convergence of $(-\Delta)^{\frac{n}{2}}\eta_k \to 0$. □

It remains to prove Theorem 4.2.1. In order to do that we recall the definition of $H^n(S^n)$.

Definition 4.2.1. Let $n \ge 3$ be an odd integer. Let $\{Y_l^m \in C^\infty(S^n) : 1 \le m \le N_l,\ l = 0,1,2,\dots\}$ be an orthonormal basis of $L^2(S^n)$ where Y_l^m is an eigenfunction of the Laplace-Beltrami operator $-\Delta_{g_0}$ (g_0 denotes the round metric on S^n) corresponding to the eigenvalue $\lambda_l = l(l+n-2)$ and N_l is the multiplicity of λ_l (see [72, p. 68]). The space $H^n(S^n)$ is defined by

$$H^n(S^n) = \left\{u \in L^2(S^n) : \|u\|_{\dot{H}^n(S^n)} < \infty\right\},$$

where for any

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} u_l^m Y_l^m$$

we set

$$\|u\|^2_{\dot{H}^n(S^n)} := \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2} \right)^2 \right) \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1))^2 (u_l^m)^2.$$

Notice that the norm $\|u\|^2_{\dot{H}^n(S^n)}$ is equivalent to the simpler norm

$$\|u\|^2 := \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \lambda_l^n (u_l^m)^2,$$

but has the advantage of taking the form

$$\|u\|_{\dot{H}^n(S^n)} = \|P^n_{g_0} u\|_{L^2(S^n)},$$

where for n odd the Paneitz operator $P^n_{g_0}$ can be defined on $H^n(S^n)$ by (see for instance [16] and the refferences there in)

$$P^n_{g_0} u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2} \right)^2 \right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1)) u_l^m Y_l^m.$$

Since the operator $P^n_{g_0}$ is positive we can define its square root, namely

$$(P^n_{g_0})^{\frac{1}{2}} u := \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2} \right)^2 \right)^{\frac{1}{4}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1))^{\frac{1}{2}} u_l^m Y_l^m, \quad u \in H^{\frac{n}{2}}(S^n),$$

where the space $H^{\frac{n}{2}}(S^n)$ is defined by

$$H^{\frac{n}{2}}(S^n) := \Bigg\{ u \in L^2(S^n) : \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2} \right)^2 \right)^{\frac{1}{2}} \times \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1)) (u_l^m)^2 < \infty \Bigg\},$$

endowed with the norm

$$\begin{aligned} \|u\|^2_{H^{\frac{n}{2}}(S^n)} &:= \|u\|^2_{L^2(S^n)} + \|u\|^2_{\dot{H}^{\frac{n}{2}}(S^n)} \\ &:= \|u\|^2_{L^2(S^n)} + \|(P^n_{g_0})^{\frac{1}{2}} u\|^2_{L^2(S^n)}. \end{aligned}$$

Definition 4.2.2. Let $f \in H^{-\frac{n}{2}}(S^n)$ be the dual of $H^{\frac{n}{2}}(S^n)$. A function $u \in H^{\frac{n}{2}}(S^n)$ is said to be a weak solution of

$$P^n_{g_0} u = f,$$

if

$$\int_{S^n} (P^n_{g_0})^{\frac{1}{2}} u (P^n_{g_0})^{\frac{1}{2}} \varphi dV_0 = \langle f, \varphi\rangle, \quad \text{for every } \varphi \in H^{\frac{n}{2}}(S^n). \tag{4.10}$$

The following estimate of Beckner is crucial in the proof of Theorem 4.2.1.

Theorem 4.2.4 ([7]). *For every $u \in H^{\frac{n}{2}}(S^n)$ one has*

$$\log\left(\frac{1}{|S^n|}\int_{S^n} e^{u-\bar{u}} dV_0\right) \le \frac{1}{2|S^n|n!}\int_{S^n} |(P^n_{g_0})^{\frac{1}{2}} u|^2 dV_0, \quad \bar{u} := \frac{1}{|S^n|}\int_{S^n} u dV_0.$$

Proof of Theorem 4.2.1. Let $\tilde{K} = K \circ \pi$, $\varphi_1 = \varphi e^{-nw_0}$ and $\tilde{\varphi_1} = \varphi_1 \circ \pi$. Define the functional J on $H^{\frac{n}{2}}(S^n)$ by

$$J(w) := \int_{S^n}\left(\frac{1}{2}|(P^n_{g_0})^{\frac{1}{2}} w|^2 + \alpha\tilde{\varphi}_1 w\right) dV_0 - \frac{\alpha\gamma_n}{n}\log\left(\int_{S^n} |\tilde{K}| e^{nw} e^{-nw_0\circ\pi} dV_0\right).$$

Using Theorem 4.2.4 we bound

$$\begin{aligned}
&\log\left(\int_{S^n} |\tilde{K}| e^{nw} e^{-nw_0\circ\pi} dV_0\right)\\
&= \log\left(\frac{1}{|S^n|}\int_{S^n} e^{nw-n\overline{w}} |\tilde{K}| e^{-nw_0\circ\pi} dV_0\right) + n\overline{w} + C\\
&\le \log\left(\frac{1}{|S^n|}\int_{S^n} e^{nw-n\overline{w}} dV_0\right) + \log\left(\|\tilde{K} e^{-nw_0\circ\pi}\|_{L^\infty}\right) + n\overline{w} + C\\
&\le \frac{n^2}{2|S^n|n!}\int_{S^n} |(P^n_{g_0})^{\frac{1}{2}} w|^2 dV_0 + n\overline{w} + C.
\end{aligned} \tag{4.11}$$

Since for any $c \in \mathbb{R}$ $J(w+c) = J(c)$ we can assume $\overline{w} = 0$. Then from (4.11) we have

$$J(w) \ge \min\left\{\frac{1}{2}, \underbrace{\left(\frac{1}{2} - \frac{\alpha\gamma_n}{n}\frac{n^2}{2|S^n|n!}\right)}_{=(2-\alpha)/4}\right\}\|w\|^2_{\dot{H}^{\frac{n}{2}}} - \varepsilon\|w\|^2_{\dot{H}^{\frac{n}{2}}} - \frac{1}{\varepsilon}\|\tilde{\varphi}_1\|^2_{L^2} - C,$$

where $0 < \varepsilon < \frac{1}{2}$ is sufficiently small so that $\frac{2-\alpha}{4} - \varepsilon > 0$ and for $\alpha < 0$ using $|K| > \delta e^{-\delta|x|^p}$ one has

$$\log\left(\int_{S^n} |\tilde{K}| e^{nw} e^{-nw_0\circ\pi} dV_0\right) \ge \frac{1}{|S^n|}\int_{S^n} \log\left(|\tilde{K}| e^{-nw_0\circ\pi}\right) dV_0 + n\overline{w} + \log|S^n| \ge -C.$$

Thus a minimizing sequence $\{w_k\}$ of J with $\overline{w}_k = 0$ is bounded in $\dot{H}^{\frac{n}{2}}(S^n)$. With the help of Poincaré's inequality

$$\|w - \overline{w}\|_{L^2(S^n)} \le \|(P^n_{g_0})^{\frac{1}{2}} w\|_{L^2(S^n)}, \quad \text{for every } w \in H^{\frac{n}{2}}(S^n),$$

which easily follows from the definition of $\|(P^n_{g_0})^{\frac{1}{2}} w\|_{L^2(S^n)}$, we conclude that the sequence $\{w_k\}$ is bounded in $H^{\frac{n}{2}}(S^n)$. Then up to a subsequence w_k converges weakly to u for some $u \in H^{\frac{n}{2}}(S^n)$. From the compactness of the map $v \mapsto e^v$ from $H^{\frac{n}{2}}(S^n)$ to $L^p(S^n)$ for any $p \in [1,\infty)$ (for a simple proof see [40, Proposition 7] which holds in higher dimension as well) we have (up to a subsequence)

$$\lim_{k\to\infty} \log\left(\int_{S^n} |\tilde{K}|e^{nw_k}e^{-nw_0\circ\pi}dV_0\right) = \log\left(\int_{S^n} |\tilde{K}|e^{nu}e^{-nw_0\circ\pi}dV_0\right).$$

Moreover from the weak convergence of w_k to u we have

$$\lim_{k\to\infty}\int_{S^n}\tilde{\varphi}_1 w_k dV_0 = \int_{S^n}\tilde{\varphi}_1 u dV_0 \quad \text{and} \quad \|u\|_{H^{\frac{n}{2}}(S^n)} \leq \liminf_{k\to\infty}\|w_k\|_{H^{\frac{n}{2}}(S^n)},$$

and from the compact embedding $H^{\frac{n}{2}}(S^n) \hookrightarrow L^2(S^n)$ we get

$$\lim_{k\to\infty}\|w_k\|_{L^2(S^n)} = \|u\|_{L^2(S^n)}.$$

Thus $\|(P^n_{g_0})^{\frac{1}{2}}u\|_{L^2(S^n)} \leq \liminf_{k\to\infty}\|(P^n_{g_0})^{\frac{1}{2}}w_k\|_{L^2(S^n)}$ which implies that u is a minimizer of J and hence u is a weak solution of (in the sense of Definition 4.2.2)

$$P^n_{g_0}u + \alpha\tilde{\varphi}_1 = \frac{\alpha\gamma_n}{\int_{S^n}\tilde{K}e^{nu}e^{-nw_0}dV_0}\tilde{K}e^{-nw_0\circ\pi}e^{nu} =: C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu}.$$

Since $\tilde{\varphi}_1 \in C^{2n+1}(S^n)$ and $\tilde{K}e^{-nw_0\circ\pi} \in C^\infty(S^n)$ we have

$$P^n_{g_0}u = C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu} - \alpha\tilde{\varphi}_1 \in L^2(S^n),$$

and by Lemma 4.2.5 below $u \in H^n(S^n)$ and a repeated use of Lemma 4.2.6 gives $u \in C^{2n+1}(S^n)$.

We set $w := u\circ\pi^{-1}$ and $w_k := u_k\circ\pi^{-1}$ where $u_k \in C^\infty(S^n)$ be such that $u_k \xrightarrow{C^{2n+1}(S^n)} u$. It is easy to see that $(-\Delta)^{\frac{n}{2}}w_k \xrightarrow{C^0(\mathbb{R}^n)} (-\Delta)^{\frac{n}{2}}w$ and $P^n_{g_0}u_k \xrightarrow{C^0(S^n)} P^n_{g_0}u$ which easily follows from

$$P^n_{g_0}u_k \xrightarrow{H^{n+1}(S^n)} P^n_{g_0}u \text{ and } H^{n+1}(S^n) \hookrightarrow C^0(S^n).$$

Now using the following identity of T. Branson [9]

$$(-\Delta)^{\frac{n}{2}}(v\circ\pi^{-1}) = e^{nw_0}(P^n_{g_0}v)\circ\pi^{-1} \quad \text{for every } v \in C^\infty(S^n),$$

we get

$$\begin{aligned}(-\Delta)^{\frac{n}{2}}w = (-\Delta)^{\frac{n}{2}}(u\circ\pi^{-1}) &= e^{nw_0}\left(C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu} - \alpha\tilde{\varphi}_1\right)\circ\pi^{-1}\\ &= C_0Ke^{nw} - \alpha\varphi = Ke^{n(w+c_w)} - \alpha\varphi.\end{aligned}$$

Since $(-\Delta)^{\frac{n}{2}}w \in L_{\frac{1}{2}}(\mathbb{R}^n)\cap C^{2n+1}(\mathbb{R}^n)$ we have

$$(-\Delta)^{\frac{n+1}{2}}w = (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n}{2}}w \in C^{2n}(\mathbb{R}^n),$$

and by bootstrap argument we conclude that $w \in C^\infty(\mathbb{R}^n)$. □

The following lemma is probably known. Since we could not find a precise refference for this, we give a proof.

Lemma 4.2.5. *Let $f \in L^2(S^n)$. Let $u \in H^{\frac{n}{2}}(S^n)$ be a weak solution (in the sense of Definition 4.2.2) of*

$$P^n_{g_0} u = f \quad \text{on } S^n.$$

Then $u \in H^n(S^n)$.

Proof. Let

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} u_l^m Y_l^m \quad \text{and } f = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} f_l^m Y_l^m.$$

Taking the test function $\varphi = Y_l^m$ in (4.10) we get

$$f_l^m = \int_{S^n} (P^n_{g_0})^{\frac{1}{2}} u (P^n_{g_0})^{\frac{1}{2}} \varphi dV_0 = \left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1)) u_l^m.$$

Hence

$$\begin{aligned} \|P^n_{g_0} u\|_{L^2(S^n)} &= \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right) \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1))^2 (u_l^m)^2 \\ &= \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} (f_l^m)^2 < \infty, \end{aligned}$$

and we conclude the proof. □

Lemma 4.2.6. *Let $u \in H^s(S^n)$ and $f \in H^{s-n+t}(S^n)$ for some $s \geq n$ and $t \geq 0$. If u solves*

$$P^n_{g_0} u = f \quad \text{on } S^n,$$

then $u \in H^{s+t}(S^n)$.

Proof. Let

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} u_l^m Y_l^m,$$

and

$$(-\Delta_{g_0})^{\frac{s-n}{2}} f =: h = \sum_{i=0}^{\infty} \sum_{j=1}^{N_i} h_i^j Y_i^j,$$

where for any $r > 0$

$$(-\Delta_{g_0})^r v = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} v_l^m \lambda_l^r Y_l^m \quad \text{for } v = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} v_l^m Y_l^m \in H^{2r}(S^n).$$

Then

$$(-\Delta_{g_0})^{\frac{s-n}{2}} P^n_{g_0} u = h \quad \text{on } S^n. \tag{4.12}$$

Multiplying both sides of (4.12) by Y_j^i and integrating on S^n one has

$$\left(\lambda_j + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_j + k(n-k-1))\lambda_j^{\frac{s-n}{2}} u_j^i = h_j^i.$$

Since $h \in H^t(S^n)$ we have

$$\sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right) \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1))^2 \lambda_l^{s-n} \lambda_l^t (u_l^m)^2 < \infty,$$

and hence $u \in H^{s+t}(S^n)$. $\square$

Chapter 5

Conformal metrics on $\mathbb{R}^n$ with arbitrary total Q-curvature

We study the existence of solution to the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu}dx < \infty,$$

where $Q \geq 0$, $\kappa \in (0,\infty)$ and $n \geq 3$. Using ODE techniques Martinazzi for $n = 6$ and Huang-Ye for $n = 4m + 2$ proved the existence of a solution to the above problem with $Q \equiv const > 0$ and for every $\kappa \in (0,\infty)$. We extend these results in every dimension $n \geq 5$, thus completely answering the problem opened by Martinazzi. Our approach also extends to the case in which Q is non-constant, and under some decay assumptions on Q we can also treat the cases $n = 3$ and 4.

5.1 Introduction and statement of the main theorems

For a function $Q \in C^0(\mathbb{R}^n)$ we consider the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu}dx < \infty, \tag{5.1}$$

where for n odd the non-local operator $(-\Delta)^{\frac{n}{2}}$ is defined in Definition 2.1.1.

Recall that solutions to (5.1) have been classified in terms of their asymptotic behavior at infinity. More precisely we have the following:

Theorem F ([19, 21, 32, 40, 45, 56, 78])**.** *Let $n \geq 1$. Let u be a solution of*

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \textit{in } \mathbb{R}^n, \quad \kappa := (n-1)!\int_{\mathbb{R}^n} e^{nu}dx < \infty. \tag{5.2}$$

Then

$$\begin{aligned} u(x) &= \frac{(n-1)!}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{|y|}{|x-y|}\right)e^{nu(y)}dy + P(x) \\ &= -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + o(\log|x|), \ \textit{as } |x| \to \infty, \end{aligned} \tag{5.3}$$

where $\gamma_n := \frac{(n-1)!}{2}|S^n|$, $\Lambda_1 := 2\gamma_n$, $o(\log|x|)/\log|x| \to 0$ *as* $|x| \to \infty$, P *is a polynomial of degree at most* $n-1$ *and* P *is bounded from above. If* $n \in \{3,4\}$ *then* $\kappa \in (0, \Lambda_1]$ *and* $\kappa = \Lambda_1$ *if and only if* u *is a spherical solution, that is,*

$$u(x) = u_{\lambda,x_0}(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \tag{5.4}$$

for some $x_0 \in \mathbb{R}^n$ *and* $\lambda > 0$*. Moreover* u *is spherical if and only if* P *is constant (which is always the case when* $n \in \{1,2\}$*).*

We recall that under the assumptions

$$deg(P) \leq n-1, \quad P(x) \xrightarrow{|x|\to\infty} -\infty \quad \text{and } \kappa \in (0, \Lambda_1),$$

a converse to Theorem F has been proven in dimension 4 by Wei-Ye [77] and extended in Chapters 3 and 4 for $n \geq 3$.

Although the assumption $\kappa \in (0, \Lambda_1]$ is a necessary condition for the existence of a solution to (5.2) for $n = 3$, 4, it is possible to have a solution for $\kappa > \Lambda_1$ arbitrarily large in higher dimension as shown by Martinazzi [51] for $n = 6$. Huang-Ye [31] extended Martinazzi's result in arbitrary even dimension n of the form $n = 4m+2$ for some $m \geq 1$, proving that for every $\kappa \in (0, \infty)$ there exists a solution to (5.2). The case $n = 4m$ remained open.

The ideas in [31, 51] are based upon ODE theory. One considers only radial solutions so that the equation in (5.2) becomes an ODE, and the result is obtained by choosing suitable initial conditions and letting one of the parameters go to $+\infty$ (or $-\infty$). However, this technique does not work if the dimension n is a multiple of 4, and things get even worse in odd dimension since $(-\Delta)^{\frac{n}{2}}$ is nonlocal and ODE techniques cannot be used.

In this chapter we extend the works of [31, 51] and completely solve the cases left open, namely we prove that when $n \geq 5$ Problem (5.2) has a solution for every $\kappa \in (0, \infty)$. In fact we do not need to assume that Q is constant, but only that it is radially symmetric with growth at infinity suitably controlled, or not even radially symmetric. Moreover, we are able to prescribe the asymptotic behavior of the solution u (as in (5.3)) up to a polynomial of degree 4 which cannot be prescribed and in particular it cannot be required to vanish when $\kappa \geq \Lambda_1$. This in turn, together with Theorem F, is consistent with the requirement $n \geq 5$, because only when $n \geq 5$ the asymptotic expansion of u at infinity admits polynomials of degree 4.

We prove the following two theorems.

Theorem 5.1.1. *Let* $n \geq 5$ *be an integer. Let* P *be a polynomial on* $\mathbb{R}^n$ *with degree at most* $n-1$*. Let* $Q \in C^0(\mathbb{R}^n)$ *be such that* $Q(0) > 0$, $Q \geq 0$, Qe^{nP} *is radially symmetric and*

$$\sup_{x\in\mathbb{R}^n} Q(x)e^{nP(x)} < \infty.$$

Then for every $\kappa > 0$ *there exists a solution* u *to* (5.1) *such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + c_1|x|^2 - c_2|x|^4 + C + o(1), \quad \textit{as } |x| \to \infty,$$

for some $c_1, c_2 > 0$ *and* $C \in \mathbb{R}$*. In fact, there exists a radially symmetric function* v *on* $\mathbb{R}^n$ *and a constant* c_v *such that*

$$v(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) + o(1), \quad \text{as } |x| \to \infty,$$

and

$$u = P + v + c_v - |x|^4, \quad x \in \mathbb{R}^n.$$

Taking $Q = (n-1)!$ and $P = 0$ in Theorem 5.1.1 one has the following corollary.

Corollary 5.1.2. *Let* $n \geq 5$*. Let* $\kappa \in (0, \infty)$*. Then there exists a radially symmetric solution* u *to* (5.2) *such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + c_1|x|^2 - c_2|x|^4 + C + o(1), \quad \text{as } |x| \to \infty,$$

for some $c_1, c_2 > 0$ *and* $C \in \mathbb{R}$*.*

Notice that the polynomial part of the solution u in Theorem 5.1.1 is not exactly the prescribed polynomial P (compare Theorems 3.1.1, 4.1.1). In general, without perturbing the polynomial part, it is not possible to find a solution for $\kappa \geq \Lambda_1$. For example, if P is non-increasing and non-constant then there is no solution u to (5.2) with $\kappa \geq \Lambda_1$ such that u has the asymptotic behavior (5.3) (see Lemma 5.3.6 below). This justifies the term $c_1|x|^2$ in Theorem 5.1.1. Then the additional term $-c_2|x|^4$ is also necessary to avoid that $u(x) \geq \frac{c_1}{2}|x|^2$ for $|x|$ large, which would contrast with the condition $\kappa < \infty$, at least if Q does not decay fast enough at infinity. In the latter case, the term $-c_2|x|^4$ can be avoided, and one obtains an existence result also in dimensions 3 and 4.

Theorem 5.1.3. *Let* $n \geq 3$*. Let* $Q \in C^0_{rad}(\mathbb{R}^n)$ *be such that* $Q \geq 0$*,* $Q(0) > 0$ *and*

$$\int_{\mathbb{R}^n} Q(x)e^{\lambda|x|^2}dx < \infty, \quad \text{for every } \lambda > 0, \quad \int_{B_1(x)} \frac{Q(y)}{|x-y|^{n-1}}dy \xrightarrow{|x|\to\infty} 0.$$

Then for every $\kappa > 0$ *there exists a radially symmetric solution* u *to* (5.1)*.*

The decay assumption on Q in Theorem 5.1.3 is sharp in the sense that if $Qe^{\lambda|x|^2} \notin L^1(\mathbb{R}^n)$ for some $\lambda > 0$, then Problem (5.1) might not have a solution for every $\kappa > 0$. For instance, if $Q = e^{-\lambda|x|^2}$ for some $\lambda > 0$, then (5.1) with $n = 3, 4$ and $\kappa > \Lambda_1$ has no solution (see Lemma 5.3.5 below).

The proof of Theorem 5.1.1 is based on the Schauder fixed point theorem, and the main difficulty is to show that the "approximate solutions" are pre-compact (see in particular Lemma 5.2.2). We will do that using blow-up analysis (see for instance [4, 53, 66]). In general, if $\kappa \geq \Lambda_1$ one can expect blow-up, but we will construct our approximate solutions carefully in a way that this does not happen. For instance in [77] (see also Chapter 3) one looks for solutions of the form $u = P + v + c_v$ where v satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)}e^{n(v(y)+c_v)}dy,$$

and c_v is a constant such that

$$\int_{\mathbb{R}^n} Qe^{n(P+v+c_v)}dx = \kappa.$$

With such a choice we would not be able to rule out blow-up. Instead, by looking for solutions of the form

$$u = P + v + P_v + c_v$$

where a posteriori $P_v = -|x|^4$, v satisfies

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{n(P(y)+P_v(y)+v(y)+c_v)}dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \quad (5.5)$$

and c_v is again a normalization constant, one can prove that the integral equation (5.5) enjoys sufficient compactness, essentially due to the term $\frac{1}{2n}|x|^2|\Delta v(0)|$ on the right-hand side. Indeed a sequence of (approximate) solutions v_k blowing-up (for simplicity) at the origin, up to rescaling, leads to a sequence (η_k) of functions satisfying for every $R>0$

$$\int_{B_R} |\Delta\eta_k - c_k|dx \le CR^{n-2} + o(1)R^{n+2}, \quad o(1) \xrightarrow{k\to\infty} 0, \quad c_k > 0,$$

and converging to η_∞ solving (for simplicity here we ignore some cases)

$$(-\Delta)^{\frac{n}{2}}\eta_\infty = e^{n\eta_\infty} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{n\eta_\infty}dx < \infty,$$

and

$$\int_{B_R} |\Delta\eta_\infty - c_\infty|dx \le CR^{n-2}, \quad c_\infty \ge 0, \quad (5.6)$$

where $c_\infty = 0$ corresponds to $\Delta\eta_\infty(0) = 0$ (see Sub-case 1.1 in Lemma 5.2.2 with $x_k = 0$). The estimate on $\|\Delta\eta_\infty\|_{L^1(B_R)}$ in (5.6) shows that the polynomial part P_∞ of η_∞ (as in (5.3)) has degree at most 2, and hence $\Delta P_\infty \le 0$ as P_∞ is bounded from above. Therefore, $c_\infty = 0 = \Delta P_\infty$, P_∞ is constant and in particular η_∞ is a spherical solution by Theorem F, that is, $\eta_\infty = u_{\lambda,x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where u_{λ,x_0} is given by (5.4). This leads to a contradiction as $\Delta\eta_\infty(0) = 0$ and $\Delta u_{\lambda,x_0} < 0$ in $\mathbb{R}^n$.

In this chapter we focus only on the case $Q \ge 0$ because the negative case is relatively well understood. For instance by a simple application of maximum principle one can show that Problem (5.1) has no solution with $Q \equiv const < 0$, $n = 2$ and $\kappa > -\infty$, but when Q is non-constant, solutions do exist, as shown by Chanillo-Kiessling in [17] under suitable assumptions. Martinazzi [52] proved that in higher even dimension $n = 2m \ge 4$ Problem (5.1) with $Q \equiv const < 0$ has solutions for some κ, and it has been shown in Theorems 3.1.2 and 4.1.2 that actually for every $\kappa \in (-\infty, 0)$ and Q negative constant (5.1) has a solution.

5.2 The case $n \ge 5$

We consider the space

$$X := \left\{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\right\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|\le 3}(1+|x|)^{|\alpha|-4}|D^\alpha v(x)| + \sum_{3<|\alpha|\le n-1}|D^\alpha v(x)|\right).$$

For $v \in X$ we set

$$A_v := \max\left\{0, \sup_{|x|\ge 10}\frac{v(x)-v(0)}{|x|^4}\right\}, \quad P_v(x) := -|x|^4 - A_v|x|^4.$$

Then

$$v(x) + P_v(x) \le v(0) - |x|^4, \quad \text{for } |x| \ge 10.$$

Let c_v be the constant determined by

$$\int_{\mathbb{R}^n} Ke^{n(v+c_v)}dx = \kappa, \quad K := Qe^{nP}e^{nP_v},$$

where the functions Q and P satisfy the hypotheses in Theorem 5.1.1. Since $Q > 0$ in a neighborhood of the origin, by a dilation argument we can assume that $Q > 0$ on B_3. More precisely, if u is a solution to (5.1) then for any $\lambda > 0$, $u_\lambda(x) := u(\lambda x) + \log\lambda$ is also a solution to (5.1) with Q replaced by Q_λ, where $Q_\lambda(x) := Q(\lambda x)$. Now for a suitable choice of $\lambda > 0$ one has $Q_\lambda > 0$ on B_3.

Then $u = P + P_v + v + c_v$ satisfies

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu}, \quad \kappa = \int_{\mathbb{R}^n} Qe^{nu}dx,$$

if and only if v satisfies

$$(-\Delta)^{\frac{n}{2}}v = Ke^{n(v+c_v)}.$$

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)K(y)e^{n(v(y)+c_v)}dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|.$$

Lemma 5.2.1. *Let v solve $tT(v) = v$ for some $0 < t \le 1$. Then*

$$v(x) = \frac{t}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)K(y)e^{n(v(y)+c_v)}dy + \frac{t}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \tag{5.7}$$

$\Delta v(0) < 0$, and $v(x) \to -\infty$ as $|x| \to \infty$. Moreover,

$$\sup_{x\in B_1^c} v(x) = v(1) = \inf_{x\in B_1} v(x),$$

and in particular $A_v = 0$.

Proof. Since v satisfies $tT(v) = v$, (5.7) follows from the definition of T. Differentiating under the integral sign and observing that $\Delta\log\frac{1}{|\cdot-y|} < 0$, from (5.7) one gets

$$\Delta v(x) < \frac{t}{2n}|\Delta v(0)|\Delta(|x|^2 - |x|^4), \quad x \in \mathbb{R}^n. \tag{5.8}$$

Taking $x = 0$ in (5.8) we obtain $\Delta v(0) < t|\Delta v(0)|$, which implies that $\Delta v(0) < 0$. Notice that the function

$$w(x) := v(x) + \frac{t}{2n}|\Delta v(0)|(|x|^4 - |x|^2)$$

is monotone decreasing as $\Delta w < 0$. This follows from (5.8) and the integral representation of radially symmetric functions given by

$$f(\xi) - f(\bar{\xi}) = \int_{\bar{\xi}}^{\xi} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta f(x) dx dr, \quad 0 \le \bar{\xi} < \xi, \quad \omega_{n-1} := |S^{n-1}|. \tag{5.9}$$

The monotonicity of w implies that $\sup_{x \in B_1^c} v(x) = v(1) = \inf_{x \in B_1} v(x)$, and hence $A_v = 0$. Finally, together with $|\Delta v(0)| > 0$, we conclude that $\lim_{|x|\to\infty} v(x) = -\infty$ as $\lim_{|x|\to\infty} w(x) \le w(1)$. ☐

Lemma 5.2.2. *Let $(v,t) \in X \times (0,1]$ satisfy $v = tT(v)$. Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{\frac{1}{8}}} w \le C, \quad w := v + c_v + \frac{1}{n}\log t.$$

Proof. Let us assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ such that $\max_{\bar{B}_{\frac{1}{8}}} w_k =: w_k(\theta_k) \to \infty$.

If θ_k is a point of local maxima of w_k then we set $x_k = \theta_k$. Otherwise, we can choose $x_k \in B_{\frac{1}{4}}$ such that x_k is a point of local maxima of w_k and $w_k(x_k) \ge w_k(x)$ for every $x \in B_{|x_k|}$. This follows from the fact that

$$\inf_{B_{\frac{1}{4}} \setminus B_{\frac{1}{8}}} w_k \not\to \infty,$$

which is a consequence of

$$\int_{\mathbb{R}^n} K e^{nw_k} dx = t_k \kappa \le \kappa, \quad K > 0 \text{ on } B_3.$$

We set $\mu_k := e^{-w_k(x_k)}$. We distinguish the following cases.

Case 1 Up to a subsequence $t_k \mu_k^2 |\Delta v_k(0)| \to c_0 \in [0, \infty)$.

We set

$$\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k) = w_k(x_k + \mu_k x) - w_k(x_k).$$

Notice that by (5.7) we have for some dimensional constant C_1

$$\begin{aligned}\Delta \eta_k(x) &= \mu_k^2 \Delta v_k(x_k + \mu_k x) \\ &= C_1 \frac{\mu_k^2}{\gamma_n} \int_{\mathbb{R}^n} \frac{K(y) e^{nw_k(y)}}{|x_k + \mu_k x - y|^2} dy + t_k \mu_k^2 \left(1 - \frac{4(n+2)}{2n}|x_k + \mu_k x|^2\right) |\Delta v_k(0)|,\end{aligned}$$

so that

$$\begin{aligned}
&\int_{B_R}\left|\Delta\eta_k(x)-t_k\mu_k^2|\Delta v_k(0)|\left(1-\frac{2(n+2)}{n}|x_k|^2\right)\right|dx\\
&\le \frac{C_1}{\gamma_n}\int_{\mathbb{R}^n}K(y)e^{nw_k(y)}\int_{B_R}\frac{\mu_k^2 dx}{|x_k+\mu_k x-y|^2}dy\\
&\quad + Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x_k\cdot x|+\mu_k^2|x|^2)dx\\
&\le \frac{C_1}{\gamma_n}t_k\kappa\int_{B_R}\frac{1}{|x|^2}dx+Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x|+\mu_k^2|x|^2)dx\\
&\le C\kappa t_k R^{n-2}+Ct_k\mu_k^2|\Delta v_k(0)|\left(\mu_k R^{n+1}+\mu_k^2R^{n+2}\right).
\end{aligned} \tag{5.10}$$

The function η_k satisfies

$$(-\Delta)^{\frac{n}{2}}\eta_k(x)=K(x_k+\mu_k x)e^{n\eta_k(x)} \quad \text{in } \mathbb{R}^n, \quad \eta_k(0)=0.$$

Moreover, $\eta_k\le C(R)$ on B_R. This follows easily if $|x_k|\le\frac{1}{9}$ as in that case $\eta_k\le 0$ on B_R for $k\ge k_0(R)$. On the other hand, for $\frac{1}{9}<|x_k|\le\frac{1}{4}$ one can use Lemma 5.2.4 (below). Therefore, by Lemma 5.4.4 (and Lemmas 5.2.6, 5.2.7 if n is odd), up to a subsequence, $\eta_k\to\eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$ where η satisfies

$$(-\Delta)^{\frac{n}{2}}\eta=K(x_\infty)e^{n\eta} \quad \text{in } \mathbb{R}^n, \quad K(x_\infty)\int_{\mathbb{R}^n}e^{n\eta}dx\le t_\infty\kappa<\infty,\ K(x_\infty)>0,$$

where (up to a subsequence) $t_k\to t_\infty$ and $x_k\to x_\infty$. Notice that $t_\infty\in(0,1]$, $x_\infty\in\bar{B}_{\frac{1}{4}}$ and for every $R>0$, by (5.10)

$$\int_{B_R}|\Delta\eta-c_0c_1|\,dx\le CR^{n-2}, \quad c_1=:1-\frac{2(n+2)}{n}|x_\infty|^2>0. \tag{5.11}$$

Hence by Theorem F we have

$$\eta(x)=P_0(x)-\alpha\log|x|+o(\log|x|), \quad \text{as } |x|\to\infty,$$

where P_0 is a polynomial of degree at most $n-1$, P_0 is bounded from above and α is a positive constant. In fact, by (5.11)

$$\int_{B_R}|\Delta P_0(x)-c_0c_1|dx\le CR^{n-2}, \quad \text{for every } R>0.$$

Since $c_0,c_1\ge 0$, it follows that P_0 is constant. This implies that η is a spherical solution and in particular $\Delta\eta<0$ on $\mathbb{R}^n$, and therefore, again by (5.11), we have $c_0=0$.

We consider the following sub-cases.

Sub-case 1.1 There exists $M>0$ such that $\frac{|x_k|}{\mu_k}\le M$.

We set $y_k:=-\frac{x_k}{\mu_k}$. Then (up to a subsequence) $y_k\to y_\infty\in B_{M+1}$. Therefore,

$$\Delta\eta(y_\infty)=\lim_{k\to\infty}\Delta\eta_k(y_k)=\lim_{k\to\infty}\mu_k^2\Delta v_k(0)=\frac{c_0}{t_\infty}=0,$$

a contradiction as $\Delta\eta<0$ on $\mathbb{R}^n$.

Sub-case 1.2 Up to a subsequence $\frac{|x_k|}{\mu_k} \to \infty$.

For any $N \in \mathbb{N}$ we can choose $\xi_{1,k}, \ldots, \xi_{N,k} \in \mathbb{R}^n$ such that $|\xi_{i,k}| = |x_k|$ for all $i = 1, \ldots, N$ and the balls $B_{2\mu_k}(\xi_{i,k})$'s are disjoint for k large enough. Since v_k's are radially symmetric, the functions $\eta_{i,k} := v_k(\xi_{i,k} + \mu_k x) - v_k(\xi_{i,k}) \to \eta_i = \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$. Therefore,

$$\lim_{k\to\infty} \int_{B_1} e^{n(v_k + c_{v_k})} dx \geq N \lim_{k\to\infty} \int_{B_{\mu_k}(\xi_{1,k})} e^{n(v_k + c_{v_k})} dx = N \frac{1}{t_\infty} \int_{B_1} e^{n\eta} dx.$$

This contradicts to the fact that

$$\int_{B_1} K e^{n(v_k + c_{v_k})} dx \leq \kappa, \quad K > 0 \text{ on } B_3.$$

Case 2 Up to a subsequence $t_k \mu_k^2 |\Delta v_k(0)| \to \infty$.

We choose $\rho_k > 0$ such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| = 1$. We set

$$\psi_k(x) = v_k(x_k + \rho_k \mu_k x) - v_k(x_k).$$

Then one can get (similar to (5.10))

$$\begin{aligned}
&\int_{B_R} \left| \Delta \psi_k(x) - \left(1 - \frac{2(n+2)}{n} |x_k|^2 \right) \right| dx \\
&\leq C_1 \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_R} \frac{\rho_k^2 \mu_k^2}{|x_k + \mu_k \rho_k x - y|^2} dx dy + C_2 \mu_k \rho_k \int_{B_R} (|x| + \mu_k \rho_k |x|^2) dx \\
&\xrightarrow{k\to\infty} 0,
\end{aligned}$$

thanks to Lemma 5.2.5 (below). Moreover, together with Lemma 5.2.4, ψ_k satisfies

$$(-\Delta)^{\frac{n}{2}} \psi_k = o(1) \quad \text{in } B_R, \quad \psi_k(0) = 0, \quad \psi_k \leq C(R) \quad \text{on } B_R.$$

Hence, by Lemma 5.4.4 (and Lemma 5.2.6 if n is odd), up to a subsequence $\psi_k \to \psi$ in $C^{n-1}_{loc}(\mathbb{R}^n)$. Then ψ must satisfy

$$\int_{B_1} |\Delta \psi - c_0| dx = 0, \quad c_0 := 1 - \frac{2(n+2)}{n} |x_\infty|^2 > 0,$$

where (up to a subsequence) $x_k \to x_\infty \in \bar{B}_{\frac{1}{4}}$. This shows that $\Delta \psi(0) = c_0 > 0$, which is a contradiction as

$$\Delta \psi(0) = \lim_{k\to\infty} \Delta \psi_k(0) = \lim_{k\to\infty} \rho_k^2 \mu_k^2 \Delta v_k(x_k) \leq 0.$$

Here, $\Delta v_k(x_k) \leq 0$ follows from the fact that x_k is a point of local maxima of v_k. $\square$

A consequence of the local uniform upper bounds of w are the following global uniform upper bounds:

Lemma 5.2.3. *There exists a constant $C > 0$ such that for all $(v, t) \in X \times (0, 1]$ with $v = tT(v)$ we have $|\Delta v(0)| \leq C$ and*

$$v(x) + c_v + \frac{1}{n}\log t \leq C, \quad \text{on } \mathbb{R}^n.$$

Proof. By Lemma 5.2.2 we have

$$\sup_{B_{\frac{1}{8}}} w := \sup_{B_{\frac{1}{8}}} \left(v + c_v + \frac{1}{n}\log t \right) \leq C.$$

Differentiating under the integral sign from (5.7), and recalling that $\Delta v(0) < 0$, we obtain

$$\begin{aligned} |\Delta v(0)| &\leq C \int_{B_{\frac{1}{8}}} \frac{1}{|y|^2} K(y) e^{nw(y)} dy + C \int_{B^c_{\frac{1}{8}}} \frac{1}{|y|^2} K(y) e^{nw(y)} dy \\ &\leq C \sup_{B_{\frac{1}{8}}} K \int_{B_{\frac{1}{8}}} \frac{1}{|y|^2} dy + C \int_{B^c_{\frac{1}{8}}} K e^{nw} dy \\ &\leq C(\kappa, K). \end{aligned}$$

By (5.8) we get

$$\Delta v(x) \leq t|\Delta v(0)| \leq C, \quad x \in \mathbb{R}^n,$$

and hence, together with (5.9)

$$v(x) = v(0) + \int_0^{|x|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta v(y) dy dr \leq v(0) + C|x|^2 \leq C + v(0), \quad x \in B_2.$$

The lemma follows from Lemmas 5.2.1 and 5.2.2. ☐

Proof of Theorem 5.1.1 Let $v \in X$ be a solution of $v = tT(v)$ for some $0 < t \leq 1$. Then $A_v = 0$ and $|\Delta v(0)| \leq C$, thanks to Lemmas 5.2.1 and 5.2.3. Hence, for $0 \leq |\beta| \leq n - 1$

$$\begin{aligned} |D^\beta v(x)| &\leq C \int_{\mathbb{R}^n} \left| D^\beta \log \left(\frac{1}{|x-y|} \right) \right| K(y) e^{n(v(y) + c_v + \frac{1}{n}\log t)} dy + C|D^\beta(|x|^2 - |x|^4)| \\ &\leq C \int_{\mathbb{R}^n} \left| D^\beta \log \left(\frac{1}{|x-y|} \right) \right| e^{-|y|^4} dy + C|D^\beta(|x|^2 - |x|^4)|, \end{aligned}$$

where in the second inequality we have used that

$$v(x) + c_v + \frac{1}{n}\log t \leq C, \quad C \text{ is independent of } v \text{ and } t,$$

which follows from Lemma 5.2.3. Now as in Lemma 5.2.8 one can show that

$$\|v\|_X \leq M,$$

and therefore, by Lemma 3.2.4, the operator T has a fixed point (say) v. Then

$$u = P + v + c_v - |x|^4,$$

is a solution to the Problem (5.1) and u has the asymptotic behavior given by

$$u(x) = P(x) - \frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) - |x|^4 + c_v + o(1), \quad \text{as } |x| \to \infty.$$

This completes the proof of Theorem 5.1.1. □

Now we give a proof of the technical lemmas used in the proof of Lemma 5.2.2.

Lemma 5.2.4. *Let $\varepsilon > 0$. Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (5.7) or (5.14) for all $k \in \mathbb{N}$. Let $x_k \in B_1 \setminus B_\varepsilon$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. Then*

$$v_k(x_k + x) - v_k(x_k) \le C(n, \varepsilon)|x|^2 t_k |\Delta v_k(0)|, \quad x \in B_1.$$

Proof. If $|x_k + x| \le |x_k|$ then $v_k(x_k + x) - v_k(x_k) \le 0$ as $v_k(x_k) \ge v_k(y)$ for every $y \in B_{|x_k|}$. For $|x_k| < |x_k + x|$, setting $a = a(k, x) := x_k + x$, and together with (5.9) we obtain

$$\begin{aligned}
v_k(x_k + x) - v_k(x_k) &= \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r \setminus B_{|x_k|}} \Delta v_k(x) dx dr \\
&\le \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_{|a|} \setminus B_{|x_k|}} t_k |\Delta v_k(0)| dx d\rho \\
&\le C(n) t_k |\Delta v_k(0)| (|B_{|a|}| - |B_{|x_k|}|) \left(\frac{1}{|x_k|^{n-2}} - \frac{1}{|a|^{n-2}} \right) \\
&\le C(n, \varepsilon) t_k |x|^2 |\Delta v_k(0)|,
\end{aligned}$$

where in the first equality we have used that

$$0 = v_k'(x_k) = \frac{1}{\omega_{n-1}|x_k|^{n-1}} \int_{B_{|x_k|}} \Delta v_k dx.$$

Hence we have the lemma. □

Lemma 5.2.5. *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (5.7) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| \le C$ and $\rho_k \mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} dx dy =: \lim_{k\to\infty} I_k = 0.$$

Proof. In order to prove the lemma we fix $R > 0$ (large). We split B_{R_0} into

$$A_1(R, y) := \{x \in B_{R_0} : |x_k + \rho_k \mu_k x - y| > R \rho_k \mu_k\}, \quad A_2(R, y) := B_{R_0} \setminus A_1(R, y).$$

Then we can write $I_k = I_{1,k} + I_{2,k}$, where

$$I_{i,k} := \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{A_i(R,y)} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} dx dy, \quad i = 1, 2.$$

Changing the variable $y \mapsto x_k + \rho_k\mu_k y$ and by Fubini's theorem one gets

$$\begin{aligned} I_{2,k} &= \rho_k^n \int_{B_{R_0}} \int_{\mathbb{R}^n} K(x_k + \rho_k\mu_k y) e^{n\eta_k(y)} \frac{1}{|x-y|^2} \chi_{|x-y|\le R} dy dx \\ &\le \rho_k^n \int_{B_{R_0}} \int_{B_{R+R_0}} K(x_k + \rho_k\mu_k y) e^{n\eta_k(y)} \frac{1}{|x-y|^2} dy dx \\ &\le C(n,\varepsilon) \left(\sup_{B_{R+R_0+1}} K e^{n\eta_k} \right) (R+R_0)^n R_0^{n-2} \rho_k^n, \end{aligned}$$

where $\eta_k(y) := w_k(x_k + \rho_k\mu_k y) - w_k(x_k)$. If $x_k \to 0$ then $\eta_k \le 0$ on B_{R+R_0+1} for k large. Otherwise, for k large $\rho_k\mu_k y \in B_1$ for every $y \in B_{R+R_0+1}$ and hence, by Lemma 5.2.4

$$\eta_k(y) = v_k(x_k + \rho_k\mu_k y) - v_k(x_k) \le C|\rho_k\mu_k y|^2 t_k |\Delta v_k(0)| \le C(R, R_0).$$

Therefore,

$$\lim_{k\to\infty} I_{2,k} = 0.$$

Using the definition of c_v we bound

$$I_{1,k} \le \frac{|B_{R_0}|}{R^2} \int_{\mathbb{R}^n} K(y) e^{nw_k(y)} dy \le C(n,\kappa,R_0) \frac{1}{R^2}.$$

Since $R > 0$ is arbitrary, we conclude the lemma. □

We need the following two lemmas only for n odd.

Lemma 5.2.6. *Let $n \ge 5$. Let v be given by (5.7). For any $r > 0$ and $\xi \in \mathbb{R}^n$ we set*

$$w(x) = v(rx + \xi), \quad x \in \mathbb{R}^n.$$

Then there exists $C > 0$ (independent of v, t, r, ξ) such that for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = n-1$ we have $\|D^\alpha w\|_{L_{\frac{1}{2}}(\mathbb{R}^n)} \le Ct(1 + r^4|\Delta v(0)|)$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ (independent of r, ξ and t) such that

$$\int_{B_R^c} \frac{|D^\alpha w(x)|}{1+|x|^{n+1}} dx < \varepsilon t(1 + r^4|\Delta v(0)|), \quad |\alpha| = n-1.$$

Proof. Differentiating under the integral sign we obtain

$$|D^\alpha w(x)| \le Ct \int_{\mathbb{R}^n} \frac{r^{n-1}}{|rx+\xi-y|^{n-1}} f(y) dy + Ctr^4|\Delta v(0)|, \quad f(y) := K(y) e^{n(v(y)+c_v)}.$$

If $n > 5$ then the above inequality is true without the term $Ctr^4|\Delta v(0)|$. Using a change of variable $y \mapsto \xi + ry$, we get

$$\begin{aligned} &\int_\Omega \frac{|D^\alpha w(x)|}{1+|x|^{n+1}} dx \\ &\le Ctr^n \int_{\mathbb{R}^n} f(\xi + ry) \int_\Omega \frac{1}{|x-y|^{n-1}} \frac{1}{1+|x|^{n+1}} dx dy + Ctr^4|\Delta v(0)| \int_\Omega \frac{dx}{1+|x|^{n+1}}. \end{aligned}$$

The lemma follows by taking $\Omega = \mathbb{R}^n$ or B_R^c. □

Lemma 5.2.7. *Let $\eta_k \to \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$. We assume that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R^c} \frac{|\Delta^{\frac{n-1}{2}}\eta_k(x)|}{1+|x|^{n+1}}dx < \varepsilon, \quad \text{for } k = 1, 2, \dots. \tag{5.12}$$

We further assume that

$$(-\Delta)^{\frac{n}{2}}\eta_k = K(x_k + \mu_k x)e^{n\eta_k} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |K(x_k + \mu_k x)|e^{n\eta_k(x)}dx \le C,$$

where $x_k \to x_\infty$, $\mu_k \to 0$, K is a continuous function and $K(x_\infty) > 0$. Then $e^{n\eta} \in L^1(\mathbb{R}^n)$ and η satisfies

$$(-\Delta)^{\frac{n}{2}}\eta = K(x_\infty)e^{n\eta} \quad \text{in } \mathbb{R}^n.$$

Proof. First notice that $\Delta^{\frac{n-1}{2}}\eta_k \to \Delta^{\frac{n-1}{2}}\eta$ in $L_{\frac{1}{2}}(\mathbb{R}^n)$, thanks to (5.12) and the convergence $\eta_k \to \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$.

We claim that η satisfies $(-\Delta)^{\frac{n}{2}}\eta = K(x_\infty)e^{n\eta}$ in $\mathbb{R}^n$ in the sense of distribution.

In order to prove the claim we let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} K(x_k + \mu_k x)e^{n\eta_k(x)}\varphi(x)dx = \int_{\mathbb{R}^n} K(x_\infty)e^{n\eta(x)}\varphi(x)dx,$$

and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}}\eta_k(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}}\eta(-\Delta)^{\frac{1}{2}}\varphi dx.$$

We conclude the claim.

To complete the lemma first notice that $e^{n\eta} \in L^1(\mathbb{R}^n)$, which follows from the fact that for any $R > 0$

$$\int_{B_R} e^{n\eta}dx = \lim_{k\to\infty} \int_{B_R} e^{n\eta_k}dx = \lim_{k\to\infty} \int_{B_R} \frac{K(x_k + \mu_k x)}{K(x_\infty)}e^{n\eta_k(x)}dx \le \frac{C}{K(x_\infty)}.$$

We fix a function $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on B_1. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we set $\varphi_k(x) = \varphi(x)\psi(\frac{x}{k})$. The lemma follows by taking $k \to \infty$, thanks to the previous claim. □

Lemma 5.2.8. *The operator $T : X \to X$ is compact.*

Proof. Let v_k be a bounded sequence in X. Then (up to a subsequence) $\{v_k(0)\}$, $\{\Delta v_k(0)\}$, $\{A_{v_k}\}$ and $\{c_{v_k}\}$ are convergent sequences. Therefore, $|\Delta v_k(0)|(|x|^2 - |x|^4)$ converges to some function in X. To conclude the lemma, it is sufficient to show that up to a subsequence $\{f_k\}$ converges in X, where f_k is defined by

$$f_k(x) = \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}dy.$$

Differentiating under the integral sign one gets

$$\begin{aligned}|D^\beta f_k(x)| &\leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}dy, \quad 0<|\beta|\leq n-1 \\ &\leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} e^{-|y|^4}dy \\ &\leq C,\end{aligned}$$

where the second inequality follows from the uniform bounds

$$|v_k(0)| \leq C,\ |c_{v_k}| \leq C,\ Qe^{nP} \leq C, \text{ and } v_k(x)+P_{v_k}(x) \leq v_k(0)-|x|^4. \tag{5.13}$$

Indeed, for $0<|\beta|\leq n-1$

$$\lim_{R\to\infty} \sup_k \sup_{x\in B_R^c} |D^\beta f_k(x)| = 0,$$

and for every $0<s<1$ we have $\|D^{n-1}f_k\|_{C^{0,s}(B_R)} \leq C(R,s)$. Finally, using (5.13) we bound

$$|f_k(x)| \leq C \int_{\mathbb{R}^n} |\log|x-y||e^{-|y|^4}dy \leq C\log(2+|x|).$$

Thus, by Ascoli's theorem, up to a subsequence, $f_k \to f$ in $C^{n-1}_{loc}(\mathbb{R}^n)$ for some $f \in C^{n-1}(R^n)$, and the global uniform estimates of f_k and $D^\beta f_k$ would imply that $f_k \to f$ in X. ☐

5.3 The case $n \geq 3$

We consider the space

$$X := \left\{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\right\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|\leq 1}(1+|x|)^{|\alpha|-2}|D^\alpha v(x)| + \sum_{1<|\alpha|\leq n-1}|D^\alpha v(x)|\right).$$

For $v \in X$, let c_v be the constant determined by

$$\int_{\mathbb{R}^n} Qe^{n(v+c_v)}dy = \kappa,$$

where Q satisfies the hypothesis in Theorem 5.1.3. Again by dilation argument we can assume that $Q>0$ on B_3.

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{n(v(y)+c_v)}dy + \frac{1}{2n}|\Delta v(0)||x|^2.$$

As in Lemma 5.2.8 one can show that the operator T is compact.

The proofs of the following two lemmas are similar to those of Lemmas 5.2.1 and 5.2.5 respectively.

Lemma 5.3.1. *Let v solve $tT(v) = v$ for some $0 < t \leq 1$. Then $\Delta v(0) < 0$, and*

$$v(x) = \frac{t}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{n(v(y)+c_v)}dy + \frac{t}{2n}|\Delta v(0)||x|^2. \tag{5.14}$$

Lemma 5.3.2. *Let $(v_k, t_k) \in X \times (0,1]$ satisfy (5.14) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $\rho_k^2 t_k \mu_k^2 |\Delta v_k(0)| \leq C$ and $\rho_k\mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} Q(y)e^{nw_k(y)}\int_{B_{R_0}} \frac{\rho_k^2\mu_k^2}{|x_k+\rho_k\mu_k x - y|^2}dxdy = 0.$$

Now we prove similar local uniform upper bounds to those in Lemma 5.2.2.

Lemma 5.3.3. *Let $(v,t) \in X \times (0,1]$ satisfy (5.14). Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{\frac{1}{8}}} w \leq C, \quad w := v + c_v + \frac{1}{n}\log t.$$

Proof. The proof is very similar to that of Lemma 5.2.2. Here we briefly sketch it.

We assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence of (v_k, t_k) and a sequence of points x_k in $B_{\frac{1}{4}}$ such that

$$w_k(x_k) \to \infty, \quad w_k \leq w_k(x_k) \text{ on } B_{|x_k|}, \quad x_k \text{ is a point of local maxima of } v_k.$$

We set $\mu_k := e^{-w_k(x_k)}$ and we distinguish following cases.

Case 1 Up to a subsequence $t_k\mu_k^2|\Delta v_k(0)| \to c_0 \in [0,\infty)$.

We set $\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k)$. Then we have

$$\int_{B_R} |\Delta \eta_k - t_k\mu_k^2|\Delta v_k(0)||dx \leq C t_k R^{n-2}.$$

Now one can proceed exactly as in Case 1 in Lemma 5.2.2.

Case 2 Up to a subsequence $t_k\mu_k^2|\Delta v_k(0)| \to \infty$.

We set $\psi_k(x) = v_k(x_k + \rho_k\mu_k x) - v_k(x_k)$ where ρ_k is determined by $t_k\rho_k^2\mu_k^2|\Delta v_k(0)| = 1$. Then by Lemma 5.3.2

$$\int_{B_R} |\Delta\psi_k - 1|dx = o(1), \quad \text{as } k \to \infty.$$

Similar to Case 2 in Lemma 5.2.2 one can get a contradiction. □

With the help of Lemma 5.3.3 we prove

Lemma 5.3.4. *There exists a constant $M > 0$ such that for all $(v,t) \in X \times (0,1]$ satisfying (5.14) we have $\|v\| \leq M$.*

Proof. Let $(v,t) \in X \times (0,1]$ satisfy (5.14). We set $w := v + c_v + \frac{1}{n}\log t$.

First we show that $|\Delta v(0)| \le C$ for some $C > 0$ independent of v and t. Indeed, differentiating under the integral sign, from (5.14), and together with Lemma 5.3.3, we get

$$\begin{aligned}|\Delta v(0)|(1+t) &\le C\int_{\mathbb{R}^n}\frac{1}{|y|^2}Q(y)e^{nw(y)}dy\\&= C\int_{B_{\frac{1}{8}}}\frac{1}{|y|^2}Q(y)e^{nw(y)}dy + C\int_{B^c_{\frac{1}{8}}}\frac{1}{|y|^2}Q(y)e^{nw(y)}dy\\&\le C\int_{B_{\frac{1}{8}}}\frac{1}{|y|^2}Q(y)dy + C\kappa\\&\le C.\end{aligned}$$

Hence $|\Delta v(0)| \le C$.

We define a function $\xi(x) := v(x) - \frac{t}{2n}|\Delta v(0)||x|^2$. Then ξ is monotone decreasing on $(0,\infty)$, which follows from the fact that $\Delta\xi \le 0$. Therefore,

$$\begin{aligned}w(x) &= \xi(x) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2\\&\le \xi(\frac{1}{8}) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2\\&\le w(\frac{1}{8}) + \frac{t}{2n}|\Delta v(0)||x|^2.\end{aligned}$$

Hence, $w(x) \le \lambda(1+|x|^2)$ on $\mathbb{R}^n$ for some $\lambda > 0$ independent of v and t. Using this in (5.14) one can show that

$$|v(x)| \le C\log(2+|x|) + C|x|^2,$$

and differentiating under the integral sign, from (5.14)

$$|D^\beta v(x)| \le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}Q(y)e^{\lambda(1+|y|^2)}dy + C|D^\beta|x|^2|, \quad 0 < |\beta| \le n-1.$$

The lemma follows easily. ☐

Proof of Theorem 5.1.3 By Schauder fixed point theorem (see Lemma 3.2.4), the operator T has a fixed point, thanks to Lemma 5.3.4. Let v be a fixed point of T. Then $u = v + c_v$ is a solution of (5.1).

This finishes the proof of Theorem 5.1.3. ☐

Now we prove the non existence results stated in the introduction of this chapter.

Lemma 5.3.5. *Let $n \in \{3,4\}$. If $Q(x) = e^{-\lambda|x|^2}$ for some $\lambda > 0$ then there is no solution to (5.1) with $\kappa > \Lambda_1$. If $Q \in C^1_{rad}(\mathbb{R}^n)$ is of the form $Q = e^{\xi}$ and it satisfies*

$$Q' \le 0, \quad |x\cdot\nabla Q(x)| \le C, \quad \frac{\xi(x)}{|x|^2} \xrightarrow{|x|\to\infty} 0,$$

then there is no radially symmetric solution to (5.1) with $\kappa > \Lambda_1$.

Proof. First we consider the case when $Q = e^{-\lambda|x|^2}$. Let u be a solution to (5.1) with $Q = e^{-\lambda|x|^2}$. Then the function $w(x) := u - \frac{\lambda}{n}|x|^2$ satisfies

$$(-\Delta)^{\frac{n}{2}} w = e^{nw}, \quad \kappa = \int_{\mathbb{R}^n} Qe^{nu}dx = \int_{\mathbb{R}^n} e^{nw}dx < \infty.$$

It follows from [40, 45] that $\kappa \le \Lambda_1$.

In order to prove the lemma for $Q = e^{\xi}$, we assume by contradiction that there is a solution u to (5.1) with $\kappa > \Lambda_1$. We set

$$v(x) := \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) Q(y)e^{nu(y)}dy, \quad h := u - v.$$

Then $v(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + o(\log|x|)$ as $|x| \to \infty$. Notice that h is radially symmetric and $(-\Delta)^{\frac{n}{2}}h = 0$ on $\mathbb{R}^n$. Therefore, $h(x) = c_1 + c_2|x|^2$ for some $c_1, c_2 \in \mathbb{R}$. This follows easily if $n = 4$. For $n = 3$, first notice that $\Delta h \in L_{\frac{1}{2}}(\mathbb{R}^3)$. Hence, by [40, Lemma 15] $\Delta h \equiv const$. Now radial symmetry of h implies that $h(x) = c_1 + c_2|x|^2$.

From a Pohozaev type identity in [78, Theorem 2.1] we get

$$\frac{\kappa}{\gamma_n}\left(\frac{\kappa}{\gamma_n} - 2\right) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} (x\cdot \nabla K(x))\, e^{nv(x)}dx, \quad K := Qe^{nh}. \tag{5.15}$$

Since $\kappa > \Lambda_1 = 2\gamma_n$, from (5.15) we deduce that $x\cdot \nabla K(x) > 0$ for some $x \in \mathbb{R}^n$. Using that $Qe^{nu} \in L^1(\mathbb{R}^n)$ and that $\xi(x) = o(|x|^2)$ at infinity, one has $c_2 \le 0$. Therefore, $x\cdot \nabla K(x) \le 0$ in $\mathbb{R}^n$, a contradiction. □

The proof of the following lemma is similar to that of Lemma 5.3.5.

Lemma 5.3.6. *Let $\kappa \ge \Lambda_1$. Let P be a non-constant and non-increasing radially symmetric polynomial of degree at most $n-1$. Then there is no solution u to (5.2) (with $n \ge 3$) such that u has the asymptotic behavior given by*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + o(\log|x|), \quad as\ |x| \to \infty.$$

5.4 Some useful results

The following identity (5.16) is due to Pizzetti [63]. A simple proof of (5.16) and (5.17) can be found in [56, Lemma 3] and [56, Proposition 4] respectively.

Lemma 5.4.1 ([56, 63])**.** *Let $\Delta^m h = 0$ in $B_{4R} \subset \mathbb{R}^n$. For any $x \in B_R$ and $0 < r < R - |x|$ we have*

$$\frac{1}{|B_r|}\int_{B_r(x)} h(z)dz = \sum_{i=0}^{m-1} c_i r^{2i}\Delta^i h(x), \tag{5.16}$$

where

$$c_0 = 1, \quad c_i = c(i,n) > 0, \quad for\ i \ge 1.$$

Moreover, for every $k \geq 0$ there exists $C = C(k, R) > 0$ such that

$$\|h\|_{C^k(B_R)} \leq C\|h\|_{L^1(B_{4R})}. \tag{5.17}$$

Lemma 5.4.2 (Proposition 22 in [40]). *Let $u \in L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0,1)$ and $(-\Delta)^\sigma u = 0$ in B_{2R}. Then for every $k \in \mathbb{N}$*

$$\|\nabla^k u\|_{C^0(B_R)} \leq C(n,\sigma,k)\frac{1}{R^k}\left(R^{2\sigma}\int_{\mathbb{R}^n\setminus B_{2R}}\frac{|u(x)|}{|x|^{n+2\sigma}}dx + \frac{\|u\|_{L^1(B_{2R})}}{R^n}\right)$$

where $\alpha \in (0,1)$ and k is a nonnegative integer.

Lemma 5.4.3 (Proposition 1.1 in [68]). *Let $\sigma \in (0,1)$. Let u be a solution of*

$$\begin{cases} (-\Delta)^\sigma u = f & \text{in } B_R \\ u = 0 & \text{in } B_R^c \end{cases}$$

Then

$$\|u\|_{C^\sigma(\mathbb{R}^n)} \leq C(R,\sigma)\|f\|_{L^\infty(B_R)}.$$

Lemma 5.4.4. *Let $R > 0$ and $B_R \subset \mathbb{R}^n$. Let $u_k \in C^{n-1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (\frac{1}{2}, 1)$ be such that*

$$u_k(0) = 0, \quad \|u_k^+\|_{L^\infty(B_R)} \leq C, \quad \|(-\Delta)^{\frac{n}{2}}u_k\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R}|\Delta u_k|dx \leq C.$$

If n is an odd integer, we also assume that $\|\Delta^{\frac{n-1}{2}}u_k\|_{L_{\frac{1}{2}}(\mathbb{R}^n)} \leq C$. Then (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{\frac{R}{8}})$.

Proof. First we prove the lemma for n even.

We write $u_k = w_k + h_k$ where

$$\begin{cases} (-\Delta)^{\frac{n}{2}}w_k = (-\Delta)^{\frac{n}{2}}u_k & \text{in } B_R \\ \Delta^j w_k = 0, & \text{on } \partial B_R, \quad j = 0, 1, \ldots, \frac{n-2}{2}. \end{cases}$$

Then by standard elliptic estimates, w_k's are uniformly bounded in $C^{n-1,\beta}(B_R)$. Therefore,

$$|h_k(0)| \leq C, \quad \|h_k^+\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R}|\Delta h_k|dx \leq C.$$

Since h_k's are $\frac{n}{2}$-harmonic, Δh_k's are $(\frac{n}{2}-1)$-harmonic in B_R, and by (5.17) we obtain

$$\|\Delta h_k\|_{C^n(B_{\frac{R}{4}})} \leq C\|\Delta h_k\|_{L^1(B_R)} \leq C.$$

Using the identity (5.16) we bound

$$\begin{aligned} \frac{1}{|B_R|}\int_{B_R(0)}h_k^-(z)dz &= \frac{1}{|B_R|}\int_{B_R(0)}h_k^+(z)dz - \frac{1}{|B_R|}\int_{B_R(0)}h_k(z)dz \\ &= \frac{1}{|B_R|}\int_{B_R(0)}h_k^+(z)dz - h_k(0) - \sum_{i=1}^{\frac{n}{2}-1}c_iR^{2i}\Delta^i h_k(0) \\ &\leq C, \end{aligned}$$

and hence

$$\int_{B_R} |h_k(z)|dz = \int_{B_R} h_k^+(z)dz + \int_{B_R} h_k^-(z)dz \leq C.$$

Again by (5.17) we obtain

$$\|h_k\|_{C^n(B_{\frac{R}{4}})} \leq C\|h_k\|_{L^1(B_R)} \leq C.$$

Thus, u_k's are uniformly bounded in $C^{n-1,\beta}(B_{\frac{R}{4}})$ and (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{\frac{R}{4}})$ for some $u \in C^{n-1}(B_{\frac{R}{4}})$.

It remains to prove the lemma for n odd.

If n is odd then $\frac{n-1}{2}$ is an integer. We split $\Delta^{\frac{n-1}{2}} u_k = w_k + h_k$ where

$$\begin{cases} (-\Delta)^{\frac{1}{2}} w_k = (-\Delta)^{\frac{1}{2}} \Delta^{\frac{n-1}{2}} u_k & \text{in } B_R \\ w_k = 0 & \text{on } B_R^c. \end{cases}$$

Then by Lemmas 5.4.2 and 5.4.3 one has $\|\Delta^{\frac{n-1}{2}} u_k\|_{C^{\frac{1}{2}}(B_{\frac{R}{2}})} \leq C$. Now one can proceed as in the case of even integer. □

Chapter 6

Large blow-up sets for the prescribed Q-curvature equation

Let $m \geq 2$ be an integer. For any open domain $\Omega \subset \mathbb{R}^{2m}$, non-positive function $\varphi \in C^\infty(\Omega)$ such that $\Delta^m \varphi \equiv 0$, and bounded sequence $(Q_k) \subset L^\infty(\Omega)$ we prove the existence of a sequence of functions $(u_k) \subset C^{2m-1}(\Omega)$ solving the Liouville equation of order $2m$

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega, \quad \limsup_{k\to\infty} \int_\Omega e^{2mu_k} dx < \infty,$$

and blowing-up exactly on the set $S_\varphi := \{x \in \Omega : \varphi(x) = 0\}$, i.e.

$$\lim_{k\to\infty} u_k(x) = +\infty \text{ for } x \in S_\varphi \text{ and } \lim_{k\to\infty} u_k(x) = -\infty \text{ for } x \in \Omega \setminus S_\varphi,$$

thus showing that a result of Adimurthi, Robert and Struwe is sharp. We extend this result to the boundary of Ω and to the case $\Omega = \mathbb{R}^{2m}$.

6.1 Introduction and main results

In several nonlinear elliptic problems of second order and "critical type", sequences of solutions are not always compact, as they can blow-up at finitely many points, see e.g [5], [11], [12], [23], [70], [73], [74]. For instance, as shown by Brézis-Merle in [12]:

Theorem G ([12]). *Given a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to the Liouville equation*

$$-\Delta u_k = Q_k e^{2u_k} \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{6.1}$$

with $\|Q_k\|_{L^\infty} \leq C$ and $\|e^{2u_k}\|_{L^1} \leq C$ for some C independent of k, there exists a finite (possibly empty) set $S_1 = \{x^{(1)}, \ldots, x^{(I)}\} \subset \Omega$ such that, up to extracting a subsequence one of the following alternatives holds:

(i) (u_k) is bounded in $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus S_1)$.

(ii) $u_k \to -\infty$ locally uniformly in $\Omega \setminus S_1$.

A similar behaviour is also found on manifolds, or in higher order and higher dimensional problems, see e.g. [50], [75], or even in 1-dimensional situations involving the operator

$(-\Delta)^{\frac{1}{2}}$, see [20], [21]. Now consider the problem

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega \subset \mathbb{R}^{2m} \tag{6.2}$$

$$\limsup_{k\to\infty} \int_\Omega e^{2mu_k}\,dx < \infty, \quad \limsup_{k\to\infty} \|Q_k\|_{L^\infty(\Omega)} < \infty. \tag{6.3}$$

Since blow-up at finitely many points appears in many problems with various critical nonlinearities and also of higher order, one might suspect that this is a general feature also holding for (6.2). On the other hand Adimurthi, Robert and Struwe [4] found an example of solutions to (6.2)-(6.3) for $m = 2$ that blow-up on a hyperplane, and showed in general that the blow-up set of a sequence (u_k) of solutions to (6.2)-(6.3) can be of Hausdorff dimension 3. This was generalized to the case of arbitrary m in [53]. More precisely for a finite set $S_1 \subset \Omega \subset \mathbb{R}^{2m}$ let us introduce

$$\mathcal{K}(\Omega, S_1) := \{\varphi \in C^\infty(\Omega \setminus S_1) : \varphi \le 0, \quad \varphi \not\equiv 0, \quad \Delta^m \varphi \equiv 0\}, \tag{6.4}$$

and for a function $\varphi \in \mathcal{K}(\Omega, S_1)$ set

$$S_\varphi := \{x \in \Omega \setminus S_1 : \varphi(x) = 0\}. \tag{6.5}$$

Theorem H ([4, 53])**.** *Let (u_k) be a sequence of solutions to (6.2)-(6.3) for some $m \ge 1$. Then the set*

$$S_1 := \left\{ x \in \Omega : \lim_{r\downarrow 0} \limsup_{k\to\infty} \int_{B_r(x)} |Q_k| e^{2mu_k} dy \ge \frac{\Lambda_1}{2} \right\}, \quad \Lambda_1 := (2m-1)!|S^{2m}|$$

is finite (possibly empty) and up to a subsequence either

(i) (u_k) is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega \setminus S_1)$, or

(ii) there exists a function $\varphi \in \mathcal{K}(\Omega, S_1)$ and a sequence $\beta_k \to \infty$ as $k \to +\infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ locally uniformly in } \Omega \setminus S_1.$$

In particular $u_k \to -\infty$ locally uniformly in $\Omega \setminus (S_\varphi \cup S_1)$.

Notice that Theorem H contains Theorem G since when $m = 1$ we have $S_\varphi = \emptyset$ for every $\varphi \in \mathcal{K}(\Omega, S_1)$ by the maximum principle. In fact the more complex blow-up behaviour of (6.2) when $m > 1$ can be seen as a consequence of the size of $\mathcal{K}(\Omega, S_1)$. A way of recovering a finite blow-up behaviour for (6.2)-(6.3) was given by Robert [65] when $m = 2$ and generalized by Martinazzi [55] when $m \ge 3$, by additionally assuming

$$\|\Delta u_k\|_{L^1(B_r(x))} \le C \quad \text{on some ball } B_r(x) \subset \Omega,$$

which is sufficient to control the "polyharmonic part" of u_k.

The first result that we will prove shows that the condition given in [4] and [53] on the set S_φ above is sharp, at least when $S_1 = \emptyset$. In fact we shall consider a slightly stronger result, by defining

$$S^*_\varphi := S_\varphi \cup \left\{ x \in \partial\Omega : \lim_{\Omega \ni y \to x} \varphi(y) = 0 \right\}, \tag{6.6}$$

namely we add to S_φ the points on $\partial\Omega$ where φ can be continuously extended to 0. Then we have

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^{2m}$, $m \geq 2$, be an open (connected) domain and let $(Q_k) \subset L^\infty(\Omega)$ be bounded. Then for every $\varphi \in \mathcal{K}(\Omega, \emptyset)$ there exists a sequence (u_k) of solutions to (6.2) with*

$$\int_\Omega e^{2mu_k}\,dx \to 0, \tag{6.7}$$

such that as $k \to \infty$

$$u_k \to -\infty \text{ loc. unif. in } \Omega \setminus S_\varphi, \quad u_k \to +\infty \text{ loc. unif. on } S_\varphi^*, \tag{6.8}$$

where S_φ and S_φ^* are as in (6.5) and (6.6). The same result holds if $m = 1$ and Ω is smoothly bounded.

The proof of Theorem 6.1.1 is based on a Schauder's fixed-point argument. The case when Ω is smoothly bounded is very elementary, as one looks for solutions of the form

$$u_k = c_k\varphi + k + v_k, \quad c_k \to \infty,$$

where v_k is a small correction term.

The general case is a priori more rigid. For instance in the case $m = 1$, when $Q_k \equiv 1$ there are few solutions to (6.2)-(6.3) when $\Omega = \mathbb{R}^2$ (see [19]) and many more when Ω is bounded (see [18]). To treat the general case we will borrow ideas from [77] (see also Chapter 3) and suitably prescribe the asymptotic behavior of u_k at infinity. More precisely we will look for solutions of the form

$$u_k = c_k\varphi + k - \alpha_k \log(1 + |x|^2) - \beta|x|^2 + v_k,$$

for some $c_k \to \infty$, $\alpha_k \to 0$, $\beta > 0$, and a function $v_k \to 0$ uniformly. If $\varphi(x) \to -\infty$ sufficiently fast as $|x| \to \infty$, or when Ω is bounded, one can choose $\beta = 0$, but the case $\Omega = \mathbb{R}^{2m}$, $\varphi(x_1, \ldots, x_{2m}) = -x_1^2$ shows that β in general must be positive when

$$\liminf_{x\in\Omega, |x|\to\infty} \varphi(x) > -\infty,$$

otherwise the condition (6.3) might fail to be satisfied.

The simplicity of the proof of Theorem 6.1.1 comes at the cost of not being able to prescribe the total Q-curvature of the metric $g_{u_k} := e^{2u_k}|dx|^2$, which will necessarily go to zero, together with the volume of g_{u_k}. Resting on variational methods from Chapter 4, going back to [15], we can extend Theorem 6.1.1 to the case in which we prescribe both the blow-up set S_φ and the total curvature of the metrics g_{u_k}. This time, though, we will have to restrict to non-negative functions Q_k.

Theorem 6.1.2. *Let $0 < \Lambda < \Lambda_1/2$, $\Omega \subset \mathbb{R}^{2m}$ open, $m \geq 2$, $\varphi \in \mathcal{K}(\Omega, \emptyset)$, and let S_φ be as in (6.5). Let further Q_k be functions for which there exists $x_0 \in S_\varphi^*$ such that*

$$\liminf_{k\to+\infty} \int_{B_\varepsilon(x_0)\cap\Omega} Q_k\,dx > 0, \quad \text{for every } \varepsilon > 0, \quad 0 \leq Q_k \leq b < \infty. \tag{6.9}$$

Then there exists a sequence $(u_k)_{k\in\mathbb{N}}$ *of solutions to* (6.2) *with*

$$\int_\Omega Q_k e^{2mu_k}dx = \Lambda, \tag{6.10}$$

such that (6.8) *holds.*

The integral assumption in (6.9) is crucial. In fact, for every $\varphi \in \mathcal{K}(\Omega, \emptyset)$ there are functions Q_k satisfying $0 \le Q_k \le b < \infty$, such that for any $\Lambda > 0$ there exists no sequence (u_k) of solution to (6.2) satisfying (6.8) and (6.10) (see Proposition 6.3.3).

As we shall see, Theorems 6.1.1 and 6.1.2 give several examples of solutions blowing-up on the boundary, already in dimension 2.

Corollary 6.1.3. *Let* $\Omega \subset \mathbb{R}^{2m}$ *with* $m \ge 1$ *be a bounded domain with smooth boundary and let* $\Gamma \subset \partial\Omega$ *be a proper closed subset. Let* (Q_k) *be as in Theorem 6.1.1. Then we can find solutions* $u_k : \Omega \to \mathbb{R}$ *to* (6.2) *such that the conclusions of Theorem 6.1.1 holds with* $S^*_\varphi = \Gamma$ *for some* $\varphi \in \mathcal{K}(\Omega, \emptyset)$. *If* $m \ge 2$ *and* (Q_k) *additionally satisfies* (6.9) *for some* $x_0 \in \Gamma$, *then we can prescribe* (6.10) *instead of* (6.7).

In the radially symmetric case we can prescribe any $\Lambda \in (0, \infty)$.

Theorem 6.1.4. *Let* $\Omega = B_{R_2} \setminus B_{R_1} \subset \mathbb{R}^{2m}$ *and* $\varphi \in \mathcal{K}(\Omega, \emptyset)$ *be radially symmetric. Let* $\Lambda > 0$ *and let* (Q_k) *be radially symmetric satisfying* (6.9). *Then there exists a sequence of radially symmetric solutions* (u_k) *to* (6.2) *such that* (6.8) *and* (6.10) *hold. For* $\Omega = B_R$ *the same conclusion holds if in addition we have* $\Delta\varphi(0) > 0$ *and* $Q_k \to 1$ *in* $L^\infty(B_\delta(0))$ *for some* $\delta > 0$.

It was open whether there exists a sequence (u_k) of solutions to (6.2)-(6.3) on some domain Ω in $\mathbb{R}^{2m}$ with 2 open regions $\Omega_0, \Omega_1 \subset \Omega$ such that

$$\|\Delta u_k\|_{L^1(\Omega_0)} = O(1), \quad \|\Delta u_k\|_{L^1(\Omega_1)} \to \infty.$$

We will prove that this is actually possible.

Theorem 6.1.5. *On* $\Omega = B_2 \subset \mathbb{R}^{2m}$ *for any* $\Lambda \in (0, \Lambda_1)$ *we can find a sequence* (u_k) *of solutions to* (6.2)-(6.3) *with* $Q_k \equiv 1$ *such that*

$$\int_{B_2} e^{2mu_k}dx = \Lambda, \tag{6.11}$$

and

$$\int_{B_1} |\Delta u_k|dx \le C, \quad \int_{B_2} (\Delta u_k)^- dx \xrightarrow{k\to\infty} \infty. \tag{6.12}$$

6.2 Blow-up with vanishing volume

In order to clarify the simple idea behind the proof we start considering the easier case when Ω is bounded and has regular boundary. The proof in the general case is more complex and only works when $m \ge 2$ (easy counterexamples can be found when $m = 1$, $\Omega = \mathbb{R}^2$, $Q_k \equiv 1$, using the classification result from [19]).

6.2.1 Case Ω smoothly bounded

In this case we can assume $m \geq 1$. The proof will be based on an application of a fixed-point argument. Consider the Banach space

$$X := C^0(\bar{\Omega}), \quad \|v\|_X = \max_{x\in\bar{\Omega}} |v(x)|.$$

For each $k \in \mathbb{N}$ choose $c_k \geq k^2$ such that

$$\|e^{2mc_k\varphi}\|_{L^2(\Omega)} \leq e^{-3mk}.$$

For $k \in \mathbb{N}$ consider the operator $T_k : X \to X$ defined by $T(v) = \bar{v}$ where $\bar{v}$ is the unique solution of

$$\begin{cases} (-\Delta)^m \bar{v} = Q_k e^{2m(k+c_k\varphi+v)} & \text{in } \Omega \\ \bar{v} = \Delta\bar{v} = \cdots = \Delta^{m-1}\bar{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

From elliptic estimates, the Sobolev embedding and Ascoli-Arzelà's theorem it follows that T_k is compact. Moreover, for every $v \in X$ we have

$$\|\bar{v}\|_X \leq C_1\|\Delta^m\bar{v}\|_{L^2(\Omega)} \leq C_2 M e^{2mk}\|e^{2mv}\|_X\|e^{2mc_k\varphi}\|_{L^2(\Omega)}, \quad \|Q_k\|_{L^\infty} \leq M.$$

This shows that

$$\|T_k(v)\|_X \leq C_3 e^{2mk}e^{-3mk}, \quad \text{for } \|v\|_X \leq 1, \quad C_3 := C_2 M. \tag{6.13}$$

Therefore $T_k(\bar{B}_1) \subset \bar{B}_{\frac{1}{2}}$ for k large enough (here B_r is a ball in X), and hence T_k has a fixed point in X. We denote it by v_k. Notice that $\|v_k\|_X \leq Ce^{-mk} \to 0$ as $k \to \infty$. Moreover, by Hölder's inequality,

$$\int_\Omega e^{2mk}e^{2mc_k\varphi}e^{2mv_k}dx \leq e^{2mk}\sqrt{|\Omega|}\|e^{2mc_k\varphi}\|_{L^2(\Omega)} \xrightarrow{k\to\infty} 0.$$

We set

$$u_k := v_k + k + c_k\varphi.$$

Then u_k satisfies

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega, \quad \int_\Omega e^{2mu_k}dx \xrightarrow{k\to\infty} 0.$$

Moreover

$$\inf_{x\in S_\varphi^*} u_k = o(1) + k \xrightarrow{k\to\infty} \infty.$$

Finally, for any compact subset $K \Subset \Omega \setminus S_\varphi$, using that $c_k \geq k^2$, we obtain

$$\max_{x\in K} u_k = o(1) + k + c_k \max_{x\in K}\varphi \leq k - \varepsilon k^2 \xrightarrow{k\to\infty} -\infty,$$

where $\varepsilon > 0$ is such that $\max_{x\in K}\varphi < -\varepsilon$. This completes the proof.

6.2.2 General case

In the general case we need to assume $m \geq 2$. We will use many ideas from [77] (see also Chapter 3). Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$. Fix $u_0 \in C^\infty(\mathbb{R}^{2m})$, $u_0 > 0$, such that $u_0(x) = \log |x|$ for $|x| \geq 2$, and notice that integration by parts yields

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m u_0 \, dx = -\gamma_{2m}, \tag{6.14}$$

where γ_{2m} is defined by

$$(-\Delta)^m \log \frac{1}{|x|} = \gamma_{2m}\delta_0 \text{ in } \mathbb{R}^{2m}, \text{ i.e. } \gamma_{2m} = \frac{\Lambda_1}{2}. \tag{6.15}$$

We will construct a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to (6.2)-(6.7) of the form

$$u_k = -\beta|x|^2 + c_k\varphi - \alpha_k u_0 + k + v_k, \quad \text{in } \Omega, \tag{6.16}$$

for some $\beta \geq 0$ and $v_k \in C^{2m-1}(\mathbb{R}^{2m})$ such that as $k \to \infty$

$$\sup_\Omega |v_k| \to 0, \quad c_k \to \infty, \quad \alpha_k \to 0.$$

In general $\beta > 0$ is an arbitrary fixed constant, but if φ satisfies

$$\int_\Omega e^{2m\varphi}|x|^{2s}\, dx < \infty, \quad \text{for some } s > 0, \tag{6.17}$$

then we can take $\beta = 0$ as well. If there exists $s > 0$ such that (6.17) holds then we set $q = s$, otherwise we take $\beta > 0$ and set $q = 1$.

We consider

$$X := C_0(\mathbb{R}^{2m}) := \left\{ v \in C^0(\mathbb{R}^{2m}) : \lim_{|x|\to\infty} v(x) = 0 \right\}, \quad \|v\|_X = \sup_{x\in\mathbb{R}^{2m}} |v(x)|.$$

For $c \in \mathbb{R}$ we set

$$F_{k,c} = \begin{cases} Q_k e^{2mk} e^{-2m\beta|x|^2} e^{2mc\varphi} & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^{2m} \setminus \Omega. \end{cases}$$

Let $\varepsilon_1 \in (0, \frac{q}{8m})$ (to be fixed later). We fix $p > 1$ and $\delta \in (-\frac{2m}{p}, \frac{2m}{p} + 1)$ such that $p(2m + \delta) < \frac{q}{4}$. For each $k \in \mathbb{N}$ we choose $c_k \geq k^2$ so that

$$\int_{\mathbb{R}^{2m}} |F_{k,c_k}(x)|(M + |x|)^q dx \leq \varepsilon_1 e^{-k} e^{-2m}, \tag{6.18}$$

$$\|F_k(M + |x|)^{\frac{q}{4}}\|_{L^p_{2m+\delta}} \leq \varepsilon_1 e^{-k}, \quad F_k := F_{k,c_k}, \tag{6.19}$$

$$\int_\Omega e^{2m(c_k\varphi+k)}(M + |x|)^q \, dx \leq e^{-k}, \tag{6.20}$$

where q is as above, $M > 0$ is such that $e^{u_0} \leq M$ on B_2 and the spcae $L^p_{2m+\delta}$ is defined in Definition 3.2.1. For each $k \in \mathbb{N}$, define a continuous function I_k on $X \times (-\frac{q}{2m}, \frac{q}{2m})$

given by

$$I_k(v,\alpha) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} F_k e^{-2m\alpha u_0} e^{2mv} dx.$$

If $I_k(v,0) > 0$ then

$$\lim_{\alpha\to 0^+} \frac{I_k(v,\alpha)}{\alpha} = \infty, \quad \frac{I_k(v,\varepsilon_1 e^{-k})}{\varepsilon_1 e^{-k}} \le 1, \quad \|v\|_X \le 1,$$

and hence there exists $\alpha \in (0, \varepsilon_1 e^{-k}]$ such that $I_k(v,\alpha) = \alpha$. Notice that

$$\sup_{\alpha\in[-\frac{q}{4m},0]} |I_k(v,\alpha)| \le e^{-k}\varepsilon_1, \quad \text{for } \|v\|_X \le 1.$$

Thus, if $I_k(v,0) < 0$ then

$$\lim_{\alpha\to 0^-} \frac{I_k(v,\alpha)}{\alpha} = \infty, \quad \frac{|I_k(v,-\varepsilon_1 e^{-k})|}{\varepsilon_1 e^{-k}} \le 1, \quad \|v\|_X \le 1,$$

and hence there exists $\alpha \in [-\varepsilon_1 e^{-k}, 0)$ such that $I_k(v,\alpha) = \alpha$. For $\|v\|_X \le 1$ we define

$$\alpha_{k,v} := \begin{cases} \inf\{\alpha > 0 : \alpha = I_k(v,\alpha)\} & \text{if } I_k(v,0) > 0 \\ \sup\{\alpha < 0 : \alpha = I_k(v,\alpha)\} & \text{if } I_k(v,0) < 0 \\ 0 & \text{if } I_k(v,0) = 0. \end{cases}$$

From the continuity of I_k it follows that $\alpha_{k,v} = I_k(v, \alpha_{k,v})$.

Lemma 6.2.1. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and for every $v \in B_1$ if*

$$I_k(v,\alpha_v) = \alpha_v \quad \textit{for some } |\alpha_v| < \frac{q}{4m},$$

then for every $w \in B_{\varepsilon^2}(v) \cap B_1$ there exists $\alpha_w \in (\alpha_v - \varepsilon, \alpha_v + \varepsilon)$ such that

$$I_k(w,\alpha_w) = \alpha_w.$$

Moreover, the map $v \mapsto \alpha_{k,v}$ is continuous on B_1.

Proof. Let $R > 0$ be such that $R^q = \frac{1}{\varepsilon^2}$. With this particular choice of R we have

$$\int_{B_R^c} |F_k| \left(1+|x|\right)^q \, dx \le C\varepsilon^2.$$

Now for $|\alpha_v - \alpha|(2m\log R)^2 < \frac{1}{2}$ we have

$$\begin{aligned}
&\frac{1}{\gamma_{2m}}\int_{B_R} F_k e^{-2m\alpha u_0} e^{2mw}\,dx \\
&= \frac{1}{\gamma_{2m}}\int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} e^{2m(w-v)} e^{2m(\alpha_v-\alpha)u_0}\,dx \\
&= \frac{1}{\gamma_{2m}}\int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv}\left(1+2m(\alpha_v-\alpha)u_0 + O(\alpha_v-\alpha)\right)\left(1+O(\varepsilon^2)\right)dx \\
&= I_k(v,\alpha_v) + \frac{2m(\alpha_v-\alpha)}{\gamma_{2m}}(1+O(\varepsilon^2))\int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv} u_0\,dx \\
&\quad + O(\alpha_v-\alpha)\int_{B_R} F_k e^{-2m\alpha_v u_0} e^{2mv}\,dx + O(\varepsilon^2) \\
&=: I_k(v,\alpha_v) + \frac{2m(\alpha_v-\alpha)}{\gamma_{2m}}(1+O(\varepsilon^2))J_1 + O(\alpha_v-\alpha)J_2 + O(\varepsilon^2).
\end{aligned}$$

Using (6.18) we get

$$\begin{aligned}
|J_1| &\le e^{2m}\int_{B_R}|F_k|e^{-2m\alpha_v u_0}u_0\,dx \le e^{2m}\int_{B_R}|F_k|(M+|x|)^{\frac{q}{2}}u_0\,dx \\
&\le C(q)e^{2m}\int_{B_R}|F_k|(M+|x|)^q\,dx \le C(q)\varepsilon_1,
\end{aligned}$$

and $J_2 = O(\varepsilon_1)$. Let $\alpha = \alpha_v + \rho$, with $|\rho| \le \frac{1}{2(2m\log R)^2}$. Then

$$I_k(w,\alpha_v+\rho) - (\alpha_v+\rho) = \rho + O(\varepsilon^2) + \rho O(\varepsilon_1).$$

We fix $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ we have $|O(\varepsilon^2)| \le \frac{\varepsilon}{4}$ and $|O(\varepsilon_1)| \le \frac{1}{4}$. Then we can choose $\bar\rho \in (-\varepsilon,\varepsilon)$ such that

$$|\bar\rho| \le \frac{1}{2(2m\log R)^2}, \quad \bar\rho + O(\varepsilon^2) + \bar\rho O(\varepsilon_1) = 0,$$

concluding the first part of the lemma.

Now we prove the continuity of the map $v \mapsto \alpha_{k,v}$ from B_1 to $\mathbb{R}$.

For $v_n \to v \in B_1$ it follows that (at least) for large n, $|\alpha_{k,v_n}| < \frac{q}{4m}$ and $|\alpha_{k,v}| < \frac{q}{4m}$. First we consider the case $\alpha_{k,v} = 0$. Then for any $\varepsilon > 0$ one has $I_k(v_n,\alpha_{v_n}) = \alpha_{v_n}$ for some $\alpha_{v_n} \in (-\varepsilon,\varepsilon)$ where $\|v - v_n\|_X < \varepsilon^2$. This follows from the first part of the lemma. Since $|\alpha_{k,v_n}| \le |\alpha_{v_n}|$, we have the continuity.

Now we consider $\alpha_{k,v} > 0$ (negative case is similar). Then $I_k(v,0) > 0$, and hence $\alpha_{k,v_n} \ge 0$ for large n. We set $\alpha_\infty := \lim_{n\to\infty}\alpha_{k,v_n}$ (this limit exists at least for a subsequence). From the continuity of the map I_k it follows that $I_k(v,\alpha_\infty) = \alpha_\infty$. Since $\alpha_\infty \ge 0$ and $I_k(v,0) > 0$, we must have $\alpha_\infty > 0$. From the definition of $\alpha_{k,v}$ we deduce that $\alpha_{k,v} \le \alpha_\infty$. We fix $\varepsilon \in (0, \frac{\alpha_{k,v}}{2})$. Then by the first part of the lemma there exists $\alpha_{v_n} \in (\alpha_{k,v} - \varepsilon, \alpha_{k,v} + \varepsilon)$ such that $I_k(v_n,\alpha_{v_n}) = \alpha_{v_n}$ for every $\|v - v_n\|_X < \varepsilon^2$. Since $\alpha_{k,v_n} \le \alpha_{v_n}$ and $\alpha_{k,v_n} \to \alpha_\infty$, we have for n large

$$\alpha_{k,v} \le \alpha_\infty \le \alpha_{k,v_n} + \varepsilon \le \alpha_{v_n} + \varepsilon \le \alpha_{k,v} + 2\varepsilon.$$

We conclude the lemma. □

Proof of Theorem 6.1.1 We define $T_k : B_1 \subset X \to X$, $v \mapsto \bar{v}$, where

$$\bar{v}(x) := \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1}{|x-y|}\right) F_k(y) e^{-2m\alpha_{k,v}u_0 + 2mv(y)}\, dy + \alpha_{k,v} u_0,$$

that is $\bar{v}$ solves

$$(-\Delta)^m \bar{v} = F_k e^{-2m\alpha_{k,v}u_0 + 2mv} + \alpha_{k,v}(-\Delta)^m u_0.$$

Notice that arguing as in Lemma 3.3.1 one gets $\bar{v} \in X$. Using (6.14) and our choice of $\alpha_{k,v}$ we have

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m \bar{v}\, dx = 0.$$

With our choice of δ and p we have $\bar{v} \in M^p_{2m,\delta}(\mathbb{R}^{2m})$, where the space $M^p_{2m,\delta}(\mathbb{R}^{2m})$ is defined in Definition 3.2.1. For $v \in \bar{B}_1 \subset X$ we bound with Lemma 3.2.2, Lemma 3.2.3 and (6.19)

$$\begin{aligned} \|T_k(v)\|_X &\le C_1 \|T_k(v)\|_{M^p_{2m,\delta}} \le C_1 \|(-\Delta)^m \bar{v}\|_{\Gamma^p_\delta}, \\ &\le C_1 \|e^{-2m\alpha_{k,v}u_0} F_k\|_{L^p_{2m+\delta}} + C_1 |\alpha_{k,v}| \|(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \\ &\xrightarrow{k\to\infty} 0. \end{aligned}$$

Therefore, for ε_1 small enough, $\|T_k(v)\|_X \le \frac{1}{2}$ and there exists a fixed point v_k for every k. Hence, thanks to (6.20), the sequence

$$u_k(x) = -\beta|x|^2 - \alpha_{k,v_k} u_0(x) + c_k \varphi(x) + k + v_k(x), \quad x \in \Omega,$$

is a sequence of solutions with the stated properties. □

6.3 Blow-up with prescribed total Q-curvature

A slightly different version of the following proposition appears in Theorem 4.2.1. For the sake of completeness we give a sketch of the proof.

Proposition 6.3.1. *Let $w_0(x) = \log \frac{2}{1+|x|^2}$ and consider two functions $K, f : \mathbb{R}^{2m} \to \mathbb{R}$ such that*

$$K \ge 0, \quad K \not\equiv 0, \quad Ke^{-2mw_0} \in L^\infty(\mathbb{R}^{2m})$$

and

$$fe^{-2mw_0} \in L^\infty(\mathbb{R}^{2m}), \quad \Lambda := \int_{\mathbb{R}^{2m}} f dx \in (0, \Lambda_1).$$

Then there exists a function $w \in C^{2m-1}(\mathbb{R}^{2m})$ and a constant c_w such that

$$(-\Delta)^m w = Ke^{2m(w+c_w)} - f \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^{2m}} Ke^{2m(w+c_w)} dx = \Lambda, \tag{6.21}$$

and $\lim_{|x|\to\infty} w(x) \in \mathbb{R}$. Moreover, if f is of the form $f = (-\Delta)^m g$ for some $g \in C^{2m}(\mathbb{R}^{2m})$ with $g(x) = O(\log|x|)$ at infinity, then w satisfies

$$w(x) = \frac{1}{\gamma_{2m}} \int_{\mathbb{R}^{2m}} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{2m(w(y)+c_w)} dy - g(x) + C,$$

for some $C \in \mathbb{R}$.

Proof. Let π be the stereographic projection from S^{2m} to $\mathbb{R}^{2m}$. We define the functional J on $H^m(S^{2m})$ given by

$$J(u) = \int_{S^{2m}} \left(\frac{1}{2}|(P^{2m}u)^{\frac{1}{2}}|^2 + \tilde{f}_1 u\right) dV_0 - \frac{\Lambda}{2m}\log\left(\int_{S^{2m}} \tilde{K}e^{-2mw_0\circ\pi}e^{2mu}dV_0\right),$$

where $f_1 := fe^{-2mw_0}$, $\tilde{f}_1 := f_1\circ\pi$, $\tilde{K} := K\circ\pi$ and P^{2m} is the Paneitz operator of order $2m$ with respect to the standard metric on S^{2m}. Then there exists $u \in H^{2m}(S^{2m})$ such that

$$P^{2m}u = \frac{\Lambda\tilde{K}e^{-2mw_0\circ\pi}e^{2mu}}{\int_{S^{2m}}\tilde{K}e^{-2mw_0\circ\pi}e^{2mu}dV_0} - \tilde{f}_1 =: C_0\tilde{K}e^{-2mw_0\circ\pi}e^{2mu} - \tilde{f}_1.$$

Notice that $P^{2m}u \in L^\infty(S^{2m})$, thanks to the embedding $H^{2m}(S^{2m}) \hookrightarrow C^0(S^{2m})$, and hence $u \in C^{2m-1}(S^{2m})$.

We set $w = u\circ\pi^{-1}$. Then $w \in C^{2m-1}(\mathbb{R}^{2m})$ and $\lim_{|x|\to\infty} w(x) \in \mathbb{R}$. Using the following identity of Branson (see [9])

$$(-\Delta)^m(v\circ\pi^{-1}) = e^{2mw_0}(P^{2m}v)\circ\pi^{-1}, \quad \text{for every } v \in C^\infty(S^{2m}),$$

and by an approximation argument, we have that

$$(-\Delta)^m w = C_0Ke^{2mw} - f =: Ke^{2m(w+c_w)} - f, \quad \text{in } \mathbb{R}^{2m}.$$

Now we set

$$\tilde{w}(x) := \frac{1}{\gamma_{2m}}\int_{\mathbb{R}^{2m}}\log\left(\frac{1+|y|}{|x-y|}\right)K(y)e^{2m(w(y)+c_w)}dy - g(x).$$

Then $\Delta^m(w-\tilde{w}) = 0$ in $\mathbb{R}^{2m}$ and $(w-\tilde{w})(x) = O(\log|x|)$ at infinity. Therefore, $w = \tilde{w} + C$ for some $C \in \mathbb{R}$.

This finishes the proof of the proposition. □

Proof of Theorem 6.1.2 Let $\varphi \in \mathcal{K}(\Omega,\emptyset)$ and let $u_0 \in C^\infty(\mathbb{R}^{2m})$ be such that $u_0 = -\log|x|$ on B_1^c. We set $f = \frac{2\Lambda}{\Lambda_1}(-\Delta)^m u_0$. For each $k \in \mathbb{N}$ we set

$$K = K_k := Q_k e^{2m(-\beta|x|^2 + k\varphi + \alpha u_0)}, \quad \alpha := \frac{2\Lambda}{\Lambda_1}, \quad \beta > 0,$$

and we extend K_k by 0 outside Ω. Then by Proposition 6.3.1 there exists a sequence of functions (w_k) satisfying

$$w_k(x) = \frac{1}{\gamma_{2m}}\int_{\mathbb{R}^{2m}}\log\left(\frac{1+|y|}{|x-y|}\right)K_k(y)e^{2m(w_k(y)+c_{w_k})}dy - \frac{2\Lambda}{\Lambda_1}u_0 + a_k,$$

for some $a_k \in \mathbb{R}$. We set

$$u_k(x) := w_k + c_{w_k} - \beta|x|^2 + k\varphi(x) + \frac{2\Lambda}{\Lambda_1}u_0(x), \quad x \in \Omega\cup S_\varphi^*.$$

Then u_k satisfies

$$u_k(x) = \frac{1}{\gamma_{2m}}\int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right)Q_k e^{2mu_k(y)}dy - \beta|x|^2 + k\varphi(x) + c_k$$

and also (6.10), where $c_k := a_k + c_{w_k}$. We conclude the proof with Lemma 6.3.2. $\square$

Lemma 6.3.2. *Let Ω be a domain in $\mathbb{R}^{2m}$. Let φ and Q_k be as in Theorem 6.1.2. Let (u_k) be a sequence of solutions to*

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy - \beta|x|^2 + k\varphi(x) + c_k, \quad x \in \Omega \cup S_\varphi^*,$$

for some $\beta > 0$. Assume that

$$\int_\Omega Q_k e^{2mu_k(y)} dy = \Lambda < \frac{\Lambda_1}{2}.$$

Then $c_k \to \infty$, $c_k = o(k)$ and

$$I_k(x) := \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy, \quad x \in \mathbb{R}^{2m},$$

is locally uniformly bounded from above on $\Omega \setminus S_\varphi$, and locally uniformly bounded from below on $\mathbb{R}^{2m}$. In particular, $u_k \to \infty$ on S_φ^* and $u_k \to -\infty$ locally uniformly on $\Omega \setminus S_\varphi$.

Proof. For any fixed $R > 0$ and $x \in B_R$ we bound

$$\begin{aligned} I_k(x) &= \int_{B_{2R}\cap\Omega} \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy + \int_{B_{2R}^c\cap\Omega} \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy \\ &\geq -C(R) + \int_{B_{2R}^c\cap\Omega} \log\left(\frac{1}{2} + \frac{1}{2|y|}\right) Q_k e^{2mu_k(y)} dy \\ &\geq -C(R). \end{aligned}$$

Since $\Lambda < \frac{\Lambda_1}{2}$, using Jensens inequality we obtain for some $p < 2m$

$$e^{2mu_k(x)} \leq e^{2mc_k} e^{-2m\beta|x|^2 + 2mk\varphi(x)} \int_{\mathbb{R}^{2m}} \left(\frac{1+|y|}{|x-y|}\right)^p Q_k(y) e^{2mu_k(y)} dy.$$

Using that

$$\int_\Omega \left(\frac{1+|y|}{|x-y|}\right)^p e^{-2m\beta|x|^2 + 2mk\varphi(x)} dx \xrightarrow{k\to\infty} 0,$$

and together with Fubini theorem, one has

$$\int_\Omega Q_k(x) e^{2mu_k(x)} dx = e^{2mc_k} o(1), \quad \text{as } k \to \infty.$$

Now $\Lambda > 0$ implies that $c_k \to \infty$.

We assume by contradiction that $c_k \neq o(k)$. Then for some $\varepsilon > 0$ we have $\frac{c_k}{k} \geq 2\varepsilon$ for k large. Let $x_0 \in S_\varphi^*$ be such that (6.9) holds. Let $\delta > 0$ be such that $\varphi(x) > -\varepsilon$ for $x \in B_\delta(x_0) \cap \Omega$. Therefore

$$u_k(x) \geq -C - k\varepsilon + c_k \geq -C + k\varepsilon, \quad x \in B_\delta(x_0) \cap \Omega,$$

and hence

$$\int_\Omega Q_k e^{2mu_k} dx \geq e^{-C+k\varepsilon} \int_{B_\delta(x_0)} Q_k dx \xrightarrow{k\to\infty} \infty,$$

a contradiction.

Now we prove that I_k is locally uniformly bounded from above on $\Omega \setminus S_\varphi$. For $\tilde{\Omega} \Subset \Omega \setminus S_\varphi$ we have

$$k\varphi + c_k \to -\infty \quad \text{uniformly on } \tilde{\Omega}.$$

Using Jensens inequality one can show that $\|e^{2mu_k}\|_{L^p(\Omega_1)} \leq C$ for some $p > 1$, where $\tilde{\Omega} \Subset \Omega_1 \Subset \Omega \setminus S_\varphi$. For $x \in \tilde{\Omega}$ we obtain by Hölder inequality

$$\begin{aligned} I_k(x) &= \frac{1}{\gamma_{2m}} \int_{\Omega_1^c \cap \Omega} \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy \\ &\quad + \frac{1}{\gamma_{2m}} \int_{\Omega_1 \cap \Omega} \log\left(\frac{1+|y|}{|x-y|}\right) Q_k e^{2mu_k(y)} dy \\ &\leq C + C \| \log|x - \cdot| \|_{L^{p'}(\Omega_1)} \|e^{2mu_k}\|_{L^p(\Omega_1)} \\ &\leq C. \end{aligned}$$

The remaining part of the lemma follows immediately. □

Proof of Corollary 6.1.3. Let $g \in C^\infty(\partial\Omega)$ be such that $g \leq 0$, $g \not\equiv 0$ on $\partial\Omega$ and $g = 0$ on Γ. Let φ be the solution to

$$\begin{cases} (-\Delta)^m \varphi = 0 & \text{in } \Omega, \\ (-\Delta)^j \varphi = 0 & \text{on } \partial\Omega, \quad j = 1, \dots, m-1 \\ \varphi = g & \text{on } \partial\Omega. \end{cases}$$

Then by maximum principle $\varphi < 0$ in Ω and hence $S_\varphi^* = \Gamma$. Then the conclusion follows by Theorem 6.1.1 and 6.1.2. □

Proposition 6.3.3. *Let Ω be a domain in $\mathbb{R}^{2m}$. Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$. Let $\tilde{\Omega} \Subset \Omega \setminus S_\varphi$ be an open set. Let Q_k be such that $Q_k \equiv 0$ on $\tilde{\Omega}^c$ and $Q_k \equiv 1$ on $\tilde{\Omega}$. Then for any $\Lambda > 0$ there exists no sequence (u_k) of solutions to (6.2) satisfying (6.8) and (6.10).*

Proof. We assume by contradiction that the statement of the proposition is not true. Then there exists a sequence of solutions (u_k) to (6.2) satisfying (6.8) and (6.10) for some $\Lambda > 0$. Therefore, by (6.8), $u_k \to -\infty$ uniformly in $\tilde{\Omega}$ and hence

$$\Lambda = \int_\Omega Q_k e^{2mu_k} dx = \int_{\tilde{\Omega}} e^{2mu_k} dx \xrightarrow{k\to\infty} 0,$$

a contradiction. □

Proof of Theorem 6.1.5 Let $m \geq 2$. We set

$$\varphi_k(r, \theta) := r^k \cos(k\theta), \quad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi.$$

We extend φ_k on $B_2 \subset \mathbb{R}^{2m}$ as a function of only two variables, that is, $\varphi_k(x) := \varphi_k(r, \theta)$ for $x \in B_2$, where (r, θ) is the polar coordinate of $\Pi(x)$ and $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^2$ is the projection map. Then φ_k is a harmonic function on B_2. Let Φ_k be the solution to the equation

$$\begin{cases} -\Delta \Phi_k = \varphi_k & \text{in } B_2, \\ \Phi_k = 0 & \text{on } \partial B_2. \end{cases}$$

We fix $0 < \Lambda < \Lambda_1$. Then by Proposition 6.3.1 there exists a sequence of solutions (w_k) to (6.21) with

$$f := \frac{2\Lambda}{\Lambda_1}(-\Delta)^m u_0, \quad K_k := \begin{cases} e^{2m(\Phi_k + \frac{2\Lambda}{\Lambda_1}u_0)} & \text{on } B_2 \\ 0 & \text{on } B_2^c, \end{cases}$$

where $u_0 \in C^\infty(\mathbb{R}^{2m})$ with $u_0 = -\log|x|$ on B_1^c. Then

$$u_k := w_k + c_{w_k} + \Phi_k + \frac{2\Lambda}{\Lambda_1}u_0$$

satisfies (6.11) and u_k is given by

$$u_k(x) = \frac{1}{\gamma_{2m}}\int_{B_2} \log\left(\frac{1+|y|}{|x-y|}\right) e^{2mu_k(y)}dy + \Phi_k(x) + c_k,$$

for some $c_k \in \mathbb{R}$. Moreover,

$$\Delta u_k = -\varphi_k + e_k,$$

where

$$|e_k(x)| \leq C\int_{B_2} \frac{e^{2mu_k(y)}}{|x-y|^2}dy.$$

Integrating, using Fubini's theorem and (6.11) we obtain $\|e_k\|_{L^1(B_2)} \leq C$. Then (6.12) follows at once from the definition of φ_k. □

6.4 Radially symmetric solutions

6.4.1 On an annulus

Let $\Omega = B_{R_2} \setminus B_{R_1}$ be an annulus. Let $X = C^0_{rad}(\bar{\Omega})$. We fix $\Lambda \in (0, \infty)$. For $k \in \mathbb{N}$ and $v \in X$ we choose $c_v = c(v,k) \in \mathbb{R}$ so that

$$\int_\Omega Q_k e^{2m(v+c_v)}dx = \Lambda.$$

Let $\varphi \in \mathcal{K}(\Omega, \emptyset)$ be radially symmetric. For $k \in \mathbb{N}$ we define an operator $T_k : X \to X$, $v \mapsto \bar{v}$ where

$$\bar{v} := \tilde{v} + k\varphi(x), \quad \tilde{v}(x) = \int_\Omega G(x,y)Q_k(y)e^{2m(v(y)+c_v)}dy,$$

and G is the Green function of $(-\Delta)^m$ on Ω with the Navier boundary conditions.

Lemma 6.4.1. *Let $k \in \mathbb{N}$ be fixed. Let $(v,t) \in X \times (0,1]$ satisfies $v = tT_k(v)$. Then there exists $M > 0$ such that $\|v\|_X \leq M$ for all such (v,t).*

Proof. We have

$$v(x) = t\int_\Omega G(x,y)Q_k(y)e^{2m(v(y)+c_v)}dy + tk\varphi(x) \geq -C(k) \quad \text{in } \Omega.$$

Hence from the definition of c_v we get

$$\Lambda = \int_\Omega Q_k e^{2m(v+c_v)}\,dx \geq e^{2m(-C(k)+c_v)} \int_\Omega Q_k\,dx > a e^{2m(-C(k)+c_v)}$$

hence $c_v \leq C(k)$. Define the cone $\mathcal{C}$ as the set

$$\mathcal{C} := \{x \in \Omega \colon |\bar{x}| \leq \rho x_1\}, \qquad \text{with } x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{2m-1}, \tag{6.22}$$

for some $\rho > 0$ to be fixed later. For some finite $M = M(\rho)$ we can write Ω as a union of (not necessarily disjoint) cones $\{\mathcal{C}_i\}_{i=1}^M$ such that for each such cone $\mathcal{C}_i$ we have

(*i*) $\mathcal{C}_i$ is congruent to $\mathcal{C}$,

(*ii*) $\int_{N(\mathcal{C}_i)} Q_k(y) e^{2m(v(y)+c_v)} dy \leq \frac{\Lambda_1}{4}, \quad N(\mathcal{C}_i) := \cup_{\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset} \mathcal{C}_j$

and we fix ρ such that (*ii*) holds. Notice that for some $\delta > 0$ we have $dist(\mathcal{C}_i, N(\mathcal{C}_i)^c) \geq \delta$ for $i = 1, \ldots, M$. Therefore, for $x \in \mathcal{C}_1$

$$v(x) \leq t \int_{N(\mathcal{C}_1)} G(x,y) Q_k(y) e^{2m(v(y)+c_v)} dy + tk\varphi(x) + C(\delta),$$

and together with Jensen's inequality, for some $p > 1$ we get

$$\int_\Omega e^{p2m(v+c_v)} dx \leq M \int_{\mathcal{C}_1} e^{p2m(v+c_v)} dx \leq C.$$

Since φ is radially symmetric and polyharmonic we have $\varphi \in C^{2m}(\bar{\Omega})$, and therefore by elliptic estimates and Sobolev embeddings

$$\|v - tk\varphi\|_X \leq C\|v - tk\varphi\|_{W^{2m,p}(\Omega)} \leq C\|(-\Delta)^m v\|_{L^p(\Omega)} \leq C,$$

concluding the proof. □

A consequence of Lemma 6.4.1 is that for every $k \in \mathbb{N}$, the operator T_k has a fixed point $v_k \in X$. We set $u_k = v_k + c_{v_k}$. Then

$$u_k(x) = \int_\Omega G(x,y) Q_k e^{2mu_k(y)} dy + k\varphi(x) + c_{v_k}, \qquad \int_\Omega Q_k e^{2mu_k(y)} dx = \Lambda. \tag{6.23}$$

We claim that $c_{v_k} \to \infty$.

Again writing Ω as a union of cones and using Jensen's inequality we obtain

$$\int_\Omega e^{2mu_k} dx \leq C e^{2mc_{v_k}} \int_\Omega e^{2mu_k(y)} dy \int_\Omega \frac{e^{2mk\varphi(x)}}{|x-y|^p} dx,$$

for some $p < 2m$. Hence, if $c_{v_k} \leq C$, then

$$\int_\Omega Q_k e^{2mu_k} dx \leq Cb \int_\Omega e^{2mu_k(y)} dy \int_\Omega \frac{e^{2mk\varphi(x)}}{|x-y|^p} dx \xrightarrow{k\to\infty} 0,$$

a contradiction. Thus $c_{v_k} \to \infty$, and hence $u_k \to \infty$ on S_φ^*.

It remains to show that $u_k \to -\infty$ in $C^0_{loc}(\Omega \setminus S_\varphi)$. Arguing as in Lemma 6.3.2 we conclude the proof. $\square$

6.4.2 On a ball

We consider

$$X = C^2_{rad}(\bar{B}_R), \quad \|v\|_X := \max_{\bar{B}_R}(|v(x)| + |v'(x)| + |v''(x)|).$$

Let $\Lambda > 0$. We fix $k \in \mathbb{N}$. For $v \in X$ define $c_v \in \mathbb{R}$ given by

$$\int_\Omega Q_k e^{2m(v+c_v)} dx = \Lambda.$$

We define $T_k : X \to X$ given by $v \mapsto \bar{v}$ where

$$\bar{v}(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1}{|x-y|}\right) Q_k(y) e^{2m(v(y)+c_v)} dy + \left(k + \frac{|\Delta v(0)|}{2\Delta\varphi(0)}\right) \varphi(x).$$

Arguing as in Lemma 5.2.2 one can show that the operator T_k has a fixed point, say v_k. We set $u_k = v_k + c_{v_k}$. Then

$$u_k(x) = \frac{1}{\gamma_{2m}} \int_\Omega \log\left(\frac{1}{|x-y|}\right) Q_k(y) e^{2mu_k(y)} dy + \left(k + \frac{|\Delta v_k(0)|}{2\Delta\varphi(0)}\right) \varphi(x) + c_{v_k},$$

and

$$\int_\Omega Q_k e^{2mu_k} dx = \Lambda.$$

Again as in Lemma 5.2.2 one can show that there exists $C > 0$ such that $u_k \le C$ on B_ε for some $\varepsilon > 0$. Using this, and as in the annulus domain case, one can show that $c_{v_k} \to \infty$. Thus $u_k(x) \to \infty$ for every $x \in S^*_\varphi$. Finally, similar to the annulus domain case, it follows that $u_k \to -\infty$ locally uniformly in $\Omega \setminus S_\varphi$. $\square$

Chapter 7

A fractional Adams-Moser-Trudinger inequality and its application

We improve the sharpness of some fractional Moser-Trudinger type inequalities, particularly those studied by Lam-Lu and Martinazzi. As an application, improving upon works of Adimurthi and Lakkis, we prove the existence of weak solutions to the problem

$$(-\Delta)^{\frac{n}{2}} u = \lambda u e^{bu^2} \quad \text{in } \Omega, \quad 0 < \lambda < \lambda_1, \quad b > 0,$$

with Dirichlet boundary condition, for any domain Ω in $\mathbb{R}^n$ with finite measure. Here λ_1 is the first eigenvalue of $(-\Delta)^{\frac{n}{2}}$ on Ω.

7.1 Introduction and statement of the main theorems

Let $n \geq 2$ and let Ω be a bounded domain in $\mathbb{R}^n$. The Sobolev embedding theorem states that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq \frac{np}{n-kp}$ and $kp < n$. However, it is not true that $W_0^{k,p}(\Omega) \subset L^\infty(\Omega)$ for $kp = n$. In the borderline case, as shown by Yudovich [79], Pohozaev [64] and Trudinger [76], $W_0^{1,n}(\Omega)$ embeds into an Orlicz space and in fact

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx < \infty, \tag{7.1}$$

for some $\alpha > 0$. Moser [61] found the best constant α in the inequality (7.1), obtaining the so called Moser-Trudinger inequality:

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_{L^n(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}} dx < \infty, \quad \alpha_n = n|S^{n-1}|^{\frac{1}{n-1}}. \tag{7.2}$$

The constant α_n in (7.2) is the best constant in the sense that for any $\alpha > \alpha_n$, the supremum in (7.1) is infinite. A generalized version of Moser-Trudinger inequality is the following theorem of Adams [2]:

Theorem I ([2]). *If k is a positive integer less than n, then there is a constant $C = C(k, n)$ such that*

$$\sup_{u \in C_c^k(\Omega),\, \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-k}}} dx \leq C|\Omega|,$$

where

$$\alpha = \alpha(k,n) = \frac{n}{|S^{n-1}|}\begin{cases}\left[\frac{\pi^{\frac{n}{2}}2^k\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)}\right]^{\frac{n}{n-k}}, & k = odd,\\ \left[\frac{\pi^{\frac{n}{2}}2^k\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}\right]^{\frac{n}{n-k}}, & k = even,\end{cases}$$

and $\nabla^k := \nabla\Delta^{\frac{k-1}{2}}$ *for* k *odd and* $\nabla^k = \Delta^{\frac{k}{2}}$ *for* k *even. Moreover the constant* α *is sharp in the sense that*

$$\sup_{u\in C_c^k(\Omega),\, \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)}\le 1}\int_\Omega f(|u|)e^{\alpha|u|^{\frac{n}{n-k}}}dx = \infty, \tag{7.3}$$

for any $f:[0,\infty)\to[0,\infty)$ *with* $\lim_{t\to\infty} f(t) = \infty$.[1]

In a recent work Martinazzi [54] has studied the Adams inequality in a fractional setting. In order to state its result first we recall that for $u \in L_s(\mathbb{R}^n)$ one can define $(-\Delta)^s u$ as a tempered distribution (see Section 2.1). Now for an open set $\Omega \subseteq \mathbb{R}^n$ (possibly $\Omega = \mathbb{R}^n$), $s > 0$ and $1 \le p \le \infty$ we define the fractional Sobolev space $\tilde{H}^{s,p}(\Omega)$ by

$$\tilde{H}^{s,p}(\Omega) := \left\{u \in L^p(\Omega) : u = 0 \text{ on } \mathbb{R}^n \setminus \Omega,\ (-\Delta)^{\frac{s}{2}}u \in L^p(\mathbb{R}^n)\right\}.$$

Theorem J ([54])**.** *For any open set* $\Omega \subset \mathbb{R}^n$ *with finite measure and for any* $p \in (1,\infty)$ *we have*

$$\sup_{u\in\tilde{H}^{\frac{n}{p},p}(\Omega),\, \|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\Omega)}\le 1}\int_\Omega e^{\alpha_{n,p}|u|^{p'}}dx \le C_{n,p}|\Omega|,$$

where the constant $\alpha_{n,p}$ *is given by*

$$\alpha_{n,p} = \frac{n}{|S^{n-1}|}\left(\frac{\Gamma(\frac{n}{2p})2^{\frac{n}{p}}\pi^{\frac{n}{2}}}{\Gamma(\frac{np-n}{2p})}\right)^{p'}. \tag{7.4}$$

Moreover, the constant $\alpha_{n,p}$ *is sharp in the sense that we cannot replace it with any larger one without making the above supremum infinite.*

Notice that condition (7.3) in Theorem I is sharper than only requiring that the constant α in the exponential is sharp, as done in Theorem J. In fact Martinazzi asked whether it is true that

$$\sup_{u\in\tilde{H}^{\frac{n}{p},p}(\Omega),\, \|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\Omega)}\le 1}\int_\Omega f(|u|)e^{\alpha_{n,p}|u|^{p'}}dx = \infty, \tag{7.5}$$

for any $f:[0,\infty)\to[0,\infty)$ with

$$\lim_{t\to\infty} f(t) = \infty, \quad f \text{ is Borel measurable}, \tag{7.6}$$

and $\alpha_{n,p}$ is given by (7.4).

The point here is that Adams constructs smooth and compactly supported test functions similar to the standard Moser functions (constant in a small ball, and decaying

[1] Identity (7.3) is proven in [2], although not explicitly stated.

logarithmically on an annulus), and then he estimates their $H_0^{k,\frac{n}{k}}$-norms in a very precise way. This becomes much more delicate when k is not integer because instead of computing partial derivatives, one has to estimate the norms of fractional Laplacians (the term $\|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\Omega)}$ in (7.5)). This is indeed done in [54], but the test functions introduced by Martinazzi are not efficient enough to prove (7.5). As we shall see this has consequences for applications to PDEs.

We shall prove that the answer to Martinazzi's question is positive, indeed in a slightly stronger form, namely the supremum in (7.5) is infinite even if we consider the full $H^{\frac{n}{p},p}$-norm on the whole space. More precisely we have:

Theorem 7.1.1. *Let Ω be an open set in $\mathbb{R}^n$ with finite measure and let $f:[0,\infty)\to[0,\infty)$ satisfy (7.6). Then*

$$\sup_{u\in\tilde{H}^{\frac{n}{p},p}(\Omega),\,\|u\|^p_{L^p(\Omega)}+\|(-\Delta)^{\frac{n}{2p}}u\|^p_{L^p(\mathbb{R}^n)}\le 1}\int_\Omega f(|u|)e^{\alpha_{n,p}|u|^{p'}}dx=\infty,\quad 1<p<\infty,$$

where the constant $\alpha_{n,p}$ is given by (7.4).

The main difficulty in the proof of Theorem 7.1.1 is to construct test and cut-off functions in a way that their fractional Laplacians of suitable orders can be estimated precisely. This will be done in section 7.2.

Here we mention that using a Green's representation formula, Iula-Maalaoui-Martinazzi [39] proved a particular case of Theorem 7.1.1 in one dimension. Their proof, though, does not extend to spaces $\tilde{H}^{\frac{n}{p},p}(\Omega)$ when $\frac{n}{p}>1$ because the function constructed using the Green representation formula do not enjoy enough smoothness at the boundary. Trying to solve this with a smooth cut-off function at the boundary allows to prove (7.5) only when f grows fast enough at infinity (for instance $f(t)\ge t^a$ for some $a>p'$).

Now we move to Moser-Trudinger type inequalities on domains with infinite measure. In this direction we refer to [43, 59, 69] and the references there in. For our purpose, here we only state the work of Lam-Lu [43].

Theorem K ([43])**.** *Let $p\in(1,\infty)$ and $\tau>0$. Then for every domain $\Omega\subset\mathbb{R}^n$ with finite measure, there exists $C=C(n,p,\tau)>0$ such that*

$$\sup_{u\in H^{\frac{n}{p},p}(\mathbb{R}^n),\,\|(\tau I-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}\le 1}\int_\Omega e^{\alpha_{n,p}|u|^{p'}}dx\le C(|\Omega|+1),$$

and

$$\sup_{u\in H^{\frac{n}{p},p}(\mathbb{R}^n),\,\|(\tau I-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}\le 1}\int_{\mathbb{R}^n}\Phi(\alpha_{n,p}|u|^{p'})dx<\infty,$$

where $\alpha_{n,p}$ is given by (7.4) and

$$\Phi(t):=e^t-\sum_{j=0}^{j_p-2}\frac{t^j}{j!},\quad j_p:=\min\{j\in\mathbb{N}:j\ge p\}.$$

Furthermore, the constant $\alpha_{n,p}$ is sharp in the above inequalities, i.e., if $\alpha_{n,p}$ is replaced by any $\alpha>\alpha_{n,p}$, then the supremums are infinite.

In the spirit of Theorem 7.1.1 we prove a stronger version of the sharpness of the constant in Theorem K, in the sense that, even without increasing the constant $\alpha_{n,p}$ we can make the two supremums in Theorem K infinite by multiplying the exponential by a function f going to infinity arbitrarily slow. Moreover it is sufficient to consider functions with compact support.

Theorem 7.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with finite measure and let $f : [0,\infty) \to [0,\infty)$ satisfy (7.6). Then for any $\tau > 0$ and for any $p \in (1,\infty)$ we have (with the notations as in Theorem K)*

$$\sup_{u\in \tilde{H}^{\frac{n}{p},p}(\Omega),\, \|(\tau I-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}\le 1} \int_\Omega f(|u|)e^{\alpha_{n,p}|u|^{p'}}dx = \infty,$$

and

$$\sup_{u\in \tilde{H}^{\frac{n}{p},p}(\Omega),\, \|(\tau I-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}\le 1} \int_\Omega f(|u|)\Phi(\alpha_{n,p}|u|^{p'})dx = \infty.$$

As an application of Theorem 7.1.1 (in the case $p = 2$ and $f(t) = t^2$, compare to (7.18)) we prove the existence of (weak) solution to a semilinear elliptic equation with exponential nonlinearity. In order to state the theorem first we need the following definition.

Definition 7.1.1. Let Ω be an open set in $\mathbb{R}^n$ with finite measure. Let $f \in L^p(\Omega)$ for some $p \in (1,\infty)$. We say that u is a weak solution of

$$(-\Delta)^{\frac{n}{2}}u = f \text{ in } \Omega,$$

if $u \in \tilde{H}^{\frac{n}{2},2}(\Omega)$ satisfies

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{n}{4}}u(-\Delta)^{\frac{n}{4}}v dx = \int_\Omega fv dx \quad \text{for every } v \in \tilde{H}^{\frac{n}{2},2}(\Omega).$$

Theorem 7.1.3. *Let Ω be an open set in $\mathbb{R}^n$ with finite measure. Let $0 < \lambda < \lambda_1$ and $b > 0$. Then there exists a nontrivial weak solution to the problem*

$$(-\Delta)^{\frac{n}{2}}u = \lambda u e^{bu^2} \ \text{ in } \Omega. \tag{7.7}$$

Due to the fact that the embedding $\tilde{H}^{\frac{n}{2},2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact for any open set Ω with finite measure (see Lemma 7.4.7), we do not need any regularity assumption or boundedness assumption on the domain Ω.

The equation (7.7) has been well studied by several authors in even and odd dimensions, with emphasis both on existence and compactness properties see e.g. [5, 23, 38, 41, 48–50, 58, 67, 75]. For instance, Lakkis [41], extending a work of Adimurthi [3], proved the existence of solution to (7.7) in any even dimension. In [42], Lam-Lu have studied equation 7.7 in even dimension with more general right hand side, namely, equation of the form $(-\Delta)^{\frac{n}{2}}u = f(x,u)$, where the function f may not satisfy the Ambrosetti-Rabinowitz condition.

In a recent work Iannizzotto-Squassina [38] have proven existence of nontrivial weak solution of (7.7) with $\Omega = (0,1)$ under an assumption, which turns out to be satisfied thanks to our Theorem 7.1.1, applied with $p = 2$ (see Lemma 7.3.5).

Finally, we mention that in a recent work Bao-Lam-Lu [6] have studied the existence of positive solutions to a polyharmonic equation on the whole space $\mathbb{R}^{2m}$, more precisely

$$(I-\Delta)^m u = f(x,u), \quad \text{in } \mathbb{R}^{2m}, \quad m \geq 1,$$

where the function f has critical growth at infinity. Moreover, under certain assumptions on f, they also discussed radial symmetry and regularity of solutions.

7.2 Moser type functions

We construct Moser type functions as follows:

First we fix two smooth functions η and φ such that $0 \leq \eta,\, \varphi \leq 1$,

$$\eta \in C_c^\infty((-1,1)), \quad \eta = 1 \text{ on } (-\frac{3}{4}, \frac{3}{4}),$$

and

$$\varphi \in C_c^\infty((-2,2)), \quad \varphi = 1, \text{ on } (-1,1).$$

For $\varepsilon > 0$, we set

$$\psi_\varepsilon(t) = \begin{cases} 1 - \varphi_\varepsilon(t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \eta(t) & \text{if } t \geq \frac{1}{2}, \end{cases}$$

and

$$v_\varepsilon(x) = \left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \left(\log\left(\frac{1}{\varepsilon}\right)\varphi_\varepsilon(|x|) + \log\left(\frac{1}{|x|}\right)\psi_\varepsilon(|x|)\right) \quad x \in \mathbb{R}^n,$$

where

$$\varphi_\varepsilon(t) = \varphi(\frac{t}{\varepsilon}).$$

Our aim is to show that the supremums (in Theorems 7.1.1 and 7.1.2) taken over the functions $\{v_\varepsilon\}_{\varepsilon>0}$ (up to a proper normalization) are infinite.

The following proposition is crucial in the proof of Theorem 7.1.1.

Proposition 7.2.1. *Let*

$$u_\varepsilon(x) := |S^{n-1}|^{-\frac{1}{p}} 2^{\frac{n}{p'}} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2p'}) \frac{1}{\Gamma(\frac{n}{2p})\gamma_n} v_\varepsilon(x).$$

Then for $1 < p < \infty$ there exists a constant $C > 0$ such that

$$\|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq \left(1 + C\left(\log\frac{1}{\varepsilon}\right)^{-1}\right)^{\frac{1}{p}}.$$

Proof. Since the proof of above proposition is quite trivial if $\frac{n}{2p}$ is an integer, from now on we only consider the case when $\frac{n}{2p}$ is not an integer.

From Lemmas 7.2.2 and 7.2.4 we have

$$\|(-\Delta)^{\frac{n}{2p}}u_\varepsilon\|^p_{L^p(B_{3\varepsilon}\cup B_2^c)} \le C\left(\log\frac{1}{\varepsilon}\right)^{-1}.$$

In order to estimate $(-\Delta)^\sigma v_\varepsilon$ on the domain $\{x : 3\varepsilon < |x| < 2\}$ we consider the function

$$R_\varepsilon(x) = v_\varepsilon(x) - \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\log\frac{1}{|x|} =: f_\varepsilon(x) + g_\varepsilon(x) \quad x\in\mathbb{R}^n,$$

where

$$\begin{aligned} f_\varepsilon(x) :&= \begin{cases} v_\varepsilon(x) - \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\log\frac{1}{|x|} & \text{if } |x|<2\varepsilon \\ 0 & \text{if } |x|\ge 2\varepsilon, \end{cases} \\ &= \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\left(\log\frac{1}{\varepsilon}-\log\frac{1}{|x|}\right)\varphi_\varepsilon(|x|) \end{aligned}$$

and

$$\begin{aligned} g_\varepsilon(x) :&= \begin{cases} v_\varepsilon(x) - \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\log\frac{1}{|x|} & \text{if } |x|>\frac{1}{2} \\ 0 & \text{if } |x|\le \frac{1}{2} \end{cases} \\ &= \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}(\eta(|x|)-1)\log\frac{1}{|x|}. \end{aligned}$$

It is easy to see that for any $\sigma>0$

$$\sup_{x\in\mathbb{R}^n}|(-\Delta)^\sigma g_\varepsilon(x)| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}. \tag{7.8}$$

With the help of Lemma 7.4.8 and the triangle inequality we bound

$$\begin{aligned} |(-\Delta)^{\frac{n}{2p}}u_\varepsilon(x)| &= \frac{1}{|S^{n-1}|^{\frac{1}{p}}\beta_{n,\frac{n}{2p}}}\left|(-\Delta)^{\frac{n}{2p}}R_\varepsilon(x) + \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}(-\Delta)^{\frac{n}{2p}}\log\frac{1}{|x|}\right| \\ &\le C|(-\Delta)^{\frac{n}{2p}}R_\varepsilon(x)| + \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\frac{1}{|S^{n-1}|^{\frac{1}{p}}}\frac{1}{|x|^{\frac{n}{p}}}. \end{aligned}$$

Using the elementary inequality

$$(a+b)^q \le a^q + C_q(b^q + a^{q-1}b), \quad 1\le q<\infty,\ a\ge 0,\ b\ge 0,$$

we get

$$\int_{3\varepsilon<|x|<2} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon(x)|^p dx$$
$$\leq \int_{3\varepsilon<|x|<2} \left(\log\frac{1}{\varepsilon}\right)^{-1} \frac{1}{|S^{n-1}|}\frac{1}{|x|^n} dx + C\int_{3\varepsilon<|x|<2} |(-\Delta)^{\frac{n}{2p}} R_\varepsilon(x)|^p dx$$
$$+ C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p'}} \int_{3\varepsilon<|x|<2} \frac{1}{|x|^{\frac{n}{p'}}} |(-\Delta)^{\frac{n}{2p}} R_\varepsilon(x)| dx$$
$$\leq 1 + C\left(\log\frac{1}{\varepsilon}\right)^{-1} + C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p'}} \int_{3\varepsilon<|x|<2} \frac{1}{|x|^{\frac{n}{p'}}} |(-\Delta)^{\frac{n}{2p}} R_\varepsilon(x)| dx,$$

where the last inequality follows from Lemma 7.2.3. Using the pointwise estimate in Lemma 7.2.3 and (7.8) one can show that

$$\int_{3\varepsilon<|x|<2} \frac{1}{|x|^{\frac{n}{p'}}} |(-\Delta)^{\frac{n}{2p}} R_\varepsilon(x)| dx \leq C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}},$$

which completes the proof. □

Lemma 7.2.2. *Let $p \in (1,\infty)$. Then there exists a constant $C = C(n,p,\sigma) > 0$ such that*

$$|(-\Delta)^\sigma v_\varepsilon(x)| \leq C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \varepsilon^{-2\sigma} \quad \textit{for } |x| \leq 3\varepsilon,\ 0 < \sigma < \frac{n}{2}.$$

Moreover,

$$\|(-\Delta)^{\frac{n}{2p}} v_\varepsilon\|^p_{L^p(B_{3\varepsilon})} \leq C\left(\log\frac{1}{\varepsilon}\right)^{-1}.$$

Proof. We claim that for every nonzero multi-index $\alpha \in \mathbb{N}^n$ there exists $C = C(n,\alpha) > 0$ such that

$$|D^\alpha v_\varepsilon(x)| \leq C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \varepsilon^{-|\alpha|}, \quad x \in \mathbb{R}^n. \tag{7.9}$$

In order to prove (7.9), we use that $\eta \in C_c^\infty((-1,1))$ and

$$v_\varepsilon(x) = \begin{cases} \left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{p'}} & \text{if } |x| \leq \varepsilon \\ \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \log\left(\frac{1}{|x|}\right)\eta(|x|) & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

Therefore, the estimate in (7.9) holds for $x \in B_\varepsilon \cup B^c_{\frac{1}{2}}$. For $x \in B_{\frac{1}{2}} \setminus B_\varepsilon$ and for a nonzero multi-index $\alpha \in \mathbb{N}^n$, we have for some constants $C_{\alpha,\beta}$

$$\left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p}} D^\alpha v_\varepsilon(x) = D^\alpha \varphi_\varepsilon(|x|) \log \frac{1}{\varepsilon} + D^\alpha \psi_\varepsilon(|x|) \log \frac{1}{|x|} + \sum_{0<\beta\le\alpha} C_{\alpha,\beta} D^\beta \left(\log \frac{1}{|x|}\right) D^{\alpha-\beta} \psi_\varepsilon(|x|)$$
$$= D^\alpha \varphi_\varepsilon(|x|) \log \frac{|x|}{\varepsilon} - \sum_{0<\beta\le\alpha} C_{\alpha,\beta} D^\beta (\log|x|)\, D^{\alpha-\beta} \psi_\varepsilon(|x|),$$

where in the second equality we used that $D^\alpha \psi_\varepsilon(|x|) = -D^\alpha \varphi_\varepsilon(|x|)$, which follows from the fact that $\varphi_\varepsilon + \psi_\varepsilon = 1$ on $B_{\frac{1}{2}}$. Moreover, from the definition of φ and φ_ε we have that

$$|D^\alpha \varphi_\varepsilon(|x|)| \le \begin{cases} C\varepsilon^{-|\alpha|} & \text{if } x \in B_{2\varepsilon} \setminus B_\varepsilon \\ 0 & \text{if } x \in B^c_{2\varepsilon} \cup B_\varepsilon. \end{cases}$$

Therefore,

$$|D^\alpha \varphi_\varepsilon(|x|)| \left|\log \frac{1}{\varepsilon} - \log \frac{1}{|x|}\right| \le C\varepsilon^{-|\alpha|}.$$

It is easy to see that for $0 < \beta \le \alpha$

$$\left|D^\beta \log(|x|)\right| \left|D^{\alpha-\beta} \psi_\varepsilon(|x|)\right| \le C(\alpha,\beta)|x|^{-|\beta|} \varepsilon^{-|\alpha-\beta|} \le C(\alpha,\beta)\varepsilon^{-|\alpha|}, \quad \text{for } x \in B_{\frac{1}{2}} \setminus B_\varepsilon.$$

This completes the proof of (7.9).

In the case when σ is not an integer then we write $\sigma = m + s$ where $0 < s < 1$ and m is a nonnegative integer. Then for $|x| \le 3\varepsilon$ we have (see Proposition 2.5.2)

$$(-\Delta)^\sigma v_\varepsilon(x) = C(n,s) \int_{\mathbb{R}^n} \frac{(-\Delta)^m v_\varepsilon(x+y) + (-\Delta)^m v_\varepsilon(x-y) - 2(-\Delta)^m v_\varepsilon(x)}{|y|^{n+2s}} dy.$$

Splitting $\mathbb{R}^n$ into

$$A_1 = \{x : |x| \le 2\varepsilon\}, \quad A_2 = \left\{x : 2\varepsilon < |x| \le \frac{1}{4}\right\} \text{ and } A_3 = \left\{x : |x| > \frac{1}{4}\right\},$$

we have

$$(-\Delta)^\sigma v_\varepsilon(x) = C(n,s) \sum_{i=1}^{3} I_i,$$

where

$$I_i := \int_{A_i} \frac{(-\Delta)^m v_\varepsilon(x+y) + (-\Delta)^m v_\varepsilon(x-y) - 2(-\Delta)^m v_\varepsilon(x)}{|y|^{n+2s}} dy.$$

For $y \in A_1$, using (7.9) we have

$$|\Delta^m v_\varepsilon(x+y) + \Delta^m v_\varepsilon(x-y) - 2\Delta^m v_\varepsilon(x)| \le |y|^2 \|D^2 \Delta^m v_\varepsilon\|_{L^\infty}$$
$$\le C|y|^2 \varepsilon^{-2m-2} \left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{p}},$$

and hence

$$|I_1| \le C\varepsilon^{-2m-2}\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\int_{A_1}\frac{dy}{|y|^{n+2s-2}} \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2\sigma}.$$

For $m \ge 1$, again by (7.9)

$$|\Delta^m v_\varepsilon(x+y) - \Delta^m v_\varepsilon(x)| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2m}.$$

Therefore,

$$|I_2+I_3| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2m}\int_{|y|>\varepsilon}\frac{dy}{|y|^{n+2s}} \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2\sigma}.$$

Since on A_2 $|x+y| \le 3\varepsilon + \frac{1}{4} < \frac{1}{2}$, one has

$$\begin{aligned}
&\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{p}}|v_\varepsilon(x+y)-v_\varepsilon(x)| \\
&= \bigg|\log\left(\frac{1}{\varepsilon}\right)(\varphi_\varepsilon(|x+y|)+\psi_\varepsilon(|x+y|)-\varphi_\varepsilon(|x|)-\psi_\varepsilon(|x|)) \\
&\quad +\log\left(\frac{\varepsilon}{|x+y|}\right)\psi_\varepsilon(|x+y|) - \log\left(\frac{\varepsilon}{|x|}\right)\psi_\varepsilon(|x|)\bigg| \\
&= \left|\log\left(\frac{\varepsilon}{|x+y|}\right)\psi_\varepsilon(|x+y|) - \log\left(\frac{\varepsilon}{|x|}\right)\psi_\varepsilon(|x|)\right| \\
&\le C + \left|\log\left(\frac{\varepsilon}{|x+y|}\right)\psi_\varepsilon(|x+y|)\right|.
\end{aligned}$$

Hence, for $m=0$, changing the variable $y \mapsto \varepsilon z$

$$\begin{aligned}
|I_2| &\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s} + C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\int_{\varepsilon<|y|<\frac{1}{4}}\frac{\left|\log\left(\frac{\varepsilon}{|x+y|}\right)\psi_\varepsilon(|x+y|)\right|}{|y|^{n+2s}}dy \\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s} + C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s}\int_{|z|>1}\frac{\left|\log|\frac{x}{\varepsilon}+z|\right|\psi_\varepsilon(\varepsilon|\frac{x}{\varepsilon}+z|)}{|z|^{n+2s}}dz \\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s} + C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s}\int_{|z|>1}\frac{\log(3+|z|)}{|z|^{n+2s}}dz \\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\varepsilon^{-2s}.
\end{aligned}$$

Finally, for $m=0$, using that $|v_\varepsilon| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}$ on $B^c_{\frac{1}{8}}$, we bound

$$|I_3| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\int_{|y|\ge\frac{1}{4}}\frac{dy}{|y|^{n+2s}} \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}.$$

The lemma follows immediately. □

Lemma 7.2.3. *For* $|x| \geq 3\varepsilon$ *we have*

$$|(-\Delta)^\sigma f_\varepsilon(x)| \leq C\frac{1}{|x|^{2\sigma}}\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\begin{cases}\left(\frac{\varepsilon}{|x|}\right)^n & \text{if } 0<\sigma<1\\ \left(\frac{\varepsilon}{|x|}\right)^{n-2m} & \text{if } 1<\sigma=m+s<\frac{n}{2},\end{cases}$$

where m *is a positive integer and* $0<s<1$*. In particular*

$$\|(-\Delta)^{\frac{n}{2p}}R_\varepsilon\|_{L^p(B_2\setminus B_{3\varepsilon})} \leq C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}.$$

Proof. Notice that for every nonzero multi-index $\alpha\in\mathbb{N}^n$ we have

$$|D^\alpha f_\varepsilon(x)| \leq C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\begin{cases}\frac{1}{|x|^{|\alpha|}} & \text{if } |x|<\varepsilon\\ \frac{1}{\varepsilon^{|\alpha|}} & \text{if } \varepsilon<|x|\leq 2\varepsilon\\ 0 & \text{if } |x|\geq 2\varepsilon.\end{cases}$$

First we consider $0<\sigma<1$. Using that $|\varphi_\varepsilon|\leq 1$, changing the variable $y\mapsto\varepsilon y$ and by Hölder inequality we obtain

$$\begin{aligned}|(-\Delta)^\sigma f_\varepsilon(x)| &= C\left|\int_{\mathbb{R}^n}\frac{f_\varepsilon(x)-f_\varepsilon(y)}{|x-y|^{n+2\sigma}}dy\right|\\ &= C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\left|\int_{|y|<2\varepsilon}\frac{\left(\log\frac{1}{\varepsilon}-\log\frac{1}{|y|}\right)\varphi_\varepsilon(|y|)}{|x-y|^{n+2\sigma}}dy\right|\\ &\leq C\varepsilon^n\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\left(\int_{|y|<2}\frac{dy}{|x-\varepsilon y|^{np+2p\sigma}}\right)^{\frac{1}{p}}\left(\int_{|y|<2}|\log|y||^{p'}dy\right)^{\frac{1}{p'}}\\ &\leq C\varepsilon^n\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\left(\frac{|x|^n}{\varepsilon^n}\frac{1}{|x|^{np+2p\sigma}}\int_{|y|<\frac{2\varepsilon}{|x|}}\frac{dy}{|\frac{x}{|x|}-y|^{np+2p\sigma}}\right)^{\frac{1}{p}},\\ &\leq C\frac{1}{|x|^{2\sigma}}\left(\frac{\varepsilon}{|x|}\right)^n\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}},\end{aligned}$$

where in the second last inequality we have used a change of variable $y\mapsto\frac{|x|}{\varepsilon}y$ and the last inequality follows from the uniform bound

$$\frac{1}{|\frac{x}{|x|}-y|^{np+2p\sigma}} \leq C \quad \text{for every } |x|\geq 3\varepsilon, \quad |y|\leq\frac{2\varepsilon}{|x|}. \tag{7.10}$$

For $\sigma > 1$, changing the variable $y \mapsto |x|y$ and by (7.10) we have

$$\begin{aligned}
|(-\Delta)^\sigma f_\varepsilon(x)| &= C\left|\int_{\mathbb{R}^n}\frac{\Delta^m f_\varepsilon(x)-\Delta^m f_\varepsilon(y)}{|x-y|^{n+2s}}dy\right|\\
&= C\left|\int_{|y|<2\varepsilon}\frac{\Delta^m f_\varepsilon(y)}{|x-y|^{n+2s}}dy\right|\\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\int_{|y|<2\varepsilon}\frac{1}{|y|^{2m}}\frac{1}{|x-y|^{n+2s}}dy\\
&\le C\frac{1}{|x|^{2\sigma}}\left(\frac{\varepsilon}{|x|}\right)^{n-2m}\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}.
\end{aligned}$$

We conclude the lemma by (7.8). □

Lemma 7.2.4. *For $0 < \sigma < \frac{n}{2}$ there exists a constant $C = C(n,\sigma)$ such that*

$$|(-\Delta)^\sigma v_\varepsilon(x)| \le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\frac{1}{|x|^{n+2\sigma}} \quad \textit{for every } x \in B_2^c.$$

Moreover,

$$\|(-\Delta)^{\frac{n}{2p}}v_\varepsilon\|^p_{L^p(B_2^c)} \le C\left(\log\frac{1}{\varepsilon}\right)^{-1}.$$

Proof. If $0 < \sigma < 1$ then

$$\begin{aligned}
|(-\Delta)^\sigma v_\varepsilon(x)| &= C\int_{|y|<1}\frac{v_\varepsilon(y)}{|x-y|^{n+2\sigma}}dy, \qquad |x|>2 \qquad (7.11)\\
&\le C\frac{1}{|x|^{n+2\sigma}}\int_{|y|<1}v_\varepsilon(y)dy\\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\frac{1}{|x|^{n+2\sigma}}\int_{|y|<1}(\log|y|+\log 2)\,dy\\
&\le C\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}}\frac{1}{|x|^{n+2\sigma}}.
\end{aligned}$$

Since the integral in the right hand side of (7.11) is a proper integral, differentiating under the integral sign one can prove the lemma in a similar way. □

Proof of Theorem 7.1.1 Without loss of generality we can assume that $B_1 \subseteq \Omega$. Let u_ε be defined as in Proposition 7.2.1. We set

$$\bar{u}_\varepsilon(x) = \frac{u_\varepsilon(x)}{\left(\|u_\varepsilon\|^p_{L^p(\Omega)} + \|(-\Delta)^{\frac{n}{2p}}u_\varepsilon\|^p_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{p}}}, \quad x \in \mathbb{R}^n.$$

Then $\bar{u}_\varepsilon \in \tilde{H}^{\frac{n}{p},p}(\Omega)$ and $\|\bar{u}\|^p_{L^p(\Omega)} + \|(-\Delta)^{\frac{n}{2p}}\bar{u}\|^p_{L^p(\mathbb{R}^n)} = 1$. We claim that there exists a constant $\delta > 0$ such that

$$\limsup_{\varepsilon\to 0}\int_{B_\varepsilon}\exp\left(\alpha_{n,p}|\bar{u}_\varepsilon|^{p'}\right)dx =: \limsup_{\varepsilon\to 0} I_\varepsilon \ge \delta. \qquad (7.12)$$

Noticing that (restriction of $\bar{u}_\varepsilon$ on B_ε is a constant function)

$$\lim_{\varepsilon\to 0}\inf_{x\in B_\varepsilon}\bar{u}_\varepsilon(x)=\infty,$$

and by (7.6), we obtain

$$\begin{aligned}
&\sup_{u\in\tilde{H}^{\frac{n}{p},p}(\Omega),\,\|u\|^p_{L^p(\Omega)}+\|(-\Delta)^{\frac{n}{2p}}u\|^p_{L^p(\mathbb{R}^n)}\le 1}\int_\Omega f(|u|)e^{\alpha_{n,p}|u|^{p'}}dx\\
&\ge \limsup_{\varepsilon\to 0}\int_\Omega f(|\bar{u}_\varepsilon|)e^{\alpha_{n,p}|\bar{u}_\varepsilon|^{p'}}dx\\
&\ge \limsup_{\varepsilon\to 0}\int_{B_\varepsilon} f(|\bar{u}_\varepsilon|)e^{\alpha_{n,p}|\bar{u}_\varepsilon|^{p'}}dx\\
&\ge \limsup_{\varepsilon\to 0}\left(I_\varepsilon\inf_{x\in B_\varepsilon}f(|\bar{u}_\varepsilon(x)|)\right)\\
&\ge \delta\limsup_{\varepsilon\to 0}\inf_{x\in B_\varepsilon}f(|\bar{u}_\varepsilon(x)|)\\
&=\infty.
\end{aligned}$$

To prove (7.12) we choose $\varepsilon=e^{-k}$. Noticing that

$$\lim_{k\to\infty}-k+k\left(1+\frac{C}{k}\right)^{-\frac{p'}{p}}=-C\frac{p'}{p},\quad \|u_\varepsilon\|^p_{L^p(\mathbb{R}^n)}\le C\left(\log\frac{1}{\varepsilon}\right)^{-1},$$

and using Proposition 7.2.1 we have

$$I_\varepsilon\ge|B_1|\varepsilon^n e^{n\log\frac{1}{\varepsilon}\left(1+C\left(\log\frac{1}{\varepsilon}\right)^{-1}\right)^{-\frac{p'}{p}}}=|B_1|e^{-kn+kn\left(1+\frac{C}{k}\right)^{-\frac{p'}{p}}}\ge\delta,$$

for some $\delta>0$. □

In order to prove Theorem 7.1.2, first we prove the following proposition which gives a similar type of estimate as in Proposition 7.2.1.

Proposition 7.2.5. *Let $\tau>0$ and $1<p<\infty$. Then there exists a constant $C>0$ such that*

$$\|(\tau I-\Delta)^{\frac{n}{2p}}u_\varepsilon\|_{L^p(\mathbb{R}^n)}\le\left(1+C\left(\log\frac{1}{\varepsilon}\right)^{-1}\right)^{\frac{1}{p}}.$$

Proof. We set

$$w_\varepsilon(x)=(\tau I-\Delta)^{\frac{n}{2p}}u_\varepsilon(x)-(-\Delta)^{\frac{n}{2p}}u_\varepsilon(x).$$

We observe that there exists $C=C(p)>0$ such that

$$h(t)=(1+t)^p-1-C(t^p+t^{p-1}+t^{\frac{1}{2}})<0,\quad\text{for every } t>0,\quad 1\le p<\infty,$$

which follows from the fact that $h(0)=0$ and $h'(t)<0$ for every $t>0$. Therefore, there holds

$$(a+b)^p\le a^p+C_p(b^p+ab^{p-1}+b^{\frac{1}{2}}a^{p-\frac{1}{2}}),\quad a\ge 0,\, b\ge 0,\, 1\le p<\infty,$$

for some constant $C_p > 0$ and using this inequality we bound

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |(\tau I - \Delta)^{\frac{n}{2p}} u_\varepsilon(x)|^p dx \\
&= \int_{\mathbb{R}^n} |w_\varepsilon(x) + (-\Delta)^{\frac{n}{2p}} u_\varepsilon(x)|^p dx \\
&\leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon(x)|^p + C \int_{\mathbb{R}^n} |w_\varepsilon(x)|^p dx + C \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon(x)||w_\varepsilon(x)|^{p-1} dx \\
&\quad + C \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon(x)|^{p-\frac{1}{2}} |w_\varepsilon(x)|^{\frac{1}{2}} dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

From Proposition 7.2.1 we have

$$I_1 \leq 1 + C \left(\log \frac{1}{\varepsilon} \right)^{-1}.$$

To estimate I_2, I_3 and I_4 we will use pointwise estimates on $(-\Delta)^\sigma u_\varepsilon$, $(-\Delta)^\sigma w_\varepsilon$ and L^p estimates on $(-\Delta)^\sigma w_\varepsilon$. For $0 < \sigma < \frac{n}{2}$ combining Lemmas 7.2.2 - 7.2.4, 7.4.8, and (7.8) we get

$$|(-\Delta)^\sigma u_\varepsilon(x)| \leq C \left(\log \frac{1}{\varepsilon} \right)^{-\frac{1}{p}} \begin{cases} \varepsilon^{-2\sigma} & \text{if } |x| < 3\varepsilon \\ |x|^{-2\sigma} & \text{if } 3\varepsilon < |x| < 2 \\ |x|^{-n-2\sigma} & \text{if } |x| > 2. \end{cases} \tag{7.13}$$

With the help of (7.13) one can verify that

$$\|(-\Delta)^\sigma u_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C(n,p,\sigma) \left(\log \frac{1}{\varepsilon} \right)^{-\frac{1}{p}}, \quad 1 \leq p < \infty,\ 0 \leq \sigma < \frac{n}{2p}, \tag{7.14}$$

and together with Lemma 7.4.2

$$I_2 \leq C \left(\log \frac{1}{\varepsilon} \right)^{-1}.$$

We conclude the proposition by showing that

$$\int_{\mathbb{R}^n} |w_\varepsilon|^q |(-\Delta)^{\frac{n}{2p}} v_\varepsilon|^{p-q} dx \leq C(n,p,q) \left(\log \frac{1}{\varepsilon} \right)^{-1}, \quad 0 < q < \frac{p^2}{p+1}. \tag{7.15}$$

It follows from Lemma 7.4.1 that

$$|w_\varepsilon(x)| \leq C \left(\log \frac{1}{\varepsilon} \right)^{-\frac{1}{p}}, \quad x \in \mathbb{R}^n,\ \frac{n}{2p} < 1,$$

and for $\frac{n}{2p} > 1$

$$|w_\varepsilon(x)| \leq C \left(\log \frac{1}{\varepsilon} \right)^{-\frac{1}{p}} \begin{cases} \varepsilon^{-\frac{n}{p}+2} & \text{if } |x| < 3\varepsilon \\ |x|^{-\frac{n}{p}+2} & \text{if } 3\varepsilon < |x| < 2 \\ 1 & \text{if } |x| > 2, \end{cases}$$

thanks to (7.13) and (7.14).

Splitting $\mathbb{R}^n$ into

$$A_1 = \{x : |x| \leq 2\} \quad \text{and } A_2 = \{x : |x| > 2\},$$

we have

$$\int_{\mathbb{R}^n} |w_\varepsilon|^q |(-\Delta)^{\frac{n}{2p}} v_\varepsilon|^{p-q} dx = \sum_{i=1}^{2} J_i, \quad J_i := \int_{A_i} |w_\varepsilon|^q |(-\Delta)^{\frac{n}{2p}} v_\varepsilon|^{p-q} dx, \; i = 1, 2.$$

Using (7.13) one can show that $J_1 \leq C \left(\log \frac{1}{\varepsilon}\right)^{-1}$ and together with $q < \frac{p^2}{p+1}$ one has $J_2 \leq C \left(\log \frac{1}{\varepsilon}\right)^{-1}$, which gives (7.15). □

Proof of Theorem 7.1.2 Here also we can assume that $B_1 \subseteq \Omega$. We choose $M > 0$ large enough such that

$$\Phi(\alpha_{n,p} t^{p'}) \geq \frac{1}{2} e^{\alpha_{n,p} t^{p'}}, \quad t \geq M.$$

We set

$$\bar{u}_\varepsilon = \frac{u_\varepsilon}{\|(\tau I - \Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(\mathbb{R}^n)}}.$$

Then we have

$$\begin{aligned} &\int_{\mathbb{R}^n} f(|\bar{u}_\varepsilon|) \Phi\left(\alpha_{n,p} |\bar{u}_\varepsilon|^{p'}\right) dx \\ &\geq \int_{u_\varepsilon \geq M} f(|\bar{u}_\varepsilon|) \Phi\left(\alpha_{n,p} |\bar{u}_\varepsilon|^{p'}\right) dx \\ &\geq \frac{1}{2} \int_{B_\varepsilon} f(|\bar{u}_\varepsilon|) e^{\alpha_{n,p} |\bar{u}_\varepsilon|^{p'}} dx, \end{aligned}$$

for $\varepsilon > 0$ small enough. Now the proof follows as in Theorem 7.1.1, thanks to Proposition 7.2.5. □

Remark: From the point of view of conductor capacity estimate (see e.g. [2, p. 393], [42, p. 2193]), it would be interesting to know whether

$$\|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(\mathbb{R}^n)} \geq 1, \quad \text{and } \|(\tau I - \Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(\mathbb{R}^n)} \geq 1,$$

or not. Here we recall that the "original" Moser functions (see [61]), that is

$$u(\varepsilon, x) := \frac{1}{\sqrt{2\pi}} \begin{cases} 0 & \text{for } x \in \mathbb{R}^2 \setminus B_1 \\ 1/\sqrt{\log(1/\varepsilon)} \log(1/|x|) & \text{for } x \in B_1 \setminus B_\varepsilon \\ \sqrt{\log(1/\varepsilon)} & \text{for } x \in B_\varepsilon, \end{cases}$$

satisfy $\|\nabla u(\varepsilon, \cdot)\|_{L^2(B_1)} = 1$.

Lemma 7.2.6. *Let u_ε be as in Proposition 7.2.1. Then there exists a constant $C = C(n, p, \tau, \varphi, \eta) > 0$ such that*

$$\|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|^p_{L^p(B_1)} \geq 1 - C\left(\log \frac{1}{\varepsilon}\right)^{-1}, \quad \|(\tau I - \Delta)^{\frac{n}{2p}} u_\varepsilon\|^p_{L^p(B_1)} \geq 1 - C\left(\log \frac{1}{\varepsilon}\right)^{-1}.$$

Moreover, if $\frac{n}{2p}$ is an integer, then $\|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(B_1)} > 1$.

Proof. Using the inequality

$$||a| - |b||^p \geq |a|^p - p|a|^{p-1}|b|, \quad p \geq 1,\ a, b \in \mathbb{R},$$

we obtain

$$\begin{aligned}
&\int_{3\varepsilon<|x|<1} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon|^p dx \\
&\geq \int_{3\varepsilon<|x|<1} \left(\log \frac{1}{\varepsilon}\right)^{-1} \frac{1}{|S^{n-1}|} \frac{1}{|x|^n} dx - C \left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{p'}} \int_{3\varepsilon<|x|<1} \frac{1}{|x|^{\frac{n}{p'}}} |(-\Delta)^{\frac{n}{2p}} R_\varepsilon| dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{3\varepsilon<|x|<1} |(\tau I - \Delta)^{\frac{n}{2p}} u_\varepsilon|^p dx \\
&\geq \int_{3\varepsilon<|x|<1} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon|^p dx - C \int_{3\varepsilon<|x|<1} |(-\Delta)^{\frac{n}{2p}} u_\varepsilon|^{p-1} |w_\varepsilon| dx,
\end{aligned}$$

where R_ε and w_ε are defined in the proof of Proposition 7.2.1 and 7.2.5 respectively.

First part of the lemma follows as in Proposition 7.2.1 and 7.2.5.

We choose $r < 1$ so that $u_\varepsilon \in C_c^\infty(B_r)$. If $\frac{n}{2p}$ is an integer, then the support of $\Delta^{\frac{n}{2p}} u_\varepsilon$ is a subset of $B_r \setminus B_\varepsilon$. Therefore, by Hölder inequality

$$\begin{aligned}
u_\varepsilon(0) &= K_{n,p} \int_{\varepsilon<|x|<r} (-\Delta)^{\frac{n}{2p}} u_\varepsilon(x) \frac{1}{|x|^{\frac{n}{p'}}} dx \\
&\leq K_{n,p} |S^{n-1}|^{\frac{1}{p'}} \|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(B_1)} \left(\log \frac{1}{\varepsilon} + \log r\right)^{\frac{1}{p'}} \\
&< K_{n,p} |S^{n-1}|^{\frac{1}{p'}} \|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(B_1)} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p'}},
\end{aligned}$$

where the first identity follows from the fact that the function

$$K_{n,p} \frac{1}{|x|^{\frac{n}{p'}}} := 2^{-\frac{n}{p}} \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2p'})}{\Gamma(\frac{n}{2p})} \frac{1}{|x|^{\frac{n}{p'}}},$$

is a fundamental solution of the operator $(-\Delta)^{\frac{n}{2p}}$. Since

$$\begin{aligned}
u_\varepsilon(0) &= |S^{n-1}|^{-\frac{1}{p}} 2^{\frac{n}{p'}} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2p'}) \frac{1}{\Gamma(\frac{n}{2p}) \gamma_n} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p'}} \\
&= K_{n,p} |S^{n-1}|^{\frac{1}{p'}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p'}},
\end{aligned}$$

we have $\|(-\Delta)^{\frac{n}{2p}} u_\varepsilon\|_{L^p(B_1)} > 1$. □

7.3 Variational arguments

Throughout this section we use the notation $\|u\| = \|(-\Delta)^{\frac{n}{4}}u\|_{L^2(\mathbb{R}^n)}$, $H = \tilde{H}^{\frac{n}{2},2}(\Omega)$ and $\alpha_0 = \alpha_{n,2}$.

To prove Theorem 7.1.3 we follow the approach in [3, 41]. First we prove that $\lambda_1 > 0$, which makes the statement of Theorem 7.1.3 meaningful.

Lemma 7.3.1. *Let Ω be an open set in $\mathbb{R}^n$ with finite measure. Then $\lambda_1 > 0$ and there exists a function $u \in H$ such that*

$$\|u\|_{L^2(\Omega)} = 1, \quad \text{and } \|u\|^2 = \lambda_1.$$

Proof. We recall that

$$\lambda_1 = \inf\left\{\|u\|^2 : u \in H,\, \|u\|_{L^2(\Omega)} = 1\right\}.$$

Let $\{u_k\}_{k=1}^{\infty} \subset H$ be a sequence such that

$$\lim_{k\to\infty} \|u_k\|^2 = \lambda_1, \quad \|u_k\|_{L^2(\Omega)} = 1 \text{ for every } k.$$

Then up to a subsequence

$$u_k \rightharpoonup u_0 \text{ in } H, \quad u_k \to u_0 \text{ in } L^2(\Omega),$$

where the latter one follows from the compact embedding $H \hookrightarrow L^2(\Omega)$ (see Lemma 7.4.7). Therefore,

$$\lambda_1 \le \|u_0\|^2 \le \liminf_{k\to\infty} \|u_k\|^2 = \lambda_1, \quad \|u_0\|_{L^2(\Omega)} = 1.$$

□

Let us now define the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \int_\Omega G(u)dx, \quad u \in H,$$

where

$$G(t) = \int_0^t g(r)dr, \quad g(r) := \lambda r e^{br^2}, \quad 0 < \lambda < \lambda_1,\, b > 0.$$

Then J is C^2 and the Fréchet derivative of J is given by

$$DJ(u)(v) = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}}u(-\Delta)^{\frac{n}{4}}v dx - \int_\Omega g(u)v dx, \quad v \in H.$$

We also define

$$F(u) = DJ(u)(u) = \|u\|^2 - \int_\Omega g(u)u dx, \quad I(u) = J(u) - \frac{1}{2}F(u),$$

$$S = \{u \in H : u \ne 0, F(u) = 0\}.$$

Observe that if $u \in H$ is a nontrivial weak solution of (7.7) then $u \in S$.

With the above notations we have:

Lemma 7.3.2. *The set S is closed in the norm topology and*

$$0 < s^2 < \frac{\alpha_0}{b}, \quad s := \sqrt{2 \inf_{u \in S} J(u)}.$$

Proof. Since F is continuous (actually F is C^1 as J is C^2), it is enough to show that 0 is an isolated point of S. If not, then there exists a sequence $\{u_k\} \subset S$ such that $\|u_k\| \to 0$ as $k \to \infty$. We set $v_k = \frac{u_k}{\|u_k\|}$. From the compactness of the embedding $H \hookrightarrow L^q(\Omega)$ for any $1 \le q < \infty$, we can assume that (up to a subsequence) $v_k \rightharpoonup v$ in H and $v_k \to v$ almost everywhere in Ω. By Lemma 7.3.4 we get

$$1 = \lambda \int_\Omega e^{bu_k^2} v_k^2 dx \xrightarrow{k\to\infty} \lambda \int_\Omega v^2 dx \le \lambda \frac{1}{\lambda_1} \|v\|^2 < 1,$$

which is a contradiction. Hence S is closed.

Since,

$$f(t) := \left(t^2 - \frac{1}{b}\right) e^{bt^2} + \frac{1}{b} > 0, \quad \text{for } t > 0,\ b > 0,$$

which follows from $f(0) = 0$ and $f'(t) > 0$ for $t > 0$, we have

$$I(u) = \frac{\lambda}{2} \int_\Omega \left(\left(u^2 - \frac{1}{b}\right) e^{bu^2} + \frac{1}{b} \right) dx > 0, \text{ if } u \in H \setminus \{0\}, \tag{7.16}$$

and in particular $J(u) = I(u) > 0$ for $u \in S$.

If possible, we assume that $s = 0$. Then there exists a sequence $\{u_k\} \subset S$ such that $J(u_k) \to 0$ as $k \to \infty$. Moreover,

$$\begin{aligned} \|u_k\|^2 = \lambda \int_\Omega u_k^2 e^{bu_k^2} dx &= \lambda \int_{u_k^2 > \frac{2}{b}} u_k^2 e^{bu_k^2} dx + \lambda \int_{u_k^2 \le \frac{2}{b}} u_k^2 e^{bu_k^2} dx \\ &\le 4 \frac{\lambda}{2} \int_{u_k^2 > \frac{2}{b}} \left(\left(u_k^2 - \frac{1}{b}\right) e^{bu_k^2} + \frac{1}{b} \right) dx + \lambda \int_{u_k^2 \le \frac{2}{b}} u_k^2 e^{bu_k^2} dx \\ &\le 4J(u_k) + \lambda \int_{u_k^2 \le \frac{2}{b}} u_k^2 e^{bu_k^2} dx, \end{aligned} \tag{7.17}$$

and hence u_k is bounded in H. Then up to a subsequence $u_k \to u$, *a.e.* in Ω and $u_k \rightharpoonup u$. Using Fatou lemma and *ii*) in Lemma 7.3.4 we obtain

$$I(u) = \frac{\lambda}{2} \int_\Omega \left(\left(u^2 - \frac{1}{b}\right) e^{bu^2} + \frac{1}{b} \right) dx \le \liminf_{k\to\infty} I(u_k) = \liminf_{k\to\infty} J(u_k) = 0,$$

and hence $u = 0$, thanks to (7.16). It follows from (7.17) that $u_k \to 0$ in H which is a contradiction as S is closed.

We prove now $s^2 < \alpha_0 b^{-1}$. First we fix $u \in H$ with $\|u\| = 1$. We consider the function

$$F_u(t) := F(tu) = \|tu\|^2 - \lambda \int_\Omega t^2 u^2 e^{bt^2 u^2} dx, \quad t \ge 0.$$

Then

$$F_u(t) \geq t^2 \left(\lambda_1 \int_\Omega u^2 dx - \lambda \int_\Omega u^2 e^{bt^2u^2} dx \right) > 0,$$

for $t > 0$ sufficiently small and $\lim_{t\to\infty} F_u(t) = -\infty$. Hence, the continuity of F_u implies that there exists $t_u > 0$ such that $F_u(t_u) = 0$, that is, $t_u u \in S$. Thus

$$\frac{s^2}{2} \leq J(t_u u) \leq \frac{1}{2}\|t_u u\|^2 = \frac{1}{2}t_u^2.$$

Again using that $t_u u \in S$ we have

$$\int_\Omega u^2 e^{bs^2u^2} dx \leq \frac{1}{\lambda t_u^2} \lambda \int_\Omega (t_u u)^2 e^{b(t_u u)^2} dx = \frac{1}{\lambda t_u^2}\|t_u u\|^2 = \frac{1}{\lambda},$$

which implies that

$$\sup_{\|u\|\leq 1,\, u\in H} \int_\Omega u^2 e^{bs^2u^2} dx < \infty, \tag{7.18}$$

and by Theorem 7.1.1 we deduce that $s^2 < \alpha_0 b^{-1}$. □

Lemma 7.3.3. *Let $u \in S$ be a minimizer of J on S. Then $DJ(u) = 0$.*

Proof. We fix a function $v \in H \setminus \{0\}$ and consider the function

$$F_{u,v}(\gamma, t) := F(\gamma u + tv), \quad \gamma > 0,\, t \in \mathbb{R}.$$

Differentiating $F_{u,v}$ with respect to γ and using that $F(u) = 0$, we get

$$\frac{\partial F_{u,v}}{\partial \gamma}(1, 0) = -2b\lambda \int_{\mathbb{R}^n} u^4 e^{bu^2} dx < 0.$$

Hence, by implicit function theorem, there exists $\delta > 0$ such that we can write $\gamma = \gamma(t)$ as a C^1 function of t on the interval $(-\delta,\, \delta)$ which satisfies

$$\gamma(0) = 1, \quad F_{u,v}(\gamma(t), t) = 0, \text{ for every } t \in (-\delta,\, \delta).$$

Moreover, choosing $\delta > 0$ smaller if necessary, we have $\gamma(t)u + tv \in S$ for every $t \in (-\delta,\, \delta)$. We write

$$\begin{aligned} DJ(u)(v) &= \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \\ &= \lim_{t\to 0} \left(\frac{J(\gamma(t)u+tv) - J(u)}{t} - \frac{J(\gamma(t)u+tv) - J(u+tv)}{t} \right). \end{aligned}$$

Since J is C^1, a first order expansion of J yields

$$\begin{aligned} J(\gamma(t)u+tv) - J(u+tv) &= J((u+tv) + (\gamma(t)-1)u) - J(u+tv) \\ &= DJ(u+tv)((\gamma(t)-1)u) + o\left((\gamma(t)-1)\|u\|\right) \\ &= (\gamma(t)-1)DJ(u+tv)(u) + (\gamma(t)-1)\|u\|o(1). \end{aligned}$$

Therefore, using that $F(u) = 0$,

$$\lim_{t\to 0} \frac{J(\gamma(t)u+tv) - J(u+tv)}{t} = \gamma'(0)DJ(u)(u) = 0.$$

On the other hand, since u is a minimizer of J on S and $\gamma(t)u + tv \in S$,

$$\frac{J(\gamma(t)u+tv)-J(u)}{t} = \begin{cases} \geq 0 & \text{if } t \geq 0 \\ \leq 0 & \text{if } t \leq 0, \end{cases}$$

implies that (since it exists)

$$\lim_{t\to 0} \frac{J(\gamma(t)u+tv)-J(u)}{t} = 0.$$

This shows that $DJ(u)(v) = 0$ for every $v \in H$, i.e., $DJ(u) = 0$. □

Proof of Theorem 7.1.3 Let $\{u_k\}$ be a sequence in S such that $\lim_{k\to\infty} J(u_k) \to \frac{s^2}{2}$. Then by (7.17) u_k is a bounded sequence in H and consequently, up to a subsequence

$$u_k \rightharpoonup u, \quad u_k \to u, \ a.e. \text{ in } \Omega, \quad \ell := \lim_{k\to\infty} \|u_k\|,$$

for some $u \in H$. First we claim that $u \neq 0$.

Assuming $u = 0$, by $ii)$ in Lemma 7.3.4 we get

$$\lim_{k\to\infty} \|u_k\|^2 = \lim_{k\to\infty} 2\left(J(u_k) + \frac{\lambda}{2b}\int_\Omega (e^{bu_k^2}-1)dx\right) = s^2 < \frac{\alpha_0}{b},$$

and hence by $i)$ in Lemma 7.3.4

$$\lim_{k\to\infty} \|u_k\|^2 = \lim_{k\to\infty} \lambda \int_\Omega u_k^2 e^{bu_k^2} dx = 0,$$

a contradiction as S is closed.

We claim that $\ell = \|u\|$. Then $u_k \to u$ in H and applying Lemmas 7.3.2 and 7.3.3 we have Theorem 7.1.3.

If the claim is false then necessarily we shall have $\ell > \|u\|$.

One has

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^2 &= \lim_{k\to\infty} 2\left(J(u_k) + \frac{\lambda}{2b}\int_\Omega (e^{bu_k^2}-1)dx\right) \\ &= 2\left(\frac{s^2}{2} + \frac{\lambda}{2b}\int_\Omega (e^{bu^2}-1)dx,\right) \\ &= s^2 - 2J(u) + \|u\|^2. \end{aligned}$$

We divide the proof in two cases, namely $J(u) \leq 0$ and $J(u) > 0$.

Case 1. We consider that $J(u) \leq 0$. Since $u \neq 0$,

$$\|u\|^2 \leq \frac{\lambda}{b}\int_\Omega (e^{bu^2}-1)dx < \lambda \int_\Omega u^2 e^{bu^2} dx,$$

where the second inequality follows from (7.16). It is easy to see that we can choose $0 < t_0 < 1$ such that

$$\|t_0 u\|^2 = \lambda \int_\Omega (t_0 u)^2 e^{b(t_0 u)^2} dx,$$

that means $t_0 u \in S$. Using that $I(tu)$ is strictly monotone increasing in t, which follows from the expression in (7.16), we obtain

$$\frac{s^2}{2} \le J(t_0 u) = I(t_0 u) < I(u) \le \liminf_{k\to\infty} J(u_k) = \frac{s^2}{2},$$

a contradiction.

Case 2. Here we assume that $J(u) > 0$. Then

$$\ell^2 = \lim_{k\to\infty} \|u_k\|^2 = s^2 - 2J(u) + \|u\|^2 < s^2 + \|u\|^2 < \frac{\alpha_0}{b} + \|u\|^2. \tag{7.19}$$

Taking $v_k = \frac{u_k}{\|u_k\|}$ we see that (up to a subsequence)

$$v_k \rightharpoonup v := \frac{u}{\ell}, \quad v_k \to v, \ a.e. \text{ in } \Omega,$$

and by Lemma 7.4.5, for every $p < (1 - \|v\|^2)^{-1}$

$$\sup_{k\in\mathbb{N}} \int_\Omega e^{p\alpha_0 v_k^2} dx < \infty.$$

Taking (7.19) into account we have

$$0 < \ell^2 - \|u\|^2 = s^2 - 2J(u) < \frac{\alpha_0}{b},$$

and therefore, we can choose $\varepsilon_0 > 0$ such that

$$1 + \varepsilon_0 = \frac{\alpha_0}{b} \frac{1}{\ell^2 - \|u\|^2}, \quad \text{i.e., } \ell^2(1+\varepsilon_0) = \frac{\alpha_0}{b}\left(1 - \frac{\|u\|^2}{\ell^2}\right)^{-1}.$$

For k large enough such that $\|u_k\|^2 \le \ell^2(1+\frac{\varepsilon_0}{2})$ holds, we observe that $b\|u_k\|^2 \le p_0\alpha_0$ for some $1 < p_0 < (1 - \|v\|^2)^{-1}$. Thus, for some $p_1 > 1$, $p_2 > 1$ with $p_1 p_2 p_0 < (1 - \|v\|^2)^{-1}$ we obtain

$$\sup_{k\in\mathbb{N}} \int_\Omega \left(u_k^2 e^{bu_k^2}\right)^{p_1} dx \le \sup_{k\in\mathbb{N}} \|u_k^{2p_1}\|_{L^{p_2'}(\Omega)} \|e^{p_1 p_0 \alpha_0 v_k^2}\|_{L^{p_2}(\Omega)} < \infty,$$

and together with Lemma 7.4.9

$$\lim_{k\to\infty} \int_\Omega u_k^2 e^{bu_k^2} dx = \int_\Omega u^2 e^{bu^2} dx.$$

Indeed,

$$\|u\|^2 < \ell^2 = \lim_{k\to\infty} \|u_k\|^2 = \lambda \lim_{k\to\infty} \int_\Omega u_k^2 e^{bu_k^2} dx = \lambda \int_\Omega u^2 e^{bu^2} dx,$$

and we can now proceed as in Case 1. □

Lemma 7.3.4. *Let u_k, $v_k \in H$ such that $u_k \rightharpoonup u$ in H, $u_k \to u$, a.e. in Ω, $v_k \rightharpoonup v$ in H and $v_k \to v$, a.e. in Ω. Then*

i) If

$$\limsup_{k\to\infty} \|u_k\|^2 < \frac{\alpha_0}{b},$$

then for every integer $\ell \geq 1$

$$\lim_{k\to\infty}\int_\Omega e^{bu_k^2} v_k^\ell dx = \int_\Omega e^{bu^2} v^\ell dx.$$

ii) If

$$\limsup_{k\to\infty}\int_\Omega u_k^2 e^{bu_k^2} dx < \infty,$$

then

$$\lim_{k\to\infty}\int_\Omega e^{bu_k^2} dx = \int_\Omega e^{bu^2} dx.$$

Proof. We prove the lemma with the help of Lemma 7.4.9.

We choose $p > 1$ such that for k large enough $p\|u_k\|^2 < \frac{\alpha_0}{b}$ holds and together with Theorem J we have

$$\sup_{k\in\mathbb{N}}\int_\Omega e^{pbu_k^2} dx < \infty.$$

Since the embedding $\tilde{H}^{\frac{n}{2},2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact (see Lemma 7.4.7) for every $1 \leq q < \infty$, we have

$$v_k^q \to v^q \text{ in } L^1(\Omega).$$

Indeed,

$$\sup_{k\in\mathbb{N}} \|e^{bu_k^2} v_k^\ell\|_{L^p(\Omega)} \leq \|v_k^\ell\|_{L^{p'}(\Omega)} \|e^{bu_k^2}\|_{L^p(\Omega)} < \infty,$$

and we conclude i).

Now ii) follows from

$$\int_{u_k^2 > M} e^{bu_k^2} dx \leq \frac{1}{M}\int_{u_k^2>M} u_k^2 e^{bu_k^2} dx \leq \frac{C}{M},$$

which implies that the function $f_k := e^{bu_k^2}$ satisfies the condition ii) in Lemma 7.4.9. □

In the following lemma we prove that the assumption $H'(v)$ in [38] is true under certain conditions.

Lemma 7.3.5. *Let* $\alpha_0 > 0$. *Let* $f(t) = e^{\alpha_0 t^2} h(t)$ *satisfies* $H(i) - (iii)$ *in [38]. Let* $h \geq 0$ *on* $[0, \infty)$ *and* $h(-t) = -h(t)$. *Let* $s\frac{f(st)}{t}$ *be a monotone increasing function with respect to* t *on* $(0, \infty)$, $s \neq 0$. *If* $\lim_{t\to\infty} h(t)t = \infty$ *then there exists* $u \in \tilde{H}^{\frac{1}{2},2}((0,1))$ *such that* $\sqrt{2\pi}\|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})} = 1$ *and*

$$\sup_{t>0} \Phi(tu) := \sup_{t>0}\left(\frac{t^2}{4\pi} - \int_0^1 F(tu)dx\right) < \frac{\omega}{2\alpha_0},$$

where

$$F(t) = \int_0^t f(s)ds,$$

and ω *is as in [38].*

Proof. For a given $M > 0$ we can choose $u \in \tilde{H}^{\frac{1}{2},2}((0,1))$ such that

$$\int_0^1 f\left(\sqrt{\frac{2\pi^2}{\alpha_0}}u\right) udx > M, \quad \sqrt{2\pi}\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R})} = 1,$$

thanks to Theorem 7.1.1. Differentiating with respect to t one has

$$\Phi'(tu) = t\left(\frac{1}{2\pi} - \int_0^1 \frac{f(tu)}{t} udx\right).$$

Hence, for $t \geq \sqrt{\frac{2\pi^2}{\alpha_0}} =: t_0$ and $2\pi M > t_0$

$$\Phi'(tu) \leq t\left(\frac{1}{2\pi} - \int_0^1 \frac{f(t_0 u)}{t_0} udx\right) < 0.$$

Thus $\Phi'(tu) \leq 0$ on $(t_0 - \varepsilon, \infty)$ for some $\varepsilon > 0$ and therefore,

$$\sup_{t>0} \Phi(tu) = \sup_{t\in(0,\,t_0-\varepsilon)} \Phi(tu) \leq \sup_{t\in(0,\,t_0-\varepsilon)} \frac{t^2}{4\pi} < \frac{\pi}{2\alpha_0}.$$

Since $\omega = \pi$, thanks to Theorem J, we conclude the lemma. □

7.4 Some useful results

Lemma 7.4.1 (Pointwise estimate)**.** *Let $s > 0$ and not an integer. Let m be the smallest integer greater than s. Then for any $\tau > 0$*

$$|(\tau I - \Delta)^s u(x) - (-\Delta)^s u(x)| \leq C\sum_{j=1}^{m-1} |(-\Delta)^{s-j}u(x)| + C\|(-\Delta)^\sigma u\|_{L^1(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where $\sigma \in \left(\max\{\frac{n}{2} - m + s, 0\}, \frac{n}{2}\right)$, the constant C depends only on n, s, σ, τ and for $m = 1$ the above sum can be interpreted as zero.

Proof. We set $f(t) = t^s$ on $\mathbb{R}^+$. By Taylor's expansion we have

$$f(t+\tau) = f(t) + \tau f'(t) + \cdots + \frac{\tau^{m-1}}{(m-1)!}f^{m-1}(t) + \frac{\tau^m}{m!}f^m(\xi_t), \quad \text{for some } t < \xi_t < t+\tau.$$

In particular

$$(\tau + t^2)^s = t^{2s} + c_1 t^{2s-2} + c_2 t^{2s-4} + \cdots + c_{m-1}t^{2s-2m+2} + E(t),$$

where the function E satisfies the estimate

$$|E(t)| \leq C(1+t)^{2s-2m}, \quad t > 0.$$

Therefore, for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\mathcal{F}((\tau I-\Delta)^s u)(\xi) &= (\tau+|\xi|^2)^s \hat{u} \\
&= \left(|\xi|^{2s} + c_1|\xi|^{2s-2} + \cdots + c_{m-1}|\xi|^{2s-2m+2} + E(|\xi|)\right)\hat{u} \\
&= \sum_{j=0}^{m-1} c_j |\xi|^{2s-2j}\hat{u} + E(|\xi|)\hat{u}(\xi) \\
&= \sum_{j=0}^{m-1} c_j \mathcal{F}((-\Delta)^{s-j}u) + E(|\xi|)\hat{u}(\xi),
\end{aligned}$$

and hence

$$(\tau I-\Delta)^s u(x) = \sum_{j=0}^{m-1} c_j(-\Delta)^{s-j}u(x) + \mathcal{F}^{-1}(E\hat{u})(x).$$

To estimate the term $\mathcal{F}^{-1}(E\hat{u})$ (uniformly in x) in terms of $L^1(\mathbb{R}^n)$ norm of (fractional) derivative of u, we observe that

$$\begin{aligned}
|E(|\xi|)\hat{u}(\xi)| &= \left| E(|\xi|)\frac{1}{|\xi|^{2\sigma}}\widehat{(-\Delta)^\sigma u}(\xi)\right| \\
&\le \frac{C}{|\xi|^{2\sigma}(1+|\xi|^2)^{m-s}}\left|\widehat{(-\Delta)^\sigma u}(\xi)\right| \\
&\le \frac{C}{|\xi|^{2\sigma}(1+|\xi|^2)^{m-s}}\|(-\Delta)^\sigma u\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Thus

$$\left|\mathcal{F}^{-1}(E\hat{u})(x)\right| \le C\|E\hat{u}\|_{L^1(\mathbb{R}^n)} \le C\|(-\Delta)^\sigma u\|_{L^1(\mathbb{R}^n)},$$

and we complete the proof. □

Lemma 7.4.2 (L^p Estimate)**.** *Let $s>0$ be a noninteger. Let $\tau>0$ be any fixed number. Then for $p \in (1,\infty)$ there exists $C=C(n,s,p,\tau)>0$ such that*

$$\|(\tau I-\Delta)^s u - (-\Delta)^s u\|_{L^p(\mathbb{R}^n)} \le C\begin{cases} \|u\|_{L^p(\mathbb{R}^n)} & \text{if } s<1 \\ \|u+(-\Delta)^{s-1}u\|_{L^p(\mathbb{R}^n)} & \text{if } s>1. \end{cases}$$

Proof. We have

$$\begin{aligned}
\mathcal{F}((\tau I-\Delta)^s u)(\xi) - \mathcal{F}((-\Delta)^s u)(\xi) &= \left((\tau+|\xi|^2)^s - |\xi|^{2s}\right)\hat{u}(\xi) \\
&= \begin{cases} \left((\tau+|\xi|^2)^s - |\xi|^{2s}\right)\hat{u}(\xi) & \text{if } s<1 \\ \frac{(\tau+|\xi|^2)^s-|\xi|^{2s}}{1+|\xi|^{2s-2}}(1+|\xi|^{2s-2})\hat{u}(\xi) & \text{if } s>1 \end{cases} \\
&=: \begin{cases} m(\xi)\hat{u}(\xi) & \text{if } s<1 \\ m(\xi)\mathcal{F}\left(u+(-\Delta)^{s-1}u\right)(\xi) & \text{if } s>1. \end{cases}
\end{aligned}$$

Now the proof follows from the Hormander multiplier theorem (see [72, p. 96]). □

The following lemma appears already in [25, p. 46], but for the sake of completeness we give a proof.

Lemma 7.4.3 (Equivalence of norms)**.** *Let $\sigma > 0$. Then for $p \in (1,\infty)$ there exists a constant $C > 0$ such that for every $u \in \mathcal{S}(\mathbb{R}^n)$*

$$\frac{1}{C}\left(\|u\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)}\right) \le \|(I-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)} \\ \le C\left(\|u\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)}\right).$$

Proof. We set

$$G_\sigma(x) = \frac{1}{(4\pi)^{\frac{\sigma}{2}}}\frac{1}{\Gamma(\frac{\sigma}{2})}\int_0^\infty e^{-\pi\frac{|x|^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{-n+\sigma}{2}}\frac{dt}{t},$$

which is the Bessel potential of order σ (see [72, p. 130]). Then

$$\int_{\mathbb{R}^n} G_\sigma(x)dx = 1, \quad \hat{G}_\sigma(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{1}{(1+|x|^2)^{\frac{\sigma}{2}}}.$$

Setting $f = (I-\Delta)^\sigma u$ we can write $u = G_{2\sigma} * f$ and by Young's inequality one has $\|u\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}$. Again writing $u = G_{2\sigma} * f$ and taking Fourier transform we obtain

$$\mathcal{F}((-\Delta)^\sigma u) = |\xi|^{2\sigma}\hat{u} = |\xi|^{2\sigma}\frac{1}{(1+|\xi|^2)^\sigma}\hat{f} =: m(\xi)\hat{f},$$

and by Hormander multiplier theorem we get $\|(-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$. Thus,

$$\|u\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)} \le C\|(I-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)}.$$

To conclude the lemma, it is sufficient to show that

$$\|(-\Delta)^s u\|_{L^p(\mathbb{R}^n)} \le C(n,s,\sigma,p)(\|u\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^\sigma u\|_{L^p(\mathbb{R}^n)}), \quad 0 < s < \sigma, \tag{7.20}$$

thanks to Lemma 7.4.2.

In order to prove (7.20) we fix a function $\varphi \in C_c^\infty(B_2)$ such that $\varphi = 1$ on B_1. Then

$$\mathcal{F}((-\Delta)^s u) = |\xi|^{2s}\hat{u} = |\xi|^{2s}\varphi\hat{u} + |\xi|^{2s}(1-\varphi)\hat{u} = m_1(\xi)\hat{u} + m_2(\xi)\mathcal{F}((-\Delta)^\sigma u),$$

where $m_1(\xi) = |\xi|^{2s}\varphi(\xi)$, $m_2(\xi) = |\xi|^{2s-2\sigma}(1-\varphi(\xi))$ are multipliers and we conclude (7.20) by Hormander multiplier theorem. ☐

Lemma 7.4.4 (Embedding to an Orlicz space)**.** *Let Ω be an open set with finite measure. Then for every $u \in \tilde{H}^{\frac{n}{2},2}(\Omega)$*

$$\int_\Omega e^{u^2}dx < \infty.$$

Proof. We set $f = (-\Delta)^{\frac{n}{4}}u$. By [54, Proposition 8] we have

$$u(x) = \int_\Omega G(x,y)f(y)dy, \quad 0 \le G(x,y) \le \frac{C_n}{|x-y|^{\frac{n}{2}}},$$

where G is a Greens function.

We choose $M > 0$ large enough such that $\|\tilde{f}\|_{L^2}C_n < \alpha_0$, where $\tilde{f} = f - f\chi_{\{|f|\le M\}}$. Then

$$|u(x)| \le C(M) + C_n I_{\frac{n}{2}}\tilde{f}(x), \quad I_{\frac{n}{2}}\tilde{f}(x) := \int_\Omega \frac{|\tilde{f}(y)|}{|x-y|^{\frac{n}{2}}}dy,$$

and by [2, Theorem 2] we conclude the proof. □

As a consequence of the above lemma one can prove a higher dimensional generalization of Lions lemma [46] (for a simple proof see e.g. [38, Lemma 2.6]), namely

Lemma 7.4.5 (Lions). *Let u_k be a sequence in $\tilde{H}^{\frac{n}{2},2}(\Omega)$ such that*

$$u_k \rightharpoonup u \text{ in } \tilde{H}^{\frac{n}{2},2}(\Omega), \quad 0 < \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} < 1, \quad \|(-\Delta)^{\frac{n}{4}} u_k\|_{L^2(\mathbb{R}^n)} = 1.$$

Then for every $0 < p < \left(1 - \|(-\Delta)^{\frac{n}{4}} u\|^2_{L^2(\mathbb{R}^n)}\right)^{-1}$, the sequence $\{e^{\alpha_0 p u_k}\}_1^\infty$ is bounded in $L^1(\Omega)$.

Lemma 7.4.6 (Poincaré inequality). *Let Ω be an open set with finite measure. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \le C\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}, \text{ for every } u \in \tilde{H}^{s,2}(\Omega).$$

Proof. We have

$$|\hat{u}(\xi)| \le \frac{1}{(2\pi)^{\frac{n}{2}}}\|u\|_{L^1(\Omega)} \le \frac{1}{(2\pi)^{\frac{n}{2}}}|\Omega|^{\frac{1}{2}}\|u\|_{L^2(\Omega)},$$

and hence

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} = \int_{\mathbb{R}^n} |\hat{u}|^2 d\xi &= \int_{|\xi|<\delta} |\hat{u}|^2 d\xi + \int_{|\xi|\ge\delta} |\hat{u}|^2 d\xi \\
&\le \frac{1}{(2\pi)^n}|\Omega|\|u\|^2_{L^2(\Omega)}|B_1|\delta^n + \delta^{-2s}\int_{|\xi|\ge\delta} |\xi|^{2s}|\hat{u}|^2 d\xi \\
&\le \frac{1}{(2\pi)^n}|\Omega||B_1|\delta^n\|u\|^2_{L^2(\Omega)} + \delta^{-2s}\int_{\mathbb{R}^n} |\mathcal{F}((-\Delta)^{\frac{s}{2}}u)(\xi)|^2 d\xi.
\end{aligned}$$

Choosing $\delta > 0$ so that $\frac{1}{(2\pi)^n}|\Omega||B_1|\delta^n = \frac{1}{2}$ we complete the proof. □

Lemma 7.4.7 (Compact embedding). *Let Ω be an open set in $\mathbb{R}^n$ with finite measure. Then the embedding $\tilde{H}^{s,2}(\Omega) \hookrightarrow \tilde{H}^{r,2}(\Omega)$ is compact for any $0 \le r < s$ (with the notation $\tilde{H}^{0,2}(\Omega) = L^2(\Omega)$). Moreover, $\tilde{H}^{\frac{n}{2},2}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $p \in [1,\infty)$.*

Proof. We prove the lemma in few steps.

Step 1 The embedding $\tilde{H}^{s,2}(\Omega) \hookrightarrow \tilde{H}^{r,2}(\Omega)$ is continuous for any $0 \le r < s$.

With the notation $\Delta^0 u = u$ we see that

$$\begin{aligned}
\|(-\Delta)^{\frac{r}{2}} u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\xi|^{2r}|\hat{u}|^2 d\xi = \int_{|\xi|\le 1} |\xi|^{2r}|\hat{u}|^2 d\xi + \int_{|\xi|>1} |\xi|^{2r}|\hat{u}|^2 d\xi \\
&\le \int_{|\xi|\le 1} |\hat{u}|^2 d\xi + \int_{|\xi|>1} |\xi|^{2s}|\hat{u}|^2 d\xi \le \|u\|^2_{L^2(\Omega)} + \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

which is Step 1, thanks to Lemma 7.4.6

Step 2 For a given $s > 0$ and a given $\varepsilon > 0$ there exists $R > 0$ such that

$$\|u\|_{L^2(\Omega\cap B_R^c)} \le \varepsilon\|u\|_{\tilde{H}^{s,2}(\Omega)}, \quad \text{for every } u \in \tilde{H}^{s,2}(\Omega).$$

To prove Step 2 it is sufficient to consider $0 < s < 1$, thanks to Step 1.

We fix $\varphi \in C_c^\infty(B_2)$ such that $\varphi = 1$ on B_1 and $0 \le \varphi \le 1$. Setting $\varphi_r(x) = \varphi(\frac{x}{r})$ we get

$$\begin{aligned}
&\|(1-\varphi_r)u\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}((1-\varphi_r)u)\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{|\xi|<R_1} |\mathcal{F}((1-\varphi_r)u)|^2 d\xi + \int_{|\xi|\ge R_1} |\mathcal{F}((1-\varphi_r)u)|^2 d\xi \\
&\le \frac{1}{(2\pi)^n}|B_{R_1}| \left(\int_{\mathbb{R}^n} |(1-\varphi_r)u| dx\right)^2 + R_1^{-2s} \int_{|\xi|\ge R_1} |\xi|^{2s} |\mathcal{F}((1-\varphi_r)u)|^2 d\xi \\
&=: I_1 + I_2.
\end{aligned}$$

Using that $supp\,(1-\varphi_r)u \subset \Omega \cap B_r^c$ and by Hölder inequality we bound

$$I_1 \le \frac{1}{(2\pi)^n}|B_{R_1}||\Omega\cap B_r^c| \int_{\Omega\cap B_r^c} |(1-\varphi_r)u|^2 dx \le \frac{1}{(2\pi)^n}|B_{R_1}||\Omega\cap B_r^c|\|u\|^2_{L^2(\Omega)}.$$

From [22, Proposition 3.4] we have

$$\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{u}|^2 d\xi = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy,$$

and hence

$$\begin{aligned}
I_2 &\le R_1^{-2s} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}((1-\varphi_r)u)|^2 d\xi \\
&= C_0 R_1^{-2s} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{((1-\varphi_r(x))u(x) - (1-\varphi_r(y))u(y))^2}{|x-y|^{n+2s}} dxdy \\
&= C_0 R_1^{-2s} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{((1-\varphi_r(x))(u(x)-u(y)) - u(y)(\varphi_r(x)-\varphi_r(y)))^2}{|x-y|^{n+2s}} dxdy \\
&\le 2C_0 R_1^{-2s} \int_{\mathbb{R}^n\times\mathbb{R}^n} \left(\frac{(1-\varphi_r(x))^2(u(x)-u(y))^2}{|x-y|^{n+2s}} + \frac{u^2(y)(\varphi_r(x)-\varphi_r(y))^2}{|x-y|^{n+2s}} \right) dxdy \\
&\le 2C_0 R_1^{-2s} \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dxdy + \int_{\mathbb{R}^n} u^2(y) \int_{\mathbb{R}^n} \frac{(\varphi_r(x)-\varphi_r(y))^2}{|x-y|^{n+2s}} dxdy \right) \\
&\le C_1 R_1^{-2s} (\|(-\Delta)^s u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\Omega)}),
\end{aligned}$$

where in the last inequality we have used that

$$\int_{\mathbb{R}^n} \frac{(\varphi_r(x)-\varphi_r(y))^2}{|x-y|^{n+2s}} dx \le C, \quad y\in\mathbb{R}^n,\ r\ge 1.$$

Thus we have Step 2 by choosing R so that $|B_{R_1}||\Omega\cap B_R^c| < \frac{\varepsilon}{2}$ where $C_1 R_1^{-2s} = \frac{\varepsilon}{2}$.

Step 3 The embedding $\tilde{H}^{s,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact for any $0 < s < 1$.

Let us consider a bounded sequence $\{u_k\}_{k=1}^\infty$ in $\tilde{H}^{s,2}(\Omega)$. Let φ, φ_ℓ be as in Step 2 (here $\ell \in \mathbb{N}$). Then for a fixed ℓ the sequence $\{\varphi_\ell u_k\}_{k=1}^\infty$ is bounded in $\tilde{H}^{s,2}(\Omega)$ (the proof is very similar to the estimate of I_2 in Step 2).

Since the embedding $\tilde{H}^{s,2}(B_r) \hookrightarrow L^2(B_r)$ is compact (see e.g. [22, Theorem 7.1]), there exists a subsequence $\{u_k^1\}_{k=1}^\infty$ such that $\varphi_1 u_k^1 \to u^1$ in $L^2(B_2)$. Inductively we will have $\varphi_\ell u_k^\ell \to u^\ell$ in $L^2(B_{2\ell})$ where $\{u_k^{\ell+1}\}_{k=1}^\infty$ is a subsequence of $\{u_k^\ell\}_{k=1}^\infty$ for $\ell \ge 1$. Moreover, we have $u^{\ell+1} = u^\ell$ on B_ℓ. Setting $u = \lim_{\ell\to\infty} u^\ell$ it follows that u_k^k converges to u in $L^2(\Omega)$, thanks to Step 2.

Step 4 The embedding $\tilde{H}^{s,2}(\Omega) \hookrightarrow \tilde{H}^{r,2}(\Omega)$ is compact for any $0 \le r < s$.

Since the composition of two compact operators is compact, we can assume that $s-r < 1$.

Let $\{u_k\}_{k=1}^\infty$ be a bounded sequence in $\tilde{H}^{s,2}(\Omega)$. Setting $v_k = (-\Delta)^{\frac{r}{2}} u_k$ we see that $\{v_k\}_{k=1}^\infty$ is a a bounded sequence in $\tilde{H}^{s-r,2}(\Omega)$. Then by Step 3 (up to a subsequence) v_k converges to some v in $L^2(\Omega)$ which is equivalent to saying that (up to a subsequence) u_k converges to some u in $\tilde{H}^{r,2}(\Omega)$.

Finally, compactness of the embedding $\tilde{H}^{\frac{n}{2},2}(\Omega) \hookrightarrow L^p(\Omega)$ follows from the compactness of $\tilde{H}^{\frac{n}{2},2}(\Omega) \hookrightarrow L^2(\Omega)$, Theorem J and Lemma 7.4.9. ☐

Lemma 7.4.8 (Exact constant)**.** *We set*

$$f(x) = \log \frac{1}{|x|}, \quad x \in \mathbb{R}^n.$$

Then

$$(-\Delta)^\sigma f(x) = \gamma_n 2^{2\sigma-n} \pi^{-\frac{n}{2}} \frac{\Gamma(\sigma)}{\Gamma(\frac{n-2\sigma}{2})} \frac{1}{|x|^{2\sigma}}, \quad 0 < \sigma < \frac{n}{2},$$

where Γ *is the gamma function and* $\gamma_n = \frac{(n-1)!}{2}|S^n|$.

Proof. From Lemma 2.5.4 we have

$$(-\Delta)^\sigma f(x) = (-\Delta)^\sigma f(e_1) \frac{1}{|x|^{2\sigma}}.$$

To compute the value of $(-\Delta)^\sigma f(e_1)$ we use the fact that $\frac{1}{\gamma_n} \log \frac{1}{|x|}$ is a fundamental solution of $(-\Delta)^{\frac{n}{2}}$ (see for instance Lemma 2.5.1), that is

$$\int_{\mathbb{R}^n} \log \frac{1}{|x|} (-\Delta)^{\frac{n}{2}} \varphi(x) dx = \gamma_n \varphi(0), \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Using integration by parts, which can be verified, we obtain

$$\begin{aligned}
\gamma_n \varphi(0) &= \int_{\mathbb{R}^n} f(x) (-\Delta)^{\frac{n}{2}} \varphi(x) dx \\
&= \int_{\mathbb{R}^n} (-\Delta)^\sigma f(x) (-\Delta)^{\frac{n}{2}-\sigma} \varphi(x) dx \\
&= \int_{\mathbb{R}^n} \frac{(-\Delta)^\sigma f(e_1)}{|x|^{2\sigma}} \left(|\xi|^{n-2\sigma} \widehat{\varphi}\right)^\vee (x) dx \\
&= (-\Delta)^\sigma f(e_1) \int_{\mathbb{R}^n} \left(\frac{1}{|x|^{2\sigma}}\right)^\vee (\xi) \left(|\xi|^{n-2\sigma} \widehat{\varphi}\right) d\xi \\
&= (-\Delta)^\sigma f(e_1) 2^{n-2\sigma-\frac{n}{2}} \frac{\Gamma(\frac{n-2\sigma}{2})}{\Gamma(\frac{2\sigma}{2})} \int_{\mathbb{R}^n} \frac{1}{|\xi|^{n-2\sigma}} \left(|\xi|^{n-2\sigma} \widehat{\varphi}\right) d\xi \\
&= (-\Delta)^\sigma f(e_1) 2^{n-2\sigma-\frac{n}{2}} \frac{\Gamma(\frac{n-2\sigma}{2})}{\Gamma(\frac{2\sigma}{2})} (2\pi)^{\frac{n}{2}} \varphi(0),
\end{aligned}$$

where in the 4th equality we have used that

$$\mathcal{F}\left(\frac{1}{|x|^{n-\alpha}}\right) = 2^{\alpha-\frac{n}{2}}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\frac{1}{|x|^{\alpha}}, \quad 0<\alpha<n, \tag{7.21}$$

in the sense of tempered distribution. Since in our case $\mathcal{F}$ is the normalized Fourier transform, the constant in the right hand side of (7.21) appears slightly different from [44, Section 5.9].

Hence we have the lemma. □

The following lemma is the Vitali's convergence theorem.

Lemma 7.4.9 (Vitali's convergence theorem)**.** *Let Ω be a measure space with finite measure μ, that is, $\mu(\Omega)<\infty$. Let f_k be a sequence of measurable function on Ω be such that*

i) $f_k \xrightarrow{k\to\infty} f$ *almost everywhere in* Ω.

ii) For $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$\int_{\tilde{\Omega}}|f_k|d\mu<\varepsilon \quad \text{for every } \tilde{\Omega}\subset\Omega \text{ with } \mu(\tilde{\Omega})<\delta.$$

Or,

ii′) There exists $p>1$ *such that*

$$\sup_{k\in\mathbb{N}}\int_{\Omega}|f_k|^p d\mu<\infty.$$

Then $f_k\to f$ *in* $L^1(\Omega)$.

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