# Blowup, Solitary Waves and Scattering for the Fractional Nonlinear Schrödinger Equation

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# Abstract

In this thesis we are concerned with the rigorous analysis for an evolution problem arising in mathematical physics: the nonlinear Schrödinger equation with power-type nonlinearity involving the fractional Laplace operator (fractional NLS). We are particularly interested in the long-time dynamics of this nonlocal equation, and study three basic problems of fundamental importance.

First, we shall deduce sufficient criteria for blowup of radial solutions of the focusing problem in the mass-supercritical and mass-critical cases. The conditions are given in terms of inequalities between a combination of the (kinetic) energy and mass of the initial datum, and that of the ground state for the corresponding elliptic equation. Using a new method to deal with the nonlocality of the fractional Laplacian, a localized virial argument enables us to conclude blowup in finite and infinite time, respectively.

Second, we consider a special class of nondispersive solutions of the focusing fractional NLS: the traveling solitary waves. Introducing an appropriate variational problem, we establish the existence of their stationary profiles (boosted ground states). In order to deal with the lack of compactness, we use the technique of compactness modulo translations adapted to the fractional Sobolev spaces. In the case of algebraic (even integer-order) nonlinearities, we derive symmetries of boosted ground states with respect to the boost direction, relying on symmetric decreasing rearrangements in Fourier space. Moreover, we show a non-optimal spatial decay of these profiles at infinity.

Third and finally, we concentrate on the asymptotics of global solutions of the defocusing problem. To have a full range of Strichartz estimates available, we restrict to the radially symmetric case. We construct the wave operator on the radial subclass of the energy space, and show asymptotic completeness. Thus we infer that any radial solution scatters to a linear solution in infinite time. Similarly to the blowup theory, this is done in the spirit of monotonicity formulae: taking a suitable virial weight and using the favourable sign of the defocusing nonlinearity, we develop a lower bound for the Morawetz action. The resulting decay estimates permit us to build a satisfactory scattering theory in the radial case.

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# 1 Introduction

## 1.1 Motivation

### Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation (NLS) with focusing power-type nonlinearity

$$\begin{cases} i\partial_t u = -\Delta u - |u|^{2\sigma} u \\ u : [0,T) \times \mathbb{R}^n \to \mathbb{C}, \quad u = u(t,x), \end{cases} \tag{NLS}$$

arises as a fundamental model in many-body quantum mechanics, as well as in nonlinear optics. In the context of nonlinear optics, one is concerned with the propagation of highly intense laser beams, the electric fields of which are assumed to be linearly polarized. Under suitable approximations and assumptions on the nature of the medium, (NLS) can then be derived rigorously from Maxwell's equations of electrodynamics. From the point of view of nonlinear optics, (NLS) thus represents the leading-order model for the propagation of intense linearly polarized continuous-wave laser beams in a homogeneous Kerr medium, and its solution is the slowly-varying amplitude of the electric field of the beam [Fib15]. Since Maxwell's equations are classical, (NLS) does not have an immediate quantum-mechanical interpretation in this respect, as one might expect upon hearing the name *Schrödinger*.

Assuming the laser beam to be propagating in the $z$-direction say, and polarized in the transversal $(x, y)$-plane (say, in $x$-direction), the nonlinear optics model also leads to the standard *physical* case of plane dimension $n = 2$ and cubic nonlinearity $\sigma = 1$, and hence the $L^2(\mathbb{R}^2)$-critical (NLS). One then often writes $z$ instead of the time variable $t$, and $-\Delta$ is the Laplacian in the transversal $(x, y)$-plane.

The monographs [SS99] and [Fib15] provide excellent references to NLS in this research area. Let us also mention that [Caz03] covers all aspects of the known NLS theory in great mathematical detail.

One is particularly interested in conditions for solutions $u$ of (NLS) to become in some sense singular in finite time $t$ (at finite distance $z$), a phenomenon called optical collapse in the language of optics (blowup in PDE language). It turns out that a crucial role for the analysis is played by so-called ground states for (NLS). These are functions $Q$ which arise as minimizers of a certain Weinstein functional, and satisfy the equation

$$-\Delta Q + Q - |Q|^{2\sigma} Q = 0 \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

In another light, considering the standing wave (stationary state/localized solution) ansatz $u(t,x) = e^{it}Q(x)$ for a solution to (NLS), the previous equation reappears, this time as a condition for the profile $Q$ of the solitary wave solution $u$. Due to a deep theorem of Gidas, Ni and Nirenberg [GNN81], it is known that any positive decaying $C^2$ solution of (1.1) is necessarily radially symmetric about some point $x_0 \in \mathbb{R}^n$, and monotonically decreasing in $r = |x - x_0|$. Moreover, a positive radial and decaying solution $Q$ to (1.1) is necessarily unique due to a result by Kwong [Kwo89]. The radial symmetry of $Q$ turns (1.1) into the ordinary differential equation

$$-Q'' - \frac{(n-1)}{r}Q' + Q - Q^{2\sigma+1} = 0.$$

This justifies that in a certain terminology one can speak of *the* ground state of (NLS). [See also [Caz03, proof of Theorem 8.1.4, p. 266] and [Rap13, p. 272] for precise statements about the uniqueness of the ground state.]

The decisive role of the ground state for the blowup analysis is illustrated for instance with the $L^2(\mathbb{R}^n)$-critical (NLS), where $\sigma = \frac{2}{n}$. In fact, Weinstein [Wei83] proved the sharp criterion that its solutions extend globally in time if the initial data has mass below the mass of the ground state $Q$, while on the other hand blowup can occur as soon as $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$ (indeed, minimal mass blowup solutions can be constructed by the lens transform or pseudo-conformal symmetry).

## Fractional Nonlinear Schrödinger Equation

The present thesis is concerned with analytic investigations on the *nonlocal version* of (NLS), called fractional nonlinear Schrödinger equation (fNLS). Replacing the Laplace operator $-\Delta$ by the fractional Laplacian $(-\Delta)^s$ leads us to the initial-value problem

$$\begin{cases} i\partial_t u = (-\Delta)^s u \pm |u|^{2\sigma} u \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^n), \quad u : [0,T) \times \mathbb{R}^n \to \mathbb{C}. \end{cases} \tag{fNLS}$$

Here $n \geqslant 1$ is the space dimension, $s \in (0,1)$ is a fractional parameter, $\sigma > 0$ determines the strength of the power-nonlinearity, and $u_0$ is a given initial datum lying in an appropriate function space like $H^s(\mathbb{R}^n)$. (fNLS) is called focusing (attractive) or defocusing (repulsive), depending on whether the minus or plus sign appears in front of the nonlinearity above, respectively. The well-posedness of (fNLS) has been analyzed by Hong and Sire [HS15b].

We will use the fractional Sobolev spaces [Caz03, p. 13]

$$H^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n); \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}(\mathcal{F}u)] \in L^p(\mathbb{R}^n) \right\}.$$

In the Hilbert space case $p = 2$, we write $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$. We will mostly work with the fractional Laplacian $(-\Delta)^s$ given as a Fourier multiplier

$$(-\Delta)^s u = \mathcal{F}^{-1}[|\xi|^{2s}(\mathcal{F}u)].$$

We refer to [DNPV12] for an exposition of various, partly interconnected, definitions of the fractional Sobolev spaces and the fractional Laplacian and their relations to each other.

The Cauchy problem (fNLS) arises for instance as an effective equation in the continuum limit of discrete models with long-range interactions. In [KLS13] Kirkpatrick, Lenzmann and Staffilani refer to models of mathematical biology, specifically for the charge transport in biopolymers like the DNA. The DNA strand is modeled by a one-dimensional lattice $h\mathbb{Z}$ of mesh size $h > 0$, the base pairs sitting at the lattice points $x_m = hm$ with $m \in \mathbb{Z}$. One then considers a discrete wave function $u_h : \mathbb{R} \times h\mathbb{Z} \to \mathbb{C}$ satisfying

$$i\frac{\mathrm{d}}{\mathrm{d}t}u_h(t,x_m) = h\sum_{n\neq m}\frac{u_h(t,x_m)-u_h(t,x_n)}{|x_m-x_n|^{1+2s}} \pm |u_h(t,x_m)|^2 u_h(t,x_m).$$

The complex twisting of DNA in 3 dimensions is a plausible reason for the base pairs interacting with all the others. This is accounted for by the sum above with the kernel decaying like an inverse power of the distance between the pairs, reminiscent of the singular integral given by the fractional Laplacian. The second term on the right side represents a cubic self-interaction of the base pair. The authors show rigorously that, as the mesh size $h > 0$ of the lattice shrinks to zero, the solution of the discrete equation tends to a solution of (fNLS) in a weak sense provided that the decay of the kernel above is not too strong (otherwise, the long-range interactions cannot survive in the limit, but rather the usual (NLS) accounting for short-range interactions appears).

Numerous applications of fractional NLS-type equations in the physical sciences could be mentioned, ranging from the description of Boson stars [FJL07] to water wave dynamics. The fractional Laplacian also appears as a natural operator when considering jump processes [Val09], which makes it valuable for Lévy processes in probability theory with applications in financial mathematics.

As in the case $s = 1$, also for $s \in (0, 1)$ one is interested in the notion of ground states for (fNLS). They arise as minimizers $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ of the Weinstein functional [FLS16]

$$\mathcal{J}(Q) = \frac{\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^{\frac{n\sigma}{s}}\|Q\|_{L^2}^{2\sigma+2-\frac{n\sigma}{s}}}{\|Q\|_{L^{2\sigma+2}}^{2\sigma+2}}$$

and equivalently satisfy with equality the corresponding Gagliardo-Nirenberg-Sobolev inequality

$$\|Q\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{\mathrm{opt}}\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^{\frac{n\sigma}{s}}\|Q\|_{L^2}^{2\sigma+2-\frac{n\sigma}{s}}.$$

Up to a suitable rescaling $Q \to \alpha Q(\beta\cdot)$, $Q$ can be shown to satisfy the equation

$$(-\Delta)^s Q + Q - |Q|^{2\sigma}Q = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Minimizers for $\mathcal{J}$ can be constructed by Lions' concentration-compactness method [Lio84]. In [FLS16] Frank, Lenzmann and Silvestre show that there exists a non-negative minimizer $Q \geqslant 0$, $Q \not\equiv 0$ for $\mathcal{J}$, which solves (1.2). Furthermore, they show that any solution $Q \in H^s(\mathbb{R}^n)$ with $Q \geqslant 0$ and $Q \not\equiv 0$ is necessarily radial about some point $x_0 \in \mathbb{R}^n$, i.e. $Q(\cdot - x_0)$ is radial, and moreover positive, strictly decreasing in $|x - x_0|$ and smooth. They completely resolve the question of uniqueness of such positive radial solutions and give a positive answer: the ground state is unique [FLS16, Theorem 3.4]. Frank and Lenzmann had previously shown the uniqueness in $n = 1$ dimension in their paper [FL13].

Similarly as before, it is known that for the $L^2(\mathbb{R}^n)$-critical focusing (fNLS), where $\sigma = \frac{2s}{n}$, one has global existence in $H^s(\mathbb{R}^n)$ provided one has mass below the ground state $\|u_0\|_{L^2} < \|Q\|_{L^2}$ [HS15b, Theorem B.1]. Global existence in turn gives rise to the question of the asymptotic behaviour of the solution.

The analysis of (fNLS) constantly makes use of the following two quantities which are conserved in time $t$.

$$M[u(t)] = \|u(t)\|_{L^2}^2 \qquad (L^2\text{-mass})$$

$$E[u(t)] = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \pm \frac{1}{2\sigma+2}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \qquad \text{(energy)}$$

Fundamental questions for (fNLS), which shall be answered in this work, are the following:

- Can we identify assumptions under which solutions to the focusing problem blow up in finite or infinite time?
- Can we prove the existence of traveling solitary wave solutions of the form $u(t,x) = e^{i\omega t}Q_v(x - vt)$ in the focusing case? Do the associated *boosted ground states* $Q_v$ possess certain symmetry, regularity and decay properties? Here $v \in \mathbb{R}^n$ is a velocity, $\omega \in \mathbb{R}$ a phase parameter.
- Do solutions to the defocusing problem exhibit an asymptotically free behaviour, i.e., do they scatter to solutions of the linear fractional Schrödinger equation as $t \to \infty$?

We summarize our main answers to these questions in the subsequent sections.

## 1.2 Blowup Result

Concerning the existence of blowup solutions for (NLS), one of the early approaches is due to Glassey [Gla77], and proves the blowing up of negative energy solutions to (NLS) in $H^1(\mathbb{R}^n)$ norm without any symmetry assumption, but under the technical assumption that the solution $u$ be of finite variance $\int_{\mathbb{R}^n} |x|^2|u(t,x)|^2 \, dx < \infty$.

Dropping the negative energy assumption, sharper conditions were found by Kutznetsov et al. [KRRT95] in supercritical cases $\sigma > \frac{2}{n}$, namely in terms of inequalities between the kinetic, respectively full energies of the given initial datum $u_0$ and the ground state $Q_N$ having the same mass $\|u_0\|_{L^2}^2 = N$ as the initial datum. However, the finite variance hypothesis remained.

In any case, these results heavily rely on the variance identity discovered by Vlasov, Petrishchev and Talanov which is not available when working with the nonlocal operator $(-\Delta)^s$, as will be pointed out in Chapter 2.

On the other hand, Ogawa and Tsutsumi [OT91] generalized Glassey's results to solutions with possibly infinite variance, but at the cost of assuming radially symmetric initial data in return. It is their strategy of so-called localized virial arguments that enables us to identify sufficient blowup conditions for (fNLS) in the $L^2(\mathbb{R}^n)$-critical and supercritical cases $\sigma \geqslant \frac{2s}{n}$. In Chapter 2 we will prove the following blowup result. (We express the mass (super-)criticality by the condition $s_c \geqslant 0$ with the scaling index $s_c = \frac{n}{2} - \frac{s}{\sigma}$.)

**Theorem** (Blowup for (super-)critical focusing fNLS with radial data [BHL16])**.** *Let $n \geqslant 2$, $s \in \left(\frac{1}{2}, 1\right)$, $0 \leqslant s_c \leqslant s$ with $\sigma < 2s$. Assume that $u \in C([0,T); H^{2s}(\mathbb{R}^n))$ is a radial solution of the focusing (fNLS). Furthermore, we suppose that either*

$$E[u_0] < 0$$

*or, if $E[u_0] \geqslant 0$, we assume that*

$$\begin{cases} E[u_0]^{s_c} M[u_0]^{s-s_c} < E[Q]^{s_c} M[Q]^{s-s_c}, \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{s-s_c} > \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s-s_c}. \end{cases}$$

*Then the following conclusions hold.*

*(i) **$L^2$-Supercritical Case:** If $0 < s_c \leqslant s$, then $u(t)$ blows up in finite time in the sense that $T < +\infty$ must hold.*

*(ii) **$L^2$-Critical Case:** If $s_c = 0$, then $u(t)$ either blows up in finite time in the sense that $T < +\infty$ must hold, or $u(t)$ blows up in infinite time such that*

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \geqslant C t^s, \quad \text{for all } t \geqslant t_*,$$

*with some constants $C > 0$ and $t_* > 0$ that depend only on $u_0$, $s$ and $n$.*

Note that $T < +\infty$ does not imply that $\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \to +\infty$ as $t \uparrow T$, as we do not necessarily have a blowup alternative. Rather, the solution ceases to be in $H^{2s}(\mathbb{R}^n)$, since the proof shows that the virial quantity does not exist indefinitely in time.

To prove this theorem, we will introduce a suitable radial cutoff function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and study the time evolution of the localized virial quantity

$$\mathcal{M}_\varphi[u(t)] = \int_{\mathbb{R}^n} \overline{u(t)} \nabla\varphi \cdot \nabla u(t) \, \mathrm{d}x.$$

However, since $(-\Delta)^s$ changed the character of the equation to a nonlocal one, we cannot immediately adapt the estimates known in the local case. To deal with the nonlocality, we will apply *Balakrishnan's formula* known from semigroup theory, which is the representation formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m}\, \mathrm{d}m.$$

This will enable us to proceed in the spirit of Ogawa and Tsutsumi. We refer to Chapter 2 for the proof.

## 1.3 Boosted Ground States and Traveling Solitary Waves Results

Making the traveling solitary wave ansatz $u(t,x) = e^{i\omega t} Q_v(x - vt)$ for solutions to focusing (fNLS), it is necessary and sufficient that the profile $Q_v$ is governed by the equation

$$(-\Delta)^s Q_v + iv \cdot \nabla Q_v + \omega Q_v - |Q_v|^{2\sigma} Q_v = 0. \tag{1.3}$$

This equation is intimately connected with interpolation estimates of Gagliardo-Nirenberg type, whose best constant can be expressed by the optimizers of an associated functional. We therefore introduce a variational approach and solve a minimization problem, based on the following Weinstein functional $\mathcal{J}^s_{v,\omega} : H^s(\mathbb{R}^n)\backslash\{0\} \to \mathbb{R}$:

$$\mathcal{J}^s_{v,\omega}(f) := \frac{(\langle f, \mathcal{T}_{s,v} f\rangle + \omega \langle f, f\rangle)^{\sigma+1}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}}}. \tag{1.4}$$

Here, $\mathcal{T}_{s,v}$ is the pseudo-differential operator $\mathcal{T}_{s,v} := (-\Delta)^s + iv \cdot \nabla$. Minimizers $Q_v$ of $\mathcal{J}^s_{v,\omega}$ on the class $H^s(\mathbb{R}^n)\backslash\{0\}$ then render equation (1.3) as their Euler-Lagrange equation (up to a suitable rescaling which leaves the functional $\mathcal{J}^s_{v,\omega}$ invariant). We call them boosted ground states in view of the appearing boost velocity $v \in \mathbb{R}^n$. In Chapter 3, we prove the following existence theorem.

**Theorem** (Existence of traveling solitary wave solutions)**.** *Let $n \geqslant 1$ and $s \in [\frac{1}{2}, 1)$. Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < 1$ for $s = \frac{1}{2}$. Then there exists a number $\omega_* \in \mathbb{R}$ such that the following holds. For any $\omega > \omega_*$, there exists a profile $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ such that*

$$\mathcal{J}^s_{v,\omega}(Q_v) = \inf_{f \in H^s(\mathbb{R}^n)\backslash\{0\}} \mathcal{J}^s_{v,\omega}(f).$$

*More generally: any minimizing sequence is relatively compact in $H^s(\mathbb{R}^n)$ up to translations. Furthermore, modulo rescaling $Q_v \to \alpha Q_v$, $Q_v$ solves equation (1.3) and thus gives rise to the traveling solitary wave solution*

$$u(t,x) = e^{i\omega t} Q_v(x - vt)$$

*of focusing* (fNLS).

We prove this theorem by adapting a compactness modulo translations argument due to Frank, Bellazzini and Visciglia [BFV14]. These authors generalized a fundamental compactness lemma due to Lieb to the fractional Sobolev spaces, which states that under appropriate assumptions a minimizing sequence admits a subsequence converging weakly to a nonzero limit after suitable translations. See also [Len06] for an adaption of Lions' concentration-compactness method [Lio84] to a constrained variational problem in order to construct boosted ground states for the Boson star equation (which corresponds to the half-wave case $\sqrt{-\Delta}$ on $\mathbb{R}^3$ with a convolution nonlinearity incorporating a Newtonian gravitational potential).

The existence theorem above will be formulated for a general pseudo-differential operator $L$, which is self-adjoint on $L^2(\mathbb{R}^n)$ with dense domain and satisfies appropriate growth conditions. The fractional Laplacian $L = (-\Delta)^s$ with $s \in (0,1)$ is then a particular instance; see Chapter 3 for more information.

We are interested in deriving symmetries for boosted ground states $Q_v$. Boulenger and Lenzmann [BL15] have recently introduced a technique to prove the existence of radially symmetric ground states for biharmonic NLS, which was previously unknown. Their method uses Schwarz rearrangement $*$ on the Fourier side, defining the symmetrization $Q^\sharp = \mathcal{F}^{-1}((\mathcal{F}Q)^*)$.

In our case, we might expect the existence of boosted ground states exhibiting symmetries with respect to the boost velocity $v \in \mathbb{R}^n$. Assuming $v \in \mathbb{R}^n$ to point into 1-direction, we define the symmetrization $Q_v^{\sharp_1} = \mathcal{F}^{-1}((\mathcal{F}Q_v)^{*_1})$, where $*_1$ is the symmetric decreasing rearrangement of a function in the last $n-1$ variables, keeping the first variable fixed. For $n=1$, we use the symmetrization $\widetilde{Q_v} = \mathcal{F}^{-1}(|\mathcal{F}Q_v|)$. Then indeed we obtain the following result.

**Theorem** (Existence of symmetric boosted ground states for integer-powers)**.** *Let $n \geqslant 1$, and $s \in [\frac{1}{2}, 1)$. Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < 1$ for $s = \frac{1}{2}$. Suppose that $\sigma \in (0, \sigma_*)$ is an integer. Then:*

*(i)* ***Case** $n \geqslant 2$**:** There exists a cylindrically symmetric minimizer of the Weinstein functional (1.4), i.e., there exists a boosted ground state $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ such that $Q_v = Q_v^{\sharp_1}$. In addition, $Q_v = Q_v^{\sharp_1}$ is continuous and bounded and has the higher Sobolev regularity $Q_v^{\sharp_1} \in H^k(\mathbb{R}^n)$ for all $k > 0$. In particular, $Q_v^{\sharp_1} \in C^\infty(\mathbb{R}^n)$ is smooth. Moreover, the functions $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Re} Q_v^{\sharp_1}(x_1, x')$ and $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Im} Q_v^{\sharp_1}(x_1, x')$ are even and odd, respectively, for any fixed $x' \in \mathbb{R}^{n-1}$.*

*(ii)* ***Case** $n = 1$**:** There exists a minimizer of the Weinstein functional (1.4), i.e. a boosted ground state $Q_v \in H^s(\mathbb{R})\backslash\{0\}$ such that $Q_v = \widetilde{Q_v}$. In addition, $Q_v = \widetilde{Q_v}$ is continuous and bounded and has the higher Sobolev regularity $\widetilde{Q_v} \in H^k(\mathbb{R})$ for all $k > 0$. In particular, $\widetilde{Q_v} \in C^\infty(\mathbb{R})$ is smooth. Moreover, the functions $\mathbb{R} \to \mathbb{R}$, $x \mapsto \operatorname{Re}\widetilde{Q_v}(x)$ and $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Im}\widetilde{Q_v}(x)$ are even and odd, respectively.*

The crucial hypothesis here is the algebraic assumption that $\sigma > 0$ is an integer.

This allows us to use the Convolution Theorem in conjunction with the Brascamp-Lieb-Luttinger Inequality for the symmetric decreasing rearrangement in order to deduce that the Weinstein functional cannot increase under symmetrization. The proof is then based on an iterative argument to guarantee that $\mathcal{F}Q_v$ lies in $L^1(\mathbb{R}^n)$; again see Chapter 3 for more information.

Finally, we establish algebraic decay at infinity for solutions to the Euler-Lagrange equation (1.3), which reads as follows.

**Theorem** (Decay of boosted ground states for the fractional Laplacian)**.** *Let $n \geqslant 1$ and $s \in [\frac{1}{2}, 1)$. Let $v \in \mathbb{R}^n$ be arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Let $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ be a solution of the Euler-Lagrange equation (1.3). Then $Q_v$ is continuous on $\mathbb{R}^n$, and there exists some constant $C > 0$ such that the following polynomial decay estimate holds:*

$$|Q_v(x)| \leqslant \frac{C}{1 + |x|^{n+1}}, \quad \textit{for all } x \in \mathbb{R}^n. \tag{1.5}$$

*In particular, any boosted ground state $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ decays polynomially according to (1.5).*

The proof rests on the Slaggie-Wichman method [His00], used to establish decay of eigenfunctions of Schrödinger operators, and then proceeds by a bootstrap argument; see Chapter 3.

## 1.4 Scattering Result

Concerning the behaviour of global solutions to (fNLS) as $t \to \pm\infty$, one is interested in situations in which nonlinear effects become asymptotically negligible. We provide a result concerning the long-time dynamics of solutions of defocusing (fNLS). To simplify the exposition, we will focus on the case of cubic nonlinearity $\sigma = 1$ in dimension $n = 3$. More specifically, we construct the so-called wave-operator $\Omega_+$ on the radial subclass $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$, and prove that it maps this class bijectively to itself. Thus all radial solutions in the defocusing case scatter to a solution of the linear equation, i.e., these solutions exhibit an asymptotically free behaviour. The sense of "free" depends on the topology in which we measure the approximation to the solution of the linear equation, and according to that choice one is led to different scattering theories [Caz03]. Here we focus on the scattering theory on the radial subclass of the energy space $H^s_x(\mathbb{R}^n)$. In Chapter 4, we will show the following result.

**Theorem** (Radial scattering and asymptotic completeness for defocusing fNLS)**.** *Let $n = 3$ and $\sigma = 1$. Let $s \in [s_0, 1)$, where $s_0 = \frac{1}{4}(7 - \sqrt{13}) \approx 0.849$. Then for every $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a unique $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that the global solution $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ of defocusing (fNLS) with initial value $u_0$ satisfies*

$$\|u(t) - e^{-it(-\Delta)^s}u_+\|_{H^s} = \|e^{it(-\Delta)^s}u(t) - u_+\|_{H^s} \to 0, \quad \textit{as } t \to +\infty. \tag{1.6}$$

*Consequently, there exists an operator* $\Omega_+ : H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$, $u_+ \mapsto u_0$. *Furthermore, the operator* $\Omega_+$ *is a continuous bijection* $H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$*; in particular, conversely for every* $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ *there exists a unique* $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ *such that the global solution* $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ *of defocusing* (fNLS) *with initial value* $u_0$ *satisfies* (1.6).

The reason for our hypothesis of radiality is that we want to use a full range of Strichartz estimates without loss of regularity; see also [HS15b]. This was established by Guo and Wang [GW14] in the radial case (Knapp counter-example for non-radial functions), and is valid for the fractional Schrödinger equation under the assumption that the powers of the fractional Laplacian are restricted to $\frac{n}{2n-1} < s \leqslant 1$ with $n \geqslant 2$. For $n = 3$, this means $\frac{3}{5} < s$. On the other hand, the development of the scattering theory requires the exponent $\sigma$ to be neither too small nor too large: for the $L^2(\mathbb{R}^3)$-supercritical exponent $\sigma = 1$ the $H^s(\mathbb{R}^3)$-subcriticality condition $\sigma < \frac{2s}{3-2s}$ means $\frac{3}{4} < s$. Both requirements are thus guaranteed for $\frac{3}{4} < s < 1$. For technical reasons, in order to avoid a continuity argument for the Strichartz norms involved, we make the further restriction $s \geqslant s_0$ above.

Next to the Strichartz estimates, scattering theory relies on decay estimates. The key to the latter is provided by the so-called Morawetz inequality; see [LS78]. To establish a Morawetz inequality for (fNLS), we will work out a monotonicity property for the virial of the solution by resorting to the extension problem related to the fractional Laplacian [CS07]; see Proposition C.6.

As for the focusing problem, we refer to the new scattering results of [GZ17] and [SWYZ17] below the energy-mass threshold and initial mass-fractional-gradient bound of the above blowup theorem in the fractional radial case; see also the results [HR08] and [DHR08] in the radial and non-radial local NLS cases, respectively. A scattering result for a Hartree type fractional NLS can be found in [Cho17]. We refer to Chapter 4 for more information.

## 1.5 Structure of the Thesis

This thesis is structured in three main chapters, each followed by an appendix. In each chapter we provide an introduction into the subject we treat, state the main results and give an outline of the course of the chapter. Many auxiliary results and technical details are given in the appendices.

### Chapter 2

In chapter 2, we prove our radial blowup theorem for focusing (fNLS). Appendix A contains various important estimates, for instance the fractional Strauss inequality, as well as Pohozaev identities for fractional ground states.

## Chapter 3

In chapter 3, we prove our existence theorem of boosted ground states and traveling solitary waves for focusing (fNLS), as well as the existence of symmetric boosted ground states. Moreover, we show properties such as regularity and spatial decay. The results of this chapter are partly formulated for a general pseudo-differential operator $L$ satisfying appropriate conditions; we will emphasize when we restrict ourselves to the special case $L = (-\Delta)^s$. Appendix B contains important theorems we use in this chapter, such as the compactness modulo translations method to solve the variational problem. We also examine the operator $(-\Delta)^s + iv \cdot \nabla$.

## Chapter 4

In chapter 4, we prove our radial scattering theorem for defocusing (fNLS). In Appendix C we collect the Strichartz estimates available in the radial case, introduce the relevant function spaces and prove Morawetz's inequality.

**Abbreviations**

| | |
|---|---|
| $C_c^\infty(\mathbb{R}^n)$ | Space of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ with compact support |
| $\mathscr{S}(\mathbb{R}^n)$ | Schwartz class of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ whose derivatives of arbitrary order are rapidly decreasing |
| $\mathscr{S}'(\mathbb{R}^n)$ | Space of tempered distributions (topological dual to $\mathscr{S}(\mathbb{R}^n)$) |
| $C(X;Y)$ | Space of continuous functions $f : X \to Y$, where $X$ and $Y$ are normed spaces |
| $L^p(\mathbb{R}^n), L^p_x(\mathbb{R}^n)$ | Space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ with $\|f\|_{L^p} < \infty$, equipped with the $p$-norm $$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^n} \lvert u(x)\rvert^p \,\mathrm{d}x\right)^{\frac{1}{p}}, & 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \lvert u(x)\rvert, & p = \infty \end{cases}$$ |
| $\langle f, g\rangle$ | Inner product on $L^2(\mathbb{R}^n)$: $\langle f, g\rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x)\,\mathrm{d}x$ |
| $L^q_t L^r_x(I \times \mathbb{R}^n)$ | Space of measurable functions $f : I \to L^r_x(\mathbb{R}^n)$, where $I \subset \mathbb{R}$ is a time interval, equipped with the mixed space-time norms $$\|f\|_{L^q_t L^r_x} = \Big\| \|f\|_{L^r_x(\mathbb{R}^n)} \Big\|_{L^q_t(I)} = \begin{cases} \left(\int_I \|f(t)\|^q_{L^r_x(\mathbb{R}^n)} \,\mathrm{d}t\right)^{\frac{1}{q}}, & q < \infty, \\ \operatorname{ess\,sup}_{t\in I} \|f(t)\|_{L^r_x(\mathbb{R}^n)}, & q = \infty \end{cases}$$ |
| $\mathcal{F}f, \hat{f}$ | Fourier transform of $f$ with the convention $$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x$$ |
| $\mathcal{F}^{-1}f$ | Inverse Fourier transform of $f$, i.e. $$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi) e^{i\xi\cdot x}\,\mathrm{d}\xi$$ |
| $H^{s,p}(\mathbb{R}^n)$ | Inhomogeneous Sobolev space of distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ with $$\mathcal{F}^{-1}[(1 + \lvert\xi\rvert^2)^{\frac{s}{2}} \mathcal{F}f] \in L^p(\mathbb{R}^n),$$ equipped with the norm $$\|f\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1 + \lvert\xi\rvert^2)^{\frac{s}{2}} \mathcal{F}f]\|_{L^p}$$ |
| $H^s(\mathbb{R}^n)$ | Hilbert space $H^{s,2}(\mathbb{R}^n)$ |

# 2 Blowup for Fractional NLS

## 2.1 Introduction and Main Result

In this chapter, we derive general criteria for blowup of solutions $u = u(t,x)$ to the initial-value problem for the fractional nonlinear Schrödinger Equation (fNLS) with a focusing power-type nonlinearity:

$$\begin{cases} i\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^n), \quad \text{for } x \in \mathbb{R}^n, \quad u : [0,T) \times \mathbb{R}^n \to \mathbb{C}. \end{cases} \tag{fNLS}$$

Here $n \geqslant 1$ denotes the space dimension, the operator $(-\Delta)^s$ stands for the fractional Laplacian with power $s \in (0,1)$, defined by its symbol $|\xi|^{2s}$ in Fourier space, and $\sigma > 0$ is a given exponent such that $\sigma \leqslant \sigma_*$ if $s < \frac{n}{2}$ and $\sigma < \sigma_*$ otherwise, where

$$\sigma_* := \begin{cases} \frac{2s}{n-2s} & \text{if } s < \frac{n}{2}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

The number $\sigma_* = \sigma_*(n,s)$ is called the $H^s$-critical exponent for (fNLS) in $n$ dimensions. The local well-posedness theory[1] for (fNLS) and our range of $s \in (0,1)$, $n \geqslant 1$, and exponents $\sigma > 0$ is not fully understood yet - see, e.g., [GH11], [HS15b], [GW11]; see also [GSWZ13] for well-posedness results in the energy-critical case under the assumption of radiality. For this reason, we shall work with sufficiently regular solutions, namely we will assume that $u \in C([0,T); H^{2s}(\mathbb{R}^n))$.

In analogy to classical NLS (i.e., $s = 1$), we have the formal conservation laws for the *energy* $E[u]$ and the $L^2$-*mass* $M[u]$ given by

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^n} |u|^{2\sigma+2} \, dx, \quad M[u] = \int_{\mathbb{R}^n} |u|^2 \, dx. \tag{2.2}$$

### Scaling Behaviour of (fNLS) and Notions of Criticality

Note that when $u(t,x)$ solves (fNLS), then[2,3] so does the rescaled version

$$u_\lambda(t,x) = \lambda^{\frac{s}{\sigma}} u(\lambda^{2s} t, \lambda x), \quad \lambda > 0 \text{ a dilation factor.}^{4} \tag{2.3}$$

[1] See [Len07] for the case of Hartree-type nonlinearities.

[2] Use the definition $(-\Delta)^s u_\lambda(t,x) = \mathcal{F}^{-1}[|\xi|^{2s} \widehat{u_\lambda(t,\cdot)}](x)$, the behaviour of the Fourier transform under dilations (see, e.g., [LP09, eq. (1.5)] and the change of variable $\eta := \frac{\xi}{\lambda}$.

[3] The initial datum is considered simultaneously transformed, $u_0 \mapsto \lambda^{\frac{s}{\sigma}} u_0(\lambda \cdot)$.

[4] (2.3) says in particular that time has $2s$ times (twice in the NLS casel $s = 1$) the dimensionality of space; see [Tao06, p. 114].

Equation (fNLS) is called $\dot{H}^\gamma(\mathbb{R}^n)$*-critical* if the scaling (2.3) leaves the homogeneous $\dot{H}^\gamma(\mathbb{R}^n)$ Sobolev norm invariant, i.e. (since a calculation shows $\|u_\lambda(t,\cdot)\|^2_{\dot{H}^\gamma} = \lambda^{\frac{2s}{\sigma}-n+2\gamma}\|u(\lambda^{2s}t,\cdot)\|^2_{\dot{H}^\gamma}$), if $\gamma = \frac{n}{2} - \frac{s}{\sigma}$ (see also [KSM14, eq. (1.8)]). Defining therefore the *scaling index*

$$s_c := \frac{n}{2} - \frac{s}{\sigma}, \tag{2.4}$$

we have $\|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}} = \|u(\lambda^{2s}t,\cdot)\|_{\dot{H}^{s_c}}$, that is, (fNLS) is $\dot{H}^{s_c}(\mathbb{R}^n)$ critical. Reflecting the scaling properties of (fNLS) and the conservation of $M[u]$, the cases $s_c < 0$, $s_c = 0$ and $s_c > 0$ are referred to as *mass-subcritical*, *mass-critical* and *mass-supercritical*, respectively. The case $s_c = 0$ corresponds to the exponent $\sigma = \frac{2s}{n}$ (equivalently, to the dispersion rate $s^* = \frac{\sigma n}{2}$).[5]

The second conserved quantity, $E[u]$, gives rise to a second notion of criticality. Namely, the case $s_c = s$ is referred to as *energy-critical*; in this case the kinetic energy $\|(-\Delta)^{\frac{s}{2}}u\|_{L^2} = \|u\|_{\dot{H}^s}$ of the solution is indeed a scale-invariant quantity of the time evolution [KSM14], as seen in the above computation. The energy-critical case corresponds to the endpoint case $\sigma = \sigma_* = \frac{2s}{n-2s}$ in $n > 2s$ dimensions (cf. also [BL15, p. 2]), equivalently, to the dispersion rate $s_* = \frac{\sigma n}{2(\sigma+1)}$. The cases $s_c < s$, $s_c = s$ and $s_c > s$ are called energy-subcritical, energy-critical and energy-supercritical, respectively. Note that the energy-critical index is always smaller than the mass-critical one, $s_* < s^*$.

One can expect blowup for (fNLS) in finite or infinite time only in the mass-supercritical and mass-critical cases $s_c \geqslant 0$. Namely, in the mass-subcritical case $s_c < 0$, one can use the conservation laws for the energy $E[u]$ and mass $M[u]$ together with the sharp fractional Gagliardo-Nirenberg inequality (A.12) to obtain an a-priori bound (depending only on the given parameters $n, s, \sigma$ and the conserved quantities $E[u_0]$ and $M[u_0]$) on the $H^s(\mathbb{R}^n)$ norm of any $H^s(\mathbb{R}^n)$-valued solution $u(t)$:

$$\begin{aligned} E[u_0] = E[u(t)] &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|^2_{L^2} - \frac{1}{2\sigma+2}\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}} \\ &\geqslant \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|^2_{L^2} - \frac{C_{n,s,\sigma}}{2\sigma+2}\|(-\Delta)^{\frac{s}{2}}u(t)\|^{\frac{n\sigma}{s}}_{L^2}\|u(t)\|^{2\sigma+2-\frac{n\sigma}{s}}_{L^2} \\ &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|^2_{L^2} - C(n,s,\sigma,M[u_0])\|(-\Delta)^{\frac{s}{2}}u(t)\|^{\frac{n\sigma}{s}}_{L^2}. \end{aligned} \tag{2.5}$$

This gives

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}u(t)\|^2_{L^2} &\leqslant 2E[u_0] + 2C(n,s,\sigma,M[u_0])\|(-\Delta)^{\frac{s}{2}}u(t)\|^{\frac{n\sigma}{s}}_{L^2} \\ &\leqslant 2E[u_0] + C(n,s,\sigma,M[u_0]) + \frac{1}{p}\|(-\Delta)^{\frac{s}{2}}u(t)\|^2_{L^2}, \end{aligned} \tag{2.6}$$

[5] Classical NLS ($s = 1$) is thus $L^2(\mathbb{R}^n)$-critical for $\sigma = 2/n$; for instance: cubic NLS in 2 dimensions, and quintic NLS in 1 dimension; cf. [KSM14].

by Young's inequality ($ab \leqslant \frac{1}{p'}a^{p'} + \frac{1}{p}b^p$), after defining the number $p = \frac{2}{(\frac{n\sigma}{s})}$, which is greater than 1 due to $s_c < 0$. Now indeed there follows the claimed a-priori bound

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \lesssim_{n,s,\sigma,M[u_0],E[u_0]} 1. \tag{2.7}$$

Hence (the $L^2$ norm is conserved) a local-in-time solution $u \in C([0,T); H^s(\mathbb{R}^n))$ (with some $T \leqslant \infty$) has no chance of blowing up in $H^s(\mathbb{R}^n)$ as $t \uparrow T$.

## Ground States

Furthermore, one expects in analogy to classical NLS theory that sufficient conditions for blowup may be found in terms of quantities of so-called *ground states* $Q \in H^s(\mathbb{R}^n)$. These are optimizers of the Gagliardo-Nirenberg inequality (A.12) (equivalently, minimizers of the associated Weinstein functional; see Appendix A.4 for more details) and satisfy the Euler-Lagrange equation

$$(-\Delta)^s Q + Q - |Q|^{2\sigma} Q = 0 \quad \text{in } \mathbb{R}^n \tag{2.8}$$

in the energy-subcritical case $s_c < s$. In the energy-critical case $s = s_c$ (which requires $n > 2s$), the relevant object $Q \in \dot{H}^s(\mathbb{R}^n)$ is the ground state, which is the optimizer for the Sobolev inequality (A.18), normalized such that it holds

$$(-\Delta)^s Q - Q^{\frac{n+2s}{n-2s}} = 0 \quad \text{in } \mathbb{R}^n. \tag{2.9}$$

Uniqueness (modulo symmetries) of ground states $Q \in H^s(\mathbb{R}^n)$ for (2.8) and all $s_c < s$ and any $n \geqslant 1$ was recently shown in [FL13, FLS16]. On the other hand, uniquenss (modulo symmetries) of ground states $Q \in \dot{H}^s(\mathbb{R}^n)$ for (2.9) is a classical fact due to Lieb [Lie83].

## Our Theorem: Sufficient Blowup Criterion

The above mentioned sufficient blowup criteria in terms of quantities of ground states for fractional NLS indeed exist, as the following main result clarifies.

**Theorem 2.1** (Blowup for (super-)critical focusing fNLS with radial initial data)**.** *Let $n \geqslant 2$, $s \in \left(\frac{1}{2}, 1\right)$, $0 \leqslant s_c \leqslant s$ with $\sigma < 2s$. Assume that $u \in C([0,T); H^{2s}(\mathbb{R}^n))$ is a radial solution of* (fNLS)*. Furthermore, we suppose that either*

$$E[u_0] < 0$$

*or, if $E[u_0] \geqslant 0$, we assume that*

$$\begin{cases} E[u_0]^{s_c} M[u_0]^{s-s_c} < E[Q]^{s_c} M[Q]^{s-s_c}, \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{s-s_c} > \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s-s_c}. \end{cases}$$

*Then the following conclusions hold.*

(i) *$L^2$-**Supercritical Case:** If $0 < s_c \leqslant s$, then $u(t)$ blows up in finite time in the sense that $T < +\infty$ must hold.*

(ii) *$L^2$-**Critical Case:** If $s_c = 0$, then $u(t)$ either blows up in finite time in the sense that $T < +\infty$ must hold, or $u(t)$ blows up in infinite time such that*

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \geqslant Ct^s, \quad \text{for all } t \geqslant t_*,$$

*with some constants $C > 0$ and $t_* > 0$ that depend only on $u_0$, $s$ and $n$.*

**Remark 2.2.** Let us make the following remarks:

(1) The condition $\sigma < 2s$ is technical; see the proof of Theorem 2.1 for details.

(2) In the energy-critical case $s = s_c$, it may happen that $Q \notin L^2(\mathbb{R}^n)$ and thus $M[Q] = +\infty$; see Appendix A.4 below. In this case, we use the convention $(+\infty)^0 = 1$. Hence the second blowup condition above becomes

$$\begin{cases} E[u_0] < E[Q] \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2} > \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2} \end{cases}$$

when $s = s_c$.

(3) In the $L^2$-critical case $s_c = 0$, the second blowup condition stated above is void, since we then get $M[u_0] < M[Q]$ and $M[u_0] > M[Q]$, which is impossible. Thus for $s_c = 0$ the only admissible condition is $E[u_0] < 0$ (i.e., energy below the ground state energy, see Remark A.9).

(4) The exclusion of the half-wave case $s = \frac{1}{2}$ is due to the lack of control for the pointwise decay of a radial function $u \in H^{\frac{1}{2}}(\mathbb{R}^n)$ with $n \geqslant 2$. In fact, the radial Sobolev inequality (Proposition A.4) is a central ingredient in our estimates, but it assumes $s \in \left(\frac{1}{2}, \frac{n}{2}\right)$. This also prevents us from going down to $n = 1$.

(5) The idea of using the scale-invariant quantity $E[u_0]^{s_c} M[u_0]^{s-s_c}$ for blowup for classical NLS comes from [HR07].

**Remark 2.3.** Let us mention that a blowup result can also be established for fractional NLS posed on a bounded star-shaped domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. In this setting, one imposes the exterior Dirichlet condition on $\mathbb{R}^n \backslash \Omega$. One is able to go down to $n = 1$ dimension, and does not have to impose any symmetry condition on the solution $u$. Under the sole assumption of negative energy, one can then conclude finite-time blowup in the $L^2$-supercritical cases $0 < s_c \leqslant s$. See [BHL16, Theorem 2] for details.

**Remark 2.4.** In the framework of classical NLS ($s = 1$), it was Ogawa and Tsutsumi [OT91] who proved finite-time blowup under the assumptions of radial symmetry for the initial datum $u_0 \in H^1(\mathbb{R}^n)$ and negative energy $E[u_0] < 0$. They thereby generalized (in the radial case) the blowup results of Glassey [Gla77] which hypothesized finite variance for the initial datum, i.e., $\int_{\mathbb{R}^n} |x|^2 |u_0(x)|^2 \, \mathrm{d}x < +\infty$. See also [SS99] for a textbook discussion.

## Comments on the Proof and the Virial Identity

By integrating (fNLS) against $i(x\cdot\nabla+\nabla\cdot x)\overline{u}(t)$ on $\mathbb{R}^n$, we make the observation that any sufficiency regular and spatially localized solution $u=u(t,x)$ of (fNLS) satisfies the following *virial identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(2\,\mathrm{Im}\int_{\mathbb{R}^n}\overline{u}(t)x\cdot\nabla u(t)\,\mathrm{d}x\right)=4\sigma nE[u_0]-2(\sigma n-2s)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2. \tag{2.10}$$

To see this, we integrate by parts: we have

$$\begin{aligned}\langle -i(x\cdot\nabla+\nabla\cdot x)u, i\partial_t u\rangle &= i\left\{\langle x\cdot\nabla u, i\partial_t u\rangle+\langle\nabla\cdot(xu), i\partial_t u\rangle\right\}\\ &= i\left\{\langle x\cdot\nabla u, i\partial_t u\rangle-\langle u, x\cdot\nabla i\partial_t u\rangle\right\},\end{aligned}$$

and

$$\begin{aligned}\langle -i(x\cdot\nabla+\nabla\cdot x)u, (-\Delta)^s u\rangle &= i\left\{\langle x\cdot\nabla u, (-\Delta)^s u\rangle+\langle\nabla\cdot(xu), (-\Delta)^s u\rangle\right\}\\ &= i\left\{2s\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2\right\},\end{aligned}$$

where we used $\langle x\cdot\nabla u,(-\Delta)^s u\rangle=\left(\frac{2s-n}{2}\right)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2$ and $\mathrm{div}\,x=n$. As for the nonlinear part, we obtain

$$\begin{aligned}&-\langle -i(x\cdot\nabla+\nabla\cdot x)u, |u|^{2\sigma}u\rangle = -i\left\{\langle x\cdot\nabla u, |u|^{2\sigma}u\rangle+\langle\nabla\cdot(xu), |u|^{2\sigma}u\rangle\right\}\\ &= -\,i\left\{\langle x\cdot\nabla u, |u|^{2\sigma}u\rangle-\langle |u|^{2\sigma}u, x\cdot\nabla u\rangle+\frac{\sigma n}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}\right\}\end{aligned}$$

by using the identity $\frac{\sigma}{\sigma+1}\nabla(|u|^{2\sigma+2})=\nabla(|u|^{2\sigma})|u|^2$. Thus

$$\begin{aligned}&\langle x\cdot\nabla u, i\partial_t u\rangle-\langle u, x\cdot\nabla i\partial_t u\rangle+\langle x\cdot\nabla u, |u|^{2\sigma}u\rangle-\langle|u|^{2\sigma}u, x\cdot\nabla u\rangle\\ =&2s\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2-\frac{\sigma n}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}=2\sigma nE[u_0]-(\sigma n-2s)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2.\end{aligned} \tag{2.11}$$

Let us use (fNLS) once again in order to rewrite the last two inner products on the left side, namely

$$\begin{aligned}\langle|u|^{2\sigma}u, x\cdot\nabla u\rangle &= \langle(-\Delta)^s u, x\cdot\nabla u\rangle-\langle i\partial_t u, x\cdot\nabla u\rangle\\ &= \left(\frac{2s-n}{2}\right)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2-\langle i\partial_t u, x\cdot\nabla u\rangle,\end{aligned}$$

and, by conjugation,

$$\langle x\cdot\nabla u, |u|^{2\sigma}u\rangle=\left(\frac{2s-n}{2}\right)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2-\langle x\cdot\nabla u, i\partial_t u\rangle.$$

Now (2.11) simplifies to

$$\langle i\partial_t u, x\cdot\nabla u\rangle-\langle u, x\cdot\nabla i\partial_t u\rangle=2\sigma nE[u_0]-(\sigma n-2s)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2,$$

that is,

$$-\, i \frac{\mathrm{d}}{\mathrm{d}t} \langle u, x \cdot \nabla u \rangle = 2\sigma n E[u_0] - (\sigma n - 2s) \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2. \tag{2.12}$$

Conjugating the last equation, we also have

$$+\, i \frac{\mathrm{d}}{\mathrm{d}t} \overline{\langle u, x \cdot \nabla u \rangle} = 2\sigma n E[u_0] - (\sigma n - 2s) \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2. \tag{2.13}$$

Now, if we sum (2.12) and (2.13), we see that (2.10) holds.

The virial identity (2.10) does not give us enough information to deduce singularity formation for solutions with negative energy $E[u_0] < 0$ in the $L^2$-critical and $L^2$-supercritical cases when $\sigma \geqslant \frac{2s}{n}$. Historically, there have been two methods which successfully yielded blowup results. They are described in the following subsections.

### 2.1.1 Coupling to a Variance Law

For classical NLS (i.e., when $s = 1$), we have the *Variance-Virial Law*, which can be expressed as

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( \int_{\mathbb{R}^n} |x|^2 |u(t)|^2 \, \mathrm{d}x \right) = 2 \, \mathrm{Im} \left( \int_{\mathbb{R}^n} \overline{u}(t) x \cdot \nabla u(t) \, \mathrm{d}x \right) \tag{2.14}$$

provided that the variance $\int_{\mathbb{R}^n} |x|^2 |u_0|^2 \, \mathrm{d}x < +\infty$ is finite. To see this, pull the time-derivative under the integral sign and insert classical NLS to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle u, |x|^2 u \rangle = \langle u, [i(-\Delta), |x|^2] u \rangle = i \langle u, [|x|^2, \nabla \cdot \nabla] u \rangle.$$

Now, use $[A, BC] = [A, B]C + B[A, C]$ and compute the remaining commutators to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle u, |x|^2 u \rangle = -2i \left\{ \langle u, x \cdot \nabla u \rangle + \langle u, \nabla \cdot (xu) \rangle \right\},$$

from which (2.14) follows by the final integration by parts

$$\langle u, \nabla \cdot (xu) \rangle = - \int_{\mathbb{R}^n} \nabla \overline{u}(t) \cdot xu \, \mathrm{d}x = -\overline{\langle u, x \cdot \nabla u \rangle}.$$

By combining (2.10) and (2.14), one obtains Glassey's celebrated blowup result for classical NLS with negative Hamiltonian $E[u_0] < 0$ and finite variance (see, e.g., [SS99, Theorem 5.3] for a proof and textbook discussion). In fact, this combination yields

$$\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle u(t), |x|^2 u(t) \rangle = 4\sigma n E[u_0] - 2(\sigma n - 2) \|\nabla u(t)\|_{L^2}^2,$$

where the second term on the right side vanishes precisely in the $L^2$-critical case. In the $L^2$-critical or $L^2$-supercritical case, this gives

$$\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle u(t), |x|^2 u(t) \rangle \leqslant 4\sigma n E[u_0],$$

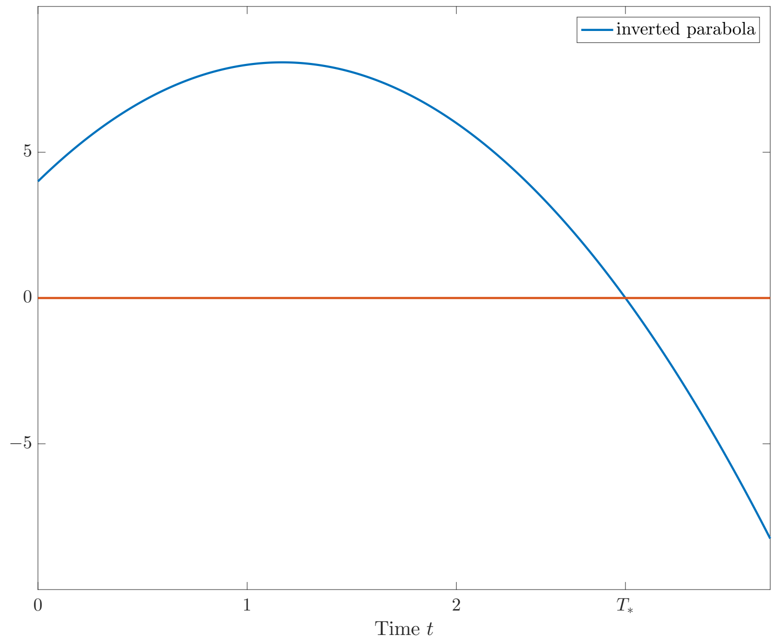


Figure 2.1: Variance lies below an inverted parabola

so that by integrating in time twice, we get (abbreviating $V(t) := \langle u(t), |x|^2 u(t)\rangle$)

$$0 \leqslant V(t) \leqslant 4\sigma n E[u_0] t^2 + V'(0)t + V(0).$$

If the energy is negative, the right side in the last inequality becomes negative in finite time (see figure 2.1). However, the variance $V(t)$ is nonnegative, hence the solution cannot extend to all times (see also [Rap13]).

But Glassey's argument breaks down in the nonlocal situation $s \neq 1$, since the identity (2.14) fails in this case, as one readily checks by dimensional analysis. Indeed, if $s = 1$, the quantity $\langle u, [i(-\Delta), |x|^2]u\rangle$ scales like length to the power 0, as required by the virial quantity on the right side of (2.14), where we note that $x \cdot \nabla$ scales in the same way. If however $s \neq 1$, the quantity $\langle u, [-i(-\Delta)^s, |x|^2]u\rangle$ scales like length to the power $-2s + 2 \neq 0$. Therefore the law (2.14) fails to hold for $s \neq 1$.

Rather, it turns out the suitable generalization of the variance for fractional NLS is given by the nonnegative quantity [BL15]

$$\mathcal{V}^{(s)}[u(t)] = \int_{\mathbb{R}^n} \overline{u}(t) x \cdot (-\Delta)^{1-s} x u(t)\, \mathrm{d}x = \|x(-\Delta)^{\frac{1-s}{2}} u(t)\|_{L^2}^2. \tag{2.15}$$

This can be justified by formal computations in the following way. Given any sufficiently regular and spatially localized solution $u(t)$ of the *free* fractional Schrödinger equation $i\partial_t u = (-\Delta)^s u$, a calculation yields the following relation of $\mathcal{V}^{(s)}[u(t)]$ to the virial:

$$\frac{1}{2s}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{(s)}[u(t)] = 2\,\mathrm{Im}\int_{\mathbb{R}^n} \overline{u}(t) x \cdot \nabla u(t)\,\mathrm{d}x. \tag{2.16}$$

Indeed, similarly as before, we obtain by inserting the free fractional NLS that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{(s)}[u(t)] &= \langle u, [i(-\Delta)^s, x \cdot (-\Delta)^{1-s} x] u\rangle \\ &= i\sum_{k=1}^n \langle u, [(-\Delta)^s, x_k(-\Delta)^{1-s}x_k]u\rangle \\ &= i\sum_{k=1}^n \langle \widehat{u}, [|\xi|^{2s}, i\partial_{\xi_k}|\xi|^{2(1-s)} i\partial_{\xi_k}]\widehat{u}\rangle\end{aligned}$$

after transforming to Fourier space with $\widehat{x_k f}(\xi) = i\partial_{\xi_k}\widehat{f}(\xi)$. Using the above commutator rule, one checks that $[|\xi|^{2s}, i\partial_{\xi_k}|\xi|^{2(1-s)} i\partial_{\xi_k}] = 2s\,(\xi_k\partial_{\xi_k} + \partial_{\xi_k}\xi_k)$. Inserting this into the previous equation and transforming back to physical space gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{(s)}[u(t)] = -2si\left(\langle u, x\cdot\nabla u\rangle + \langle u, \nabla\cdot(xu)\rangle\right),$$

from which (2.16) follows by a final integration by parts as before. Note however that when one considers the nonlinear equation, then for $s = 1$ the appearing commutator $[|u|^{2\sigma}, |x|^2]$ clearly vanishes, while for $s \neq 1$ the appearance of $[|u|^{2\sigma}, x\cdot(-\Delta)^{1-s}x]$ significantly complicates the situation. Namely, in the latter situation, identity (2.16) breaks down and the correct equation acquires highly nontrivial error terms due to the nonlinearity. In particular, for $s \in (0,1)$, these error terms seem very hard to control for local nonlinearities $f(u) = -|u|^{2\sigma}u$, even in the class of radial solutions. So far, the cases where the application of $\mathcal{V}^{(s)}[u(t)]$ has turned out to be successful to prove blowup results for fractional NLS deal with radial solutions and focusing Hartree-type nonlinearities, e.g., $f(u) = -(|x|^{-\gamma} * |u|^2)u$ with $\gamma \geqslant 1$; see, e.g., [FL07]. In the context of biharmonic NLS ($s = 2$) with local nonlinearity, a localized version of $\mathcal{V}^{(s)}[u(t)]$ has been used to prove blowup results, by using some smoothing properties of $(-\Delta)^{\frac{1-s}{2}}$ when $s > 1$; see [BL15].

### 2.1.2 Localized Virial Law

In the context of classical NLS ($s = 1$), Ogawa and Tsutsumi [OT91] argued that it is reasonable to drop the finite-variance assumption and pass from the space $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2\,\mathrm{d}x)$ to the more natural energy space $H^1(\mathbb{R}^n)$. They were able to prove blowup for radial negative-energy solutions with infinite variance of $L^2$-supercritical and $L^2$-critical focusing classical NLS. Their method by-passes the use of a variance-type quantity, and replaces the unbounded function $x$ by a suitable cut-off function $\varphi_R$ such that $\nabla\varphi_R(x) \equiv x$ for $|x| \leqslant R$ and $\nabla\varphi_R(x) \equiv 0$ for $|x| \gg R$.

It is their strategy of so-called localized virial identities that we implement for fractional NLS to prove Theorem 2.1. However, when one tries to directly apply the arguments in [OT91] to study the time evolution of the localized virial $\mathcal{M}_{\varphi_R}[u(t)]$ for fractional NLS, one encounters severe difficulties due to the nonlocal nature of the fractional Laplacian $(-\Delta)^s$. In particular, the nonnegativity of certain error terms due the localization, which are pivotal in the arguments of [OT91], seem to be elusive. To overcome this difficulty, we employ the representation formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} \, \mathrm{d}m, \tag{2.17}$$

valid for all $s \in (0,1)$, which is also known as *Balakrishnan's formula* used in semi-group theory (see formula (2.1) in [Bal60, p. 420]). In fact, by means of (2.17), we are able to derive the differential estimate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_{\varphi_R}[u(t)] \leqslant 4\sigma n E[u_0] - 2\delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \\ + o_R(1) \cdot \left(1 + \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{(\frac{\sigma}{s})+}\right) \end{aligned} \tag{2.18}$$

for any sufficiently regular and radial solution $u(t,x)$ of (fNLS) in dimensions $n \geqslant 2$ and $s \in \left(\frac{1}{2}, 1\right)$. Here $\delta = \sigma n - 2s > 0$ is a strictly positive constant in the mass-supercritical case $s_c > 0$, and the error term $o_R(1)$ tends to 0 as $R \to \infty$ uniformly in time $t$. With the help of the key estimate (2.18), we can then apply a standard ODE comparison argument to show that $u(t)$ cannot exist for all times $t \geqslant 0$ under the assumptions of Theorem 2.1. For the mass-critical case $s_c = 0$, we have $\delta = 0$, and the differential estimate (2.18) needs to be refined and leads only to the weaker conclusion of possible infinite time blowup as stated in Theorem 2.1 (ii).

Furthermore, we can exploit the idea of using Balakrishnan's formula (2.17) to obtain *Morawetz-Lin-Strauss estimates* for fractional NLS and thereby establish scattering results for *defocusing (repulsive)* fractional NLS. This is done in chapter 4.

## 2.2 Localized Virial Estimate for Fractional NLS

In this section, we derive localized virial estimates for radial solutions of fractional NLS. First, we derive a general formula for solutions $u(t,x)$ that are not necessarily radial. Then we sharpen the estimates in the class of radial solutions.

### 2.2.1 A General Virial Identity

Let $n \geqslant 1$, $s \in \left[\frac{1}{2}, 1\right)$, and $\sigma > 0$. Throughout this section, we assume that

$$u \in C([0,T); H^{2s}(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n))$$

is a solution of (fNLS). Note that, at this point, we do not impose any symmetry assumption on the solution $u(t,x)$. Note also that for $u(t) \in H^{2s}(\mathbb{R}^n)$, conservation of mass $M[u]$ and energy $E[u]$ follows directly by integrating (fNLS) against $\overline{u}(t)$ (and taking the imaginary part [Caz03, p. 56]) and $\partial_t \overline{u}(t)$ (and taking the real part [Caz03, p. 56]), respectively. There is no need for an approximation argument in order to have well-defined pairings.

If the exponent $\sigma$ is not $H^{2s}$-supercritical (in particular if $s_c \leqslant s$), the condition $u \in C([0,T); L^{2\sigma+2}(\mathbb{R}^n))$ is redundant by Sobolev embeddings. Furthermore, we remark that the following localized virial identities could be extended to $u \in C([0,T); H^s(\mathbb{R}^n))$, provided we have a decent local well-posedness theory in $H^s(\mathbb{R}^n)$. However, as pointed out before, we prefer to work with strong $H^{2s}$-valued solutions for (fNLS) in order to guarantee that the following calculations are well-defined a-priori.

Let us assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function with $\nabla\varphi \in (W^{3,\infty}(\mathbb{R}^n))^n$. We define the **localized virial** of $u = u(t,x)$ to be the quantity given by

$$\mathcal{M}_\varphi[u(t)] := 2\,\mathrm{Im} \int_{\mathbb{R}^n} \overline{u}(t) \nabla\varphi \cdot \nabla u(t)\, \mathrm{d}x = 2\,\mathrm{Im} \int_{\mathbb{R}^n} \overline{u}(t) \partial_k \varphi \partial_k u(t)\, \mathrm{d}x. \tag{2.19}$$

The localized virial $\mathcal{M}_\varphi[u(t)]$ is well-defined, since $u(t) \in H^s(\mathbb{R}^n)$ with $s \geqslant \frac{1}{2}$ and Lemma A.1 immediately gives the bound

$$|\mathcal{M}_\varphi[u(t)]| \lesssim C\left(\|\nabla\varphi\|_{L^\infty}, \|\Delta\varphi\|_{L^\infty}\right) \|u(t)\|_{H^{\frac{1}{2}}}^2.$$

The idea is now to study the time evolution of the localized virial. To do that, we introduce the auxiliary function $u_m = u_m(t,x)$ defined as

$$u_m(t) := c_s \frac{1}{-\Delta + m} u(t) = c_s \mathcal{F}^{-1}\left(\frac{\widehat{u}(t,\xi)}{|\xi|^2 + m}\right) \quad \text{with } m > 0, \tag{2.20}$$

where the constant

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}} \tag{2.21}$$

will be a convenient normalization factor. By the smoothing properties of $(-\Delta + m)^{-1}$, we have that $u_m(t) \in H^{\alpha+2}(\mathbb{R}^n)$ holds for any $t \in [0,T)$ whenever $u(t) \in H^\alpha(\mathbb{R}^n)$.

**Lemma 2.5** (General virial identity). *For any $t \in [0,T)$, we have the identity*

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_\varphi[u(t)] = &\int_0^\infty m^s \int_{\mathbb{R}^n} \left\{ 4 \overline{\partial_k u_m} (\partial^2_{kl}\varphi) \partial_l u_m - (\Delta^2\varphi) |u_m|^2 \right\} \mathrm{d}x\, \mathrm{d}m \\ &- \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^n} (\Delta\varphi) |u|^{2\sigma+2}\, \mathrm{d}x, \end{aligned}}$$

*where $u_m = u_m(t,x)$ is defined in (2.20) above.*

**Remark 2.6.** (1) If we make the formal substitution and take the unbounded function $\varphi(x) = \frac{|x|^2}{2}$, so that $\nabla\varphi(x) = x$, we have $\partial_r^2\varphi \equiv 1$ and $\Delta^2\varphi \equiv 0$. By applying the identity

$$\int_0^\infty m^s \int_{\mathbb{R}^n} |\nabla u_m|^2 \, \mathrm{d}x \, \mathrm{d}m = s\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2,$$

valid for any $u \in \dot{H}^s(\mathbb{R}^n)$ (see (2.33) below), we recover the formal virial identity (2.10) by an elementary calculation.

(2) From the proof given below (see (2.32)) and Lemma A.2, we deduce the bound

$$\begin{aligned} &\left| \int_0^\infty m^s \int_{\mathbb{R}^n} \left\{ 4\overline{\partial_k u_m}(\partial_{kl}^2\varphi)\partial_l u_m - (\Delta^2\varphi)|u_m|^2 \right\} \, \mathrm{d}x \, \mathrm{d}m \right| \\ &\lesssim \|\nabla^2\varphi\|_{L^\infty}\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \|\Delta^2\varphi\|_{L^\infty}^s\|\Delta\varphi\|_{L^\infty}^{1-s}\|u\|_{L^2}^2 \lesssim C\|u\|_{H^s}^2, \end{aligned}$$

where $C > 0$ only depends on $\|\nabla\varphi\|_{W^{3,\infty}}$.

(3) The usage of the auxiliary function $u_m$ and Balakrishnan's representation formula (2.17) for the fractional Laplacian $(-\Delta)^s$ is inspired by the joint work [KLR13] of Krieger, Lenzmann and Raphaël.

*Proof of Lemma 2.5.* Given $\varphi : \mathbb{R}^n \to \mathbb{R}$, let us define the (formally) self-adjoint operator

$$\Gamma_\varphi := -i(\nabla\varphi \cdot \nabla + \nabla \cdot \nabla\varphi) \tag{2.22}$$

acting on functions via

$$\Gamma_\varphi f := -i(\nabla\varphi \cdot \nabla f + \operatorname{div}((\nabla\varphi) f)) = -2i\nabla\varphi \cdot \nabla f - i(\Delta\varphi) f. \tag{2.23}$$

Integrating by parts, we readily check

$$\mathcal{M}_\varphi[u(t)] = \langle u(t), \Gamma_\varphi u(t) \rangle.$$

By taking the time derivative and using that $u(t)$ solves (fNLS), we get

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\varphi[u(t)] = \langle u(t), [(-\Delta)^s, i\Gamma_\varphi] u(t) \rangle + \langle u(t), [-|u|^{2\sigma}, i\Gamma_\varphi] u(t) \rangle} \tag{2.24}$$

where $[X, Y] \equiv XY - YX$ denotes the commutator of $X$ and $Y$. Indeed, as $\Gamma_\varphi$ is time independent and linear, $u(t)$ solves (fNLS), $(-\Delta)^s$ is self-adjoint and $|u|^{2\sigma}$ is real, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\varphi[u(t)] &= \langle \partial_t u, \Gamma_\varphi u \rangle + \langle u, \Gamma_\varphi \partial_t u \rangle \\ &= i\langle (-\Delta)^s u - |u|^{2\sigma} u, \Gamma_\varphi u \rangle - i\langle u, \Gamma_\varphi((-\Delta)^s u - |u|^{2\sigma} u) \rangle \\ &= i\Big\{ \langle (-\Delta)^s u, \Gamma_\varphi u \rangle - \langle |u|^{2\sigma} u, \Gamma_\varphi u \rangle - \langle u, \Gamma_\varphi((-\Delta)^s u) \rangle + \langle u, \Gamma_\varphi(|u|^{2\sigma} u) \rangle \Big\} \\ &= i\Big\{ \langle u, [(-\Delta)^s, \Gamma_\varphi] u \rangle - \langle u, [|u|^{2\sigma}, \Gamma_\varphi] u \rangle \Big\} \\ &= \langle u, [(-\Delta)^s, i\Gamma_\varphi] u \rangle + \langle u, [-|u|^{2\sigma}, i\Gamma_\varphi] u \rangle. \end{aligned}$$

[Alternatively, directly use the self-adjointness of $\Gamma_\varphi$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\varphi[u(t)] = \langle \partial_t u, \Gamma_\varphi u\rangle + \langle u, \Gamma_\varphi \partial_t u\rangle = \langle \partial_t u, \Gamma_\varphi u\rangle + c.c.$$

Then also use the self-adjointness of $(-\Delta)^s$ and the real-valuedness of $|u|^{2\sigma}$ to derive (2.24).]

In the language of quantum mechanics, (2.24) can be seen in the light of Heisenberg's formula, which states that the evolution of the expectation of a quantum-mechanical observable $A$ is related to the expectation of the commutator of that observable with the Hamiltonian $H$ of the system by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), Au(t)\rangle = i\langle u(t), [H, A]u(t)\rangle + \langle u(t), \frac{\partial A}{\partial t}u(t)\rangle$$

whenever $u(t)$ satisfies the Schrödinger equation $i\partial_t u = Hu$; cf. [Gri95, p. 123]. In our case, the expectation of the time-independent operator $\Gamma_\varphi$ is precisely the virial $\mathcal{M}_\varphi[u(t)]$, and we have the Hamiltonian $H = (-\Delta)^s + V$ with the nonlinear potential $V = -|u|^{2\sigma}$.

By our regularity assumptions on $u(t)$, we have $(-\Delta)^s u(t) \in L^2(\mathbb{R}^n)$ and $\Gamma_\varphi u(t) \in H^{2s-1}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $s \geqslant \frac{1}{2}$. In particular, the terms above are well-defined a-priori. We discuss the terms on the right side of (2.24) as follows.

**Step 1 (Dispersive Term).** For $s \in (0, 1)$, we have the so-called *Balakrishnan's formula*:

$$(-\Delta)^s = \frac{\sin \pi s}{\pi}\int_0^\infty m^{s-1}\frac{-\Delta}{-\Delta + m}\,\mathrm{d}m. \tag{2.25}$$

This formula follows from the spectral calculus applied to the self-adjoint operator $-\Delta$ and the formula[6] $x^s = \frac{\sin \pi s}{\pi}\int_0^\infty m^{s-1}\frac{x}{x+m}\,\mathrm{d}m$ valid for any real number $x > 0$ and $s \in (0, 1)$. The following commutator identity valid for operators $A \geqslant 0$ and $B$ (with $m > 0$ a positive number) is easily verified:

$$\left[\frac{A}{A+m}, B\right] = \left[\mathbb{1} - \frac{m}{A+m}, B\right] = -m\left[\frac{1}{A+m}, B\right] = m\frac{1}{A+m}[A, B]\frac{1}{A+m}. \tag{2.26}$$

[6]Indeed, a consequence of the Residue Theorem is the formula [FB00, Satz 7.12]

$$\int_0^\infty m^{\lambda-1}R(m)\,\mathrm{d}m = \frac{\pi}{\sin \pi\lambda}\sum_{a\in\mathbb{C}_+}\mathrm{Res}(f; a), \quad \text{where } f(z) := (-z)^{\lambda-1}R(z), \tag{$*$}$$

whenever $\lambda > 0, \lambda \notin \mathbb{Z}$, $R(m) = \frac{P(m)}{Q(m)}$ (with $Q \neq 0$ on $\mathbb{R}_{\geqslant 0}$) is a rational function such that $R(0) \neq 0$ and $\lim_{m\to\infty} m^\lambda |R(m)| = 0$. Here $\mathbb{C}_+ = \mathbb{C}\backslash\{x \in \mathbb{R}; x \geqslant 0\}$ is the complex plane slit along the positive reals. For us, the hypotheses are satisfied for $\lambda = s \in (0, 1)$ and $R(m) = \frac{x}{x+m}$. We see that $R$ and hence $f$ has a simple pole at $-x \in \mathbb{C}_+$ with residue $\mathrm{Res}(f; -x) = \lim_{z\to -x}(z - (-x))f(z) = x^s$.

Combining (2.25) and (2.26) with $A = -\Delta$, we obtain the formal commutator identity

$$[(-\Delta)^s, B] = \frac{\sin \pi s}{\pi} \int_0^\infty m^s \frac{1}{-\Delta + m} [-\Delta, B] \frac{1}{-\Delta + m} \, dm. \tag{2.27}$$

Next, we apply this identity to $B = i\Gamma_\varphi$ and we use that

$$[-\Delta, i\Gamma_\varphi] = -4\partial_k(\partial^2_{kl}\varphi)\partial_l - \Delta^2\varphi \tag{2.28}$$

(apply the commutator to a test function and verify this equality by a direct calculation[7]). Let us now apply the formal identities above to the situation at hand. We first assume that $u \in C_c^\infty(\mathbb{R}^n)$ holds. We claim

$$\langle u, [(-\Delta)^s, i\Gamma_\varphi]u\rangle = \int_0^\infty m^s \int_{\mathbb{R}^n} \left\{ 4\overline{\partial_k u_m}(\partial^2_{kl}\varphi)\partial_l u_m - (\Delta^2\varphi)|u_m|^2 \right\} dx \, dm, \tag{2.29}$$

where $u_m = c_s(-\Delta + m)^{-1}u$ with $m > 0$ and the constant $c_s > 0$ defined in (2.21). Indeed, for $u \in C_c^\infty(\mathbb{R}^n)$, we can apply Balakrishnan's formula (2.25) (where the $m$-integral is a convergent Bochner integral) to express $(-\Delta)^s u$. Using (2.27), Fubini's theorem and (2.28), we obtain

$$\begin{aligned}
\langle u, [(-\Delta)^s, i\Gamma_\varphi]u\rangle &= \left\langle u, \left( \frac{\sin \pi s}{\pi} \int_0^\infty m^s \frac{1}{-\Delta + m} [-\Delta, i\Gamma_\varphi] \frac{1}{-\Delta + m} \, dm \right) u \right\rangle \\
&= \frac{\sin \pi s}{\pi} \int_0^\infty m^s \left\langle u, \frac{1}{-\Delta + m} [-\Delta, i\Gamma_\varphi] \frac{1}{-\Delta + m} u \right\rangle dm \\
&= \int_0^\infty m^s \left\langle c_s \frac{1}{-\Delta + m} u, [-\Delta, i\Gamma_\varphi] c_s \frac{1}{-\Delta + m} u \right\rangle dm \\
&= \int_0^\infty m^s \int_{\mathbb{R}^n} \left\{ \overline{u_m} \left( -4\partial_k((\partial^2_{kl}\varphi)\partial_l u_m) - (\Delta^2\varphi)u_m \right) \right\} dx \, dm \\
&= \int_0^\infty m^s \int_{\mathbb{R}^n} \left\{ 4\overline{\partial_k u_m}(\partial^2_{kl}\varphi)\partial_l u_m - (\Delta^2\varphi)|u_m|^2 \right\} dx \, dm,
\end{aligned}$$

integrating by parts in the last step. This yields (2.29) for $u \in C_c^\infty(\mathbb{R}^n)$.

The next step is to extend (2.29) to any $u \in H^{2s}(\mathbb{R}^n)$ by the following approximation argument. Let $(u_j)_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence such that $u_j \to u$ strongly in $H^{2s}(\mathbb{R}^n)$. We have

$$\langle u_j, [(-\Delta)^s, i\Gamma_\varphi]u_j\rangle \to \langle u, [(-\Delta)^s, i\Gamma_\varphi]u\rangle. \tag{2.30}$$

To see this, write

$$\begin{aligned}
&\langle u_j, [(-\Delta)^s, i\Gamma_\varphi]u_j\rangle - \langle u, [(-\Delta)^s, i\Gamma_\varphi]u\rangle \\
=&\langle u_j - u, [(-\Delta)^s, i\Gamma_\varphi]u_j\rangle + \langle u, [(-\Delta)^s, i\Gamma_\varphi](u_j - u)\rangle,
\end{aligned}$$

[7] We have $[-\Delta, i\Gamma_\varphi]\psi = -\Delta(\nabla \cdot ((\nabla\varphi)\psi) + \nabla\varphi \cdot \nabla\psi) + \nabla \cdot (\nabla\varphi\Delta\psi) + \nabla\varphi \cdot \nabla\Delta\psi$.

and show that both terms on the right side tend to zero as $j \to \infty$. Namely, using the self-adjointness of $(-\Delta)^s$ and $\Gamma_\varphi$, the first term reads

$$\langle u_j - u, [(-\Delta)^s, i\Gamma_\varphi]u_j\rangle = \langle (-\Delta)^s(u_j-u), i\Gamma_\varphi u_j\rangle + \langle i\Gamma_\varphi(u_j-u), (-\Delta)^s u_j\rangle.$$

Hence for the first term the Cauchy-Schwarz inequality gives

$$\begin{aligned}&|\langle u_j - u, [(-\Delta)^s, i\Gamma_\varphi]u_j\rangle| \\ \leqslant&\|(-\Delta)^s(u_j-u)\|_{L^2}\|i\Gamma_\varphi u_j\|_{L^2} + \|i\Gamma_\varphi(u_j-u)\|_{L^2}\|(-\Delta)^s u_j\|_{L^2} \\ \lesssim&\|u_j-u\|_{H^{2s}}\|u_j\|_{H^{2s}} \lesssim \|u_j-u\|_{H^{2s}} \to 0.\end{aligned}$$

Here we also used that the linear operator $i\Gamma_\varphi$ is bounded $H^{2s}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which follows from Hölder's inequality, the regularity $\nabla\varphi \in (W^{1,\infty}(\mathbb{R}^n))^n$ and the continuity $H^{2s}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$ for $s \geqslant \frac{1}{2}$:

$$\begin{aligned}\|i\Gamma_\varphi f\|_{L^2} &\lesssim \|\nabla\varphi\cdot\nabla f\|_{L^2} + \|(\Delta\varphi)f\|_{L^2} \\ &\lesssim \|\nabla\varphi\|_{L^\infty}\|\nabla f\|_{L^2} + \|\Delta\varphi\|_{L^\infty}\|f\|_{L^2} \\ &\lesssim C(\|\nabla\varphi\|_{W^{1,\infty}})\|f\|_{H^1} \lesssim C(\|\nabla\varphi\|_{W^{1,\infty}})\|f\|_{H^{2s}}, \quad \text{for all } f\in H^{2s}(\mathbb{R}^n).\end{aligned}$$

Analogously, the second term is estimated by

$$|\langle u, [(-\Delta)^s, i\Gamma_\varphi](u_j-u)\rangle| \lesssim \|u\|_{H^{2s}}\|u_j-u\|_{H^{2s}} \lesssim \|u_j-u\|_{H^{2s}} \to 0.$$

Now (2.30) follows, and this yields the left-hand side of (2.29). Next we claim

$$\lim_{j\to\infty} G[u_j,u_j] = G[u,u] \tag{2.31}$$

where we define the bilinear form

$$G[f,g] := \int_0^\infty m^s \int_{\mathbb{R}^n} \overline{\partial_k f_m}(\partial^2_{kl}\varphi)\partial_l g_m \,\mathrm{d}x\,\mathrm{d}m$$

with $f_m = c_s(-\Delta+m)^{-1}f$ and $g_m = c_s(-\Delta+m)^{-1}g$. Since $u_j \to u$ strongly in $H^{2s}(\mathbb{R}^n)$, the convergence (2.31) follows from

$$|G[f,g]| \lesssim \|\partial^2_{kl}\varphi\|_{L^\infty}\|(-\Delta)^{\frac{s}{2}}f\|_{L^2}\|(-\Delta)^{\frac{s}{2}}g\|_{L^2}.^{8} \tag{2.32}$$

To prove (2.32), first note that, by using Plancherel's and Fubini's theorem,

$$\begin{aligned}\int_0^\infty m^s\int_{\mathbb{R}^n}|\nabla f_m|^2\,\mathrm{d}x\,\mathrm{d}m &= \int_{\mathbb{R}^n}\left(\frac{\sin\pi s}{\pi}\int_0^\infty\frac{m^s\,\mathrm{d}m}{(|\xi|^2+m)^2}\right)|\xi|^2|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n}(s|\xi|^{2s-2})|\xi|^2|\widehat{f}(\xi)|^2\,\mathrm{d}\xi = s\|(-\Delta)^{\frac{s}{2}}f\|^2_{L^2}\end{aligned} \tag{2.33}$$

[8] Write $G[u_j,u_j]-G[u,u] = G[u_j-u,u_j]+G[u,u_j-u]$ and use the strong convergence $u_j\to u$ in $H^{2s}(\mathbb{R}^n)$ (in particular the boundedness $\sup_{j\geqslant1}\|(-\Delta)^{\frac{s}{2}}u_j\|_{L^2}\lesssim 1$) to see that (2.32) is sufficient for (2.31).

for arbitrary $f \in \dot{H}^s(\mathbb{R}^n)$.[9] Next, we introduce the bilinear form

$$H[f,g] := G[f,g] + \mu s \int_{\mathbb{R}^n} \overline{f}(-\Delta)^s g \, \mathrm{d}x \quad \text{with } \mu := \text{ess-sup}_{x\in\mathbb{R}^n} \|(\partial^2_{kl}\varphi)(x)\|,$$

where $\|A\|$ denotes the operator norm of a matrix $A \in \mathbb{R}^{n\times n}$. Thus from (2.33) and by using the pointwise lower bound $\overline{\partial_k f_m}(\partial^2_{kl}\varphi)\partial_l f_m \geqslant -\mu|\nabla f_m|^2$ we get

$$H[f,f] \geqslant -\mu \int_0^\infty m^s \int_{\mathbb{R}^n} |\nabla f_m|^2 \, \mathrm{d}x \, \mathrm{d}m + \mu s \|(-\Delta)^{\frac{s}{2}} f\|^2_{L^2} = 0,$$

that is, the positive semidefiniteness of $H[f,g]$. On the other hand $\mu \lesssim \|\partial^2_{kl}\varphi\|_{L^\infty}$ and thus

$$\begin{aligned} H[f,f] &\leqslant |G[f,f]| + \mu s \int_{\mathbb{R}^n} \overline{f}(-\Delta)^s f \, \mathrm{d}x \\ &\leqslant \mu \int_0^\infty m^s \int_{\mathbb{R}^n} |\nabla f_m|^2 \, \mathrm{d}x \, \mathrm{d}m + \mu s \int_{\mathbb{R}^n} \overline{f}(-\Delta)^s f \, \mathrm{d}x \\ &\lesssim \|\partial^2_{kl}\varphi\|_{L^\infty} \|(-\Delta)^{\frac{s}{2}} f\|^2_{L^2}. \end{aligned}$$

Since $H[f,g]$ is positive semidefinite, we have the Cauchy-Schwarz inequality $|H[f,g]| \leqslant \sqrt{H[f,f]}\sqrt{H[g,g]}$. Consequently, we deduce

$$\begin{aligned} |G[f,g]| &\leqslant \sqrt{H[f,f]}\sqrt{H[g,g]} + \mu s \|(-\Delta)^{\frac{s}{2}} f\|_{L^2} \|(-\Delta)^{\frac{s}{2}} g\|_{L^2} \\ &\lesssim \|\partial^2_{kl}\varphi\|_{L^\infty} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2} \|(-\Delta)^{\frac{s}{2}} g\|_{L^2}, \end{aligned}$$

which is the desired bound (2.32).

To complete the proof of (2.29) for $u \in H^{2s}(\mathbb{R}^n)$, it remains to show that

$$\lim_{j\to\infty} K[u_j, u_j] = K[u,u] \tag{2.34}$$

for the bilinear form

$$K[f,g] := \int_0^\infty m^s \int_{\mathbb{R}^n} (\Delta^2\varphi)\overline{f_m} g_m \, \mathrm{d}x \, \mathrm{d}m.$$

But, by following the proof of Lemma A.2, we obtain

$$|K[f,g]| \lesssim \|\Delta^2\varphi\|^s_{L^\infty} \|\Delta\varphi\|^{1-s}_{L^\infty} \|f\|_{L^2} \|g\|_{L^2},$$

from which we immediately deduce (2.34).

[9] Apply (∗) in footnote [6] - but this time with $\lambda = s+1 \in (1,2)$ and $R(m) = \frac{1}{(|\xi|^2+m)^2}$. Since $R$, hence $f$ has a pole of order 2 at $-|\xi|^2$, we have [FB00, Bemerkung 6.4] that $\mathrm{Res}(f; -|\xi|^2) = \tilde{f}^{(1)}(-|\xi|^2)$, with $\tilde{f}(z) = (z+|\xi|^2)^2 f(z) = (-z)^s$, so that $\mathrm{Res}(f; -|\xi|^2) = -s|\xi|^{2s-2}$. Together with $\sin \pi\lambda = -\sin \pi s$, the claimed formula follows from (∗).

**Step 2 (Nonlinear Term).** This part of the proof is analogous to the classical NLS. In fact, we compute

$$\begin{aligned}\langle u, [-|u|^{2\sigma}, i\Gamma_\varphi]u\rangle &= -\langle u, [|u|^{2\sigma}, \nabla\varphi\cdot\nabla + \nabla\cdot\nabla\varphi]u\rangle = 2\langle u, \nabla\varphi\cdot\nabla(|u|^{2\sigma})u\rangle \\ &= 2\int_{\mathbb{R}^n} \nabla\varphi\cdot\nabla(|u|^{2\sigma})|u|^2\,\mathrm{d}x = -\frac{2\sigma}{\sigma+1}\int_{\mathbb{R}^n}(\Delta\varphi)|u|^{2\sigma+2}\,\mathrm{d}x,\end{aligned}$$

where the last step follows from integration by parts after inserting the identity $\nabla(|u|^{2\sigma+2}) = \frac{\sigma+1}{\sigma}\nabla(|u|^{2\sigma})|u|^2$, which is easily verified.[10] The proof of Lemma 2.5 is now complete. □

### 2.2.2 Localized Virial Estimate for Radial Solutions

In this subsection, we will apply the general virial identity Lemma 2.5 for the localized virial $\mathcal{M}_\varphi[u(t)]$ when $\varphi$ is a suitable approximation of the unbounded function $a(x) = \frac{1}{2}|x|^2$ and hence $\nabla a(x) = x$. This choice will result in a localized virial identity that will be exploited to prove blowup for radial solutions of (fNLS).

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be as above, that is $\varphi$ is a real-valued function such that $\nabla\varphi \in (W^{3,\infty}(\mathbb{R}^n))^n$. In addition, we assume that $\varphi = \varphi(r)$ is radially symmetric and satisfies

$$\varphi(r) = \begin{cases} \frac{r^2}{2}, & \text{for } r \leqslant 1 \\ \text{const.} & \text{for } r \geqslant 10 \end{cases} \quad \text{and} \quad \varphi''(r) \leqslant 1 \text{ for } r \geqslant 0. \tag{2.35}$$

For given $R > 0$, we define the rescaled function $\varphi_R : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\varphi_R(r) := R^2\varphi\left(\frac{r}{R}\right). \tag{2.36}$$

The following inequalities hold:

$$1 - \varphi_R''(r) \geqslant 0, \quad 1 - \frac{\varphi_R'(r)}{r} \geqslant 0, \quad n - \Delta\varphi_R(r) \geqslant 0 \quad \text{for all } r \geqslant 0. \tag{2.37}$$

Indeed, the first inequality follows from $\varphi_R''(r) = \varphi''\left(\frac{r}{R}\right) \leqslant 1$, while the second inequality follows by integrating the first inequality on $[0, r]$ and using $\varphi'(0) = 0$. Finally, the third inequality follows from $n - \Delta\varphi_R(r) = 1 - \varphi_R''(r) + (n-1)\{1 - \frac{\varphi_R'(r)}{r}\} \geqslant 0$ thanks to the first and the second inequality.

For later use, we record the following properties of $\varphi_R$, which can be easily checked:

$$\begin{cases} \nabla\varphi_R(r) = R\varphi'\left(\frac{r}{R}\right)\frac{x}{|x|} = \begin{cases} x & \text{for } r \leqslant R, \\ 0 & \text{for } r \geqslant 10R \end{cases}; \\ \|\nabla^j\varphi_R\|_{L^\infty} \lesssim R^{2-j} \text{ for } 0 \leqslant j \leqslant 4; \\ \operatorname{supp}(\nabla^j\varphi_R) \subset \begin{cases} \{|x| \leqslant 10R\} & \text{for } j = 1, 2 \\ \{R \leqslant |x| \leqslant 10R\} & \text{for } 3 \leqslant j \leqslant 4 \end{cases}. \end{cases} \tag{2.38}$$

[10] Indeed, from $\nabla(|u|^{2\sigma})|u|^2 = \sigma(|u|^2)^{\sigma-1}\nabla(|u|^2)|u|^2 = \sigma|u|^{2\sigma}\nabla(|u|^2)$, we see that $\nabla(|u|^{2\sigma+2}) = \nabla(|u|^{2\sigma})|u|^2 + |u|^{2\sigma}\nabla(|u|^2) = \left(1 + \frac{1}{\sigma}\right)\nabla(|u|^{2\sigma})|u|^2$.

For such a radial $\varphi_R$, we prove the following differential inequality for the time evolution of the localized virial $\mathcal{M}_{\varphi_R}[u(t)]$.

**Lemma 2.7** (Localized radial virial estimate)**.** *Let $n \geqslant 2$, $s \in \left(\frac{1}{2}, 1\right)$, and assume in addition that $u = u(t,x)$ is a radial solution of* (fNLS)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\varphi_R}[u(t)] \leqslant 4\sigma n E[u_0] - 2(\sigma n - 2s)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 + C \cdot \left( R^{-2s} + C R^{-\sigma(n-1)+\varepsilon s} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} \right),$$

*for any $0 < \varepsilon < \frac{(2s-1)\sigma}{s}$. Here $C > 0$ is some constant that depends only on $M[u_0], n, s, \sigma$ and $\varepsilon$.*

**Remark 2.8.** We assume the strict inequality $s > \frac{1}{2}$ in order to have the radial Sobolev (generalized Strauss) inequality (2.39). In the limiting case $s = \frac{1}{2}$, this inequality is no longer valid, and we cannot control the error induced by the nonlinearity.

*Proof of Lemma 2.7.* We shall often omit the time variable $t$ in the argument of $u(t,x)$ for notational convenience. First, recall the Hessian of a radial function $f : \mathbb{R}^n \to \mathbb{C}$ may be written as

$$\partial_{kl}^2 f = \left(\delta_{kl} - \frac{x_k x_l}{r^2}\right)\frac{\partial_r f}{r} + \frac{x_k x_l}{r^2}\partial_r^2 f.$$

Thus, we can rewrite the first term on the right-hand side of Lemma 2.5 as follows[11]

$$4\int_0^\infty m^s \int_{\mathbb{R}^n} \overline{\partial_k u_m}(\partial_{kl}^2\varphi_R)\partial_l u_m \,\mathrm{d}x\,\mathrm{d}m = 4\int_0^\infty m^s \int_{\mathbb{R}^n} (\partial_r^2 \varphi_R)|\nabla u_m|^2 \,\mathrm{d}x\,\mathrm{d}m.$$

From (2.33) and the first inequality in (2.37) we thus deduce

$$\begin{aligned} &4\int_0^\infty m^s \int_{\mathbb{R}^n} \overline{\partial_k u_m}(\partial_{kl}^2\varphi_R)\partial_l u_m \,\mathrm{d}x\,\mathrm{d}m \\ &= 4s\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 - 4\int_0^\infty m^s \int_{\mathbb{R}^n} (1-\partial_r^2\varphi_R)|\nabla u_m|^2 \,\mathrm{d}x\,\mathrm{d}m \\ &\leqslant 4s\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2. \end{aligned}$$

Moreover, from Lemma A.2 we have the bound on the Bi-Laplacian term

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^n} (\Delta^2\varphi_R)|u_m|^2 \,\mathrm{d}x\,\mathrm{d}m \right| \lesssim \|\Delta^2\varphi_R\|_{L^\infty}^s \|\Delta\varphi_R\|_{L^\infty}^{1-s}\|u\|_{L^2}^2 \lesssim R^{-2s},$$

[11] Recall that, by hypothesis, $u(t)$ is radial. Hence, so is $u_m$ and we have $\partial_l u_m = \partial_r u_m \frac{x_l}{r}$, $\overline{\partial_k u_m} = \overline{\partial_r u_m}\frac{x_k}{r}$.

where in the last estimate we used the conservation of mass, i.e., $\|u(t)\|_{L^2}^2 \equiv \|u_0\|_{L^2}^2$, and the properties (2.38) of $\varphi_R$. The last term on the right-hand side of the general virial identity Lemma 2.5 is handled by $\Delta\varphi_R(r) - n \equiv 0$ on $\{r \leqslant R\}$, which gives

$$-\frac{2\sigma}{\sigma+1}\int_{\mathbb{R}^n}(\Delta\varphi_R)|u|^{2\sigma+2}\,\mathrm{d}x = -\frac{2\sigma n}{\sigma+1}\int_{\mathbb{R}^n}|u|^{2\sigma+2}\,\mathrm{d}x - \frac{2\sigma}{\sigma+1}\int_{|x|\geqslant R}(\Delta\varphi_R - n)|u|^{2\sigma+2}\,\mathrm{d}x.$$

Next, we recall the fractional radial Sobolev (generalized Strauss) inequality [CO09] (restated in Proposition A.4 here)

$$\sup_{x\in\mathbb{R}^n\setminus\{0\}} |x|^{\frac{n}{2}-\alpha}|u(x)| \leqslant C(n,\alpha)\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2} \tag{2.39}$$

for all radially symmetric functions $u \in \dot{H}^\alpha(\mathbb{R}^n)$ provided that $\frac{1}{2} < \alpha < \frac{n}{2}$. Now, let $0 < \varepsilon < \frac{(2s-1)\sigma}{s}$ and set $\alpha = \frac{1}{2} + \varepsilon\frac{s}{2\sigma}$, which implies that $\frac{1}{2} < \alpha < s < \frac{n}{2}$. From the interpolation inequality [BCD13, Proposition 1.32] $\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2} \leqslant \|u\|_{L^2}^{1-\frac{\alpha}{s}}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{\alpha}{s}} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{\alpha}{s}}$ and the generalized Strauss inequality (2.39), we deduce

$$\begin{aligned}\int_{|x|\geqslant R}|u|^{2\sigma+2}\,\mathrm{d}x &\leqslant \|u\|_{L^2}^2\|u\|_{L^\infty(|x|\geqslant R)}^{2\sigma} \lesssim C(n,\alpha,\varepsilon)R^{-2\sigma\left(\frac{n}{2}-\alpha\right)}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^{2\sigma} \\ &\lesssim C(n,\alpha,\varepsilon)R^{-2\sigma\left(\frac{n}{2}-\alpha\right)}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{2\sigma\alpha}{s}} \\ &= C(n,\alpha,\varepsilon)R^{-\sigma(n-1)+\varepsilon s}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{\sigma}{s}+\varepsilon}.\end{aligned}$$

Collecting all previous estimates, we realize that we have shown [recall also the property $n - \Delta\varphi_R \geqslant 0$ from (2.37), as well as the properties (2.38), namely that $\|\Delta\varphi_R\|_{L^\infty} \lesssim 1$ and $\Delta\varphi_R \equiv 0$ for $r \geqslant 10R$]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\varphi_R}[u(t)] \leqslant& 4s\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 - \frac{2\sigma n}{\sigma+1}\int_{\mathbb{R}^n}|u(t,x)|^{2\sigma+2}\,\mathrm{d}x \\ &+ C\cdot\left(R^{-2s} + CR^{-\sigma(n-1)+\varepsilon s}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{\sigma}{s}+\varepsilon}\right) \\ =& 4\sigma nE[u_0] - 2(\sigma n - 2s)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \\ &+ C\cdot\left(R^{-2s} + CR^{-\sigma(n-1)+\varepsilon s}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{\sigma}{s}+\varepsilon}\right),\end{aligned}$$

for any $0 < \varepsilon < \frac{2(s-1)\sigma}{s}$, with some constant $C > 0$ that depends only on $M[u_0], n, \varepsilon, s$ and $\sigma$. In the last step, we inserted the definition of energy $E[u(t)]$ and used its conservation, i.e., $E[u(t)] \equiv E[u_0]$. The proof of Lemma 2.7 is now complete. □

For the proof of part (ii) of Theorem 2.1 (blowup in the $L^2$-critical case) one needs to use the following refined version of the localized radial virial estimate Lemma 2.7 involving the nonnegative (see (2.37)) radial functions

$$\psi_{1,R}(r) := 1 - \partial_r^2\varphi_R(r) \geqslant 0 \quad \text{and} \quad \psi_{2,R}(r) := n - \Delta\varphi_R(r) \geqslant 0. \tag{2.40}$$

**Lemma 2.9** (Refined localized radial virial estimate)**.** *Under the hypotheses of Lemma 2.7 and additionally $\sigma = \frac{2s}{n}$, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\varphi_R}[u(t)] \leqslant 8sE[u_0] - 4\int_0^\infty m^s \int_{\mathbb{R}^n} \left\{\psi_{1,R} - c(\eta)\psi_{2,R}^{\frac{n}{2s}}\right\} |\nabla u_m|^2 \,\mathrm{d}x\,\mathrm{d}m + \mathcal{O}\left((1+\eta^{-\beta})R^{-2s} + \eta(1+R^{-2}+R^{-4})\right),$$

*for every $\eta > 0$ and $R > 0$, where $c(\eta) = \frac{\eta}{n+2s}$ and $\beta = \frac{2s}{n-2s}$.*

*Proof, see [BHL16, Lemma 2.3].* For notational convenience, we write $\psi_1 = \psi_{1,R}$ and $\psi_2 = \psi_{2,R}$ in the following. Recall $\psi_2 \equiv 0$ on $\{r \leqslant R\}$. Inspecting the proof of Lemma 2.7, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\varphi_R}[u(t)] &= 8sE[u_0] - 4\int_0^\infty m^s \int_{\mathbb{R}^n} \psi_1 |\nabla u_m|^2 \,\mathrm{d}m\,\mathrm{d}x \\ &\quad + \frac{4s}{n+2s}\int_{\mathbb{R}^n} \psi_2 |u|^{\frac{4s}{n}+2}\,\mathrm{d}x - \int_0^\infty m^s \int_{\mathbb{R}^n} (\Delta^2\varphi_R)|u_m|^2\,\mathrm{d}x\,\mathrm{d}m \\ &= 8sE[u_0] - 4\int_0^\infty m^s \int_{\mathbb{R}^n} \psi_1 |\nabla u_m|^2 \,\mathrm{d}m\,\mathrm{d}x \\ &\quad + \frac{4s}{n+2s}\int_{\mathbb{R}^n} \psi_2 |u|^{\frac{4s}{n}+2}\,\mathrm{d}x + \mathcal{O}(R^{-2s}).\end{aligned} \tag{2.41}$$

We divide the rest of the proof into the following steps.

**Step 1 (Control of Nonlinearity).** Recall that supp $\psi_2 \subset \{|x| \geqslant R\}$. We apply the radial Sobolev inequality (2.39) to the radial function $\psi_2^{\frac{n}{4s}} u \in H^s(\mathbb{R}^n)$ and use that $\|u\|_{L^2} \lesssim 1$, which together yields

$$\begin{aligned}\int_{\mathbb{R}^n} \psi_2 |u|^{\frac{4s}{n}+2}\,\mathrm{d}x &= \int_{|x|\geqslant R} (\psi_2^{\frac{n}{4s}}|u|)^{\frac{4s}{n}} |u|^2\,\mathrm{d}x \leqslant \|\psi_2^{\frac{n}{4s}} u\|_{L^\infty(|x|\geqslant R)}^{\frac{4s}{n}} \|u\|_{L^2}^2 \\ &\lesssim R^{-\frac{2s}{n}(n-2s)} \|(-\Delta)^{\frac{s}{2}}(\psi_2^{\frac{n}{4s}} u)\|_{L^2}^{\frac{4s}{n}} \\ &\leqslant \eta \|(-\Delta)^{\frac{s}{2}}(\psi_2^{\frac{n}{4s}} u)\|_{L^2}^2 + \mathcal{O}(\eta^{-\beta}R^{-2s}), \quad \beta = \frac{2s}{n-2s},\end{aligned} \tag{2.42}$$

where in the last step we used Young's inequality $ab \lesssim \eta a^q + \eta^{-\frac{p}{q}} b^p$ with $\frac{1}{p}+\frac{1}{q} = 1$ such that $q = \frac{n}{2s}$, $\beta = \frac{p}{q}$, and $\eta > 0$ is an arbitrary number. For notational convenience, let us define $\chi := \psi_2^{\frac{n}{4s}}$. From the identity (2.33) we recall that

$$s\|(-\Delta)^{\frac{s}{2}}(\chi u)\|_{L^2}^2 = \int_0^\infty m^s \int_{\mathbb{R}^n} |\nabla(\chi u)_m|^2 \,\mathrm{d}x\,\mathrm{d}m, \tag{2.43}$$

where we denote

$$(\chi u)_m = c_s \frac{1}{-\Delta + m}(\chi u)$$

for $m > 0$ and $c_s$ as in (2.21) above. To estimate the right-hand side of (2.43), we split the $m$-integral into the regions $\{0 < m \leqslant 1\}$ (low frequencies) and $\{m \geqslant 1\}$ (high frequencies), respectively.

To estimate the contribution in the low-frequency region, we notice that

$$\int_0^1 m^s \int_{\mathbb{R}^n} \left| \frac{\nabla}{-\Delta + m} (\chi u) \right|^2 \, dx \, dm \leqslant \int_0^1 m^{s-1} \|\chi u\|_{L^2}^2 \, dm \lesssim 1, \tag{2.44}$$

where we make use of the bounds $\|\frac{\nabla}{-\Delta+m}\|_{L^2 \to L^2} \leqslant m^{-\frac{1}{2}}$, and $\|\chi\|_{L^\infty} \lesssim 1$. To control the right-hand side of (2.43) in the high-frequency region, we need a more elaborate argument worked out in the next step.

**Step 2 (Control of High Frequencies $m \geqslant 1$).** Note the commutator identity $\left[\frac{1}{-\Delta+m}, \chi\right] = \frac{1}{-\Delta+m}[\Delta, \chi]\frac{1}{-\Delta+m}$, which in fact follows by inserting the identity operator and then using the distributive law for operators:

$$\begin{aligned}
\left[\frac{1}{-\Delta+m}, \chi\right] &= \frac{1}{-\Delta+m}\chi(-\Delta+m)\frac{1}{-\Delta+m} - \frac{1}{-\Delta+m}(-\Delta+m)\chi\frac{1}{-\Delta+m} \\
&= \frac{1}{-\Delta+m}\chi(-\Delta)\frac{1}{-\Delta+m} + m\frac{1}{-\Delta+m}\chi\frac{1}{-\Delta+m} \\
&\quad - \frac{1}{-\Delta+m}(-\Delta)\chi\frac{1}{-\Delta+m} - m\frac{1}{-\Delta+m}\chi\frac{1}{-\Delta+m} \\
&= \frac{1}{-\Delta+m}[\Delta, \chi]\frac{1}{-\Delta+m}.
\end{aligned}$$

From this identity, we conclude

$$\begin{aligned}
\nabla(\chi u)_m &= c_s \nabla\left(\frac{1}{-\Delta+m}(\chi u)\right) = c_s \nabla\left(\left[\frac{1}{-\Delta+m}, \chi\right] u + \chi\frac{1}{-\Delta+m}u\right) \\
&= \nabla(\chi u_m) + c_s \nabla\left[\frac{1}{-\Delta+m}, \chi\right] u = \chi \nabla u_m + \nabla\chi u_m + \frac{\nabla}{-\Delta+m}[\Delta, \chi] u_m,
\end{aligned}$$

with $c_s = \sqrt{\frac{\sin \pi s}{\pi}}$ defined in (2.21). Thus we get

$$\begin{aligned}
&\int_1^\infty m^s \int_{\mathbb{R}^n} \left| \frac{\nabla}{-\Delta+m}[\Delta, \chi] u_m \right|^2 \, dx \, dm \lesssim \int_1^\infty m^{s-1} (\|\nabla\chi \cdot \nabla u_m\|_{L^2}^2 + \|\Delta\chi u_m\|_{L^2}^2) \, dm \\
&\lesssim \int_1^\infty m^{s-1} \left\{ \|\nabla\chi\|_{L^\infty}^2 \left\| \frac{\nabla}{-\Delta+m} u \right\|_{L^2}^2 + \|\Delta\chi\|_{L^\infty}^2 \left\| \frac{1}{-\Delta+m} u \right\|_{L^2}^2 \right\} dm \\
&\lesssim \int_1^\infty (m^{s-2} \|\nabla\chi\|_{L^\infty}^2 + m^{s-3} \|\Delta\chi\|_{L^\infty}^2) \, dm \lesssim \frac{\|\nabla\chi\|_{L^\infty}^2}{1-s} + \frac{\|\Delta\chi\|_{L^\infty}^2}{2-s},
\end{aligned}$$

where we used that $[\Delta, \chi] = 2(\nabla\chi) \cdot \nabla + \Delta\chi$ as well as the estimates $\|\frac{\nabla}{-\Delta+m}\|_{L^2 \to L^2} \leqslant m^{-\frac{1}{2}}$ and $\|\frac{1}{-\Delta+m}\|_{L^2 \to L^2} \leqslant m^{-1}$ and conservation of mass in the last line. Similarly, we get

$$\int_1^\infty m^s \int_{\mathbb{R}^n} |\nabla\chi u_m|^2 \, dx \, dm \lesssim \frac{\|\nabla\chi\|_{L^\infty}^2}{1-s}.$$

Recalling that $\chi = \psi_2^{\frac{n}{4s}}$ with $\psi_2 = n - \Delta\varphi_R$, the properties (2.38) are seen to imply that $\|\nabla\chi\|_{L^\infty} \lesssim R^{-1}$ and $\|\Delta\chi\|_{L^\infty} \lesssim R^{-2}$. Thus we can summarize the estimates found above and (2.44) to conclude that

$$s\|(-\Delta)^{\frac{s}{2}}(\chi u)\|_{L^2}^2 = \int_1^\infty m^s \int_{\mathbb{R}^n} \chi^2 |\nabla u_m|^2 \, \mathrm{d}x \, \mathrm{d}m + \mathcal{O}(1 + R^{-2} + R^{-4}). \tag{2.45}$$

**Step 3 (Conclusion).** If we now combine (2.45) with (2.42), we obtain

$$\int_{\mathbb{R}^n} \psi_2 |u|^{\frac{4s}{n}+2} \, \mathrm{d}x = \frac{\eta}{s} \int_1^\infty m^s \int_{\mathbb{R}^n} \chi^2 |\nabla u_m|^2 \, \mathrm{d}x \, \mathrm{d}m + \mathcal{O}(\eta^{-\beta} R^{-2s} + \eta(1 + R^{-2} + R^{-4})).$$

By inserting this back into (2.41) and setting $c(\eta) = \frac{\eta}{n+2s}$, we complete the proof of Lemma 2.9. □

## 2.3 Radial Blowup in $\mathbb{R}^n$: Proof of Theorem 2.1

In this section, we prove the blowup Theorem 2.1. We start with the proof in the case (i), i.e., the $L^2(\mathbb{R}^n)$-supercritical case $s_c > 0$ (equivalently, $\sigma > \frac{2s}{n}$).

### 2.3.1 Proof of Theorem 2.1, Case (i)

Let $n \geqslant 2$ and $s \in \left(\frac{1}{2}, 1\right)$. We consider the $L^2$-supercritical case $0 < s_c \leqslant s$ and impose the extra (technical) condition that $\sigma < 2s$ holds (see below for details on this condition). Recall that we suppose that

$$u \in C([0, T); H^{2s}(\mathbb{R}^n))$$

is a radially symmetric solution to (fNLS). Let $\varphi_R(r)$ with $R > 0$ be a radially symmetric cutoff function on $\mathbb{R}^n$ as introduced in subsection 2.2.2. We shortly write

$$\mathcal{M}_R[u(t)] := \mathcal{M}_{\varphi_R}[u(t)]$$

for the localized virial of $u(t)$.

#### Case of negative energy: $E[u_0] < 0$

We will now exploit the localized radial virial estimate Lemma 2.7. As in that Lemma, for any $0 < \varepsilon < \frac{(2s-1)\sigma}{s}$, we have $\alpha := \frac{1}{2} + \varepsilon\frac{s}{2\sigma} \in \left(\frac{1}{2}, \frac{n}{2}\right)$ and hence that $-\sigma(n-1) + \varepsilon s = -2\sigma\left(\frac{n}{2} - \alpha\right) < 0$. Thus the number

$$-\gamma := \max\{-2s, -\sigma(n-1) + \varepsilon s\} < 0$$

is strictly negative and therefore we obtain for the error term in Lemma 2.7

$$C \cdot \left( R^{-2s} + C R^{-\sigma(n-1)+\varepsilon s} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} \right) \lesssim R^{-\gamma} \cdot \left( 1 + \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} \right)$$

for all $R \geqslant 1$. Now we define the (by $L^2$-supercriticality $s_c > 0$) positive number $\delta := \sigma n - 2s > 0$. Since $R^{-\gamma} \to 0$ as $R \to +\infty$, we can write down Lemma 2.7 in the form (with $o_R(1) \to 0$ as $R \to +\infty$ uniformly in time $t$)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_R[u(t)] &\leqslant 4\sigma n E[u_0] - 2\delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 + o_R(1) \cdot \left( 1 + \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} \right) \\ &\leqslant 2\sigma n E[u_0] - \delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2, \quad \text{for all } t \in [0,T), \end{aligned} \tag{2.46}$$

provided that $R \gg 1$ is taken sufficiently large. In the last step, we used the strict negativity $E[u_0] < 0$, Young's inequality, and that $\frac{\sigma}{s} + \varepsilon < 2$ when $\varepsilon > 0$ is sufficiently small. [This explains the condition $\sigma < 2s$, since such an $\varepsilon > 0$ exists precisely in this case.[12]]

Estimate (2.46) is the key inequality for adapting the strategy of Ogawa-Tsutsumi [OT91] to the setting of fractional NLS with focusing $L^2$-supercritical nonlinearity. Suppose now by contradiction that $u(t)$ exists for all times, i.e., we can take $T = +\infty$. From (2.46) and $E[u_0] < 0$ it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] \leqslant -c$ with some constant $c > 0$. By integrating this bound, we conclude that $\mathcal{M}_R[u(t)] < 0$ for all $t \geqslant t_1$ with some sufficiently large time $t_1 \gg 1$. Thus, if we integrate (2.46) on $[t_1, t]$, we obtain

$$\mathcal{M}_R[u(t)] \leqslant -\delta \int_{t_1}^t \|(-\Delta)^{\frac{s}{2}} u(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leqslant 0 \quad \text{for all } t \geqslant t_1. \tag{2.47}$$

On the other hand, from Lemma A.1 and $L^2$-mass conservation we deduce

$$\begin{aligned} |\mathcal{M}_R[u(t)]| &\lesssim C(\varphi_R) \left( \||\nabla|^{\frac{1}{2}} u(t)\|_{L^2}^2 + \||\nabla|^{\frac{1}{2}} u(t)\|_{L^2} \right) \\ &\lesssim C(\varphi_R) \left( \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{1}{s}} + \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{1}{2s}} \right) \end{aligned} \tag{2.48}$$

where we also used the interpolation estimate [BCD13, Proposition 1.32] $\||\nabla|^{\frac{1}{2}} u\|_{L^2} \leqslant \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{\frac{1}{2s}} \|u\|_{L^2}^{1-\frac{1}{2s}}$ for $s > \frac{1}{2}$. Next, we claim a lower bound on the kinetic energy:

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \gtrsim 1 \quad \text{for all } t \geqslant 0. \tag{2.49}$$

[12] We estimate

$$\begin{aligned} o_R(1) + o_R(1) \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} &\leqslant o_R(1) + C_\delta o_R(1)^{\left(\frac{2}{\frac{\sigma}{s}+\varepsilon}\right)'} + \delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \\ &\leqslant -2n\sigma E[u_0] + \delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \end{aligned}$$

for $R \gg 1$ sufficiently large. The first inequality uses Young's inequality $ab \leqslant C_\delta a^{p'} + \delta b^p$, for which we need that $p = \frac{2}{\frac{\sigma}{s}+\varepsilon} > 1$, the second inequality requires $E[u_0] < 0$.

Indeed, suppose this bound was not true. Then we have that $\|(-\Delta)^{\frac{s}{2}} u(t_k)\|_{L^2} \to 0$ for some sequence of times $t_k \in [0,\infty)$. However, by $L^2$-mass conservation and the Gagliardo-Nirenberg inequality (A.12), this implies that $\|u(t_k)\|_{L^{2\sigma+2}} \to 0$ as well. Hence $E[u(t_k)] \to 0$, which is a contradiction to $E[u(t)] \equiv E[u_0] < 0$. Thus (2.49) is true. [If $s_c = s$, we conclude with the same argument that (2.49) holds, but without using $L^2$-mass conservation, and we replace the Gagliardo-Nirenberg inequality by the Sobolev inequality (A.18).]

If we now combine the lower bound (2.49) with (2.48), we find

$$|\mathcal{M}_R[u(t)]| \lesssim C(\varphi_R) \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{1}{s}}. \tag{2.50}$$

Thus we conclude from (2.47)

$$\mathcal{M}_R[u(t)] \leqslant -\delta \int_{t_1}^{t} \|(-\Delta)^{\frac{s}{2}} u(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \lesssim -C(\varphi_R) \int_{t_1}^{t} |\mathcal{M}_R[u(\tau)]|^{2s} \, \mathrm{d}\tau \quad \text{for all } t \geqslant t_1. \tag{2.51}$$

This nonlinear integral inequality serves as the basis for a standard ODE comparison argument. Indeed, the ODE

$$\begin{cases} \dot{v} = C v^{2s}, \\ v(t_1) = -\mathcal{M}_R[u(t_1)] \equiv a > 0 \end{cases} \tag{2.52}$$

has the exact solution

$$v(t) = a \left( \frac{1}{1 - a^{2s-1} C(2s-1)(t - t_1)} \right)^{\frac{1}{2s-1}},$$

which satisfies

$$v(t) \to +\infty, \quad \text{as } t \uparrow t_1 + \tfrac{1}{a^{2s-1} C(2s-1)} =: t_* > 0.$$

In other words, $w(t) := -v(t)$ is the exact solution to the ODE

$$\begin{cases} \dot{w} = -C(-w)^{2s} \equiv f(w), \\ w(t_1) = \mathcal{M}_R[u(t_1)] < 0 \end{cases}$$

and it satisfies

$$w(t) \to -\infty, \quad \text{as } t \uparrow t_1 + \tfrac{1}{a^{2s-1} C(2s-1)} =: t_* > 0.$$

But since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_R[u(t)] &\leqslant -\delta \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \lesssim -\delta C(\varphi_R)^{-2s} |\mathcal{M}_R[u(t)]|^{2s} \\ &= -\delta C(\varphi_R)^{-2s} (-\mathcal{M}_R[u(t)])^{2s} = f(\mathcal{M}_R[u(t)]) \end{aligned}$$

by (2.46), (2.50) and (2.47), and initially $\mathcal{M}_R[u(t_1)] \leqslant w(t_1)$, we conclude from Growall's Lemma A.6 that $\mathcal{M}_R[u(t)] \to -\infty$ as $t \uparrow t_*$ for some finite $t_* < +\infty$. Hence the solution $u(t)$ cannot exist for all times $t \geqslant 0$ and consequently we must have that $T < +\infty$ holds.[13]

**Case of nonnegative energy:** $E[u_0] \geqslant 0$

Suppose that $E[u_0] \geqslant 0$ and that we have

$$\begin{cases} E[u_0]^{s_c} M[u_0]^{s-s_c} < E[Q]^{s_c} M[Q]^{s-s_c}, \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{s-s_c} > \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s-s_c}. \end{cases} \tag{2.53}$$

Recall our convention that for the energy-critical case $s_c = s$, we set $M[Q]^{s-s_c} = M[Q]^0 = 1$ although, the ground state $Q$ may fail to be in $L^2(\mathbb{R}^n)$ for $s = s_c$; see Appendix A.4 below. Recall also (2) of Remark 2.2.

From the conservation of energy $E[u_0]$ and $L^2$-mass $M[u_0]$ combined with the Gagliardo-Nirenberg inequality (A.12) (when $s_c < s$) or Sobolev's inequality (A.18) (when $s = s_c$), we get

$$\begin{aligned} E[u_0] = E[u(t)] &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{1}{2\sigma+2}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &\geqslant \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{C_{n,s,\sigma}}{2\sigma+2} M[u_0]^{\sigma+1-\frac{n\sigma}{2s}} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{n\sigma}{s}} \\ &= F(\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}), \end{aligned} \tag{2.54}$$

where the function $F : [0, \infty) \to \mathbb{R}$ is defined by

$$F(y) := \frac{1}{2}y^2 - \frac{C_{n,s,\sigma}}{2\sigma+2} M[u_0]^{\frac{\sigma}{s}(s-s_c)} y^{2+\frac{2\sigma s_c}{s}}, \tag{2.55}$$

and $C_{n,s,\sigma} > 0$ denotes the optimal constant for the Gagliardo-Nirenberg inequality (A.12) if $s < s_c$ or Sobolev's inequality (A.18) if $s_c = s$. One checks that $F(y)$ attains a unique global maximum[14]

$$F(y_{\max}) = \frac{s_c}{n} y_{\max}^2 \tag{2.56}$$

at the point

$$y_{\max} = K_{n,s,\sigma}^{\frac{1}{s_c}} M[u_0]^{-\frac{s-s_c}{2s_c}} \quad \text{with } K_{n,s,\sigma} = \left(\frac{2s(\sigma+1)}{n\sigma C_{n,s,\sigma}}\right)^{\frac{s}{2\sigma}}. \tag{2.57}$$

[13] In fact, this shows that the localized virial $\mathcal{M}_R[u(t)]$ blows up at the latest at time $t_* < +\infty$ defined above. By (2.50) then, so does the kinetic energy $\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2$, provided it exists up to that time.

[14] $F(y) = \alpha y^2 - \beta y^{2+\frac{2\sigma s_c}{s}}$ with $s_c > 0$ satisfies $F(y) \to 0$ as $y \downarrow 0$ and $F(y) \to -\infty$ as $y \to \infty$. $F$ is differentiable (in particular, continuous) on $(0, \infty)$ and we have $F(y) > 0$ for small $y > 0$ (take $y > 0$ such that $y^{\frac{2\sigma s_c}{s}} < \frac{\alpha}{\beta}$). Thus $F$ has a global positive maximum. But $F'(y) = 0$ if and only if $y = y_{\max}$.

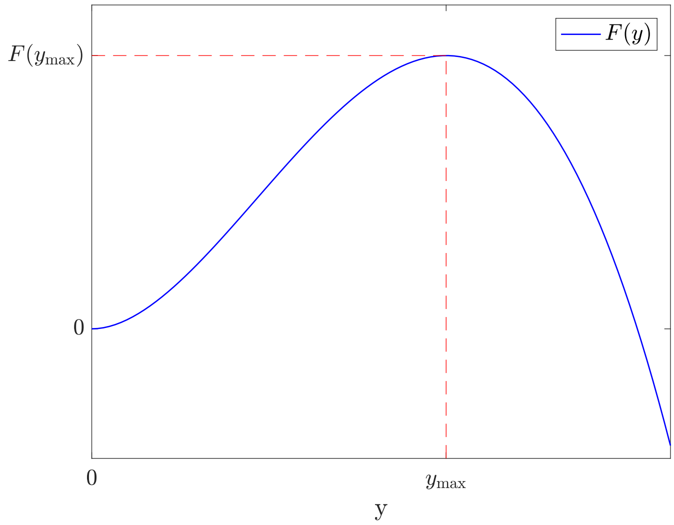


Figure 2.2: The function $F(y)$

The constant $K_{n,s,\sigma}$ is expressed by the Pohozaev identities of Lemma A.7, namely

$$K_{n,s,\sigma} = \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s-s_c} = \left(\frac{s_c}{n}\right)^{-\frac{s_c}{2}} E[Q]^{\frac{s_c}{2}} M[Q]^{\frac{s-s_c}{2}}.$$

[Note that also the correct formulae appear in the energy-critical case $s_c = s$; see Proposition A.11.] Thus hypotheses (2.53) read

$$\begin{cases} E[u_0] < F(y_{\max}), \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2} > y_{\max}. \end{cases}$$

This initial barrier on the kinetic energy can never be crossed. Namely, by a continuity-in-time argument, we deduce that

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} > y_{\max}, \quad \text{for all } t \in [0, T). \tag{2.58}$$

Indeed, suppose this bound was not true. Then there must be some $\tilde{t} \in (0, T)$ such that $\|(-\Delta)^{\frac{s}{2}} u(\tilde{t})\|_{L^2} \leqslant y_{\max}$[15] and it follows from $u \in C([0, T); H^{2s}(\mathbb{R}^n))$ that there exists some $t_* \in (0, \tilde{t}]$ such that $\|(-\Delta)^{\frac{s}{2}} u(t_*)\|_{L^2} = y_{\max}$. Consequently

$$F(y_{\max}) > E[u_0] \overset{(2.54)}{\geqslant} F(\|(-\Delta)^{\frac{s}{2}} u(t_*)\|_{L^2}) = F(y_{\max}),$$

a contradiction. Therefore the lower bound (2.58) holds.

Next, we pick $\eta > 0$ so small that still

$$E[u_0]^{s_c} M[u_0]^{s-s_c} \leqslant (1-\eta)^{s_c} E[Q]^{s_c} M[Q]^{s-s_c}.$$

[15]The case $\tilde{t} = 0$ cannot occur due to the hypothesis $\|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2} > y_{\max}$.

From (2.58), we then obtain by an elementary calculation that

$$2\delta(1-\eta)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \geqslant 4\sigma n E[u_0] \quad \text{for all } t \in [0,T),$$

where we recall that $\delta = \sigma n - 2s > 0$. By inserting this bound into the differential inequality of the localized radial virial estimate Lemma 2.7, we get (with $o_R(1) \to 0$ as $R \to \infty$ uniformly in $t$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] &\leqslant 4\sigma n E[u_0] - 2\delta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 + o_R(1)\cdot\left(1+\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon}\right)\\ &\leqslant -2\delta\eta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 + o_R(1)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} + o_R(1)\\ &\leqslant -(\delta\eta + o_R(1))\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 + o_R(1),\end{aligned} \tag{2.59}$$

for $R \gg 1$ large enough, where we have chosen $\varepsilon > 0$ small enough such that $\frac{\sigma}{s}+\varepsilon < 2$ (which is possible, since $\sigma < 2s$ by assumption) and used Young's inequality similarly as before.[16] Choosing $R \gg 1$ sufficiently large and using (2.58) again [that is, the estimate $o_R(1)(1-\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2) \leqslant o_R(1)(1-y_{\max}^2) \leqslant \frac{\delta\eta}{2}y_{\max}^2 \leqslant \frac{\delta\eta}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2$ for $R \gg 1$ large enough], we thus conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] \leqslant -\frac{\delta\eta}{2}\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \quad \text{for all } t \in [0,T). \tag{2.60}$$

Suppose now that $T = +\infty$ holds. Since $\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2} > y_{\max} > 0$ for all $t \geqslant 0$, we see from (2.60) by integration that $\mathcal{M}_R[u(t)] < 0$ for all $t \geqslant t_1$ with some sufficiently large time $t_1 \gg 1$. Hence, by integrating (2.60) on $[t_1, t]$, we obtain

$$\mathcal{M}_R[u(t)] \leqslant -\frac{\delta\eta}{2}\int_{t_1}^t \|(-\Delta)^{\frac{s}{2}}u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leqslant 0 \quad \text{for all } t \geqslant t_1.$$

By following exactly the steps after (2.47) above (now, the lower bound (2.58) on the kinetic energy is the substitute of the lower bound (2.49) from before), we deduce that $u(t)$ cannot exist for all times $t \geqslant 0$.

The proof of Theorem 2.1, case (i) is now complete. □

[16] Indeed, the last estimate in (2.59) can be written as

$$o_R(1)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{\sigma}{s}+\varepsilon} + o_R(1)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \leqslant \delta\eta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2. \tag{$*$}$$

Taking $p = \frac{2}{\frac{\sigma}{s}+\varepsilon} > 1$ and denoting $p'$ its dual, Young's inequality with $\frac{\delta\eta}{2}$ bounds the left side of ($*$) from above by

$$C_{\frac{\delta\eta}{2}} o_R(1)^{p'} + \left(\frac{\delta\eta}{2} + o_R(1)\right)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2.$$

Now taking $R \gg 1$ so big that both $o_R(1) \leqslant \frac{\delta\eta}{4}$ and $C_{\frac{\delta\eta}{2}} o_R(1)^{p'} \leqslant \frac{\delta\eta}{4}y_{\max}^2$, the previous expression is bounded from above by $\delta\eta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2$. Hence ($*$).

### 2.3.2 Proof of Theorem 2.1, Case (ii)

Let $n \geqslant 2$, $s \in \left(\frac{1}{2}, 1\right)$, and we consider the $L^2(\mathbb{R}^n)$-critical exponent $\sigma = \frac{2s}{n}$. We assume that

$$u \in C([0,T); H^{2s}(\mathbb{R}^n))$$

is a radially symmetric solution of (fNLS) with negative energy

$$E[u_0] < 0.$$

Let $\varphi_R(r)$ with $R > 0$ be a radially symmetric cutoff function on $\mathbb{R}^n$ as introduced in subsection 2.2.2. Recall the definitions of the functions $\psi_{1,R}(r)$ and $\psi_{2,R}(r)$ from (2.40), depending on the function $\varphi_R(r)$. As in Lemma 2.9, define $c(\eta) = \frac{\eta}{n+2s}$. As shown in Appendix A.4 below, we can choose $\varphi_R(r)$ and $\eta > 0$ sufficiently small such that

$$\psi_{1,R}(r) - c(\eta)(\psi_{2,R}(r))^{\frac{n}{2s}} \geqslant 0 \quad \text{for all } r > 0,$$

and for all $R > 0$.

Thus if we choose $\eta \ll 1$ small enough to achieve this, and then $R \gg 1$ large enough, we can apply Lemma 2.9 to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] \leqslant 4sE[u_0] \quad \text{for all } t \in [0,T), \tag{2.61}$$

where we write $\mathcal{M}_{\varphi_R}[u(t)] = \mathcal{M}_R[u(t)]$ again for notational convenience. Now suppose that $u(t)$ exists for all times $t \geqslant 0$, i.e., we can take $T = +\infty$. From (2.61) we infer that (by negativity of energy $E[u_0]$)

$$\mathcal{M}_R[u(t)] \leqslant -ct \quad \text{for all } t \geqslant t_0 \tag{2.62}$$

with some sufficiently large time $t_0 > 0$ and some constant $c > 0$ depending only on $s$ and $E[u_0] < 0$. On the other hand, if we invoke Lemma A.1, we see that

$$\begin{aligned} |\mathcal{M}_R[u(t)]| &\lesssim C(\varphi_R)\left(\||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}\||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}\right) \\ &\lesssim C(\varphi_R)\left(\||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}^2\right) \\ &\lesssim C(\varphi_R)\left(\||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}^2 + 1\right) \lesssim C(\varphi_R)\left(\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{s}} + 1\right), \end{aligned} \tag{2.63}$$

where we also used the conservation of $L^2$-mass of $u(t)$ together with the interpolation estimate $\||\nabla|^{\frac{1}{2}}u\|_{L^2} \leqslant \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{1}{2s}}\|u\|_{L^2}^{1-\frac{1}{2s}}$ for $s > \frac{1}{2}$. By combining (2.63) and (2.62), we get

$$+ct \leqslant -\mathcal{M}_R[u(t)] = |\mathcal{M}_R[u(t)]| \lesssim C(\varphi_R)\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{s}} + C(\varphi_R) \quad \text{for all } t \geqslant t_0.$$

Thus

$$\begin{aligned} \left(\frac{c}{C(\varphi_R)}\right)^s t^s &\lesssim \left(\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{s}} + 1\right)^s \\ &\leqslant \|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2} + 1, \end{aligned}$$

using also the inequality $(a+b)^s \leqslant a^s + b^s$ for $0 \leqslant s \leqslant 1$. Since the left side tends to $+\infty$ as $t \to +\infty$, so does the right side. Hence there must exist $t_* > t_0$ such that for all $t \geqslant t_*$, we have $1 < \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}$. It follows that

$$Ct^s \leqslant \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \quad \text{for all } t \geqslant t_*$$

with some sufficiently large time $t_* > 0$ and some constant $C > 0$ that depends only on $u_0$, $s$ and $n$.

The proof of Theorem 2.1, case (ii) is now complete. □

# A Blowup

## A.1 Various Estimates

**Lemma A.1** (Bound on localized virial). *Let $n \geqslant 1$, and suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is such that $\nabla\varphi \in (W^{1,\infty}(\mathbb{R}^n))^n$. Then, for all $u \in H^{\frac{1}{2}}(\mathbb{R}^n)$, we have the estimate*

$$|\langle u, \nabla\varphi \cdot \nabla u\rangle| \leqslant C\left(\||\nabla|^{\frac{1}{2}}u\|_{L^2}^2 + \|u\|_{L^2}\||\nabla|^{\frac{1}{2}}u\|_{L^2}\right), \tag{A.1}$$

*with some constant $C > 0$ that depends only on $\|\nabla\varphi\|_{W^{1,\infty}}$ and $n$. In particular, this yields a bound on the localized virial $\mathcal{M}_\varphi[u(t)] = 2\,\mathrm{Im}\int_{\mathbb{R}^n} \overline{u(t)}\nabla\varphi \cdot \nabla u(t)\,\mathrm{d}x$ of a solution $u \in C([0,T); H^{2s}(\mathbb{R}^n))$ to (fNLS) with $s \geqslant \frac{1}{2}$, namely*

$$|\mathcal{M}_\varphi[u(t)]| \lesssim C(\|\nabla\varphi\|_{W^{1,\infty}})\|u(t)\|_{H^{\frac{1}{2}}}^2.$$

*Proof.* We rewrite the gradient as $\nabla = |\nabla|^{\frac{1}{2}}\frac{\nabla}{|\nabla|}|\nabla|^{\frac{1}{2}}$ and use the Cauchy-Schwarz inequality to estimate ($\varphi$ is real-valued)

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \overline{u}(x)\nabla\varphi(x)\cdot\nabla u(x)\,\mathrm{d}x\right| &= \left|\left\langle |\nabla|^{\frac{1}{2}}((\nabla\varphi)u), \frac{\nabla}{|\nabla|}|\nabla|^{\frac{1}{2}}u\right\rangle\right| \\ &\leqslant \||\nabla|^{\frac{1}{2}}((\nabla\varphi)u)\|_{L^2}\left\|\frac{\nabla}{|\nabla|}|\nabla|^{\frac{1}{2}}u\right\|_{L^2} \\ &\lesssim \||\nabla|^{\frac{1}{2}}((\nabla\varphi)u)\|_{L^2}\||\nabla|^{\frac{1}{2}}u\|_{L^2}\end{aligned}$$

where in the last step we used the fact that the Riesz projector is a bounded operator (multiplier) on $L^2(\mathbb{R}^n)$; in fact (see [LP09, Exercise 2.11, page 41] or [Ste93]), more generally

$$\left\|\frac{\nabla}{|\nabla|}g\right\|_{L^p} \leqslant C_p\|g\|_{L^p}, \quad 1 < p < \infty.$$

Now we claim that

$$\||\nabla|^{\frac{1}{2}}((\nabla\varphi)u)\|_{L^2} \lesssim \|\nabla\varphi\|_{W^{1,\infty}}\left(\||\nabla|^{\frac{1}{2}}u\|_{L^2} + \|u\|_{L^2}\right). \tag{A.2}$$

This estimate is a consequence of the fact that $\nabla\varphi$ is bounded with bounded derivatives and $u \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$. In fact, the proof of [LL01, Theorem 7.16] can be adapted as follows. Recall the relation of the homogeneous Sobolev norms to the Gagliardo semi-norms, i.e., $\|\cdot\|_{\dot{H}^s} = C[\cdot]_{H^s}$, where $C > 0$ is some constant depending only on $n$ and $s$. Note the inequality

$$|ab-cd|^2 = \frac{1}{4}|(a-c)(b+d)+(a+c)(b-d)|^2 \leqslant |a-c|^2(|b|^2+|d|^2)+(|a|^2+|c|^2)|b-d|^2.$$[1]

[1] It is clear by the elementary inequality $|z+w|^2 \leqslant 2(|z|^2+|w|^2)$.

Now estimate [also use the symmetry of some of the appearing integrals in $x$ and $y$ and Fubini's Theorem]

$$
\begin{aligned}
&\||\nabla|^{\frac{1}{2}}((\nabla\varphi)u)\|_{L^2}^2 = C[(\nabla\varphi)u]_{H^{\frac{1}{2}}}^2 = C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\nabla\varphi(x)u(x)-\nabla\varphi(y)u(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\nabla\varphi(y)|^2\frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\nabla\varphi(x)-\nabla\varphi(y)|^2\frac{|u(x)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \|\nabla\varphi\|_{L^\infty}^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y + \|\nabla^2\varphi\|_{L^\infty}^2\int\int_{|x-y|\leqslant 1}\frac{1}{|x-y|^{n-1}}|u(x)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \|\nabla\varphi\|_{L^\infty}^2\int\int_{|x-y|>1}\frac{1}{|x-y|^{n+1}}|u(x)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \|\nabla\varphi\|_{W^{1,\infty}}^2\left(\||\nabla|^{\frac{1}{2}}u\|_{L^2}^2 + \|u\|_{L^2}^2\right).
\end{aligned}
$$

Taking the square root proves (A.2) and hence the lemma. □

**Lemma A.2** (Bound on Bi-Laplacian term). *Let $n \geqslant 1$, $s \in (0,1)$, and suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\Delta\varphi \in W^{2,\infty}(\mathbb{R}^n)$. Then, for all $u \in L^2(\mathbb{R}^n)$, we have*

$$
\left|\int_0^\infty m^s\int_{\mathbb{R}^n}(\Delta^2\varphi)|u_m|^2\,\mathrm{d}x\,\mathrm{d}m\right| \lesssim \|\Delta^2\varphi\|_{L^\infty}^s\|\Delta\varphi\|_{L^\infty}^{1-s}\|u\|_{L^2}^2.
$$

**Remark A.3.** A direct application of Hölder's inequality yields the bound

$$
\left|\int_0^\infty m^s\int_{\mathbb{R}^n}(\Delta^2\varphi)|u_m|^2\,\mathrm{d}x\,\mathrm{d}m\right| \lesssim \|\Delta^2\varphi\|_{L^\infty}\||\nabla|^{s-1}u\|_{L^2}^2,
$$

by using that $\frac{\sin\pi s}{\pi}\int_0^\infty \frac{m^s}{(|\xi|^2+m)^2}\,\mathrm{d}m = s|\xi|^{2s-2}$ as in (2.33). However, such a bound in terms of the negative order Sobolev norm $\|u\|_{\dot{H}^{s-1}}$ would be of no use to us.

*Proof of Lemma A.2.* This is an extension of the proof of [KLR13, Lemma B.3] to $n \geqslant 1$ and $s \in (0,1)$. Let us split the $m$-integral into $\int_0^\Lambda \cdots + \int_\Lambda^\infty \ldots$, where $\Lambda > 0$ will be fixed in a moment. For the first part, we integrate by parts in $x$ twice and use Hölder's inequality to find

$$
\begin{aligned}
&\left|\int_0^\Lambda m^s\int_{\mathbb{R}^n}(\Delta^2\varphi)|u_m|^2\,\mathrm{d}x\,\mathrm{d}m\right| \\
&= \left|\int_0^\Lambda m^s\int_{\mathbb{R}^n}(\Delta\varphi)\left\{(\Delta\overline{u_m})u_m + 2\nabla\overline{u_m}\cdot\nabla u_m + \overline{u_m}(\Delta u_m)\right\}\,\mathrm{d}x\,\mathrm{d}m\right| \\
&\lesssim \|\Delta\varphi\|_{L^\infty}\int_0^\Lambda m^s(\|\Delta u_m\|_{L^2}\|u_m\|_{L^2} + \|\nabla u_m\|_{L^2}^2)\,\mathrm{d}m \\
&\lesssim \|\Delta\varphi\|_{L^\infty}\|u\|_{L^2}^2\left(\int_0^\Lambda m^{s-1}\,\mathrm{d}m\right) \lesssim \|\Delta\varphi\|_{L^\infty}\|u\|_{L^2}^2.
\end{aligned}
\tag{A.3}
$$

Here, we have also used the bounds

$$\|\Delta u_m\|_{L^2} \lesssim \|u\|_{L^2}, \quad \|\nabla u_m\|_{L^2} \lesssim m^{-\frac{1}{2}}\|u\|_{L^2}, \quad \|u_m\|_{L^2} \lesssim m^{-1}\|u\|_{L^2},$$

which immediately follow from the definition $u_m = c_s \cdot (-\Delta + m)^{-1} u$ (as in (2.20), (2.21) above) and Plancherel's Theorem. For the second part, we find

$$\begin{aligned} &\left| \int_\Lambda^\infty m^s \int_{\mathbb{R}^n} (\Delta^2 \varphi) |u_m|^2 \, \mathrm{d}x \, \mathrm{d}m \right| \lesssim \|\Delta^2 \varphi\|_{L^\infty} \left( \int_\Lambda^\infty m^s \|u_m\|_{L^2}^2 \, \mathrm{d}m \right) \\ &\lesssim \|\Delta^2 \varphi\|_{L^\infty} \|u\|_{L^2}^2 \left( \int_\Lambda^\infty m^{s-2} \, \mathrm{d}m \right) \lesssim \|\Delta^2 \varphi\|_{L^\infty} \|u\|_{L^2}^2 \Lambda^{s-1}. \end{aligned} \tag{A.4}$$

Combining (A.3) and (A.4) yields the estimate

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^n} (\Delta^2 \varphi) |u_m|^2 \, \mathrm{d}x \, \mathrm{d}m \right| \lesssim (\|\Delta \varphi\|_{L^\infty} \Lambda^s + \|\Delta^2 \varphi\|_{L^\infty} \Lambda^{s-1}) \|u\|_{L^2}^2 \tag{A.5}$$

for arbitrary $\Lambda > 0$. Minimizing this bound with respect to $\Lambda$ gives the optimal choice $\Lambda = \frac{1-s}{s} \frac{\|\Delta^2 \varphi\|_{L^\infty}}{\|\Delta \varphi\|_{L^\infty}}$. By evaluating (A.5) with this particular $\Lambda$ the lemma is proved. □

## A.2 Fractional Radial Sobolev Inequality

Let $\dot{H}^s_{\mathrm{rad}}(\mathbb{R}^n) := \{u \in \dot{H}^s(\mathbb{R}^n);\ u \text{ is radially symmetric}\}$. Cho and Ozawa have proved the following inequality; see also [Str77, Radial Lemma 1, p. 155] for the original inequality due to Strauss.

**Proposition A.4** (Generalized Strauss inequality; see [CO09])**.** *Let $n \geqslant 2$ and $s \in \left(\frac{1}{2}, \frac{n}{2}\right)$. Then there exists some constant $C(n,s)$, depending only on $n$ and $s$, such that*

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{\frac{n}{2}-s} |u(x)| \leqslant C(n,s) \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}, \quad \forall u \in \dot{H}^s_{\mathrm{rad}}(\mathbb{R}^n). \tag{A.6}$$

*Proof (see [CO09]).* In [Tay11a, p. 264, formula (6.8)],[2] it is argued that the Fourier transform of a radial function $f$ on $\mathbb{R}^n$ can be represented in terms of Bessel functions $J_\nu$ by

$$\mathcal{F}f(\xi) = |\xi|^{1-\frac{n}{2}} \int_0^\infty f(\varrho) J_{\frac{n}{2}-1}(\varrho|\xi|) \varrho^{\frac{n}{2}} \, \mathrm{d}\varrho. \tag{A.7}$$

Let $u \in \dot{H}^s_{\mathrm{rad}}(\mathbb{R}^n)$. We obtain

$$u(x) = |x|^{1-\frac{n}{2}} \int_0^\infty \widehat{u}(\varrho) J_{\frac{n}{2}-1}(\varrho|x|) \varrho^{\frac{n}{2}} \, \mathrm{d}\varrho. \tag{A.8}$$

[2] See also [Gra08, Appendix B.4/B.5].

Hence, by Cauchy-Schwarz,

$$\begin{aligned}
|x|^{\frac{n}{2}-s}|u(x)| &\leqslant |x|^{1-s}\int_0^\infty |\widehat{u}(\varrho)||J_{\frac{n}{2}-1}(\varrho|x|)|\varrho^{\frac{n}{2}}\,\mathrm{d}\varrho \\
&\leqslant |x|^{1-s}\left(\int_0^\infty |J_{\frac{n}{2}-1}(\varrho|x|)|^2\varrho^{1-2s}\,\mathrm{d}\varrho\right)^{\frac12}\left(\int_0^\infty |\widehat{u}(\varrho)|^2\varrho^{2s+(n-1)}\,\mathrm{d}\varrho\right)^{\frac12} \\
&= \left(\int_0^\infty |J_{\frac{n}{2}-1}(\vartheta)|^2\vartheta^{1-2s}\,\mathrm{d}\vartheta\right)^{\frac12}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^\infty |\widehat{u}(\xi)|^2|\xi|^{2s}\,\mathrm{d}\xi\right)^{\frac12} \\
&= C(n,s)\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}.
\end{aligned}$$

Here, we substituted $\vartheta = \varrho|x|$ and noticed that

$$\int_0^\infty |\widehat{u}(\varrho)|^2\varrho^{2s+(n-1)}\,\mathrm{d}\varrho = \frac{1}{n\omega_n}\int_0^\infty\int_{\partial B_\varrho(0)}|\widehat{u}(\varrho)|^2\varrho^{2s}\,\mathrm{d}S\,\mathrm{d}\varrho = \frac{1}{n\omega_n}\int_{\mathbb{R}^n}|\widehat{u}(\xi)|^2|\xi|^{2s}\,\mathrm{d}\xi$$

using polar coordinates, and $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$. The constant is

$$C(n,s) = \left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^\infty |J_{\frac{n}{2}-1}(\vartheta)|^2\vartheta^{1-2s}\,\mathrm{d}\vartheta\right)^{\frac12} = \left(\frac{\Gamma(2s-1)\Gamma\left(\frac{n}{2}-s\right)\Gamma\left(\frac{n}{2}\right)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)^2\Gamma\left(\frac{n}{2}-1+s\right)}\right)^{\frac12},$$

using that [Wat95, p. 403, formula (2)]

$$\int_0^\infty |J_{\frac{n}{2}-1}(\vartheta)|^2\vartheta^{1-2s}\,\mathrm{d}\vartheta = \frac{\Gamma(2s-1)\Gamma\left(\frac{n}{2}-s\right)}{2^{2s-1}\Gamma(s)^2\Gamma\left(\frac{n}{2}-1+s\right)}.$$

□

## A.3 ODE Comparison Principle

**Lemma A.5** (Gronwall's Inequality in differential form [Eva97, p. 624])**.** *Let $\eta$ be a nonnegative, absolutely continuous function on $[t_0, T]$, which satisfies for a.e. $t$ the differential inequality*

$$\dot{\eta}(t) \leqslant \phi(t)\eta(t) + \psi(t),$$

*where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[t_0, T]$. Then*

$$\eta(t) \leqslant e^{\int_{t_0}^t \phi(s)\,\mathrm{d}s}\left(\eta(t_0) + \int_{t_0}^t \psi(s)\,\mathrm{d}s\right) \quad \text{for all } t_0 \leqslant t \leqslant T.$$

*Proof [Eva97].* The claim follows from

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\eta(s)e^{-\int_{t_0}^s \phi(r)\,\mathrm{d}r}\right) = e^{-\int_{t_0}^t \phi(r)\,\mathrm{d}r}(\dot{\eta}(s) - \phi(s)\eta(s)) \leqslant e^{-\int_{t_0}^t \phi(r)\,\mathrm{d}r}\psi(s) \leqslant \psi(s)$$

and integration on $[t_0, t]$. □

**Lemma A.6** (A version of Gronwall's inequality)**.** *Let $f(t,u)$ be continuous in $t$ and Lipschitz continuous in $u$. Suppose that $u(t), v(t)$ are $C^1$ for $t \geqslant t_0$ and satisfy*

$$\dot{u}(t) \leqslant f(t,u(t)), \quad \dot{v}(t) = f(t,v(t))$$

*and initially $u(t_0) \leqslant v(t_0)$. Then $u(t) \leqslant v(t)$ for all $t \geqslant t_0$.*

*Proof.* Assume by contradiction that $u(T) > v(T)$ for some $T > t_0$, and set

$$t_1 = \sup\{t;\ t_0 \leqslant t < T \text{ and } u(t) \leqslant v(t)\}.$$

By continuity of $u - v$, we have $t_0 \leqslant t_1 < T$, $u(t_1) = v(t_1)$, and $u(t) > v(t)$ for all $T > t > t_1$. Thus for $t_1 \leqslant t \leqslant T$, we have $|u(t) - v(t)| = u(t) - v(t)$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t) - v(t)) \leqslant f(t,u(t)) - f(t,v(t)) \leqslant L|u(t) - v(t)| = L(u(t) - v(t))$$

by Lipschitz continuity of $f(t,u)$ in $u$. Applying Lemma A.5 with the nonnegative functions $\eta \equiv u - v$, $\phi \equiv L$, $\psi \equiv 0$ and the interval $[t_1, T]$ gives $u(t) - v(t) \leqslant e^{L(t-t_1)}(u(t_1) - v(t_1)) = 0$ on $[t_1, T]$, a contradiction. □

## A.4 Ground States and Cutoff Functions

Let $n \geqslant 1$, $s \in (0,1)$ and $\sigma > 0$. Recall the definition of the scaling index $s_c = \frac{n}{2} - \frac{s}{\sigma}$.

### A.4.1 Pohozaev Identities: the Energy-Subcritical Case $s_c < s$

Making the solitary wave solution ansatz $u(t,x) = e^{i\omega t}\omega^{\frac{1}{2\sigma}}Q(\omega^{\frac{1}{2s}}x)$ [FL13, p. 263], it is easily checked that $u$ solves (fNLS) if and only if the profile $Q$ solves the stationary problem

$$(-\Delta)^s Q + Q - |Q|^{2\sigma}Q = 0 \quad \text{in } \mathbb{R}^n \tag{A.9}$$

[simply use the behaviour of the Fourier transform $\mathcal{F}$ when acting on dilations and a change of variable to see that $((-\Delta)^s f)(x) = \omega((-\Delta)^s Q)(\omega^{\frac{1}{2s}}x)$, where $f(x) := (\delta_{\omega^{\frac{1}{2s}}}Q)(x) := Q(\omega^{\frac{1}{2s}}x)$.]

Pohozaev-type identities show that the energy-subcriticality condition $s_c < s$ is necessary for (A.9) to possess nontrivial solutions $Q \in H^s(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$; see Remark A.8 below and also [FLS16, p. 1681]. Conversely, $s_c < s$ is also sufficient for the existence of such nontrivial solutions $Q$ [FLS16, p. 1681]. Indeed, solutions can be constructed variationally, namely by considering the associated Weinstein functional

$$\mathcal{W}^s_{n,\sigma}[u] := \frac{\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{n\sigma}{s}}\|u\|_{L^2}^{2\sigma+2-\frac{n\sigma}{s}}}{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}} \tag{A.10}$$

and solving the corresponding minimization problem

$$C_{n,s,\sigma}^{-1} := \inf_{0 \not\equiv u \in H^s(\mathbb{R}^n)} \mathcal{W}_{n,\sigma}^s[u]. \tag{A.11}$$

The infimum is attained, thus some nontrivial solution $Q$ exists, and moreover, $Q$ is also unique modulo symmetries [FLS16]. Therefore $C_{n,s,\sigma} > 0$ is the sharp constant for the Gagliardo-Nirenberg inequality[3]

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{n,s,\sigma} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{\frac{n\sigma}{s}} \|u\|_{L^2}^{2\sigma+2-\frac{n\sigma}{s}}, \quad u \in H^s(\mathbb{R}^n). \tag{A.12}$$

Nontrivial optimizers $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ of (A.12) (they turn (A.12) into an equality), equivalently, nontrivial minimizers $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ of the Weinstein functional (A.10) are called *ground states*. From the invariance of the Weinstein functional under the rescaling $Q \mapsto \mu Q(\lambda \cdot)$, it can be checked that any ground state $Q$ necessarily solves the Euler-Lagrange equation (A.9) after being rescaled in this way with some appropriate constants $\mu$ and $\lambda$. Moreover, as shown in [FL13, FLS16], the function $Q$ is smooth, and we can choose $Q = Q(|x|) > 0$ to be radially symmetric, strictly positive, and strictly decreasing in $|x|$.

We have the following identities for real-valued solutions of (A.9) and the ground state $Q$ [BHL16, Proposition B.1]; see also [BL15, Proposition A.1] and [Caz03, Lemma 8.1.2] for Pohozaev identities in the context of biharmonic and classical NLS, respectively.

**Proposition A.7** (Pohozaev-type identities for $s_c < s$)**.** *Let $n \geqslant 1$, $s \in (0,1)$ and $0 < \sigma < \sigma_*$ (equivalently,[4] $s_c < s$). Then any real-valued solution $Q \in H^s(\mathbb{R}^n)$ of (A.9) necessarily satisfies the Pohozaev identities*

$$\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 + \|Q\|_{L^2}^2 - \|Q\|_{L^{2\sigma+2}}^{2\sigma+2} = 0, \tag{A.13}$$

$$(2s-n)\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 - n\|Q\|_{L^2}^2 + \frac{2n}{2\sigma+2}\|Q\|_{L^{2\sigma+2}}^{2\sigma+2} = 0, \tag{A.14}$$

*and consequently*

$$\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 = \left(\frac{n}{n+2(s-s_c)}\right) \|Q\|_{L^{2\sigma+2}}^{2\sigma+2} = \left(\frac{n}{2(s-s_c)}\right) \|Q\|_{L^2}^2. \tag{A.15}$$

[3]In classical ($s = 1$) NLS theory one makes use of the classical Gagliardo-Nirenberg inequality, namely

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{n,\sigma} \|\nabla u\|_{L^2}^{n\sigma} \|u\|_{L^2}^{2\sigma+2-n\sigma}, \quad u \in H^1(\mathbb{R}^n), \ 0 < \sigma < \frac{2}{n-2} \text{ for } n \geqslant 2;$$

see, for example [Wei83, inequality (I.2)].

[4]Note that when $n \geqslant 1$, $s \in (0,1)$, and $\sigma > 0$ are fixed numbers, the hypotheses $s_c < s$ and $\sigma < \sigma_*$ are equivalent in any of the cases $s < \frac{n}{2}$, $s = \frac{n}{2}$ (i.e., $n = 1$ and $s = \frac{1}{2}$), and $s > \frac{n}{2}$ (i.e., $n = 1$ and $s > \frac{1}{2}$).

*If moreover $Q \in H^s(\mathbb{R}^n)$ is a ground state, then*

$$K_{n,s,\sigma} = \|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{s-s_c} = \left(\frac{s_c}{n}\right)^{-\frac{s_c}{2}} E[Q]^{\frac{s_c}{2}} M[Q]^{\frac{s-s_c}{2}}, \tag{A.16}$$

*where*

$$K_{n,s,\sigma} := \left(\frac{2s(\sigma+1)}{n\sigma C_{n,s,\sigma}}\right)^{\frac{s}{2\sigma}}. \tag{A.17}$$

*Proof.* Integrating (A.9) against $Q$ gives (A.13). Next, integrating (A.9) against $x \cdot \nabla Q$ yields (A.14): one uses that $\langle x \cdot \nabla Q, (-\Delta)^s Q\rangle = \left(\frac{2s-n}{2}\right)\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^2$, and $\langle x \cdot \nabla Q, Q\rangle = -\frac{n}{2}\|Q\|_{L^2}^2$, and again the identity $\frac{\sigma+1}{\sigma}\nabla(|Q|^{2\sigma})|Q|^2 = \nabla(|Q|^{2\sigma+2})$ to handle the nonlinearity.

To see the former, note that the commutator identity [FLS16, p. 1703] $[x \cdot \nabla, (-\Delta)^s] = -2s(-\Delta)^s$ immediately implies

$$\begin{aligned}\langle x \cdot \nabla Q, (-\Delta)^s Q\rangle &= \langle (-\Delta)^{\frac{s}{2}}(x \cdot \nabla Q), (-\Delta)^{\frac{s}{2}}Q\rangle \\ &= -\langle [x \cdot \nabla, (-\Delta)^{\frac{s}{2}}]Q, (-\Delta)^{\frac{s}{2}}Q\rangle + \langle x \cdot \nabla(-\Delta)^{\frac{s}{2}}Q, (-\Delta)^{\frac{s}{2}}Q\rangle \\ &= s\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^2 - \frac{n}{2}\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^2.\end{aligned}$$

The commutator identity itself is easily seen on the Fourier side via the identification $xf(x) \leftrightarrow i\partial_\xi \widehat{f}(\xi)$, $\partial_x f(x) \leftrightarrow i\xi\widehat{f}(\xi)$, so that for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have (sum over $j$)

$$\begin{aligned}\mathcal{F}([x \cdot \nabla, (-\Delta)^s]\phi)(\xi) &= i\partial_{\xi_j}(i\xi_j|\xi|^{2s}\widehat{\phi}) - |\xi|^{2s}\left(i\partial_{\xi_j}(i\xi_j\widehat{\phi})\right) \\ &= -2s\xi_j^2|\xi|^{2s-2}\widehat{\phi} = -2s|\xi|^{2s}\widehat{\phi} = -2s\mathcal{F}((-\Delta)^s\phi)(\xi).\end{aligned}$$

Combining (A.13) and (A.14), the equalities (A.15) immediately follow. Finally, using that a ground state $Q$ turns the Gagliardo-Nirenberg inequality (A.12) into an equality and expressing $\|Q\|_{L^{2\sigma+2}}^{2\sigma+2}$ in this equality through $\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^2$ via (A.15) yields the first equality in (A.16). Expressing $\|Q\|_{L^{2\sigma+2}}^{2\sigma+2}$ in the definition of the energy $E[Q]$ again through $\|(-\Delta)^{\frac{s}{2}}Q\|_{L^2}^2$ via (A.15) immediately implies the second equality in (A.16). This completes the proof of Proposition A.7. □

**Remark A.8** (No nontrivial solutions in $H^s(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$ for $\sigma \geqslant \sigma_*$)**.** Let $n \geqslant 1$, $s \in (0,1)$, and $\sigma > 0$. We mention that the condition $\sigma < \sigma_*$ (i.e., $s_c < s$) is necessary for the existence of nontrivial solutions $Q \in H^s(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$ to equation (A.9). In fact, let $Q$ be such a solution. If $\sigma \geqslant \sigma_*$ (this can only happen if $s < \frac{n}{2}$, since otherwise $\sigma_* = +\infty$, and it is equivalent to $s_c \geqslant s$), identities (A.13) and (A.14) of Proposition A.7 remain valid and are combined to give $\|Q\|_{L^2} = 0$, so that $Q \equiv 0$.

**Remark A.9** (Ground state energies)**.** From (A.15), observe the negativity, vanishing, and positivity of the ground state energy $E[Q]$ in the mass-subcritical, mass-critical and mass-supercritical cases $s_c < 0$, $s_c = 0$ and $s_c > 0$, respectively.

### A.4.2 Pohozaev Identities: the Energy-Critical Case $s_c = s$

Let us consider the energy-critical case $s_c = s$, i.e., $\sigma = \sigma_* := \frac{2s}{n-2s}$, which requires that we are in space dimension $n > 2s$. In this case we are lead (see how the exponents of the Gagliardo-Nirenberg inequality (A.12) above collapse to the following ones) to the Sobolev inequality

$$\|u\|_{L^{2\sigma_*+2}}^{2\sigma_*+2} \leqslant C_{n,s} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{2\sigma_*+2} \tag{A.18}$$

valid for all $u \in \dot{H}^s(\mathbb{R}^n)$, where $C_{n,s} > 0$ denotes the best constant.

Existence and uniqueness (modulo symmetries) of optimizers for (A.18) are classical facts. In fact, for the dual problem of optimizing the weak Young inequality (Hardy-Littlewood-Sobolev inequality), the set of optimizers are known in closed form [Lie83].

**Lemma A.10** (Explicit form of HLS-optimizers)**.** *For $n > 2s$, $Q \in \dot{H}^s(\mathbb{R}^n)$ optimizes (A.18) if and only if*

$$Q = Q_{\lambda,\mu,a}(x) = \lambda \cdot \left( \frac{1}{\mu^2 + |x-a|^2} \right)^{\frac{n-2s}{2}} \tag{A.19}$$

*with some parameters $\lambda \in \mathbb{C}\backslash\{0\}$, $\mu > 0$, and $a \in \mathbb{R}^n$.*

Without loss of generality we can take $a = 0$ and choose $\lambda$ real-valued and positive and pick $\mu > 0$, so that $Q(x) = Q(|x|) > 0$ is a radial and positive optimizer of (A.18).

Observe that any optimizer of (A.18) solves the Euler-Lagrange equation (possibly after a suitable rescaling $Q \to \alpha Q$)

$$(-\Delta)^s Q - |Q|^{2\sigma_*} Q = 0 \quad \text{in } \mathbb{R}^n.$$

Indeed, let us define the functional

$$\mathcal{W}[Q] := \frac{\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{2\sigma_*+2}}{\|Q\|_{L^{2\sigma_*+2}}^{2\sigma_*+2}}.$$

Let $Q$ be an optimizer, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ and $i(\varepsilon) := \mathcal{W}[Q + \varepsilon\phi]$. Then necessarily $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} i(\varepsilon) = 0$, which leads to

$$0 = \mathrm{Re}\langle (-\Delta)^s Q - \beta |Q|^{2\sigma_*} Q, \phi \rangle, \quad \text{where } \beta = \frac{\mathcal{W}[Q]}{\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^{2\sigma_*}} > 0.$$

Testing this equation with $i\phi$ instead of $\phi$ and using $\mathrm{Re}\langle \psi, i\phi \rangle = -\mathrm{Im}\langle \psi, \phi \rangle$ gives the same formula for the imaginary part. Thus, by arbitrariness of $\phi$,

$$(-\Delta)^s Q - \beta |Q|^{2\sigma_*} Q = 0, \quad \text{in } \mathbb{R}^n.$$

$\beta > 0$ can easily be scaled away. Namely, since $\mathcal{W}$ is clearly invariant under the rescaling $Q \to \alpha Q$, with $\alpha > 0$, we see that $\tilde{Q} := \beta^{\frac{1}{2\sigma_*}} Q$ is still an optimizer, but it solves

$$(-\Delta)^s \widetilde{Q} - |\widetilde{Q}|^{2\sigma_*} \widetilde{Q} = 0, \quad \text{in } \mathbb{R}^n.$$

In particular, when $Q$ is chosen as above, i.e., $Q(x) > 0$, and suitably rescaled, we get

$$(-\Delta)^s Q - Q^{\frac{n+2s}{n-2s}} = 0, \quad \text{in } \mathbb{R}^n. \tag{A.20}$$

Furthermore, having merely $n > 2s$, an optimizer $Q$ may fail to be in $L^2(\mathbb{R}^n)$, since we have $Q \in L^2(\mathbb{R}^n)$ if and only if $n > 4s$. To see this, note that (A.19) implies by changing variables $z = x - a$ that

$$\|Q\|_{L^2}^2 = |\lambda|^2 \int_{\mathbb{R}^n} \left( \frac{1}{\mu^2 + |z|^2} \right)^{n-2s} \mathrm{d}z = C + c_n \int_1^\infty \left( \frac{1}{\mu^2 + \varrho^2} \right)^{n-2s} \varrho^{n-1} \,\mathrm{d}\varrho$$

Let $n > 4s$. We use $\frac{1}{\mu^2+\varrho^2} \leqslant \frac{1}{\varrho^2}$ and see that the integral at infinity is majorized by the integral $\int_1^\infty \varrho^{-n+4s-1} \,\mathrm{d}\varrho$, which converges for $n > 4s$, so that $Q \in L^2(\mathbb{R}^n)$. Conversely, let $Q \in L^2(\mathbb{R}^n)$, so that the integral at infinity must converge. But since there exist $C > 0$ and $R > 0$ such that $\frac{1}{\mu^2+\varrho^2} \geqslant \frac{1}{C\varrho^2}$ for all $\varrho \geqslant R$, we have

$$\int_R^\infty \left( \frac{1}{\mu^2 + \varrho^2} \right)^{n-2s} \varrho^{n-1} \,\mathrm{d}\varrho \gtrsim \int_R^\infty \varrho^{-n+4s-1} \,\mathrm{d}\varrho$$

and the last integral does not converge for $n \leqslant 4s$.

**Proposition A.11** (Pohozaev-type identity for $s_c = s$). *For the Sobolev optimizer $Q \in \dot{H}^s(\mathbb{R}^n)$ as above, we have*

$$K_{n,s} = \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^s = \left( \frac{s}{n} \right)^{-\frac{s}{2}} E[Q]^{\frac{s}{2}} \quad \text{with } K_{n,s} = \left( \frac{1}{C_{n,s}} \right)^{\frac{n-2s}{4}}.$$

*Proof.* If we integrate (A.20) against $Q$, we find $\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 = \|Q\|_{L^{2\sigma_*+2}}^{2\sigma_*+2}$ with $\sigma_* = \frac{2s}{n-2s}$. Since $Q$ optimizes (A.18), we insert the previous identity into (A.18) (with equality sign) to get the first claimed identity $K_{n,s} = \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^s$. Finally, by definition of energy

$$\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 = 2E[Q] + \frac{1}{\sigma_* + 1} \|Q\|_{L^{2\sigma_*+2}}^{2\sigma_*+2} = 2E[Q] + \frac{1}{\sigma_* + 1} \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2,$$

from which the second claimed identity follows by respecting $s = \frac{n}{2} - \frac{s}{\sigma_*}$. □

### A.4.3 Cutoff Function for the $L^2$-Critical Case

To construct a suitable virial function $\varphi(r)$ for the $L^2$-critical case, we can adapt the choice made in [OT91, p. 325] used for classical NLS. Let $g \in W^{3,\infty}(\mathbb{R}^n)$ be a radial function such that

$$g(r) = \begin{cases} r & \text{for } 0 \leqslant r \leqslant 1, \\ r-(r-1)^3 & \text{for } 1 < r \leqslant 1+\frac{1}{\sqrt{3}}, \\ g(r) \text{ smooth and } g'(r) < 0 & \text{for } 1+\frac{1}{\sqrt{3}} < r < 10, \\ 0 & \text{for } r \geqslant 10. \end{cases} \tag{A.21}$$

We define the radial function $\varphi(r)$ by setting

$$\varphi(r) := \int_0^r g(s)\,\mathrm{d}s. \tag{A.22}$$

It is elementary to check that $\varphi(r)$ defined above satisfies assumption (2.35) of Subsection 2.2.2. Recall that we set $\varphi_R(r) = R^2\varphi\left(\frac{r}{R}\right)$ for $R > 0$ given. Furthermore, recall the definitions of the nonnegative functions $\psi_{1,R}(r) = 1 - \partial_r^2\varphi_R(r) \geqslant 0$ and $\psi_{2,R}(r) = n - \Delta\varphi_R(r) \geqslant 0$ from (2.40). Let $c(\eta) = \frac{\eta}{n+2s}$ for $\eta > 0$. We claim that if $\eta > 0$ is sufficiently small, and $R > 0$ arbitrary, we have

$$\psi_{1,R}(r) - c(\eta)(\psi_{2,R}(r))^{\frac{n}{2s}} \geqslant 0 \quad \text{for all } r \geqslant 0. \tag{A.23}$$

To prove (A.23), we argue as follows. First, by scaling, we can assume $R = 1$ without loss of generality.[5] Let us put $\psi_1(r) = \psi_{1,R=1}(r)$ and $\psi_2(r) = \psi_{2,R=1}(r)$. Note that $\psi_{1,R}(r) \equiv \psi_{2,R}(r) \equiv 0$ for $0 \leqslant r \leqslant R$ and hence (A.23) is trivially true in that region. Next, we observe that

$$\psi_1(r) = 1 - g'(r) \geqslant 1, \quad |\psi_2(r)| = |n - \Delta\varphi(r)| \leqslant C \quad \text{for } r \geqslant 1 + \tfrac{1}{\sqrt{3}}$$

with some constant $C > 0$ [recall that $\Delta\varphi(r) = g'(r) + \frac{n-1}{r}g(r)$, as well as the smoothness of $g(r)$ in this region and that $g(r) \equiv 0$ for $r \geqslant 10$]. Thus we can choose $\eta > 0$ sufficiently small such that (A.23) holds for $r \geqslant 1+\frac{1}{\sqrt{3}}$. Finally, a computation yields that

$$\psi_1(r) = 3(r-1)^2, \quad |\psi_2(r)|^{\frac{n}{2s}} = |n - \Delta\varphi(r)|^{\frac{n}{2s}} \leqslant C(r-1)^{\frac{n}{s}} \quad \text{for } 1 \leqslant r \leqslant 1 + \tfrac{1}{\sqrt{3}},$$

with some constant $C > 0$. Herein, we computed

$$\begin{aligned} n - \Delta\varphi(r) &= n - \left(\partial_r^2\varphi(r) + \frac{n-1}{r}\partial_r\varphi(r)\right) = n - \left(g'(r) + \frac{n-1}{r}g(r)\right) \\ &= (r-1)^2\left(3 + (n-1)\frac{r-1}{r}\right) \leqslant C(r-1)^2 \quad \text{for } 1 \leqslant r \leqslant 1 + \tfrac{1}{\sqrt{3}}. \end{aligned}$$

Since $\frac{n}{s} \geqslant 2$, we deduce that (A.23) holds in the region $1 \leqslant r \leqslant 1+\frac{1}{\sqrt{3}}$, too, provided that $\eta > 0$ is sufficiently small. Indeed, if $r = 1$, then $\psi_1(1) = \psi_2(1) = 0$, so any

[5] That is, we have $\psi_{1,1}\left(\frac{r}{R}\right) = \psi_{1,R}(r)$ and $\psi_{2,1}\left(\frac{r}{R}\right) = \psi_{2,R}(r)$. Hence, once (A.23) is proved for $R = 1$, it follows for arbitrary $R > 0$.

$\eta > 0$ can be chosen. On the other hand, if $1 < r \leqslant 1 + \frac{1}{\sqrt{3}}$, we see from the previous compuations that (A.23) follows in the region $1 < r \leqslant 1 + \frac{1}{\sqrt{3}}$ if we choose $\eta > 0$ such that $c(\eta) \leqslant \frac{3}{C}(r-1)^{2-\frac{n}{s}}$ for all $1 < r \leqslant 1 + \frac{1}{\sqrt{3}}$. But since $\frac{n}{s} \geqslant 2$ and $0 < r - 1 \leqslant \frac{1}{\sqrt{3}}$, we have $\frac{3}{C}(r-1)^{2-\frac{n}{s}} \geqslant \frac{3}{C}(\sqrt{3})^{\frac{n}{s}-2} =: \tilde{c} > 0$. Thus we conclude by taking $\eta > 0$ so small that $c(\eta) \leqslant \tilde{c}$.

# 3 Boosted Ground States and Traveling Solitary Waves

## 3.1 Introduction and Main Results

In this chapter, we consider NLS-type equations with focusing power-type nonlinearity

$$i\partial_t u = Lu - |u|^{2\sigma} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{3.1}$$

in $n \geqslant 1$ spatial dimensions. Here, $L$ is a pseudodifferential operator defined by its symbol $m(\xi)$ in Fourier space. For the real-valued function $m : \mathbb{R}^n \to \mathbb{R}$ we make the general assumption that there exist constants $\lambda, A, B > 0$ such that

$$A(1+|\xi|^2)^s \leqslant m(\xi) + \lambda \leqslant B(1+|\xi|^2)^s \quad \text{for all } \xi \in \mathbb{R}^n.^{1} \tag{A}$$

The magnitude of the power-nonlinearity is given by a number $\sigma \in (0, \sigma_*)$ in the $H^s(\mathbb{R}^n)$-subcritical regime, i.e. $\sigma_* = \frac{2s}{n-2s}$ if $s < \frac{n}{2}$ and $\sigma_* = +\infty$ if $s \geqslant \frac{n}{2}$.

Evidently, by the Cauchy-Schwarz inequality, the function $\xi \mapsto (m(\xi) - v \cdot \xi)$ is bounded from below, provided that $s > \frac{1}{2}$ and $v \in \mathbb{R}^n$ arbitrary, or $s = \frac{1}{2}$ and $|v| < A$.[2]

### Existence

We will prove the existence of a special class of solutions to equation (3.1), namely the class of traveling solitary waves of the form

$$u(t,x) = e^{i\omega t} Q_v(x - vt), \tag{3.2}$$

where $v \in \mathbb{R}^n$ is a given velocity parameter and $\omega \in \mathbb{R}$ is a phase parameter. By pluggin the ansatz (3.2) into (3.1) we see that $u = u(t,x)$ of the form (3.2) solves (3.1) if and only if the profile $Q_v$ solves the pseudo-differential equation

$$LQ_v + iv \cdot \nabla Q_v + \omega Q_v - |Q_v|^{2\sigma} Q_v = 0. \tag{3.3}$$

[1] Assumption (A) is easily verified if $L = (-\Delta)^s$, i.e. $m(\xi) = |\xi|^{2s}$, with $s > 0$; see page 82. In this case it is necessary that $A \leqslant 1$, since $\frac{|\xi|^{2s}+\lambda}{(1+|\xi|^2)^s} \to 1$ as $|\xi| \to \infty$.

[2] In fact, if $L = \sqrt{-\Delta}$, i.e. $s = \frac{1}{2}$, we may let $A = 1$; see page 82.

For $n = 1$ space dimension, cubic ($\sigma = 1$) nonlinearity and $L = (-\Delta)^s$ being the fractional Laplacian, the existence of traveling soliton solutions to (3.1) within the range $s \in (\frac{1}{2}, 1)$ of fractional parameters has recently been shown by Hong and Sire [HS15a]. In their paper, they overcome the lack of Galilean invariance of fractional NLS by introducing an ansatz function arising from so-called pseudo-Galilean transformations under which fractional NLS is almost invariant (i.e. invariant up to some controllable error term).

Our approach here is based on variational arguments involving the Weinstein functional $\mathcal{J}^s_{v,\omega} : H^s(\mathbb{R}^n)\backslash\{0\} \to \mathbb{R}$ defined by

$$\mathcal{J}^s_{v,\omega}(f) := \frac{(\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle)^{\sigma+1}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}}}, \tag{3.4}$$

where $\mathcal{T}_{s,v}$ is the pseudo-differential operator

$$\mathcal{T}_{s,v} := L + iv \cdot \nabla.$$

Our first main result gives the existence of traveling solitary wave solutions.

**Theorem 3.1** (Existence of traveling solitary wave solutions)**.** *Let $n \geqslant 1$, $s \geqslant \frac{1}{2}$, and $L$ be a pseudo-differential operator satisfying assumption (A). Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < A$ for $s = \frac{1}{2}$. Then there exists a number $\omega_* \in \mathbb{R}$ such that the following holds. For any $\omega > \omega_*$, there exists a profile $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ such that*

$$\mathcal{J}^s_{v,\omega}(Q_v) = \inf_{f \in H^s(\mathbb{R}^n)\backslash\{0\}} \mathcal{J}^s_{v,\omega}(f).$$

*More generally: any minimizing sequence is relatively compact in $H^s(\mathbb{R}^n)$ up to translations. Furthermore, modulo rescaling $Q_v \to \alpha Q_v$, $Q_v$ solves the pseudo-differential equation (3.3) and thus gives rise to the traveling solitary wave solution*

$$u(t, x) = e^{i\omega t} Q_v(x - vt)$$

*of* (3.1).

The number $\omega_*$ appearing above is defined by $-\omega_* = \inf_{\xi \in \mathbb{R}^n}(m(\xi) - v \cdot \xi)$.[3] Note that for $\omega > \omega_*$, the expression $\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle$ in the numerator of the Weinstein functional (3.4) is always positive when $f \not\equiv 0$.

Reflecting the fact that the functions $Q_v$ arise as minimizers of the functional $\mathcal{J}^s_{v,\omega}$ incorporating a velocity $v$, we often refer to them as boosted ground states and to the corresponding solitary waves as traveling (ground state) solitary waves.

Note that when $L = (-\Delta)^s$, in the unboosted case $v = 0$ and for fractional parameters $s \in (0, 1)$, we can immediately construct a solitary wave solution to (3.1)

[3] When $L = (-\Delta)^s$, $\frac{1}{2} \leqslant s < 1$, is the fractional Laplacian, this is precisely the Legendre transform [Eva97, p. 121] of the convex function $\xi \mapsto |\xi|^{2s}$ evaluated at the point $v \in \mathbb{R}^n$; see Lemma B.8.

as follows. As in [FLS16], a related variational problem has a ground state solution $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ which solves the equation

$$(-\Delta)^s Q + Q - |Q|^{2\sigma} Q = 0.$$

Existence of such $Q$ can be established by concentration-compactness arguments. Frank, Lenzmann and Silvestre [FLS16] prove uniqueness up to symmetries. Using the invariance of (3.1) under the rescaling

$$u \to u_\alpha(t,x) = \alpha^{\frac{s}{\sigma}} u(\alpha^{2s} t, \alpha x), \quad \alpha > 0,$$

we see that $Q_{v=0}(x) := \omega^{\frac{1}{2\sigma}} Q(\omega^{\frac{1}{2s}} x)$ solves (3.3) with $v = 0$ and thus gives rise to the solitary wave solution $u(t,x) = e^{i\omega t} Q_{v=0}(x)$ of (3.1).

## Symmetries

Our second main result establishes symmetry properties of boosted ground states for $s \geqslant \frac{1}{2}$. The method of proof is due to Boulenger and Lenzmann [BL15]. Under the assumption that the exponent $\sigma > 0$ in the power-nonlinearity is an integer, the authors are able to prove radiality of (unboosted) ground states for biharmonic NLS ($L = \Delta^2$). Their idea is to use the symmetric decreasing rearrangement $*$ (Schwarz symmetrization) in Fourier space, that is, to define the well-behaved operation $\sharp$ by $Q^\sharp = \mathcal{F}^{-1}((\mathcal{F}Q)^*)$.

Similarly, we obtain existence of cylindrically symmetric boosted ground states in $n \geqslant 2$ dimensions for integer $\sigma$. The symmetry axis is induced by the boost velocity $v$. Up to a rotation of the coordinate system, we can assume $v$ to point into 1-direction, $v = (v_1, 0, \ldots, 0)$. Then the symmetric decreasing rearrangement with respect to the last $n-1$ variables $*_1$ (Steiner symmetrization in codimension $n-1$) in Fourier space is the appropriate notion. That is, we define the operation $\sharp_1$ by $Q_v^{\sharp_1} = \mathcal{F}^{-1}((\mathcal{F}Q_v)^{*_1})$. In that context, we will often write $(\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$ for a vector $\xi \in \mathbb{R}^n$. We refer to section 3.5 for precise definitions and notations.

In $n = 1$ dimension the operation $\sharp_1$ loses its meaning. Instead, we consider the symmetrization $\widetilde{Q_v} := \mathcal{F}^{-1}(|\mathcal{F}Q_v|)$. Our method yields the existence of a boosted ground state $Q_v \in H^s(\mathbb{R})\backslash\{0\}$ such that $Q_v = \widetilde{Q_v}$, whose real-part is an even, and whose imaginary part is an odd function, respectively.

When working with a general operator $L$ as above, as a further technical ingredient we assume the following "monotonicity property" in $n \geqslant 2$ dimensions:

$$m(\xi_1, \xi') \geqslant m(\xi_1, \eta'), \quad \text{if } |\xi'| \geqslant |\eta'|.^{4} \tag{B}$$

Theorem 3.1 has already established existence of a minimizer of the functional $\mathcal{J}^s_{v,\omega}$ on the class $H^s(\mathbb{R}^n)\backslash\{0\}$. Our symmetry results described above are valid under

[4] Assumption (B) is clear for $L = (-\Delta)^s$, i.e. $m(\xi) = |\xi|^{2s}$ with $s > 0$ (see Remark 3.13).

the additional assumption that there is a minimizer which belongs also to $L^\infty(\mathbb{R}^n)$. In the case $s > \frac{n}{2}$, this assumption becomes redundant by Sobolev's embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

We can now state our second main result.

**Theorem 3.2** (Existence of symmetric boosted ground states for integer-powers)**.** *Let $n \geqslant 1$, $s \geqslant \frac{1}{2}$, and $L$ be a pseudo-differential operator satisfying assumptions (A) and (B). Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < A$ for $s = \frac{1}{2}$. Suppose that $\sigma \in (0, \sigma_*)$ is an integer. Furthermore, assume that there exists a minimizer of $\mathfrak{J}^s_{v,\omega}$ on the class $H^s(\mathbb{R}^n)\backslash\{0\}$ which is also in $L^\infty(\mathbb{R}^n)$. Then:*

*(i)* ***Case** $n \geqslant 2$**:** There exists a cylindrically symmetric minimizer of the Weinstein functional (3.4), i.e., there exists a boosted ground state $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ such that $Q_v = Q_v^{\sharp_1}$. In addition, $Q_v = Q_v^{\sharp_1}$ is continuous and bounded and has the higher Sobolev regularity $Q_v^{\sharp_1} \in H^k(\mathbb{R}^n)$ for all $k > 0$. In particular, $Q_v^{\sharp_1} \in C^\infty(\mathbb{R}^n)$ is smooth. Moreover, the functions $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Re} Q_v^{\sharp_1}(x_1, x')$ and $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Im} Q_v^{\sharp_1}(x_1, x')$ are even and odd, respectively, for any fixed $x' \in \mathbb{R}^{n-1}$.*

*(ii)* ***Case** $n = 1$**:** There exists a minimizer of the Weinstein functional (3.4), i.e. a boosted ground state $Q_v \in H^s(\mathbb{R})\backslash\{0\}$ such that $Q_v = \widetilde{Q_v}$. In addition, $Q_v = \widetilde{Q_v}$ is continuous and bounded and has the higher Sobolev regularity $\widetilde{Q_v} \in H^k(\mathbb{R})$ for all $k > 0$. In particular, $\widetilde{Q_v} \in C^\infty(\mathbb{R})$ is smooth. Moreover, the functions $\mathbb{R} \to \mathbb{R}$, $x \mapsto \operatorname{Re} \widetilde{Q_v}(x)$ and $\mathbb{R} \to \mathbb{R}$, $x_1 \mapsto \operatorname{Im} \widetilde{Q_v}(x)$ are even and odd, respectively.*

Note the range of application of Theorem 3.2 for given $s \geqslant \frac{1}{2}$. If $s \geqslant \frac{n}{2}$ (in particular, in $n = 1$ dimension), any $\sigma \in \mathbb{N}$ can be chosen, while if $s < \frac{n}{2}$, we can take any $\sigma \in \mathbb{N}$ such that $\sigma < \frac{2s}{n-2s}$.

As a direct application of Theorem 3.2, we get the analogous statement for the fractional Laplacian $L = (-\Delta)^s$ with $\frac{1}{2} \leqslant s < 1$. In this case, (A) and (B) are automatically fulfilled. Moreover, we will explicitly verify that any $H^s(\mathbb{R}^n)$-solution of (3.3) automatically belongs to $L^\infty(\mathbb{R}^n)$; see section 3.6 for the details. Thus we can drop the additional hypothesis of Theorem 3.2 and formulate the following result.

**Theorem 3.2′** (Version of Theorem 3.2 for the fractional Laplacian)**.** *Let $n \geqslant 1$, $s \in [\frac{1}{2}, 1)$, and $L = (-\Delta)^s$. Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < 1$ for $s = \frac{1}{2}$. Suppose that $\sigma \in (0, \sigma_*)$ is an integer. Then the conclusions of Theorem 3.2 hold.*

Note the range of application of Theorem 3.2′ for given $s \in [\frac{1}{2}, 1)$:

| Dimension $n$ | Permitted nonlinearities $\sigma \in \mathbb{N}$ | Corresponding $s \in [\frac{1}{2}, 1)$ |
|---|---|---|
| 1 | all $\sigma \in \mathbb{N}$ | all $s \in [\frac{1}{2}, 1)$ |
| 2 | $\sigma \in \mathbb{N}$ such that $\sigma < \frac{s}{1-s}$ | $s > \frac{\sigma}{\sigma+1}$ |
| 3 | only $\sigma = 1$ (cubic) | $s > \frac{3}{4}$ |
| 4 and higher | none | none |

For $n = 2$ we can thus take $\sigma \in \mathbb{N}$ as large as we like provided that $s < 1$ is sufficiently close to 1.

## Decay at infinity

Focusing on the case of the fractional Laplacian $L = (-\Delta)^s$, $\frac{1}{2} \leqslant s < 1$, we prove a decay result for critical points of the functional $\mathcal{J}^s_{v,\omega}$. The Euler-Lagrange equation (3.3) may be written in the form

$$Q_v = R_{\mathcal{T}_{s,v}}(-\omega)(|Q_v|^{2\sigma} Q_v).$$

Here, the resolvent $R_{\mathcal{T}_{s,v}}(-\omega) = (\mathcal{T}_{s,v} + \omega)^{-1}$ is well-defined by the spectral properties of the operator $\mathcal{T}_{s,v} = (-\Delta)^s + iv \cdot \nabla$; see Appendix B.2. Equivalently, we may write down the integral equation

$$Q_v = G^{(s)}_{v,\omega} \star (|Q_v|^{2\sigma} Q_v), \tag{3.5}$$

where $G^{(s)}_{v,\omega}$ is the fundamental solution (Green's function) associated to (3.3), which is a kernel with the Fourier representation

$$\mathcal{F} G^{(s)}_{v,\omega}(\xi) = \frac{1}{|\xi|^{2s} - v \cdot \xi + \omega}. \tag{3.6}$$

In section 3.6, we prove that $G^{(s)}_{v,\omega}$ decays like $|x|^{-(n+1)}$. In section 3.7, we show that at infinity, any critical point of the functional $\mathcal{J}^s_{v,\omega}$ decays in space at least as fast as the Green's function. In particular, this is true for any boosted ground state.

Our third main result reads as follows.

**Theorem 3.3** (Decay of boosted ground states for the fractional Laplacian)**.** *Let $n \geqslant 1$, $s \in [\frac{1}{2}, 1)$, and $L = (-\Delta)^s$. Let $v \in \mathbb{R}^n$ be arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Let $Q_v \in H^s(\mathbb{R}^n) \setminus \{0\}$ be a solution of the Euler-Lagrange equation (3.3). Then $Q_v$ is continuous on $\mathbb{R}^n$, and there exists some constant $C > 0$ such that the following polynomial decay estimate holds:*

$$|Q_v(x)| \leqslant \frac{C}{1 + |x|^{n+1}}, \quad \text{for all } x \in \mathbb{R}^n. \tag{3.7}$$

*In particular, any boosted ground state $Q_v \in H^s(\mathbb{R}^n) \setminus \{0\}$ decays polynomially according to (3.7).*

However, the optimal rate is expected to be $|Q_v(x)| \lesssim (1 + |x|^{n+2s})^{-1}$; see also Remark 3.23.

## Outline of chapter 3

In section 3.2 we introduce the relevant variational problem. We define an equivalent norm on the Sobolev space $H^s(\mathbb{R}^n)$, which will be useful in the proof that the infimum is attained. Also, this norm equivalence immediately yields the strict positivity of the infimum. Section 3.3 gives the Euler-Lagrange equation associated to the functional appearing in the variational problem, which any minimizer necessarily must satisfy after a suitable rescaling. In section 3.4 we prove Theorem 3.1 with the help of compactness modulo translations in $\dot{H}^s(\mathbb{R}^n)$, thereby obtaining the existence of boosted ground states and traveling solitary wave solutions to (3.1). In section 3.5 we find that boosted ground states exhibit symmetry properties with respect to the boost axis given by the boost velocity $v$, proving Theorem 3.2. From section 3.6 on, we focus on the case $L = (-\Delta)^s$ of the fractional Laplacian. We give the proof that solutions to the Euler-Lagrange equation are in $L^\infty(\mathbb{R}^n)$. Then Theorem 3.2′ follows as a corollary of Theorem 3.2 and Theorem 3.1. Finally, in section 3.7 we justify the decay estimate of Theorem 3.3.

# 3.2 The Variational Problem

## 3.2.1 An Equivalent Norm on $H^s(\mathbb{R}^n)$

For later use, let us observe that the operator $\mathcal{T}_{s,v}$ induces an equivalent norm on $H^s(\mathbb{R}^n)$ via its symbol in Fourier space. For $\omega > \omega_*$, we define

$$\|\cdot\| : H^s(\mathbb{R}^n) \to \mathbb{R},\; f \mapsto \|f\| := \|\widehat{f}\|_{L^2_\mu} := \sqrt{\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\,(m(\xi) - v\cdot\xi + \omega)\,\mathrm{d}\xi}.$$

**Remark 3.4.** The condition $\omega > \omega_*$ guarantees that $\mathrm{d}\mu(\xi) = (m(\xi) - v\cdot\xi + \omega)\,\mathrm{d}\xi$ is a positive measure. Clearly, $\|\cdot\|$ maps $H^s(\mathbb{R}^n)$ to $\mathbb{R}$ due to

$$\begin{aligned}\|f\|^2 &\leqslant B\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2(1+|\xi|^2)^s\,\mathrm{d}\xi + |v|\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2|\xi|\,\mathrm{d}\xi + \omega\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\\ &\leqslant B\|f\|^2_{H^s} + |v|\|f\|^2_{H^{1/2}} + \omega\|f\|^2_{L^2} \lesssim \|f\|^2_{H^s},\end{aligned}$$

using assumption (A) on the symbol $m$, the Cauchy-Schwarz inequality, Plancherel's Theorem, and the embedding $H^s(\mathbb{R}^n) \hookrightarrow H^{1/2}(\mathbb{R}^n)$ for $s \geqslant \frac{1}{2}$. Moreover, $\|\cdot\|$ clearly defines a norm on $H^s(\mathbb{R}^n)$ by linearity of the Fourier transform and the norm properties of $\|\cdot\|_{L^2_\mu}$.

**Lemma 3.5** (An equivalent norm on $H^s(\mathbb{R}^n)$)**.** *Let $n \geqslant 1$, $s \geqslant \frac{1}{2}$, and $L$ be a pseudo-differential operator satisfying assumption (A). Let $v \in \mathbb{R}^n$ be arbitrary for $s > \frac{1}{2}$, and $|v| < A$ for $s = \frac{1}{2}$. Suppose $\omega > \omega_*$. Then the norms $\|\cdot\|$ and $\|\cdot\|_{H^s}$ are equivalent on the Hilbert space $H^s(\mathbb{R}^n)$, i.e., for any $f \in H^s(\mathbb{R}^n)$ we have*

$$C_1\|f\|_{H^s} \leqslant \|f\| \leqslant C_2\|f\|_{H^s}$$

*for some constants $C_2 \geqslant C_1 > 0$ independent of $f$.*

*Proof.* We show that $\varphi(\xi) := m(\xi) - v \cdot \xi + \omega$ satisfies

$$(1 + |\xi|^2)^s \lesssim \varphi(\xi) \lesssim (1 + |\xi|^2)^s.$$

Indeed, on the one hand, by assumption (A) and Cauchy-Schwarz we have

$$\frac{\varphi(\xi)}{(1 + |\xi|^2)^s} \leqslant B + \frac{|v||\xi| + |\omega|}{(1 + |\xi|^2)^s} \leqslant (B + |\omega|) + \frac{|v||\xi|}{(1 + |\xi|^2)^s} \lesssim 1,$$

using that

$$\Psi(\xi) := \frac{|\xi|}{(1 + |\xi|^2)^s} = \frac{|\xi|^{1-2s}}{\left(\frac{1}{|\xi|^2} + 1\right)^s} \overset{|\xi| \to \infty}{\longrightarrow} \begin{cases} 1, & \text{if } s = \frac{1}{2}, \\ 0, & \text{if } s > \frac{1}{2}, \end{cases}$$

and the continuity of $\Psi : \mathbb{R}^n \to \mathbb{R}$ to get the last boundedness. Conversely, we estimate

$$\frac{\varphi(\xi)}{(1 + |\xi|^2)^s} \geqslant A + \frac{(\omega - \lambda) - |v||\xi|}{(1 + |\xi|^2)^s} =: A + \psi(\xi),$$

where

$$\psi(\xi) = \frac{\omega - \lambda}{(1 + |\xi|^2)^s} - |v|\Psi(\xi) \overset{|\xi| \to \infty}{\longrightarrow} \begin{cases} -|v|, & \text{if } s = \frac{1}{2}, \\ 0, & \text{if } s > \frac{1}{2}. \end{cases}$$

For $s = \frac{1}{2}$, the hypothesis $|v| < A$ thus guarantees that we can find $R > 0$ so large that, say, $A + \psi(\xi) \geqslant \frac{A - |v|}{2}$ for all $|\xi| \geqslant R$, while for $s > \frac{1}{2}$ we can find $R > 0$ so large that, say, $A + \psi(\xi) \geqslant \frac{A}{2}$ for all $|\xi| \geqslant R$. Thus in any case ($s = \frac{1}{2}$ or $s > \frac{1}{2}$), there exists $R > 0$ and some positive constant $C > 0$ such that

$$\frac{\varphi(\xi)}{(1 + |\xi|^2)^s} \geqslant C \quad \text{for all } |\xi| \geqslant R.$$

On the ball $\overline{B_R(0)}$ however, we have the positive lower bound

$$\frac{\varphi(\xi)}{(1 + |\xi|^2)^s} \geqslant \frac{\omega - \omega_*}{(1 + |\xi|^2)^s} \geqslant \min_{\xi \in \overline{B_R(0)}} \frac{\omega - \omega_*}{(1 + |\xi|^2)^s}.$$

The proof of Lemma 3.5 is now complete. □

### 3.2.2 The Weinstein Functional

With the operator $\mathcal{T}_{s,v}$ and $\omega > \omega_*$ as above, let us consider the Weinstein functional $\mathcal{J}^s_{v,\omega} : H^s(\mathbb{R}^n) \backslash \{0\} \to \mathbb{R}$ given by

$$\mathcal{J}^s_{v,\omega}(f) := \frac{(\langle f, \mathcal{T}_{s,v} f \rangle + \omega \langle f, f \rangle)^{\sigma+1}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}}}. \tag{3.8}$$

This functional is well-defined on $H^s(\mathbb{R}^n)\backslash\{0\}$. Indeed, in the $H^s(\mathbb{R}^n)$-subcritical regime $0 < \sigma < \sigma_*$ that we consider, Sobolev embedding guarantees that for $f \in H^s(\mathbb{R}^n)\backslash\{0\}$ we have $f \in L^{2\sigma+2}(\mathbb{R}^n)$ with $\|f\|_{L^{2\sigma+2}} < \infty$ and $\|f\|_{L^{2\sigma+2}} \neq 0$ as $f \not\equiv 0$. Furthermore, by Lemma 3.5 we have[5] $\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle = \|f\|^2 \lesssim \|f\|^2_{H^s}$. Hence $\mathcal{J}^s_{v,\omega}(f) < \infty$, as claimed. Next, let us show that that $\mathcal{J}^s_{v,\omega}$ is bounded from below by a strictly positive constant. Namely, for $f \in H^s(\mathbb{R}^n)\backslash\{0\}$, Sobolev embedding gives $\|f\|_{L^{2\sigma+2}} \leqslant C_{\mathrm{Sob}}\|f\|_{H^s}$, where $C_{\mathrm{Sob}} > 0$. On the other hand, by Lemma 3.5 there exists $C_1 > 0$ such that

$$\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle = \|f\|^2 \geqslant C_1^2\|f\|^2_{H^s}.$$

Combination of these facts yields the strictly positive lower bound

$$\mathcal{J}^s_{v,\omega}(f) = \frac{(\|f\|^2)^{\sigma+1}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}}} \geqslant \left(\frac{C_1}{C_{\mathrm{Sob}}}\right)^{2\sigma+2}.$$

We wish to solve the following (unconstrained) minimization problem on the nonempty admissible class $H^s(\mathbb{R}^n)\backslash\{0\}$:

$$\mathcal{J}^{s,*}_{v,\omega} := \inf\{\mathcal{J}^s_{v,\omega}(f); f \in H^s(\mathbb{R}^n)\backslash\{0\}\}. \tag{3.9}$$

By the consideration above, $\mathcal{J}^{s,*}_{v,\omega} > 0$. Finding that this infimum is attained will in particular establish the validity of the corresponding Gagliardo-Nirenberg-Sobolev (GNS) inequality (involving a boost term)

$$\|f\|^{2\sigma+2}_{L^{2\sigma+2}} \leqslant C_{\mathrm{opt}}(\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle)^{\sigma+1}, \quad f \in H^s(\mathbb{R}^n). \tag{3.10}$$

Here, $C_{\mathrm{opt}} > 0$ is the sharp constant for this inequality, which is given by the optimizers $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ of (3.9), namely

$$\frac{1}{C_{\mathrm{opt}}} = \mathcal{J}^s_{v,\omega}(Q_v), \quad \text{if } \mathcal{J}^s_{v,\omega}(Q_v) = \mathcal{J}^{s,*}_{v,\omega}. \tag{3.11}$$

However, by the scaling property $\mathcal{J}^s_{v,\omega}(\alpha f) = \mathcal{J}^s_{v,\omega}(f)$, $\alpha > 0$, it is clear that problem (3.9) is equivalent to the (constrained) minimization problem

$$\inf\{\mathcal{J}^s_{v,\omega}(f); f \in H^s(\mathbb{R}^n), \|f\|^{2\sigma+2}_{L^{2\sigma+2}} = \Lambda\}, \quad \Lambda > 0. \tag{3.12}$$

We let without loss $\Lambda = 1$ and are therefore concerned with the (constrained) minimization problem

$$\mathcal{J}^{s,*}_{v,\omega} = \inf\{\mathcal{J}^s_{v,\omega}(f); f \in H^s(\mathbb{R}^n), \|f\|^{2\sigma+2}_{L^{2\sigma+2}} = 1\}. \tag{3.13}$$

[5] Recall also that by $\omega > \omega_*$ and Plancherel, $\langle f, \mathcal{T}_{s,v} f\rangle + \omega\langle f, f\rangle > 0$ for $f \not\equiv 0$.

## 3.3 The Euler-Lagrange Equation

**Lemma 3.6** (Euler-Lagrange equation). *Let $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ be a minimizer of the functional $\mathcal{J}^s_{v,\omega}$ on the class $H^s(\mathbb{R}^n)\backslash\{0\}$, i.e.*

$$\mathcal{J}^s_{v,\omega}(Q) = \inf\{\mathcal{J}^s_{v,\omega}(f);\ f \in H^s(\mathbb{R}^n)\backslash\{0\}\}.$$

*Then $Q$ necessarily solves the Euler-Lagrange equation (3.3), i.e.*

$$LQ + iv \cdot \nabla Q + \omega Q - |Q|^{2\sigma} Q = 0,$$

*possibly only after the suitable rescaling $Q \to \alpha Q$, where $\alpha = \left(\frac{\mathcal{J}^{s,*}_{v,\omega}}{\langle Q, (\mathcal{T}_{s,v}+\omega)Q\rangle^\sigma}\right)^{\frac{1}{2\sigma}}$.*

*Proof.* For any $\varphi \in C^\infty_c(\mathbb{R}^n)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{J}^s_{v,\omega}(Q + \varepsilon\varphi)\bigg|_{\varepsilon=0} = \lim_{h\to 0}\frac{\mathcal{J}^s_{v,\omega}(Q + h\varphi) - \mathcal{J}^s_{v,\omega}(Q)}{h}.$$

Respecting chain and product rule for the Fréchet derivative, and using the self-adjointness of the operator $\mathcal{T}_{s,v} = L + iv\cdot\nabla$, which follows from Plancherel's Theorem and the real-valuedness of $m(\xi)$, we deduce

$$\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma \operatorname{Re}\langle\varphi, (\mathcal{T}_{s,v} + \omega)Q\rangle - \mathcal{J}^{s,*}_{v,\omega} \operatorname{Re}\langle\varphi, |Q|^{2\sigma}Q\rangle = 0.$$

Dividing by $\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma = (\mathcal{J}^{s,*}_{v,\omega})^{\frac{\sigma}{\sigma+1}}\|Q\|^{2\sigma}_{L^{2\sigma+2}} > 0$ gives

$$\operatorname{Re}\langle\varphi, (\mathcal{T}_{s,v} + \omega)Q - \frac{\mathcal{J}^{s,*}_{v,\omega}}{\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma}|Q|^{2\sigma}Q\rangle = 0, \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n).$$

Testing the last equation with $i\varphi$ instead of $\varphi$ and using $\operatorname{Re}(-iz) = \operatorname{Im} z$ for $z \in \mathbb{C}$ yields the analogous statement for the imaginary part:

$$\operatorname{Im}\langle\varphi, (\mathcal{T}_{s,v} + \omega)Q - \frac{\mathcal{J}^{s,*}_{v,\omega}}{\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma}|Q|^{2\sigma}Q\rangle = 0, \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n).$$

Hence $Q$ solves

$$(\mathcal{T}_{s,v} + \omega)Q - \frac{\mathcal{J}^{s,*}_{v,\omega}}{\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma}|Q|^{2\sigma}Q = 0.$$

Recall the invariance of our functional $\mathcal{J}^s_{v,\omega}(f)$ under rescaling $f \to \alpha f$, $\alpha > 0$. Therefore (cf. [Wei83, p. 571]) the rescaled version $\widetilde{Q}$ defined by

$$\widetilde{Q} = \left(\frac{\mathcal{J}^{s,*}_{v,\omega}}{\langle Q, (\mathcal{T}_{s,v} + \omega)Q\rangle^\sigma}\right)^{\frac{1}{2\sigma}} Q$$

is still a minimizer, but satisfies the Euler-Lagrange equation $(\mathcal{T}_{s,v} + \omega)\widetilde{Q} - |\widetilde{Q}|^{2\sigma}\widetilde{Q} = 0$, as claimed. ∎

## 3.4 Traveling Solitons: Proof of Theorem 3.1

Let $(u_j)_{j\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{J}^{s,*}_{v,\omega}$, that is

$$u_j \in H^s(\mathbb{R}^n),\ \|u_j\|^{2\sigma+2}_{L^{2\sigma+2}} = 1\ \forall j \in \mathbb{N}, \quad \lim_{j\to\infty} \mathcal{J}^s_{v,\omega}(u_j) = \mathcal{J}^{s,*}_{v,\omega} \in (0,\infty). \tag{3.14}$$

Then necessarily we have the boundedness $\sup_{j\in\mathbb{N}} |\langle u_j, \mathcal{T}_{s,v} u_j\rangle + \omega\langle u_j, u_j\rangle| \lesssim 1$, i.e., $\sup_{j\in\mathbb{N}} \|u_j\| \lesssim 1$. Thanks to Lemma 3.5 this means that $(u_j)_{j\in\mathbb{N}}$ is also bounded in $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s})$, i.e.

$$\sup_{j\in\mathbb{N}} \|u_j\|_{H^s} \lesssim 1 \tag{3.15}$$

or equivalently

$$\sup_{j\in\mathbb{N}} (\|u_j\|_{\dot{H}^s} + \|u_j\|_{L^2}) \lesssim 1. \tag{3.16}$$

The proof now proceeds in three steps.

**Step 1: Application of the pqr Lemma.** We take $p = 2$, $q = 2\sigma + 2$ and the number $r > 2\sigma + 2$ given as follows. By Sobolev's embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ [we let $2^* = \frac{2n}{n-2s}$ for $s < \frac{n}{2}$, $2^* = \infty$ for $s > \frac{n}{2}$, and in case of $s = \frac{n}{2}$, we replace $2^*$ by a fixed number $\alpha \in (2\sigma + 2, \infty)$ in the following argument], and of course $H^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. Thus by interpolation $H^s(\mathbb{R}^n) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^n)$ for all $H^s(\mathbb{R}^n)$-subcritical $0 < \sigma < \sigma_*$. Interpolation between $L^{2\sigma+2}(\mathbb{R}^n)$ and $L^{2^*}(\mathbb{R}^n)$ thus gives

$$\|f\|_{L^r} \leqslant \|f\|^{\theta}_{L^{2\sigma+2}} \|f\|^{1-\theta}_{L^{2^*}} \quad \text{for all } f \in H^s(\mathbb{R}^n), \text{ where } \tfrac{1}{r} = \tfrac{\theta}{2\sigma+2} + \tfrac{1-\theta}{2^*}.$$

In particular, for our minimizing sequence with normalized $L^{2\sigma+2}(\mathbb{R}^n)$ norms

$$\sup_{j\in\mathbb{N}} \|u_j\|_{L^r} \leqslant \sup_{j\in\mathbb{N}} \|u_j\|^{1-\theta}_{L^{2^*}} \lesssim \sup_{j\in\mathbb{N}} \|u_j\|^{1-\theta}_{H^s} \lesssim 1, \quad \text{where } \tfrac{1}{r} = \tfrac{\theta}{2\sigma+2} + \tfrac{1-\theta}{2^*}, \tag{3.17}$$

using Sobolev's embedding and (3.15) in the last two estimates, respectively. Take now $2\sigma + 2 < r < 2^*$. Then (3.16), the fact that $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j$ and (3.17) yield the existence of constants $C_2, C_{2\sigma+2}, C_r > 0$ such that for all $j$

$$\|u_j\|_{L^2} \leqslant C_2, \quad \|u_j\|_{L^{2\sigma+2}} \geqslant C_{2\sigma+2}, \quad \|u_j\|_{L^r} \leqslant C_r.$$

From the pqr Lemma B.2 follows the existence of constants $\eta, c > 0$ such that

$$\inf_{j\in\mathbb{N}} |\{x \in \mathbb{R}^n;\ |u_j(x)| > \eta\}| \geqslant c. \tag{3.18}$$

**Step 2: Application of generalized Lieb's compactness.** According to (3.16) and (3.18) our minimizing sequence $(u_j)_{j\in\mathbb{N}}$, $u_j \in \dot{H}^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ satisfies the hypotheses of the generalized Lieb Lemma B.1. Consequently there exists a sequence of vectors $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that the translated sequence $(\widetilde{u_j})_{j\in\mathbb{N}}$, where $\widetilde{u_j}(x) := u_j(x + x_j)$, has a subsequence that converges weakly in $H^s(\mathbb{R}^n)$ to some nonzero function $u \in H^s(\mathbb{R}^n)\backslash\{0\}$.[6] The function $u$ is a candidate for a solution to our minimization problem.

**Step 3: Proof that $u \not\equiv 0$ minimizes $\mathcal{J}^s_{v,\omega}$.** Since $(u_j)_{j\in\mathbb{N}}$ is still a minimizing sequence for $\mathcal{J}^{s,*}_{v,\omega}$ in the sense of (3.14) and the functional $\mathcal{J}^s_{v,\omega}$ is invariant with respect to translations, also $(\widetilde{u_j})_{j\in\mathbb{N}}$ is minimizing:

$$\widetilde{u_j} \in H^s(\mathbb{R}^n), \ \|\widetilde{u_j}\|^{2\sigma+2}_{L^{2\sigma+2}} = 1 \ \forall j \in \mathbb{N}, \quad \lim_{k\to\infty} \mathcal{J}^s_{v,\omega}(\widetilde{u_j}) = \mathcal{J}^{s,*}_{v,\omega}. \tag{3.19}$$

We may therefore assume w.l.o.g. that $x_j = 0$ for all $j$ and rename $(\widetilde{u_j})_{j\in\mathbb{N}}$ to $(u_j)_{j\in\mathbb{N}}$. From (3.19) we obtain

$$\lim_{j\to\infty} \left(\langle u_j, \mathcal{T}_{s,v} u_j\rangle + \omega\langle u_j, u_j\rangle\right) = (\mathcal{J}^{s,*}_{v,\omega})^{\frac{1}{\sigma+1}}. \tag{3.20}$$

Up to extracting a further subsequence, we may assume that (due to Corollary B.4)

$$\lim_{j\to\infty} u_j(x) = u(x) \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{3.21}$$

Corollary B.4 was applicable because the embedding $H^s(\mathbb{R}^n) \hookrightarrow \dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ is continuous, so from $u_j \rightharpoonup u$ weakly in $H^s(\mathbb{R}^n)$ we get $u_j \rightharpoonup u$ weakly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$; similarly, we get $u_j \rightharpoonup u$ weakly in $L^2(\mathbb{R}^n)$. Since the $u_j$'s are uniformly $L^{2\sigma+2}$-bounded functions, (3.21) allows us to apply the Brézis-Lieb improvement of Fatou's Lemma, Theorem B.5.[7] This theorem implies (when inserting $\|u_j\|^{2\sigma+2}_{L^{2\sigma+2}} = 1$ for all $j$), written in little 'o' notation,

$$\|u_j - u\|^{2\sigma+2}_{L^{2\sigma+2}} + \|u\|^{2\sigma+2}_{L^{2\sigma+2}} = 1 + o(1). \tag{3.22}$$

We claim:

$$\langle u_j - u, \mathcal{T}_{s,v}(u_j - u)\rangle + \omega\langle u_j - u, u_j - u\rangle + \langle u, \mathcal{T}_{s,v} u\rangle + \omega\langle u, u\rangle = (\mathcal{J}^{s,*}_{v,\omega})^{\frac{1}{\sigma+1}} + o(1). \tag{3.23}$$

To prove (3.23), notice that after expanding the scalar product and exploiting (3.20) it is sufficient to show

$$A_j + B_j := \left(\langle u, \mathcal{T}_{s,v} u\rangle - \mathrm{Re}\langle u_j, \mathcal{T}_{s,v} u\rangle\right) + \omega\left(\langle u, u\rangle - \mathrm{Re}\langle u_j, u\rangle\right) = o(1).$$

Indeed, by $u_j \rightharpoonup u$ weakly in $H^s(\mathbb{R}^n)$, $(L^2(\mathbb{R}^n))^* \subset (H^s(\mathbb{R}^n))^*$ and $\langle u, \cdot\rangle \in (L^2(\mathbb{R}^n))^*$, we have $\langle u, u_j\rangle \to \langle u, u\rangle$, and hence $B_j = o(1)$. As for $A_j$ we argue as follows. Consider $L^2_\mu(\mathbb{R}^n)$ with the measure $\mu$ given by $\mathrm{d}\mu(\xi) = (m(\xi) - v\cdot\xi + \omega)\,\mathrm{d}\xi$. The linear operator $\mathcal{F} : H^s(\mathbb{R}^n) \to L^2_\mu(\mathbb{R}^n)$ is continuous due to the estimate

$$\|\mathcal{F} f\|_{L^2_\mu} = \|f\| \leqslant C_2\|f\|_{H^s},$$

where the constant $C_2 > 0$ is independent of $f \in H^s(\mathbb{R}^n)$; see Lemma 3.5. Clearly, if $\varphi \in (L^2_\mu(\mathbb{R}^n))^*$ then $\varphi \circ \mathcal{F} \in (H^s(\mathbb{R}^n))^*$. Since $u_j \rightharpoonup u$ weakly in $H^s(\mathbb{R}^n)$, we get

[6] We continue to call the index $j$ when passing to further subsequences.

[7] See also [BL83], where it is pointed out how this result can be used in the calculus of variations to prove existence of optimizers in cases in which compactness is not available.

$(\varphi \circ \mathcal{F})(u_j) \to (\varphi \circ \mathcal{F})(u)$ for all $\varphi \in (L^2_\mu(\mathbb{R}^n))^*$. Again, since $(L^2(\mathbb{R}^n))^* \subset (H^s(\mathbb{R}^n))^*$ and $\langle \widehat{u}, \cdot \rangle_{L^2_\mu} \in (L^2_\mu(\mathbb{R}^n))^*$, this gives $\langle \widehat{u}, \widehat{u_j} \rangle_{L^2_\mu} \to \langle \widehat{u}, \widehat{u} \rangle_{L^2_\mu}$. Equivalently by Plancherel this means $\langle u_j, \mathcal{T}_{s,v} u \rangle \to \langle u, \mathcal{T}_{s,v} u \rangle$, and hence $A_j = o(1)$. This proves (3.23).

With the operator $H := \mathcal{T}_{s,v} + \omega \mathrm{Id}$ (which is positive by our choice $\omega > \omega_*$) we shortly write this as

$$\langle u_j - u, H(u_j - u) \rangle + \langle u, Hu \rangle = (\mathcal{J}^{s,*}_{v,\omega})^{\frac{1}{\sigma+1}} + o(1). \tag{3.24}$$

From (3.22) and (3.24) it follows that

$$\begin{aligned} &\mathcal{J}^{s,*}_{v,\omega} \left\{ \|u_j - u\|^{2\sigma+2}_{L^{2\sigma+2}} + \|u\|^{2\sigma+2}_{L^{2\sigma+2}} + o(1) \right\} = \mathcal{J}^{s,*}_{v,\omega} \cdot 1 \\ &= \{ \langle u_j - u, H(u_j - u) \rangle + \langle u, Hu \rangle + o(1) \}^{\sigma+1} \\ &\geqslant \langle u_j - u, H(u_j - u) \rangle^{\sigma+1} + \langle u, Hu \rangle^{\sigma+1} + o(1) \\ &\geqslant \mathcal{J}^{s,*}_{v,\omega} \|u_j - u\|^{2\sigma+2}_{L^{2\sigma+2}} + \langle u, Hu \rangle^{\sigma+1} + o(1). \end{aligned} \tag{3.25}$$

In the second to last step in (3.25), we used the elementary inequality (see Lemma B.7)

$$(\alpha + \beta)^{\sigma+1} \geqslant \alpha^{\sigma+1} + \beta^{\sigma+1}, \quad \text{for } \alpha, \beta, \sigma \geqslant 0,$$

which is applicable by positivity of $H$. In the last step of (3.25) the definition of $\mathcal{J}^{s,*}_{v,\omega}$ was inserted. Simplifying in (3.25) and taking the limit $j \to \infty$, it follows that $\mathcal{J}^{s,*}_{v,\omega} \|u\|^{2\sigma+2}_{L^{2\sigma+2}} \geqslant \langle u, Hu \rangle^{\sigma+1}$, which gives with $u \not\equiv 0$ that

$$\mathcal{J}^{s,*}_{v,\omega} \geqslant \frac{\langle u, Hu \rangle^{\sigma+1}}{\|u\|^{2\sigma+2}_{L^{2\sigma+2}}} = \mathcal{J}^s_{v,\omega}(u).$$

The converse inequality $\mathcal{J}^{s,*}_{v,\omega} \leqslant \mathcal{J}^s_{v,\omega}(u)$ is clear by $u \in H^s(\mathbb{R}^n) \backslash \{0\}$. Thus $u$ is minimizing, as desired. The proof of Theorem 3.1 is now complete. □

**Remark 3.7.** The above proof actually shows that all minimizing sequences $(u_j)_{j\in\mathbb{N}}$ for $\mathcal{J}^{s,*}_{v,\omega}$ are relatively compact in $H^s(\mathbb{R}^n)$ up to translations.

*Proof.* Let $(u_j)_{j\in\mathbb{N}}$ be a minimizing sequence as in (3.14), and let $u \in H^s(\mathbb{R}^n) \backslash \{0\}$ be as above, namely, up to translation and passing to subsequences if necessary, $u_j \rightharpoonup u$ weakly in $H^s(\mathbb{R}^n)$ and $u$ minimizes the functional $\mathcal{J}^s_{v,\omega}$ on $H^s(\mathbb{R}^n) \backslash \{0\}$. Let $\alpha = \|u\|_{L^{2\sigma+2}}$. Since $\lim_{j\to\infty} \mathcal{J}^s_{v,\omega}(u_j) = \mathcal{J}^{s,*}_{v,\omega} = \mathcal{J}^s_{v,\omega}(u)$, we have

$$\lim_{j\to\infty} \mathcal{J}^s_{v,\omega}(u_j)^{\frac{1}{\sigma+1}} = \frac{\langle u, \mathcal{T}_{s,v} u \rangle + \omega \langle u, u \rangle}{\alpha^2}.$$

In other words, using $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j$, we have $\lim_{j\to\infty} \|\widehat{u_j}\|_{L^2_\mu} = \|\frac{1}{\alpha} \widehat{u}\|_{L^2_\mu}$. By [LL01, Theorem 2.11], we get $\widehat{u_j} \to \frac{\widehat{u}}{\alpha}$ strongly in $L^2_\mu(\mathbb{R}^n)$. Thus from Lemma 3.5 it follows that

$$\|u_j - \frac{u}{\alpha}\|_{H^s} \leqslant \frac{1}{C_1} \left\| u_j - \frac{u}{\alpha} \right\| = \frac{1}{C_1} \left\| \widehat{u_j} - \frac{\widehat{u}}{\alpha} \right\|_{L^2_\mu} \to 0.$$

This proves the remark. □

## 3.5 Symmetries

### 3.5.1 Schwarz and Steiner Symmetrization

We begin by recalling some important notions of symmetrization of a given function; see in particular the textbook references [LL01, Chapter 3] and [Kes06]. For $u : \mathbb{R}^n \to \mathbb{C}$ measurable and vanishing at infinity in the weak sense that

$$d_u(t) := |\{x \in \mathbb{R}^n;\ |u(x)| > t\}| < \infty \quad \text{for all } t > 0,$$

we define its **($n$-dimensional) Schwarz symmetrization**, also called its symmetric decreasing rearrangement (with respect to $n$ variables), to be the function

$$u^*(x) := \int_0^\infty \chi_{\{|u|>t\}^*}(x)\,\mathrm{d}t. \tag{3.26}$$

Here, a rearranged level set $\{x \in \mathbb{R}^n;\ |u(x)| > t\}^*$ is defined to be the open ball $B_{R_t}(0)$ of radius $R_t$ centered at the origin, where $R_t \geqslant 0$ is chosen such that

$$|\{|u| > t\}| = |B_{R_t}(0)| = \omega_n R_t^n$$

(with the understanding $B_0(0) = \varnothing$). Here $\omega_n$ is the volume of the $n$ dimensional unit ball. We call $d_u(t)$ the distribution function of $u$. Clearly, $d_u : (0, \infty) \to \mathbb{R}_+$ is non-negative (because the measure is so), non-increasing (because the measure is monotone) and continuous from the right (because the measure is continuous from below); cf. also [Hun66].

**Definition 3.8** (Equimeasurability)**.** Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be measurable and vanishing at infinity. $f$ and $g$ are called equimeasurable on $\mathbb{R}^n$ if they have the same distribution function, i.e.,

$$d_f(t) = d_g(t), \quad \text{for all } t > 0.$$

Recall some basic properties of the ($n$-dimensional) Schwarz symmetrization $u^*$ (we refer to [Kes06, p. 17] and [LL01, p. 81] for proofs and further details):

1. $u^*$ is nonnegative, radially symmetric as well as radially decreasing in $\mathbb{R}^n$,
2. the level sets of $u^*$ are the rearranged level sets of $|u|$, i.e.

$$\{x \in \mathbb{R}^n;\ u^*(x) > t\} = \{x \in \mathbb{R}^n;\ |u(x)| > t\}^*,$$

   and, as consequences [the balls $\{|u| > t\}^*$ are open by definition and the family $\mathcal{A} = \{(a, \infty); a > 0\} \subset \mathrm{Pot}(\mathbb{R}_+)$ generates the Borel $\sigma$-algebra on $\mathbb{R}_+$]
3. $u^*$ is measurable and lower semi-continuous,
4. $u^*$ and $|u|$ are equimeasurable, in particular we have $\|u^*\|_{L^p} = \|u\|_{L^p}$ for all $1 \leqslant p \leqslant \infty$,

5. if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, then $(\varphi \circ |u|)^* = \varphi \circ u^*$ (see [Kes06, Proposition 1.1.4] for the proof of an analogous statement).

In particular, for equimeasurable functions $f, g$ the balls $\{|f| > t\}^*$ and $\{|g| > t\}^*$ are equal for all $t > 0$. Thus also the Schwarz symmetrizations are equal, i.e. $f^* = g^*$, since

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)\,\mathrm{d}t = \int_0^\infty \chi_{\{|g|>t\}^*}(x)\,\mathrm{d}t = g^*(x).$$

As a second notion, for $n \geqslant 2$ and a function $u$ as above, we introduce the **Steiner symmetrization in codimension** $n-1$, denoted $u^{*_1} : \mathbb{R}^n \to \mathbb{R}_+$, as follows (see also [Cap14]). For a vector $x \in \mathbb{R}^n$, let us write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then the value of $u^{*_1}$ at $x$ is defined to be the value of the $(n-1)$-dimensional Schwarz symmetrization of the function $u(x_1, \cdot) : \mathbb{R}^{n-1} \to \mathbb{C}$ at $x'$. In formulae,

$$u^{*_1} : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}_+, \quad u^{*_1}(x_1, x') := u(x_1, \cdot)^*(x'), \tag{3.27}$$

where $*$ is to be understood as the $(n-1)$-dimensional Schwarz symmetrization. Note that $u^{*_1}$ is a nonnegative function because for any $x_1$ the function $u(x_1, \cdot)^*$ is nonnegative. We list some elementary properties of Steiner symmetrization.

**Lemma 3.9** (Elementary properties of $*_1$). *Let $n \geqslant 2$ and $u : \mathbb{R}^n \to \mathbb{C}$ be measurable and vanishing at infinity. Then the following holds:*

*(i) Steiner symmetrization $*_1$ preserves the $L^p$ norm, i.e., if $u \in L^p(\mathbb{R}^n)$, then also $u^{*_1} \in L^p(\mathbb{R}^n)$ with $\|u^{*_1}\|_{L^p} = \|u\|_{L^p}$.*

*(ii) The Steiner symmetrization $u^{*_1}$ of $u$ is cylindrically symmetric with respect to 1-axis, i.e., for any $(x_1, x'), (x_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$ we have*

$$u^{*_1}(x_1, x') = u^{*_1}(x_1, y'), \quad \text{if } |x'| = |y'|.$$

*Proof.* (i) follows from Fubini's Theorem and the fact that for any $x_1 \in \mathbb{R}$ the functions $|u(x_1, \cdot)|$ and $u(x_1, \cdot)^*$ are equimeasurable on $\mathbb{R}^{n-1}$. That is, we check

$$\begin{aligned}\|u^{*_1}\|_{L^p}^p &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |u(x_1, \cdot)^*(x')|^p \,\mathrm{d}x_1\,\mathrm{d}x' = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |u(x_1, \cdot)^*(x')|^p \,\mathrm{d}x'\,\mathrm{d}x_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \,\mathrm{d}x'\,\mathrm{d}x_1 = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |u(x_1, x')|^p \,\mathrm{d}x_1\,\mathrm{d}x' = \|u\|_{L^p}^p.\end{aligned}$$

(ii) is clear, since the $(n-1)$-dimensional Schwarz symmetrization of $u(x_1, \cdot)$ is radially symmetric on $\mathbb{R}^{n-1}$, hence for $|x'| = |y'|$ we have

$$u^{*_1}(x_1, x') = u(x_1, \cdot)^*(x') = u(x_1, \cdot)^*(y') = u^{*_1}(x_1, y'). \qquad \square$$

### 3.5.2 Properties of Steiner Symmetrization in Fourier space

Recently, in [BL15], $n$-dimensional Schwarz symmetrization $*$ in Fourier space has been used to prove existence of radially symmetric ground states for biharmonic

NLS. Here we are concerned with boosted ground states with boost velocity $v \in \mathbb{R}^n$, where $v$ (up to rotations of the coordinate system) can always be assumed to point into 1-direction. In this situation Steiner symmetrization $*_1$ in codimension $n-1$ gives rise to the analogous operation, i.e.,

$$u^{\sharp_1} := \mathcal{F}^{-1}((\mathcal{F}u)^{*_1}), \quad \text{provided that } n \geqslant 2. \tag{3.28}$$

Analogously this will enable us to establish cylindrical symmetry of boosted ground states with respect to the cylinder axis given by $v$, at least provided that $\sigma > 0$ is an integer.

The operation $\sharp_1$ makes sense only if $n \geqslant 2$. If $n = 1$ we consider the modulus of the Fourier transform. More precisely, we work with the operation

$$\widetilde{u} := \mathcal{F}^{-1}(|\mathcal{F}u|), \quad \text{provided that } n = 1. \tag{3.29}$$

Some properties of $*_1$ are inherited on the Fourier side and give the following properties of $\sharp_1$.

**Lemma 3.10** (Properties of $\sharp_1$)**.** *Let $n \geqslant 2$ and $u : \mathbb{R}^n \to \mathbb{C}$ be measurable and vanishing at infinity. Then the following holds:*

*(i) $\sharp_1$ preserves the $L^2$ norm, i.e., if $u \in L^2(\mathbb{R}^n)$, then also $u^{\sharp_1} \in L^2(\mathbb{R}^n)$ with $\|u^{\sharp_1}\|_{L^2} = \|u\|_{L^2}$.*

*(ii) The function $u^{\sharp_1}$ is cylindrically symmetric with respect to 1-axis, i.e., for any $(x_1, x'), (x_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$ we have*

$$u^{\sharp_1}(x_1, x') = u^{\sharp_1}(x_1, y'), \quad \text{if } |x'| = |y'|.$$

*(iii) For $u$ with the additional assumption that $\mathcal{F}u \in L^1(\mathbb{R}^n)$, we have that $u^{\sharp_1}$ is bounded and continuous, and the following properties hold:*

$$u^{\sharp_1}(-x) = \overline{u^{\sharp_1}(x)}, \quad \forall x \in \mathbb{R}^n, \tag{3.30}$$

$$u^{\sharp_1}(0) \geqslant |u^{\sharp_1}(x)|, \quad \forall x \in \mathbb{R}^n. \tag{3.31}$$

*Proof.* (i) follows immediately from Lemma 3.9 (i) and Plancherel's Theorem, namely

$$\|u^{\sharp_1}\|_{L^2}^2 = \|\mathcal{F}^{-1}((\mathcal{F}u)^{*_1})\|_{L^2}^2 = \|(\mathcal{F}u)^{*_1}\|_{L^2}^2 = \|(\mathcal{F}u)\|_{L^2}^2 = \|u\|_{L^2}^2.$$

(ii) follows directly from the fact that $u^{\sharp_1}$ is the inverse Fourier transform of the (according to Lemma 3.9 (ii)) with respect to 1-axis cylindrically symmetric function $(\mathcal{F}u)^{*_1}$. In detail, let $x'$, $y' \in \mathbb{R}^{n-1}$ with $|x'| = |y'|$. Choose an orthogonal transformation $R \in O(n-1, \mathbb{R})$ of $\mathbb{R}^{n-1}$ such that $Rx' = y'$. Denote $R^\dagger$ its adjoint.[8] By Fubini's Theorem

$$\begin{aligned} u^{\sharp_1}(x_1, x') &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\mathcal{F}u)^{*_1}(\xi_1, \xi') e^{ix_1\xi_1} e^{ix'\cdot\xi'} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi' \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} e^{ix_1\xi_1} \left( \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1, \xi') e^{ix'\cdot\xi'} \, \mathrm{d}\xi' \right) \mathrm{d}\xi_1. \end{aligned} \tag{3.32}$$

[8] Since $R$ is orthogonal and real, we have $R^{-1} = R^t = \overline{R}^t = R^\dagger$.

We compute the inner integral, using $x' = R^\dagger y'$ and the change of variables $\phi(\xi') = R\xi'$. Since $|\mathrm{Det}J(\phi;\xi')| = |\mathrm{Det}R| = 1$ for the Jacobian determinant, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,\xi')e^{ix'\cdot\xi'}\,\mathrm{d}\xi' = \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,\xi')e^{iR^\dagger y'\cdot\xi'}\,\mathrm{d}\xi'\\ &= \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,\xi')e^{iy'\cdot R\xi'}\,\mathrm{d}\xi' = \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,R^{-1}\phi(\xi'))e^{iy'\cdot\phi(\xi')}|\mathrm{Det}J(\phi;\xi')|\,\mathrm{d}\xi'\\ &= \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,R^{-1}\eta')e^{iy'\cdot\eta'}\,\mathrm{d}\eta' = \int_{\mathbb{R}^{n-1}} (\mathcal{F}u)^{*_1}(\xi_1,\eta')e^{iy'\cdot\eta'}\,\mathrm{d}\eta',\end{aligned} \tag{3.33}$$

where the last equality is due to the cylindrical symmetry of $(\mathcal{F}u)^{*_1}$ with respect to 1-axis. Putting (3.33) into (3.32) and applying Fubini again gives

$$u^{\sharp_1}(x_1,x') = u^{\sharp_1}(x_1,y').$$

(iii) follows from Bochner's theorem (e.g. [RS75]). First, by Fubini's Theorem, the definition of the Steiner symmetrization in codimension $n-1$ and the equimeasurability of the functions $|\mathcal{F}u(\xi_1,\cdot)|$ and $\mathcal{F}u(\xi_1,\cdot)^*$ on $\mathbb{R}^{n-1}$, we see that the hypothesis $\mathcal{F}u \in L^1(\mathbb{R}^n)$ is equivalent to $(\mathcal{F}u)^{*_1} \in L^1(\mathbb{R}^n)$. But $(\mathcal{F}u)^{*_1} \in L^1(\mathbb{R}^n)$ is a nonnegative function on $\mathbb{R}^n$. Bochner's Theorem then implies that $\mathcal{F}^{-1}((\mathcal{F}u)^{*_1}) = u^{\sharp_1}$ is a positive definite function. This means that (see [BL15, p. 32] or also [Str03, p. 131]) it is a bounded and continuous function with the following property:

$$\forall m \in \mathbb{N}, \forall x^1,\dots,x^m \in \mathbb{R}^n : \overline{\zeta}^t U_{\mathbf{x}}^{\sharp_1}\zeta = \sum_{i,j=1}^m u^{\sharp_1}(x^i - x^j)\overline{\zeta_i}\zeta_j \geqslant 0, \quad \forall\zeta \in \mathbb{C}^m. \tag{3.34}$$

The matrix $U_{\mathbf{x}}^{\sharp_1} = \left(u^{\sharp_1}(x^i - x^j)\right)_{i,j=1,\dots,m}$ is associated to $\mathbf{x} = (x^1,\dots,x^m) \in \mathbb{R}^{n\times m}$. Taking $m = 1$, $x^1 = x \in \mathbb{R}^n$ arbitrary (e.g. $x = 0$) and $\zeta = 1$ yields that $u^{\sharp_1}(0) \geqslant 0$, in particular $u^{\sharp_1}(0)$ must be real. Proceed now as in [BL15]: take $m = 2$ with $x^1 = 0$, $x^2 = x \in \mathbb{R}^n$ (arbitrary). Then (3.34) reads

$$(|\zeta_1|^2 + |\zeta_2|^2)u^{\sharp_1}(0) + \overline{\zeta_1}\zeta_2 u^{\sharp_1}(-x) + \overline{\zeta_2}\zeta_1 u^{\sharp_1}(x) \geqslant 0, \quad \forall\zeta = (\zeta_1,\zeta_2) \in \mathbb{C}^2. \tag{3.35}$$

We draw several conclusions from (3.35).

1. Let $\zeta = u^{\sharp_1}(x)\cdot(1,i) = (u^{\sharp_1}(x), iu^{\sharp_1}(x)) \in \mathbb{C}^2$. Then (3.35) gives

$$|u^{\sharp_1}(x)|^2\left\{2u^{\sharp_1}(0) + i(u^{\sharp_1}(-x) - u^{\sharp_1}(x))\right\} \geqslant 0.$$

Since $u^{\sharp_1}(0) \in \mathbb{R}$, this implies $|u^{\sharp_1}(x)|^2\,\mathrm{Im}[i(u^{\sharp_1}(-x) - u^{\sharp_1}(x))] = 0$, which yields ($\mathrm{Im}[iz] = \mathrm{Re}\,z$ for $z \in \mathbb{C}$)

$$|u^{\sharp_1}(x)|^2\,\mathrm{Re}\,u^{\sharp_1}(-x) = |u^{\sharp_1}(x)|^2\,\mathrm{Re}\,u^{\sharp_1}(x). \tag{3.36}$$

2. Let $\zeta = u^{\sharp_1}(x) \cdot (1,1) = (u^{\sharp_1}(x), u^{\sharp_1}(x)) \in \mathbb{C}^2$. Then (3.35) gives

$$|u^{\sharp_1}(x)|^2 \left\{2u^{\sharp_1}(0) + u^{\sharp_1}(-x) + u^{\sharp_1}(x)\right\} \geqslant 0.$$

Since $u^{\sharp_1}(0) \in \mathbb{R}$, this implies $|u^{\sharp_1}(x)|^2 \operatorname{Im}[u^{\sharp_1}(-x) + u^{\sharp_1}(x)] = 0$, that is

$$|u^{\sharp_1}(x)|^2 \operatorname{Im} u^{\sharp_1}(-x) = -|u^{\sharp_1}(x)|^2 \operatorname{Im} u^{\sharp_1}(x). \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$|u^{\sharp_1}(x)|^2 u^{\sharp_1}(-x) = |u^{\sharp_1}(x)|^2 \overline{u^{\sharp_1}(x)}. \tag{$*$}$$

From ($*$), we deduce (3.30) provided that $u^{\sharp_1}(x) \neq 0$. However, if $u^{\sharp_1}(x) = 0$, we repeat the previous derivation with the vectors $\zeta = (u^{\sharp_1}(-x), iu^{\sharp_1}(-x))$ and $\zeta = (u^{\sharp_1}(-x), u^{\sharp_1}(-x))$ instead, and arrive at the similar formula

$$|u^{\sharp_1}(-x)|^2 u^{\sharp_1}(-x) = |u^{\sharp_1}(-x)|^2 \overline{u^{\sharp_1}(x)}. \tag{$**$}$$

Since $u^{\sharp_1}(x) = 0$, ($**$) reads $u^{\sharp_1}(-x) = 0$, hence we also get (3.30).

3. Finally, it remains to prove (3.31). If $u^{\sharp_1}(x) = 0$, then (3.31) is clear by $u^{\sharp_1}(0) \geqslant 0$. If $u^{\sharp_1}(x) \neq 0$, we let (see [Kat04, p. 150]) $\zeta = (-\frac{|u^{\sharp_1}(x)|}{u^{\sharp_1}(x)}, 1) \in \mathbb{C}^2$. Then (3.35) gives, after inserting (3.30), that (3.31) holds. (Notice that by (3.30) the assumption $u^{\sharp_1}(x) \neq 0$ is equivalent to $u^{\sharp_1}(-x) \neq 0$.) The proof of Lemma 3.10 is now complete. □

### 3.5.3 Symmetrization Decreases the Kinetic Energy

In this subsection we investigate how our symmetrization affects the energy terms appearing in the Weinstein functional. First, consider the functional

$$\mathcal{G}_v : \dot{H}^{\frac{1}{2}}(\mathbb{R}) \to \mathbb{R}, \quad \mathcal{G}_v(u) := -\frac{1}{2} \int_{\mathbb{R}^n} (v \cdot \xi) |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

As $s \geqslant \frac{1}{2}$, we have $H^s(\mathbb{R}^n) \hookrightarrow \dot{H}^{1/2}(\mathbb{R}^n)$, so $\mathcal{G}_v$ is well-defined on $H^s(\mathbb{R}^n)$. Also, by Lemma 3.12 below, $u \in H^s(\mathbb{R}^n)$ gives $u^{\sharp_1} \in H^s(\mathbb{R}^n)$ (respectively, $\widetilde{u} \in H^s(\mathbb{R}^n)$ if $n = 1$), so that $\mathcal{G}_v(u^{\sharp_1})$ (respectively, $\mathcal{G}_v(\widetilde{u})$ if $n = 1$) is well-defined. Let us confirm that this boost term is invariant under our symmetrization.

**Lemma 3.11** (Invariance of boost term under symmetrization). *Let $n \geqslant 1$ and $u \in H^s(\mathbb{R}^n)$. Then:*

*(i) If $n = 1$, then $\mathcal{G}_v(\widetilde{u}) = \mathcal{G}_v(u)$.*

*(ii) If $n \geqslant 2$ and $v = (v_1, 0, \ldots, 0)$ points into 1-direction, then $\mathcal{G}_v(u^{\sharp_1}) = \mathcal{G}_v(u)$.*

*Proof.* (i) is clear. To prove (ii), we understand again $*$ as the decreasing rearrangement in the last $n-1$ variables. We know that the functions $\mathcal{F}u(\xi_1,\cdot)^*$ and $|\mathcal{F}u(\xi_1,\cdot)|$ are equimeasurable on $\mathbb{R}^{n-1}$, for any fixed $\xi_1 \in \mathbb{R}$. It follows that

$$\begin{aligned}
\mathcal{G}_v(u^{\sharp_1}) &= -\frac{1}{2}\int_{\mathbb{R}^n} (v_1\xi_1)|\widehat{u^{\sharp_1}}(\xi)|^2 \,\mathrm{d}\xi \\
&= -\frac{1}{2}\int_{\mathbb{R}} (v_1\xi_1)\left(\int_{\mathbb{R}^{n-1}} |(\mathcal{F}u)^{*_1}(\xi_1,\xi')|^2 \,\mathrm{d}\xi'\right) \mathrm{d}\xi_1 \\
&= -\frac{1}{2}\int_{\mathbb{R}} (v_1\xi_1)\left(\int_{\mathbb{R}^{n-1}} |(\mathcal{F}u)(\xi_1,\cdot)^*(\xi')|^2 \,\mathrm{d}\xi'\right) \mathrm{d}\xi_1 \\
&= -\frac{1}{2}\int_{\mathbb{R}} (v_1\xi_1)\left(\int_{\mathbb{R}^{n-1}} |(\mathcal{F}u)(\xi_1,\cdot)(\xi')|^2 \,\mathrm{d}\xi'\right) \mathrm{d}\xi_1 \\
&= -\frac{1}{2}\int_{\mathbb{R}^n} (v_1\xi_1)|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi = \mathcal{G}_v(u).
\end{aligned}$$ □

Second, we consider the functional

$$\mathcal{T} : H^s(\mathbb{R}^n) \to \mathbb{R}, \quad \mathcal{T}(u) := \frac{1}{2}\int_{\mathbb{R}^n} m(\xi)|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

By our general assumption (A), we have

$$A\|u\|_{H^s}^2 \leqslant \int_{\mathbb{R}^n} (m(\xi)+\lambda)|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi \leqslant B\|u\|_{H^s}^2, \quad u \in H^s(\mathbb{R}^n),$$

so $\mathcal{T}$ is well-defined on $H^s(\mathbb{R}^n)$.

**Lemma 3.12** (Kinetic energy decreases under symmetrization)**.** *Let $n \geqslant 1$ and $u \in H^s(\mathbb{R}^n)$. Then:*

*(i) If $n = 1$, then $\mathcal{T}(\widetilde{u}) = \mathcal{T}(u)$.*

*(ii) If $n \geqslant 2$ and we suppose the "monotonicity property" (B), i.e.,*

$$m(\xi_1,\xi') \geqslant m(\xi_1,\eta'), \quad \text{if } |\xi'| \geqslant |\eta'|, \tag{3.38}$$

*then*

$$\mathcal{T}(u^{\sharp_1}) \leqslant \mathcal{T}(u). \tag{3.39}$$

**Remark 3.13.** If $L = (-\Delta)^s$ with $s > 0$, then clearly (3.38) is true:

$$m(\xi_1,\xi') = \left(\sqrt{\xi_1^2+|\xi'|^2}\right)^{2s} \geqslant \left(\sqrt{\xi_1^2+|\eta'|^2}\right)^{2s} = m(\xi_1,\eta'), \quad \text{if } |\xi'| \geqslant |\eta'|.$$

In this case $\mathcal{T}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2$.

*Proof (cf. [BL15, Proof of Lemma A.1]).* (i) is clear. We prove (ii). Since $u \in L^2(\mathbb{R}^n)$, so is $\mathcal{F}u$ and thus $(\mathcal{F}u)^{*_1}$. Consequently, by definition of $u^{\sharp_1}$ as the inverse Fourier transform of an $L^2$ function, we have $u^{\sharp_1} \in L^2(\mathbb{R}^n)$ with $\mathcal{F}(u^{\sharp_1}) = (\mathcal{F}u)^{*_1}$. It is a fundamental property of the symmetric decreasing rearrangement that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, then $(\varphi \circ |g|)^* = \varphi \circ g^*$ for any Borel measurable function $g : \mathbb{R}^{n-1} \to \mathbb{C}$ vanishing at infinity (see [LL01, p. 81]). Hence (take $\varphi(x) = x^2$) we have $(|g|^2)^* = (g^*)^2$. This means $(|h(\xi_1, \cdot)|^2)^* = (h(\xi_1, \cdot)^*)^2$, a.e. $\xi_1 \in \mathbb{R}$, for $h : \mathbb{R}^n \to \mathbb{C}$ measurable and vanishing at inifinity; in other words, $(|h|^2)^{*_1} = (h^{*_1})^2$ a.e. Thus (take $h = \mathcal{F}u$), claim (3.39) is equivalent to

$$\begin{aligned}\mathcal{T}(u^{\sharp_1}) &= \frac{1}{2}\int_{\mathbb{R}^n} m(\xi)|(\mathcal{F}u)^{*_1}(\xi)|^2 \, \mathrm{d}\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^n} m(\xi)(|(\mathcal{F}u)(\xi)|^2)^{*_1} \, \mathrm{d}\xi \leqslant \frac{1}{2}\int_{\mathbb{R}^n} m(\xi)|(\mathcal{F}u)(\xi)|^2 \, \mathrm{d}\xi.\end{aligned}$$

It is therefore sufficient to prove

$$\int_{\mathbb{R}^n} m(\xi)h^{*_1}(\xi) \, \mathrm{d}\xi \leqslant \int_{\mathbb{R}^n} m(\xi)h(\xi) \, \mathrm{d}\xi$$

for any nonnegative measurable function $h : \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity. By definition of $*_1$, this is equivalent to ($*$ is again the symmetric decreasing rearrangement on $\mathbb{R}^{n-1}$)

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')h(\xi_1, \cdot)^*(\xi') \, \mathrm{d}\xi' \right) \mathrm{d}\xi_1 \leqslant \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')h(\xi_1, \xi') \, \mathrm{d}\xi' \right) \mathrm{d}\xi_1. \tag{3.40}$$

Therefore, it suffices to prove

$$\int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')h(\xi_1, \cdot)^*(\xi') \, \mathrm{d}\xi' \leqslant \int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')h(\xi_1, \xi') \, \mathrm{d}\xi', \quad \text{a.e. } \xi_1 \in \mathbb{R}, \tag{3.41}$$

and then integrate this inequality over $\mathbb{R}$ to get (3.40). Let us prove (3.41). By the layer-cake representation, write $h(\xi_1, \xi') = \int_0^\infty \chi_{\{h(\xi_1, \cdot) > t\}}(\xi') \, \mathrm{d}t$ for a.e. $\xi' \in \mathbb{R}^{n-1}$. By this, (3.26) (in dimension $n-1$) and Fubini's theorem, it is clear that it suffices to show

$$\int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')\chi_{A^*}(\xi') \, \mathrm{d}\xi' \leqslant \int_{\mathbb{R}^{n-1}} m(\xi_1, \xi')\chi_A(\xi') \, \mathrm{d}\xi' \tag{3.42}$$

for any measurable set $A \subset \mathbb{R}^{n-1}$ of finite measure. Here $A^*$ is the rearrangement of $A$, i.e., the open ball $B_R(0)$ in $\mathbb{R}^{n-1}$ centered at the origin with measure $\mu(A)$, where $\mu$ shall denote the measure (with the convention that $A^* = \varnothing$ if $\mu(A) = 0$). By additivity of the measure, we have $\mu(A \backslash A^*) = \mu(A) - \mu(A \cap A^*)$ and $\mu(A^* \backslash A) = \mu(A^*) - \mu(A^* \cap A)$. Subtracting these two equations from one another and using $\mu(A) = \mu(A^*)$ (by definition of rearrangement of a set) yields $\mu(A \backslash A^*) = \mu(A^* \backslash A)$.

Thus, by using the monotonicity property (3.38), letting $\eta' = (R, 0, \dots, 0) \in \mathbb{R}^{n-1}$,

$$\begin{aligned}\int_{A\backslash A^*} m(\xi_1, \xi')\,\mathrm{d}\xi' \geqslant \int_{A\backslash A^*} m(\xi_1, \eta')\,\mathrm{d}\xi' &= m(\xi_1, \eta')\mu(A\backslash A^*)\\ &= m(\xi_1, \eta')\mu(A^*\backslash A) = \int_{A^*\backslash A} m(\xi_1, \eta')\,\mathrm{d}\xi'\\ &\geqslant \int_{A^*\backslash A} m(\xi_1, \xi')\,\mathrm{d}\xi',\end{aligned}$$

since $\xi' \notin A^*$ if and only if $|\xi'| \geqslant R$. Therefore

$$\begin{aligned}\int_A m(\xi_1, \xi')\,\mathrm{d}\xi' &= \int_{A\backslash A^*} m(\xi_1, \xi')\,\mathrm{d}\xi' + \int_{A\cap A^*} m(\xi_1, \xi')\,\mathrm{d}\xi'\\ &\geqslant \int_{A^*\backslash A} m(\xi_1, \xi')\,\mathrm{d}\xi' + \int_{A\cap A^*} m(\xi_1, \xi')\,\mathrm{d}\xi' = \int_{A^*} m(\xi_1, \xi')\,\mathrm{d}\xi'.\end{aligned}$$

This is exactly (3.42), from which (3.41) and hence (3.39) follows. The assertion $u^{\sharp_1} \in H^s(\mathbb{R}^n)$ follows from the estimate

$$\begin{aligned}\|u^{\sharp_1}\|_{H^s}^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{u^{\sharp_1}}(\xi)|^2\,\mathrm{d}\xi \leqslant \frac{1}{A}\int_{\mathbb{R}^n} (m(\xi) + \lambda)|\widehat{u^{\sharp_1}}(\xi)|^2\,\mathrm{d}\xi\\ &= \frac{1}{A}\mathcal{T}(u^{\sharp_1}) + \lambda\|u^{\sharp_1}\|_{L^2}^2 \leqslant \frac{1}{A}\mathcal{T}(u) + \lambda\|u\|_{L^2}^2 \lesssim \|u\|_{H^s}^2,\end{aligned}$$

using assumption (A), Plancherel, (3.39) and Lemma 3.10 (i), and again assumption (A). The proof of Lemma 3.12 is now complete. □

### 3.5.4 Symmetrization Increases the Potential Energy

In the next step, we show that the potential energy goes into the right direction under our symmetrization, at least provided that $\sigma \geqslant 0$ is an integer. We keep in mind that $p = 2\sigma + 2$ is then even. In the case of non-even $p$, their may or may not exist counterexamples to the following behaviour of potential energy.

**Lemma 3.14** (Potential energy increases under symmetrization)**.** *Let $n \geqslant 1$ and $\sigma \geqslant 0$ be an integer. Let $u \in L^2(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$ and suppose that $\mathcal{F}u \in L^1(\mathbb{R}^n)$. Then*

$$\begin{cases} u^{\sharp_1} \in L^{2\sigma+2}(\mathbb{R}^n) & \text{if } n \geqslant 2,\\ \widetilde{u} \in L^{2\sigma+2}(\mathbb{R}) & \text{if } n = 1,\end{cases}$$

*and we have the estimate*

$$\begin{cases} \|u\|_{L^{2\sigma+2}} \leqslant \|u^{\sharp_1}\|_{L^{2\sigma+2}} & \text{if } n \geqslant 2,\\ \|u\|_{L^{2\sigma+2}} \leqslant \|\widetilde{u}\|_{L^{2\sigma+2}} & \text{if } n = 1.\end{cases}$$

### Preliminary Lemmata

To prove Lemma 3.14, we shall make use of the Brascamp-Lieb-Luttinger inequality, which reads as follows (see [BLL74] and also [LL01]).

**Theorem 3.15** (Brascamp-Lieb-Luttinger inequality; see [LL01, Theorem 3.8]). *Let $n, m \geqslant 1$ and $u_1, u_2, \ldots, u_m : \mathbb{R}^n \to \mathbb{R}_+$ be non-negative measurable functions vanishing at infinity. Let $k \leqslant m$ and $B = (b_{ij})$ be a $k \times m$ matrix (with $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$). Define*

$$I_n[u_1, \ldots, u_m] := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m} u_j \Big( \sum_{i=1}^{k} b_{ij} y^i \Big) \, \mathrm{d}y^1 \cdots \mathrm{d}y^k. \tag{3.43}$$

*Then*

$$I_n[u_1, \ldots, u_m] \leqslant I_n[u_1^*, \ldots, u_m^*].$$

The following lemma shows how an $(m-1)$-fold convolution evaluated at zero can be related to a Brascamp-Lieb-Luttinger quantity $I_{n-1}$ like in (3.43).

**Lemma 3.16.** *Let $n, m \geqslant 2$. For functions $u_1, \ldots, u_m$ and $x = (x_1, x') \in \mathbb{R}^n$ we have*

$$\begin{aligned}(u_1 \star \cdots \star u_m)(x) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m-1} u_j(y^j) u_m \Big( x - \sum_{i=1}^{m-1} y^i \Big) \, \mathrm{d}y^1 \cdots \mathrm{d}y^{m-1} \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}^{n-1}} \cdots \int_{\mathbb{R}^{n-1}} \prod_{j=1}^{m-1} u_j(y_1^j, y^{j\prime}) \\ &\quad \times u_m \Big( x_1 - \sum_{i=1}^{m-1} y_1^i, x' - \sum_{i=1}^{m-1} y^{i\prime} \Big) \, \mathrm{d}y^{1\prime} \cdots \mathrm{d}y^{m-1\prime} \Big\} \, \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1}.\end{aligned} \tag{3.44}$$

*In particular, at $x = 0$, we have that*

$$\begin{aligned}&(u_1 \star \cdots \star u_m)(0) \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\Big[ u_1(y_1^1, \cdot), \ldots, u_{m-1}(y_1^{m-1}, \cdot), u_m \Big( - \sum_{i=1}^{m-1} y_1^i, \cdot \Big) \Big] \, \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1},\end{aligned} \tag{3.45}$$

*where $I_{n-1}$ is defined according to (3.43) with the $(m-1) \times m$ matrix*

$$\left( \begin{array}{cccc|c} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{array} \right).$$

*Here, the matrix in the left block is the $(m-1) \times (m-1)$ unit matrix.*

*Proof.* The second equality in (3.44) is Fubini's Theorem, while the first equality follows by induction on $m$. Indeed, for $m = 2$, the formula is correct by definition of convolution of two functions. For the inductive step, assume the formula is correct for $m \in \mathbb{N}$. For $m + 1 \in \mathbb{N}$, we have by hypothesis

$$\begin{aligned}
&(u_1 \star \cdots \star u_m \star u_{m+1})(x) = \int_{\mathbb{R}^n} (u_1 \star \cdots \star u_m)(z^m) u_{m+1}(x - z^m) \, \mathrm{d}z^m \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m-1} u_j(y^j) u_m\left( z^m - \sum_{i=1}^{m-1} y^i \right) \mathrm{d}y^1 \cdots \mathrm{d}y^{m-1} \right\} u_{m+1}(x - z^m) \, \mathrm{d}z^m \\
&= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m-1} u_j(y^j) \left\{ \int_{\mathbb{R}^n} u_m\left( z^m - \sum_{i=1}^{m-1} y^i \right) u_{m+1}(x - z^m) \, \mathrm{d}z^m \right\} \mathrm{d}y^1 \cdots \mathrm{d}y^{m-1},
\end{aligned} \tag{3.46}$$

the last equality by Fubini's Theorem. Changing variables $y^m := z^m - \sum_{i=1}^{m-1} y^i$ yields for the inner integral

$$\int_{\mathbb{R}^n} u_m\left( z^m - \sum_{i=1}^{m-1} y^i \right) u_{m+1}(x - z^m) \, \mathrm{d}z^m = \int_{\mathbb{R}^n} u_m(y^m) u_{m+1}\left( x - \sum_{i=1}^{m} y^i \right) \mathrm{d}y^m. \tag{3.47}$$

Inserting (3.47) into (3.46) and using Fubini again yields (3.44) for $m + 1 \in \mathbb{N}$. □

**Corollary 3.17.** *Let $n, m \geqslant 2$. For non-negative measurable functions $u_1, u_2, \ldots, u_m : \mathbb{R}^n \to \mathbb{R}_+$ vanishing at infinity, we have*

$$(u_1 \star \cdots \star u_m)(0) \leqslant (u_1^{*_1} \star \cdots \star u_m^{*_1})(0).$$

*Proof.* Using (3.45), Brascamp-Lieb-Luttinger (Theorem 3.15), the definition of Steiner symmetrization in codimension $n - 1$ and (3.45) again, we get

$$\begin{aligned}
&(u_1 \star \cdots \star u_m)(0) \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[ u_1(y_1^1, \cdot), \ldots, u_{m-1}(y_1^{m-1}, \cdot), u_m\left( -\sum_{i=1}^{m-1} y_1^i, \cdot \right) \right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1} \\
&\leqslant \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[ u_1(y_1^1, \cdot)^*, \ldots, u_{m-1}(y_1^{m-1}, \cdot)^*, u_m\left( -\sum_{i=1}^{m-1} y_1^i, \cdot \right)^* \right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1} \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[ u_1^{*_1}(y_1^1, \cdot), \ldots, u_{m-1}^{*_1}(y_1^{m-1}, \cdot), u_m^{*_1}\left( -\sum_{i=1}^{m-1} y_1^i, \cdot \right) \right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1} \\
&= (u_1^{*_1} \star \cdots \star u_m^{*_1})(0).
\end{aligned}$$

□

We need one more technical lemma which will be applied in the proof of Lemma 3.14.

**Lemma 3.18** (Some rearrangement relations)**.** *Let $n \geqslant 2$ and $g, u : \mathbb{R}^n \to \mathbb{C}$ be measurable functions vanishing at infinity. Then:*

*(i) For fixed $x_1 \in \mathbb{R}$, the functions $g(x_1, \cdot)$ and $|g(x_1, \cdot)|$ are equimeasurable on $\mathbb{R}^{n-1}$. Consequently, their Steiner symmetrizations coincide, i.e. $|g|^{*_1} = g^{*_1}$. In particular, we have $|\mathcal{F}u|^{*_1} = (\mathcal{F}u)^{*_1}$ and $|\mathcal{F}\overline{u}|^{*_1} = (\mathcal{F}\overline{u})^{*_1}$.*

*(ii) The functions $g$ and $\overline{g}$ are equimeasurable on $\mathbb{R}^n$. Consequently, they have the same Schwarz rearrangements, i.e. $g^* = \overline{g}^*$.*

*(iii) The functions $g$ and $g(-\cdot)$ are equimeasurable on $\mathbb{R}^n$. Consequently, they have the same Schwarz rearrangements, i.e. $g^* = g(-\cdot)^*$.*

*(iv) We have $|\mathcal{F}\overline{u}|^{*_1} = (\mathcal{F}u)^{*_1}(-\cdot)$, i.e.*

$$|\mathcal{F}\overline{u}|^{*_1}(\xi) = (\mathcal{F}u)^{*_1}(-\xi), \quad \textit{for a.e. } \xi \in \mathbb{R}^n.$$

*Proof.* (i) and (ii) are clear.

(iv) immediately follows from (i), (ii) and (iii) via the formula

$$\mathcal{F}\overline{g}(\xi) = \overline{\mathcal{F}g(-\xi)}. \tag{3.48}$$

Namely,

$$\begin{aligned}|\mathcal{F}\overline{u}|^{*_1}(\xi_1, \xi') &\overset{(i)}{=} (\mathcal{F}\overline{u})^{*_1}(\xi_1, \xi') = \mathcal{F}\overline{u}(\xi_1, \cdot)^*(\xi') \overset{(3.48)}{=} \overline{\mathcal{F}u(-\xi_1, -\cdot)}^*(\xi') \\ &\overset{(ii)}{=} \mathcal{F}u(-\xi_1, -\cdot)^*(\xi') \overset{(iii)}{=} \mathcal{F}u(-\xi_1, \cdot)^*(\xi') = (\mathcal{F}u)^{*_1}(-\xi_1, \xi') \\ &= (\mathcal{F}u)^{*_1}(-\xi_1, -\xi'),\end{aligned}$$

the last equality by cylindrical symmetry of $(\mathcal{F}u)^{*_1}$ with respect to 1-axis (Lemma 3.9 (ii)). Here, the particular case of (i) with $\mathcal{F}\overline{u}$ was used and (ii) and (iii) were applied with the decreasing rearrangement $*$ on $\mathbb{R}^{n-1}$ to the function $g = g_{\xi_1} = \mathcal{F}u(-\xi_1, -\cdot)$ depending on $n-1$ variables.

It remains to show (iii). To this matter, we note that

$$y \in \{y \in \mathbb{R}^n;\ |g(-y)| > t\} \Longleftrightarrow -y \in \{y \in \mathbb{R}^n;\ |g(y)| > t\}.$$

In other words, the set $\{y \in \mathbb{R}^n;\ |g(-y)| > t\}$ is just the reflection of the set $\{y \in \mathbb{R}^n;\ |g(y)| > t\}$ at the origin $0 \in \mathbb{R}^n$, i.e.,

$$\{y \in \mathbb{R}^n;\ |g(-y)| > t\} = \mathcal{R}_0(\{y \in \mathbb{R}^n;\ |g(y)| > t\}),$$

where $\mathcal{R}_0(S) := \{R_0 y; y \in S\}$ for a set $S \subset \mathbb{R}^n$ with the linear operator $R_0 = -\mathrm{Id}$. Lebesgue measure is invariant under transformations $y \mapsto a + Ty$ with $a \in \mathbb{R}^n$ and $T \in O(n, \mathbb{R})$, in particular under the reflection $y \mapsto R_0 y$, $R_0 \in O(n, \mathbb{R})$. Thus the sets $\{y \in \mathbb{R}^n;\ |g(y)| > t\}$ and $\mathcal{R}_0(\{y \in \mathbb{R}^n;\ |g(y)| > t\}) = \{y \in \mathbb{R}^n;\ |g(-y)| > t\}$ have equal measure. Therefore $g$ and $g(-\cdot)$ are equimeasurable and thus their Schwarz rearrangements coincide. This proves (iii). The proof of Lemma 3.18 is now complete. □

### Proof of Lemma 3.14 (cf. [BL15, Lemma A.1])

We proceed in two steps.

**Step 1: $u^{\sharp_1} \in L^{2\sigma+2}(\mathbb{R}^n)$.** Let first $n \geqslant 2$. By Plancherel, $\mathcal{F}u \in L^2(\mathbb{R}^n)$, hence by Lemma 3.9 (i) $(\mathcal{F}u)^{*_1} \in L^2(\mathbb{R}^n)$. Since we suppose $\mathcal{F}u \in L^1(\mathbb{R}^n)$, Lemma 3.9 (i) gives that also $(\mathcal{F}u)^{*_1} \in L^1(\mathbb{R}^n)$. Hence $(\mathcal{F}u)^{*_1} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which implies ($\mathcal{F}^{-1} : L^1 \to L^\infty$ is bounded and $\mathcal{F}^{-1} : L^2 \to L^2$ is isometric) that $\mathcal{F}^{-1}((\mathcal{F}u)^{*_1}) = u^{\sharp_1} \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. By interpolation, $u^{\sharp_1} \in L^{2\sigma+2}(\mathbb{R}^n)$.

Let now $n = 1$. Similarly as above, we obtain from Plancherel and the hypothesis $\mathcal{F}u \in L^1(\mathbb{R})$ that $\mathcal{F}u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Equivalently, $|\mathcal{F}u| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. As above, this implies $\tilde{u} = \mathcal{F}^{-1}(|\mathcal{F}u|) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^{2\sigma+2}(\mathbb{R})$.

**Step 2: Conclusion with Brascamp-Lieb-Luttinger inequality.** The convolution theorem in our convention of the Fourier transform reads [RS75, Theorem IX.3]

$$\mathcal{F}(f \star g) = (2\pi)^{\frac{n}{2}} \mathcal{F}(f)\mathcal{F}(g),$$

and conversely,

$$\mathcal{F}(fg) = (2\pi)^{-\frac{n}{2}} \mathcal{F}(f) \star \mathcal{F}(g). \tag{3.49}$$

We have

$$\begin{aligned} \|u\|_{L^{2\sigma+2}}^{2\sigma+2} &= (2\pi)^{\frac{n}{2}} \mathcal{F}((u\overline{u})^{\sigma+1})(0) = (2\pi)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}\sigma} (\mathcal{F}(u\overline{u}) \star \cdots \star \mathcal{F}(u\overline{u}))(0) \\ &= (2\pi)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}\sigma} (2\pi)^{-\frac{n}{2}(\sigma+1)} (\mathcal{F}u \star \mathcal{F}\overline{u} \star \cdots \star \mathcal{F}u \star \mathcal{F}\overline{u})(0) \\ &\leqslant (2\pi)^{-n\sigma} (|\mathcal{F}u| \star |\mathcal{F}\overline{u}| \star \cdots \star |\mathcal{F}u| \star |\mathcal{F}\overline{u}|)(0) \end{aligned} \tag{3.50}$$

by definition of the Fourier transform, applying the convolution theorem (3.49) $\sigma$ times (here, the condition that $\sigma$ is an integer enters), and then again on each of the remaining $\sigma + 1$ factors $\mathcal{F}(u\overline{u})$. Using Corollary 3.17, we see that

$$\begin{aligned} &(|\mathcal{F}u| \star |\mathcal{F}\overline{u}| \star \cdots \star |\mathcal{F}u| \star |\mathcal{F}\overline{u}|)(0) \leqslant (|\mathcal{F}u|^{*_1} \star |\mathcal{F}\overline{u}|^{*_1} \star \cdots \star |\mathcal{F}u|^{*_1} \star |\mathcal{F}\overline{u}|^{*_1})(0) \\ =&((\mathcal{F}u)^{*_1} \star (\mathcal{F}u)^{*_1}(-\cdot) \star \cdots \star (\mathcal{F}u)^{*_1} \star (\mathcal{F}u)^{*_1}(-\cdot))(0). \end{aligned}$$

Here, the second step uses statements (i) and (iv) of Lemma 3.18. By definition of $u^{\sharp_1}$, we have $(\mathcal{F}u)^{*_1} = \mathcal{F}(u^{\sharp_1})$. Finally observe that $(\mathcal{F}u)^{*_1}(-\cdot) = \mathcal{F}(\overline{u^{\sharp_1}})$, since in general by the definition of Fourier transform one has $\mathcal{F}\overline{g}(\xi) = \overline{\mathcal{F}g(-\xi)}$. Consequently

$$\mathcal{F}(\overline{u^{\sharp_1}})(\xi) = \overline{\mathcal{F}u^{\sharp_1}(-\xi)} = \overline{(\mathcal{F}u)^{*_1}(-\xi)} = (\mathcal{F}u)^{*_1}(-\xi),$$

where the last step uses the fact that $g^{*_1}$ is real-valued for any function $g$. Using these last facts and applying the convolution theorem backwards, we finally get

$$\begin{aligned} \|u\|_{L^{2\sigma+2}}^{2\sigma+2} &\leqslant (2\pi)^{-n\sigma} (\mathcal{F}(u^{\sharp_1}) \star \mathcal{F}(\overline{u^{\sharp_1}}) \star \cdots \star \mathcal{F}(u^{\sharp_1}) \star \mathcal{F}(\overline{u^{\sharp_1}}))(0) \\ &= (2\pi)^{-n\sigma} (2\pi)^{\frac{n}{2}(\sigma+1)} (\mathcal{F}(u^{\sharp_1}\overline{u^{\sharp_1}}) \star \cdots \star \mathcal{F}(u^{\sharp_1}\overline{u^{\sharp_1}}))(0) \\ &= (2\pi)^{-n\sigma} (2\pi)^{\frac{n}{2}(\sigma+1)} (2\pi)^{\frac{n}{2}\sigma} \mathcal{F}((u^{\sharp_1}\overline{u^{\sharp_1}})^{\sigma+1})(0) \\ &= (2\pi)^{\frac{n}{2}} \mathcal{F}((u^{\sharp_1}\overline{u^{\sharp_1}})^{\sigma+1})(0) = \|u^{\sharp_1}\|_{L^{2\sigma+2}}^{2\sigma+2}. \end{aligned} \tag{3.51}$$

This completes the proof in the case $n \geqslant 2$.

Let now $n = 1$. Again, by the convolution theorem, we have as above (see (3.50))

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant (2\pi)^{-n\sigma}(|\mathcal{F}u| \star |\mathcal{F}\overline{u}| \star \cdots \star |\mathcal{F}u| \star |\mathcal{F}\overline{u}|)(0)$$
$$= (2\pi)^{-n\sigma}(\mathcal{F}(\widetilde{u}) \star |\mathcal{F}\overline{u}| \star \cdots \star \mathcal{F}(\widetilde{u}) \star |\mathcal{F}\overline{u}|)(0),$$

where we inserted the definition $\widetilde{u} = \mathcal{F}^{-1}(|\mathcal{F}u|)$. Now notice that

$$\mathcal{F}(\overline{\widetilde{u}})(\xi) = \overline{\mathcal{F}(\widetilde{u})(-\xi)} = \overline{|\mathcal{F}u(-\xi)|} = |\mathcal{F}u(-\xi)| = |\overline{\mathcal{F}u(-\xi)}| = |\mathcal{F}\overline{u}(\xi)|.$$

Hence (applying the convolution theorem backwards as in (3.51))

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant (2\pi)^{-n\sigma}(\mathcal{F}(\widetilde{u}) \star \mathcal{F}(\overline{\widetilde{u}}) \star \cdots \star \mathcal{F}(\widetilde{u}) \star \mathcal{F}(\overline{\widetilde{u}}))(0) = \|\widetilde{u}\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

The proof of Lemma 3.14 is now complete. □

### 3.5.5 Proof of Theorem 3.2

We come to the proof of Theorem 3.2. Recall that we suppose that $\sigma > 0$ is an integer. Let $Q_v \in H^s(\mathbb{R}^n)$ be a boosted ground state, whose existence is guaranteed by Theorem 3.1. Furthermore, when $L$ is a general pseudo-differential operator satisfying assumption (A) and the monotonicity property (B), we make the additional assumption that $Q_v \in L^\infty(\mathbb{R}^n)$.[9] All of this is true for the fractional Laplacian $L = (-\Delta)^s$, $\frac{1}{2} \leqslant s < 1$, and in this case we prove $Q_v \in L^\infty(\mathbb{R}^n)$ in the next section.

Now we give the proof of Theorem 3.2 in four steps.

**Step 1: For $\sigma \geqslant 1$ an integer, $Q_v \in H^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.** Recall that up to rescaling $Q_v$ satisfies the equation

$$Q_v = (L + iv \cdot \nabla + \omega)^{-1}(|Q_v|^{2\sigma} Q_v).$$

The operators

$$\frac{(-\Delta)^s}{L + iv \cdot \nabla + \omega}, \quad (L + iv \cdot \nabla + \omega)^{-1}$$

are bounded multipliers $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, since for $\varphi(\xi) = m(\xi) - v \cdot \xi + \omega > 0$ $(\omega > \omega_*)$ we get the boundedness

$$0 \leqslant \frac{|\xi|^{2s}}{\varphi(\xi)} \lesssim \frac{|\xi|^{2s}}{(1 + |\xi|^2)^s} \leqslant 1, \quad 0 \leqslant \frac{1}{\varphi(\xi)} \lesssim \frac{1}{(1 + |\xi|^2)^s} \leqslant 1$$

[9] If $s > \frac{n}{2}$, this is automatically fulfilled by Sobolev's embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

from the proof of Lemma 3.5. Consequently, for each $\gamma \in \mathbb{R}$, $(L + iv \cdot \nabla + \omega)^{-1}$ defines a smoothing $H^{\gamma}(\mathbb{R}^n) \to H^{\gamma+2s}(\mathbb{R}^n)$. Indeed, if $f \in H^{\gamma}(\mathbb{R}^n)$, then

$$\begin{aligned}\|(L + iv \cdot \nabla + \omega)^{-1} f\|_{\dot{H}^{\gamma+2s}} &= \|(-\Delta)^{\frac{\gamma+2s}{2}}(L + iv \cdot \nabla + \omega)^{-1} f\|_{L^2} \\ &= \|\frac{(-\Delta)^s}{L + iv \cdot \nabla + \omega}(-\Delta)^{\frac{\gamma}{2}} f\|_{L^2} \\ &\lesssim \|(-\Delta)^{\frac{\gamma}{2}} f\|_{L^2} = \|f\|_{\dot{H}^{\gamma}}\end{aligned}$$

and

$$\|(L + iv \cdot \nabla + \omega)^{-1} f\|_{L^2} \lesssim \|f\|_{L^2},$$

so that $\|(L + iv \cdot \nabla + \omega)^{-1} f\|_{H^{\gamma+2s}} \lesssim \|f\|_{H^{\gamma}} < \infty$.

In case of $s > \frac{n}{2}$ there is an algebra structure on $H^s(\mathbb{R}^n)$ (see [LP09, Theorem 3.4]), so that for $Q_v \in H^s(\mathbb{R}^n)$ also $(Q_v\overline{Q_v})^{\sigma} Q_v \in H^s(\mathbb{R}^n)$. Using $Q_v \in H^s(\mathbb{R}^n)$ and the algebra and above smoothing properties, we'd be done in case of $s > \frac{n}{2}$.

In the case $s \leqslant \frac{n}{2}$, we exploit $Q \in L^{\infty}(\mathbb{R}^n)$ and a Moser inequality. Namely, it is still true that for fixed integer $k \in \mathbb{N}$, the space $H^k(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ has an algebra structure as it holds that [Tay11b, Proposition 3.7, p. 11]

$$\|fg\|_{H^k} \lesssim \|f\|_{L^{\infty}}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_{L^{\infty}}, \quad \text{if } f, g \in H^k(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n).$$

For any fixed $k \in \mathbb{N}$, one easily uses this to show by induction that $(Q_v\overline{Q_v})^{\sigma} Q_v \in H^k(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ for all $\sigma \in \mathbb{N}$, provided that $Q_v \in H^k(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. This in turn implies the assertion by the following iteration. By Lemma 3.19 below we know that $Q_v \in H^2(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. By the previous argument $(Q_v\overline{Q_v})^{\sigma} Q_v \in H^2(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. Since the operator $((-\Delta)^s + iv \cdot \nabla + \omega)^{-1}$ defines a smoothing $H^2(\mathbb{R}^n) \to H^{2+2s}(\mathbb{R}^n)$, we conclude that $Q_v \in H^3(\mathbb{R}^n)$, since $s \geqslant \frac{1}{2}$. Hence $Q_v \in H^3(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. Continuing this iteration, we deduce that $Q_v \in H^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. In particular, by Sobolev embedding [Str03, Theorem 8.1.2] $Q_v$ is smooth, $Q_v \in C^{\infty}(\mathbb{R}^n)$. [See also [LB96, p. 727] for similar reasoning.]

**Step 2: $\mathcal{F}Q_v \in L^1(\mathbb{R}^n)$.** Pick $k \in \mathbb{N}$ such that $k > \frac{n}{2}$. This is sufficient for $\xi \mapsto \langle\xi\rangle^{-k}$ to be in $L^2(\mathbb{R}^n)$. By step 1, $Q_v \in H^k(\mathbb{R}^n)$. Now Hölder's inequality gives

$$\begin{aligned}\|\widehat{Q_v}\|_{L^1} &= \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^{\frac{k}{2}}}(1 + |\xi|^2)^{\frac{k}{2}}|\widehat{Q_v}(\xi)|\, d\xi \\ &\leqslant \|\langle\xi\rangle^{-k}\|_{L^2}\|\langle\xi\rangle^k\widehat{Q_v}\|_{L^2} \lesssim \|\langle\xi\rangle^k\widehat{Q_v}\|_{L^2} = \|Q_v\|_{H^k} < \infty.\end{aligned}$$

**Step 3: Existence of a symmetric boosted ground state.** As $Q_v \in H^s(\mathbb{R}^n) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^n)$, we have $Q_v \in L^2(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$ and by step 2, $\mathcal{F}Q_v \in L^1(\mathbb{R}^n)$. By Lemma 3.14

$$\begin{cases} Q_v^{\sharp_1} \in L^{2\sigma+2}(\mathbb{R}^n) & \text{if } n \geqslant 2, \\ \widetilde{Q_v} \in L^{2\sigma+2}(\mathbb{R}) & \text{if } n = 1, \end{cases}$$

and we have the estimate

$$\begin{cases} \|Q_v\|_{L^{2\sigma+2}} \leqslant \|Q_v^{\sharp_1}\|_{L^{2\sigma+2}} & \text{if } n \geqslant 2, \\ \|Q_v\|_{L^{2\sigma+2}} \leqslant \|\widetilde{Q_v}\|_{L^{2\sigma+2}} & \text{if } n = 1. \end{cases} \tag{3.52}$$

From

$$\mathcal{J}^s_{v,\omega}(f) = \frac{\big(2\mathcal{T}(f) + 2\mathcal{G}_v(f) + \omega\|f\|^2_{L^2}\big)^{\sigma+1}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}}},$$

(3.52) and the preceding Lemmata 3.11 and 3.12 it follows that

$$\begin{cases} \mathcal{J}^s_{v,\omega}(Q^{\sharp_1}_v) \leqslant \mathcal{J}^s_{v,\omega}(Q_v) & \text{if } n \geqslant 2, \\ \mathcal{J}^s_{v,\omega}(\widetilde{Q_v}) \leqslant \mathcal{J}^s_{v,\omega}(Q_v) & \text{if } n = 1. \end{cases}$$

Hence $Q^{\sharp_1}_v$, respectively $\widetilde{Q_v}$, is minimizing, too.

**Step 4: Additional symmetries of boosted ground states.** It remains to verify the additional symmetry properties of a symmetric ground state given by step 3. Let first $n \geqslant 2$. We know from Lemma 3.10 (iii) that $Q^{\sharp_1}_v$ is continuous, and bounded with the estimate

$$Q^{\sharp_1}_v(0) \geqslant |Q^{\sharp_1}_v(x)|, \quad \text{for all } x \in \mathbb{R}^n.$$

Now, by definition of the inverse Fourier transform and the fact that $(\mathcal{F}Q_v)^{*_1}$ is real-valued, we have

$$Q^{\sharp_1}_v(x) = \mathcal{F}^{-1}((\mathcal{F}Q_v)^{*_1})(x) = \mathcal{F}^{-1}(\overline{(\mathcal{F}Q_v)^{*_1}})(x) = \overline{\mathcal{F}^{-1}((\mathcal{F}Q_v)^{*_1})(-x)} = \overline{Q^{\sharp_1}_v(-x)}.$$

Comparing real and imaginary parts in this equation, $Q^{\sharp_1}_v(x) = \overline{Q^{\sharp_1}_v(-x)}$, we find

$$\begin{cases} \operatorname{Re} Q^{\sharp_1}_v(x) = \operatorname{Re} Q^{\sharp_1}_v(-x) \\ \operatorname{Im} Q^{\sharp_1}_v(x) = -\operatorname{Im} Q^{\sharp_1}_v(-x) \end{cases}$$

Finally, using the fact that $Q^{\sharp_1}_v$ is cylindrically symmetric with respect to 1-axis yields: for any $x' \in \mathbb{R}^{n-1}$

$$\begin{cases} \text{the function } \mathbb{R} \to \mathbb{R},\ x_1 \mapsto \operatorname{Re} Q^{\sharp_1}_v(x_1, x') \text{ is even,} \\ \text{the function } \mathbb{R} \to \mathbb{R},\ x_1 \mapsto \operatorname{Im} Q^{\sharp_1}_v(x_1, x') \text{ is odd.} \end{cases}$$

This completes the proof of Theorem 3.2 for $n \geqslant 2$.

Let now $n = 1$. Since $|\mathcal{F}Q_v| \in L^1(\mathbb{R}^n)$, we know by Fourier inversion that $\widetilde{Q_v} \in C_0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is continuous and bounded, vanishing at infinity. [$C_0(\mathbb{R}^n)$ denotes the space of functions $f$ vanishing at infinity in the sense that for all $t > 0$ the set $\{x \in \mathbb{R}^n; |f(x)| \geqslant t\}$ is compact; see [Wer07, p. 6].] Since $|\mathcal{F}Q_v|$ is real, we get analogously to the above

$$\widetilde{Q_v}(x) = \mathcal{F}^{-1}(|\mathcal{F}Q_v|)(x) = \mathcal{F}^{-1}(\overline{|\mathcal{F}Q_v|})(x) = \overline{\mathcal{F}^{-1}(|\mathcal{F}Q_v|)(-x)} = \overline{\widetilde{Q_v}(-x)}.$$

Comparing real and imaginary parts in this equation, $\widetilde{Q_v}(x) = \overline{\widetilde{Q_v}(-x)}$, we find

$$\begin{cases} \text{the function } \mathbb{R} \to \mathbb{R},\ x \mapsto \operatorname{Re} \widetilde{Q_v}(x) \text{ is even,} \\ \text{the function } \mathbb{R} \to \mathbb{R},\ x \mapsto \operatorname{Im} \widetilde{Q_v}(x) \text{ is odd.} \end{cases}$$

The proof of Theorem 3.2 is now complete. □

## 3.6 Regularity

Provided that $L = (-\Delta)^s$ is the fractional Laplacian with $\frac{1}{2} \leqslant s < 1$, this section will provide the outstanding proof that solutions to the Euler-Lagrange equation (3.3) are essentially bounded.

### 3.6.1 Higher Sobolev Regularity

However, for the moment, we still let $L$ be a general pseudo-differential operator satisfying our standard assumptions. We first give a conditional result that under the essential boundedness condition a higher Sobolev regularity holds.

**Lemma 3.19** (Higher Sobolev regularity). *Let $n \geqslant 1$, $s \geqslant \frac{1}{2}$, and $L$ be a pseudo-differential operator satisfying assumption (A). Let $v \in \mathbb{R}^n$ arbitrary for $s > \frac{1}{2}$, and $|v| < A$ for $s = \frac{1}{2}$. Let $Q_v \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ solve the Euler-Lagrange equation (3.3) with $\omega > \omega_*$. Then the higher Sobolev regularity $Q_v \in H^{2s+1}(\mathbb{R}^n)$ holds, consequently $Q_v \in H^2(\mathbb{R}^n)$. In particular, any minimizer $Q_v \in (H^s(\mathbb{R}^n)\backslash\{0\}) \cap L^\infty(\mathbb{R}^n)$ of problem (3.9) is likewise regular.*

Directly from the Euler-Lagrange equation, this can be shown similarly to [FL13, Lemma B.2] as follows.

*Proof.* **Step 1: $Q_v \in H^{2s}(\mathbb{R}^n)$.** According to the Euler-Lagrange equation (3.3),

$$Q_v = (L + iv \cdot \nabla + \omega)^{-1}(|Q_v|^{2\sigma} Q_v), \quad \omega > \omega_*. \tag{3.53}$$

Note that $Q_v \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ gives $Q_v \in L^{4\sigma+2}$ by interpolation. In other words, $|Q_v|^{2\sigma} Q_v \in L^2(\mathbb{R}^n)$. Thus

$$\|Q_v\|_{\dot{H}^{2s}} = \|(-\Delta)^s Q_v\|_{L^2} = \|\frac{(-\Delta)^s}{L + iv \cdot \nabla + \omega}(|Q_v|^{2\sigma} Q_v)\|_{L^2} \lesssim \||Q_v|^{2\sigma} Q_v\|_{L^2},$$

since $(-\Delta)^s(L + iv \cdot \nabla + \omega)^{-1}$ is a bounded multiplier $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (see the first step in the proof of Theorem 3.2). Hölder's inequality gives

$$\||Q_v|^{2\sigma} Q_v\|_{L^2} \lesssim \|Q_v\|_{L^\infty}^{2\sigma} \|Q_v\|_{L^2},$$

hence $\|Q_v\|_{\dot{H}^{2s}} < \infty$. Since $Q_v \in L^2(\mathbb{R}^n)$, we find $Q_v \in H^{2s}(\mathbb{R}^n)$.

**Step 2: $Q_v \in H^{2s+1}(\mathbb{R}^n)$.** Since $s \geqslant \frac{1}{2}$, we have $H^{2s}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$, thus $Q_v \in H^1(\mathbb{R}^n)$ by step 1. Then

$$\begin{aligned} \|Q_v\|_{\dot{H}^{2s+1}} &= \|(-\Delta)^{s+\frac{1}{2}} Q_v\|_{L^2} = \|\frac{(-\Delta)^s}{L + iv \cdot \nabla + \omega}(-\Delta)^{\frac{1}{2}}(|Q_v|^{2\sigma} Q_v)\|_{L^2} \\ &\lesssim \|(-\Delta)^{\frac{1}{2}}(|Q_v|^{2\sigma} Q_v)\|_{L^2} \lesssim \|\nabla(|Q_v|^{2\sigma} Q_v)\|_{L^2} \end{aligned} \tag{3.54}$$

Check that $|\nabla(|Q_v|^{2\sigma} Q_v)| \leqslant (2\sigma + 1)|Q_v|^{2\sigma}|\nabla Q_v|$ almost everywhere in $\mathbb{R}^n$. Then Hölder's inequality gives

$$\|\nabla(|Q_v|^{2\sigma} Q_v)\|_{L^2} \lesssim \|Q_v\|_{L^\infty}^{2\sigma} \|\nabla Q_v\|_{L^2},$$

hence $\|Q_v\|_{\dot{H}^{2s+1}} < \infty$. Since $Q_v \in L^2(\mathbb{R}^n)$, we find $Q_v \in H^{2s+1}(\mathbb{R}^n)$. □

### 3.6.2 A Proof for $Q_v \in L^\infty(\mathbb{R}^n)$

The above proof of higher Sobolev regularity was shown under the additional hypothesis that $Q_v \in L^\infty(\mathbb{R}^n)$, which we have yet to verify. Observe that (3.53) reads

$$\begin{aligned} Q_v &= (L + iv \cdot \nabla + \omega)^{-1}(|Q_v|^{2\sigma} Q_v) \\ &= \mathcal{F}^{-1}\left[\frac{1}{\varphi(\cdot)}\mathcal{F}(|Q_v|^{2\sigma}Q_v)\right] = \mathcal{F}^{-1}\left[\frac{1}{\varphi}\right] \star (|Q_v|^{2\sigma}Q_v) \\ &= \int_{\mathbb{R}^n} G^{(s)}_{v,\omega}(\cdot - y)(|Q_v|^{2\sigma}Q_v)(y)\,\mathrm{d}y. \end{aligned} \tag{3.55}$$

$1/\varphi$ is the multiplier in Fourier space associated to the resolvent $(L + iv\cdot\nabla + \omega)^{-1}$, where $\varphi(\xi) = m(\xi) - v \cdot \xi + \omega$. The integral kernel $G^{(s)}_{v,\omega} = \mathcal{F}^{-1}[1/\varphi]$ denotes the associated Green's function.

There are now three cases:

**Case** $s > \frac{n}{2}$**:** We directly conclude from Sobolev's embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ (see [Caz03, Remark 1.4.1, (v)]) that $Q_v \in L^\infty(\mathbb{R}^n)$ holds.

**Case** $s = \frac{n}{2}$**:** Since $(1 + |\xi|^2)^s \lesssim \varphi(\xi)$, we have

$$0 \leqslant (\mathcal{F}G^{(s)}_{v,\omega})(\xi) = \frac{1}{\varphi(\xi)} \lesssim \frac{1}{(1+|\xi|^2)^s} \leqslant 1,$$

so that $\mathcal{F}G^{(s)}_{v,\omega} \in L^\infty(\mathbb{R}^n)$. Moreover, this estimate implies that $\mathcal{F}G^{(s)}_{v,\omega} \in L^{1+\varepsilon}(\mathbb{R}^n)$ for all $\varepsilon > 0$. Indeed, an integration with polar coordinates shows

$$\begin{aligned} \|\mathcal{F}G^{(s)}_{v,\omega}\|^p_{L^p} &= \int_{B_1(0)} |\mathcal{F}G^{(s)}_{v,\omega}(\xi)|^p\,\mathrm{d}\xi + \int_{\mathbb{R}^n\setminus B_1(0)} |\mathcal{F}G^{(s)}_{v,\omega}(\xi)|^p\,\mathrm{d}\xi \\ &\lesssim 1 + \int_{\mathbb{R}^n\setminus B_1(0)} \frac{1}{|\xi|^{2sp}}\,\mathrm{d}\xi \lesssim 1 + \int_1^\infty \rho^{-2sp+n-1}\,\mathrm{d}\rho < \infty \end{aligned}$$

provided that $-2sp + n = -np + n < 0$, i.e. $p > 1$. Thus $\mathcal{F}G^{(s)}_{v,\omega} \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for all $\varepsilon > 0$. Hence the Hausdorff-Young inequality (see [LP09, Theorem 2.3]) yields that $G^{(s)}_{v,\omega} \in L^p(\mathbb{R}^n)$ for all $2 \leqslant p < \infty$. On the other hand, since $H^s(\mathbb{R}^n)$ continuously embeds into $L^{2\sigma+2}(\mathbb{R}^n)$ for $\sigma \in (0, \sigma_*)$ subcritical, we have $|Q_v|^{2\sigma}Q_v \in L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^n)$. Since $q := \frac{2\sigma+2}{2\sigma+1} \in (1,2)$, we may let $p = q' \in (2,\infty)$ be its dual, and conclude from (3.55) and Young's inequality (see [LP09, Theorem 2.2]) that $Q_v \in L^\infty(\mathbb{R}^n)$ holds.

**Case** $s < \frac{n}{2}$**:** This final case will be established by an iterative argument using the weak Young inequality; see Proposition 3.24 below. We first develop some preliminaries in the following exhibition.

#### Introduction of a Kernel

Fix some $0 < c_0 < -\omega_* + \omega$ and define $a(\xi) := \varphi(\xi) - c_0$. We observe that there exists $c > 0$ such that

$$a(\xi) \geqslant c|\xi|^{2s}, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3.56}$$

Indeed, since $\varphi(\xi) \gtrsim (1+|\xi|^2)^s \gtrsim |\xi|^{2s}$, there is $C>0$ such that $\varphi(\xi) \geqslant C|\xi|^{2s}$ for all $\xi \in \mathbb{R}^n$. Let $R>0$ be so big that $\frac{C}{2}R^{2s} \geqslant c_0$, and hence

$$a(\xi) = \varphi(\xi) - c_0 \geqslant C|\xi|^{2s} - c_0 \geqslant \frac{C}{2}|\xi|^{2s}, \quad \text{for all } |\xi| \geqslant R.$$

Let then $c_1 > 0$ be so small that $\frac{-\omega_* + \omega - c_0}{c_1} \geqslant R^{2s}$, and hence

$$a(\xi) = \varphi(\xi) - c_0 \geqslant -\omega_* + \omega - c_0 \geqslant c_1 R^{2s} \geqslant c_1|\xi|^{2s}, \quad \text{for all } |\xi| \leqslant R.$$

Take $c = \min\{c_1, \frac{C}{2}\}$ to verify (3.56). From $\int_0^\infty e^{-t\gamma}\,\mathrm{d}t = \frac{1}{\gamma}$ for $\gamma > 0$, we have

$$\frac{1}{\varphi(\xi)} = \int_0^\infty e^{-tc_0} e^{-ta(\xi)}\,\mathrm{d}t,$$

or by functional calculus

$$(L + iv\cdot\nabla + \omega)^{-1} = \int_0^\infty e^{-tc_0} e^{-t(L+iv\cdot\nabla+\omega-c_0)}\,\mathrm{d}t.$$

Hence

$$G^{(s)}_{v,\omega}(x) = \mathcal{F}^{-1}[1/\varphi](x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{1}{\varphi(\xi)}\,\mathrm{d}\xi = \int_0^\infty e^{-tc_0} P^{(s)}_{v,\omega}(x,t)\,\mathrm{d}t, \tag{3.57}$$

where we used Fubini and defined the kernel

$$P^{(s)}_{v,\omega}(x,t) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)}\,\mathrm{d}\xi.$$

### Pointwise Bounds on the Kernel

The next lemma establishes pointwise bounds on the kernel $P^{(s)}_{v,\omega}$. In the proof we differentiate $a(\xi)$. To do this and make things explicit, from now on we look at the particular case $L = (-\Delta)^s$, i.e. $m(\xi) = |\xi|^{2s}$, with $\frac{1}{2} \leqslant s < 1$.

Assumption (A) is clearly true for the fractional Laplacian $L = (-\Delta)^s$ with $s > 0$. Namely, we simply fix some $\lambda > 0$ and consider the continuous positive function

$$f : [0,\infty) \to \mathbb{R}_{>0}, \quad f(x) := \frac{x^{2s} + \lambda}{(1+x^2)^s}.$$

$f$ satisfies $f(x) \to \lambda$ as $x \downarrow 0$, and $f(x) \to 1$ as $x \to \infty$, so that by continuity and positivity the numbers $B = \sup_{x\geqslant 0} f(x)$ and $A = \inf_{x \geqslant 0} f(x) > 0$ exist. The limit $f(x) \to 1$ as $x \to \infty$ also clarifies that (A) can only be true if $A \leqslant 1$. In the case $s = \frac{1}{2}$, we recall that the condition $|v| < A$ is sufficient for $\xi \mapsto (|\xi|^{2s} - v\cdot\xi)$ to be bounded from below. We can choose the least restrictive of these sufficient conditions, namely $A = 1$. Indeed, since $(1+|\xi|^2)^{\frac{1}{2}} \leqslant |\xi| + 1$, we can take $A = 1$, any $\lambda \geqslant 1$ and then determine $B$ as above to see that (A) holds.

**Lemma 3.20** (Pointwise bounds on the kernel $P^{(s)}_{v,\omega}$ for $L = (-\Delta)^s$)**.** *Let $n \geqslant 1$, $s \in [\frac{1}{2}, 1)$, and $L = (-\Delta)^s$. Let $v \in \mathbb{R}^n$ be arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Then we have the pointwise bound*

$$|P^{(s)}_{v,\omega}(x,t)| \leqslant C \min\{t^{-\frac{n}{2s}}, |x|^{-n}\} \tag{3.58}$$

*with some constant $C > 0$ depending only on $n, s, v$ and $\omega$.*

*Proof.* From (3.56) we see that the kernel $P^{(s)}_{v,\omega}$ obeys the pointwise bound

$$\begin{aligned} |P^{(s)}_{v,\omega}(x,t)| &\lesssim_{n,s,v,\omega} \int_{\mathbb{R}^n} e^{-ct|\xi|^{2s}} \,\mathrm{d}\xi \lesssim_{n,s,v,\omega} \int_0^\infty \int_{\partial B_\varrho(0)} e^{-ct\varrho^{2s}} \,\mathrm{d}S \,\mathrm{d}\varrho \\ &\lesssim_{n,s,v,\omega} \int_0^\infty e^{-ct\varrho^{2s}} \varrho^{n-1} \,\mathrm{d}\varrho \lesssim_{n,s,v,\omega} t^{-\frac{n}{2s}} \int_0^\infty e^{-u} u^{\frac{n}{2s}-1} \,\mathrm{d}u \\ &\lesssim_{n,s,v,\omega} t^{-\frac{n}{2s}} \Gamma\left(\frac{n}{2s}\right) \lesssim_{n,s,v,\omega} t^{-\frac{n}{2s}}, \end{aligned} \tag{3.59}$$

where we switched to polar coordinates and substituted $u = ct\varrho^{2s}$. Now, let $j = 1, \dots, n$ be fixed. Using $x_j e^{ix\cdot\xi} = -i\partial_{\xi_j}(e^{ix\cdot\xi})$, an integration by parts reveals

$$\begin{aligned} (x_j - itv_j) P^{(s)}_{v,\omega}(x,t) &= \frac{i}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} (-2st|\xi|^{2s-2}\xi_j + tv_j) \,\mathrm{d}\xi - itv_j P^{(s)}_{v,\omega}(x,t) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} \{-i(2st|\xi|^{2s-2}\xi_j)\} \,\mathrm{d}\xi. \end{aligned} \tag{3.60}$$

Consequently

$$\begin{aligned} |(x_j - itv_j) P^{(s)}_{v,\omega}(x,t)| &\lesssim t \int_{\mathbb{R}^n} e^{-ta(\xi)} |\xi|^{2s-1} \,\mathrm{d}\xi \overset{(3.56)}{\lesssim} t \int_{\mathbb{R}^n} e^{-ct|\xi|^{2s}} |\xi|^{2s-1} \,\mathrm{d}\xi \\ &\lesssim t \int_0^\infty e^{-ct\varrho^{2s}} \varrho^{2s+n-2} \,\mathrm{d}\varrho \lesssim t^{\frac{1-n}{2s}} \int_0^\infty e^{-u} u^{1+\frac{n-1}{2s}-1} \,\mathrm{d}u \\ &\lesssim t^{\frac{1-n}{2s}} \Gamma\left(1 + \frac{n-1}{2s}\right) \lesssim t^{\frac{1-n}{2s}}. \end{aligned}$$

We claim that this holds more generally:

$$|(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t)| \lesssim t^{\frac{k-n}{2s}}, \quad k = 1, \dots, n. \tag{3.61}$$

Once this is proved, we get in particular ($k = n$)

$$|(x_j - itv_j)^n P^{(s)}_{v,\omega}(x,t)| \lesssim 1, \quad \text{for all } j = 1, \dots, n, \tag{3.62}$$

thus

$$\begin{aligned} |x|^n |P^{(s)}_{v,\omega}(x,t)| &\leqslant \left(\sum_{j=1}^n |x_j - itv_j|\right)^n |P^{(s)}_{v,\omega}(x,t)| \\ &\leqslant \left(n \max_{j=1,\dots,n} |x_j - itv_j|\right)^n |P^{(s)}_{v,\omega}(x,t)| \overset{(3.62)}{\lesssim} 1. \end{aligned} \tag{3.63}$$

From (3.63) and (3.59) it follows that the pointwise bound (3.58) holds, which proves the lemma. We have thus reduced the lemma to claim (3.61), which we prove now. Namely, introduce the map

$$f \mapsto K_f, \quad K_f(x,t) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} f(\xi)\, \mathrm{d}\xi. \tag{3.64}$$

Mimicking the computation of (3.60), we see that

$$\begin{aligned}(x_j - itv_j)K_f(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} \{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j}) f(\xi)\}\, \mathrm{d}\xi \\ &= K_{\{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j})\}f}(x,t)\end{aligned}$$

which immediately implies by induction that

$$(x_j - itv_j)^k K_f(x,t) = K_{\{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j})\}^k f}(x,t), \quad \text{for all } k \in \mathbb{N}. \tag{3.65}$$

Notice that $K_1(x,t) = P^{(s)}_{v,\omega}(x,t)$, so that letting $f \equiv 1$ in (3.65) gives

$$(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t) = K_{\{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j})\}^k 1}(x,t), \quad k = 1, 2, \ldots.$$

To use this formula for an estimate, let us analyze the behaviour of the expression on the right side, i.e., $\{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j})\}^k 1$. Setting

$$f_0 = 1, \quad f_k := \{-i(2st|\xi|^{2s-2}\xi_j - \partial_{\xi_j})\} f_{k-1}, \quad k = 1, 2, \ldots,$$

we may write

$$(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t) = K_{f_k}(x,t), \quad k = 1, 2, \ldots. \tag{3.66}$$

Writing down more of the $f_k$'s, we observe that the pattern is the following[10]:

$$f_k = \begin{cases} \sum_{\nu=1}^{\frac{k}{2}} t^\nu \sum_{\ell=\frac{k}{2}}^{k} a_{\nu\ell} |\xi|^{2s\nu - 2\ell} \xi_j^{2\ell - k} + \sum_{\nu=\frac{k}{2}+1}^{k} t^\nu \sum_{\ell=\nu}^{k} a_{\nu\ell} |\xi|^{2s\nu-2\ell} \xi_j^{2\ell-k}, \\ \sum_{\nu=1}^{\frac{k+1}{2}} t^\nu \sum_{\ell=\frac{k+1}{2}}^{k} a_{\nu\ell} |\xi|^{2s\nu - 2\ell} \xi_j^{2\ell - k} + \sum_{\nu=\frac{k+1}{2}+1}^{k} t^\nu \sum_{\ell=\nu}^{k} a_{\nu\ell} |\xi|^{2s\nu-2\ell} \xi_j^{2\ell-k}, \end{cases} \tag{3.67}$$

for any even $k \geqslant 2$ and any odd $k \geqslant 3$, respectively. The coefficients $a_{\nu\ell}$ depend on $s$. Therefore

$$|f_k| \overset{(3.67)}{\lesssim_{s,k}} \sum_{\nu=1}^{k} t^\nu |\xi|^{2s\nu - k} = t|\xi|^{2s-k} + \cdots + t^k |\xi|^{2sk-k}, \quad k = 1, 2, \ldots. \tag{3.68}$$

[10]That (3.67) is true can be proved by induction. The coefficients $a_{\nu\ell}$ depend on $s$.

Let now $k = 1, \dots, n$ be fixed. Then

$$\begin{aligned}
|(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t)| &\overset{(3.66)}{=} |K_{f_k}(x,t)| \overset{(3.64)}{\lesssim} \int_{\mathbb{R}^n} e^{-ta(\xi)} |f_k(t,\xi)| \,\mathrm{d}\xi \\
&\overset{(3.56)}{\lesssim} \int_{\mathbb{R}^n} e^{-ct|\xi|^{2s}} |f_k(t,\xi)| \,\mathrm{d}\xi \\
&\overset{(3.68)}{\lesssim} \sum_{\nu=1}^{k} t^\nu \int_{\mathbb{R}^n} e^{-ct|\xi|^{2s}} |\xi|^{2s\nu - k} \,\mathrm{d}\xi \\
&\lesssim \sum_{\nu=1}^{k} t^\nu \int_0^\infty e^{-ct\varrho^{2s}} \varrho^{2s\nu - k + (n-1)} \,\mathrm{d}\varrho \\
&\lesssim t^{\frac{k-n}{2s}} \sum_{\nu=1}^{k} \Gamma\left(\nu - \frac{k-n}{2s}\right) \lesssim t^{\frac{k-n}{2s}}.
\end{aligned} \tag{3.69}$$

Here, we substituted $u = ct\rho^{2s}$ as before and noted the convergence of the $\Gamma$-expression, since $k \in \{1, \dots, n\}$ and $\nu \in \{1, \dots, k\}$. The proof of Lemma 3.20 is now complete. □

**Remark 3.21** (Extension and pointwise bound for Green's function in the strict case $s > \frac{1}{2}$)**.** Inspecting the proof of Lemma 3.20, we see that if $s > \frac{1}{2}$ then (3.61) still holds in the case $k = n + 1$, that is,

$$|(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t)| \lesssim t^{\frac{k-n}{2s}}, \quad k = 1, \dots, n+1. \tag{3.70}$$

Indeed, fixing $k \in \{1, \dots, n+1\}$, nothing changes in estimate (3.69): we still deduce

$$|(x_j - itv_j)^k P^{(s)}_{v,\omega}(x,t)| \lesssim t^{\frac{k-n}{2s}} \sum_{\nu=1}^{k} \Gamma\left(\nu - \frac{k-n}{2s}\right) \lesssim t^{\frac{k-n}{2s}}, \tag{3.71}$$

because if $s > \frac{1}{2}$, then $k \leqslant n+1$ implies $\nu - \frac{k-n}{2s} \geqslant \nu - \frac{1}{2s} > 0$ for all $\nu \in \{1, \dots, k\}$. This is no longer true in the limiting case $s = \frac{1}{2}$ in which the previous estimate fails (take $k = n+1$ and $\nu = 1$). Thus the hypothesis $s > \frac{1}{2}$ is crucial. Having now proved (3.70), we deduce in a similar fashion as getting (3.63) above that

$$\begin{aligned}
|x|^{n+1} |P^{(s)}_{v,\omega}(x,t)| &\leqslant \left(\sum_{j=1}^{n} |x_j|\right)^{n+1} |P^{(s)}_{v,\omega}(x,t)| \leqslant \left(\sum_{j=1}^{n} |x_j - itv_j|\right)^{n+1} |P^{(s)}_{v,\omega}(x,t)| \\
&\leqslant \left(n \max_{j=1,\dots,n} |x_j - itv_j|\right)^{n+1} |P^{(s)}_{v,\omega}(x,t)| \overset{(3.70)}{\lesssim} t^{\frac{1}{2s}}.
\end{aligned}$$

This pointwise bound $|P^{(s)}_{v,\omega}(x,t)| \lesssim t^{\frac{1}{2s}} |x|^{-(n+1)}$ implies a corresponding bound for the Green's function $G^{(s)}_{v,\omega}(x)$ (substitute $\theta = c_0 t$):

$$\begin{aligned}
|G^{(s)}_{v,\omega}(x)| &\overset{(3.57)}{\lesssim} |x|^{-(n+1)} \int_0^\infty e^{-c_0 t} t^{\frac{1}{2s}} \,\mathrm{d}t \lesssim |x|^{-(n+1)} \int_0^\infty e^{-\theta} \theta^{\frac{1}{2s}} \,\mathrm{d}\theta \\
&\lesssim |x|^{-(n+1)} \Gamma\left(1 + \frac{1}{2s}\right) \lesssim |x|^{-(n+1)}.
\end{aligned} \tag{3.72}$$

The following kernel-estimates are analogous to [Mar02, Lemma 7].

**Lemma 3.22** ($L^p$-estimates on the kernel $G^{(s)}_{v,\omega}$, cf. [Mar02, Lemma 7]). *Let $n \geqslant 1$, $s \in \left[\frac{1}{2}, 1\right)$, and $L = (-\Delta)^s$. Let $v \in \mathbb{R}^n$ be arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Then:*

*(i) If $n \geqslant 2$ (i.e. $s < \frac{n}{2}$), we have*

$$|G^{(s)}_{v,\omega}(x)| \lesssim \begin{cases} |x|^{-(n-2s)} & \text{for } |x| \leqslant 1, \\ |x|^{-(n+1)} & \text{for } |x| \geqslant 1. \end{cases} \tag{3.73}$$

*If $n = 1$ and $s = \frac{1}{2}$ (i.e. $s = \frac{n}{2}$), we have*

$$|G^{(s)}_{v,\omega}(x)| \lesssim \begin{cases} -\log|x| + 1 & \text{for } |x| \leqslant 1, \\ |x|^{-2} & \text{for } |x| \geqslant 1. \end{cases} \tag{3.74}$$

*If $n = 1$ and $s > \frac{1}{2}$ (i.e. $s > \frac{n}{2}$), the kernel $G^{(s)}_{v,\omega}(x)$ is continuous and bounded, and we have*

$$|G^{(s)}_{v,\omega}(x)| \lesssim \begin{cases} 1 & \text{for } |x| \leqslant 1, \\ |x|^{-2} & \text{for } |x| \geqslant 1. \end{cases} \tag{3.75}$$

*(ii) $|x|^{n+1}G^{(s)}_{v,\omega}(x) \in L^\infty(\mathbb{R}^n)$ and for $1 \leqslant p < \infty$ we have $|x|^\alpha G^{(s)}_{v,\omega}(x) \in L^p(\mathbb{R}^n)$ provided that*

$$\begin{cases} (n-2s) - \frac{n}{p} < \alpha < n + 1 - \frac{n}{p} & \text{for } n \geqslant 2 \text{ and } s \in [\frac{1}{2}, 1), \\ -\frac{n}{p} < \alpha < n + 1 - \frac{n}{p} & \text{for } n = 1 \text{ and } s \in [\frac{1}{2}, 1). \end{cases} \tag{3.76}$$

*In particular, $G^{(s)}_{v,\omega} \in L^p(\mathbb{R}^n)$ provided that*

$$\begin{cases} 1 \leqslant p < \frac{n}{n-2s} & \text{for } n \geqslant 2 \text{ and } s \in [\frac{1}{2}, 1), \\ 1 \leqslant p < \infty & \text{for } n = 1 \text{ and } s = \frac{1}{2}, \\ 1 \leqslant p \leqslant \infty & \text{for } n = 1 \text{ and } s \in (\frac{1}{2}, 1). \end{cases} \tag{3.77}$$

*(iii) Furthermore, if $n \geqslant 2$, then $G^{(s)}_{v,\omega}$ is in $L^{\frac{n}{n-2s},\infty}(\mathbb{R}^n)$ ("weak-$L^{\frac{n}{n-2s}}$").*

*Proof.* (i) Let $n \geqslant 2$. We have by Lemma 3.20

$$\begin{aligned} |G^{(s)}_{v,\omega}(x)| &\leqslant \int_0^\infty |P^{(s)}_{v,\omega}(x,t)|\,\mathrm{d}t \lesssim \int_0^\infty \min\{t^{-\frac{n}{2s}}, |x|^{-n}\}\,\mathrm{d}t \\ &\lesssim \int_0^{|x|^{2s}} |x|^{-n}\,\mathrm{d}t + \int_{|x|^{2s}}^\infty t^{-\frac{n}{2s}}\,\mathrm{d}t \overset{(s<\frac{n}{2})}{\lesssim} |x|^{-(n-2s)}. \end{aligned} \tag{3.78}$$

If $s > \frac{1}{2}$ strictly, we obtain the estimate $|G^{(s)}_{v,\omega}(x)| \lesssim |x|^{-(n+1)}$ from Remark 3.21. If however $s = \frac{1}{2}$ and $n \geqslant 1$, we use the explicit formula

$$G^{(s)}_{v,\omega}(x) = C_n \int_0^\infty e^{-t\omega} \frac{t}{(t^2 + (x - itv)^2)^{\frac{n+1}{2}}}\,\mathrm{d}t.$$

This formula follows from $G^{(s)}_{v,\omega}(x) = \mathcal{F}^{-1}\left[\frac{1}{\varphi}\right](x) = \int_0^\infty e^{-t\omega}\mathcal{F}^{-1}(e^{-t(|\xi|-v\cdot\xi)})(x)\,\mathrm{d}t$ (use Fubini as in (3.57)), together with the explicit formula for the inverse Fourier transform of $e^{-t|\xi|}$ on $\mathbb{R}^n$ (see [Str03, p. 54]), that is,

$$\mathcal{F}^{-1}(e^{-t|\xi|})(x) = C_n\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}},$$

plus an analytic continuation for dealing with the additional term $e^{tv\cdot\xi}$, which corresponds to a shift $x \mapsto x - itv$ (i.e. $\mathcal{F}^{-1}(e^{-t(|\xi|-v\cdot\xi)})(x) = \mathcal{F}^{-1}(e^{-t|\xi|})(x-itv)$).

Since $z := t^2 + (x-itv)^2 = (1-v^2)t^2 + |x|^2 - 2itv\cdot x$ and $|z| \geqslant |\operatorname{Re} z|$, we get

$$|G^{(s)}_{v,\omega}(x)| \lesssim \int_0^\infty e^{-t\omega}\frac{t}{((1-v^2)t^2+|x|^2)^{\frac{n+1}{2}}}\,\mathrm{d}t \lesssim \frac{1}{|x|^{n+1}}\int_0^\infty e^{-t\omega}t\,\mathrm{d}t \lesssim \frac{1}{|x|^{n+1}}.$$

[Note that $\omega > 0$, since $\omega > \omega_*$ and $\omega_* \geqslant 0$; see Lemma B.8. Hence the previous integral converges.] We have thus proved (3.73) and the second estimates in (3.74), (3.75), respectively.

Let now $n = 1$ and $s = \frac{1}{2}$. We have for $|x| \leqslant 1$

$$\begin{aligned}|G^{(s)}_{v,\omega}(x)| &\lesssim \int_0^\infty e^{-t\omega}\frac{t}{(1-v^2)t^2+|x|^2}\,\mathrm{d}t\\ &= \frac{1}{2(1-v^2)}\left(-\log|x|^2 + \omega\int_0^\infty e^{-t\omega}\log((1-v^2)t^2+|x|^2)\,\mathrm{d}t\right)\\ &\leqslant -\frac{1}{1-v^2}\log|x| + \frac{\omega}{2}\int_0^\infty e^{-t\omega}t^2\,\mathrm{d}t \lesssim -\frac{1}{1-v^2}\log|x| + 1,\end{aligned}$$

where we first integrated by parts and then used the elementary inequality $\log((1-v^2)t^2+|x|^2) \leqslant \log((1-v^2)t^2+1) \leqslant (1-v^2)t^2$. This proves the first estimate of (3.74) [notice that its constant blows up as $|v| \uparrow 1$].

Finally, when $n = 1$ and $s > \frac{1}{2}$, it is easy to check that $\widehat{G^{(s)}_{v,\omega}} = \frac{1}{|\xi|^{2s}-v\cdot\xi+\omega}$ is in $L^1(\mathbb{R}^n)$. [Simply recall $(1+|\xi|^2)^s \lesssim \varphi(\xi)$.] Hence $G^{(s)}_{v,\omega}(x) = (\mathcal{F}^{-1}\widehat{G^{(s)}_{v,\omega}})(x) = (\mathcal{F}\widehat{G^{(s)}_{v,\omega}})(-x)$ is bounded and continuous, proving the first estimate of (3.75).

(ii) is an easy consequence of (i). For instance, use $|x|^2(-\log|x|+1) \to 0$ as $|x| \to 0$ to see that $|x|^2G^{(s)}_{v,\omega}(x) \in L^\infty(\mathbb{R})$, and use

$$\int_{-1}^1 |x|^{\alpha p}(-\log|x|+1)^p\,\mathrm{d}x \lesssim \int_{-1}^1 |x|^{\alpha p}(-\log|x|)^p\,\mathrm{d}x + \int_{-1}^1 |x|^{\alpha p}\,\mathrm{d}x$$

together with

$$\begin{aligned}\int_{-1}^1 |x|^{\alpha p}(-\log|x|)^p\,\mathrm{d}x &= 2\int_0^1 x^{\alpha p}(-\log x)^p\,\mathrm{d}x = 2\int_0^\infty y^p e^{-y(1+\alpha p)}\,\mathrm{d}y\\ &= \frac{2}{(1+\alpha p)^{p+1}}\int_0^\infty e^{-z}z^p\,\mathrm{d}z = \frac{2}{(1+\alpha p)^{p+1}}\Gamma(p+1)\end{aligned}$$

to see that $|x|^\alpha G^{(s)}_{v,\omega}(x) \in L^p(\mathbb{R})$ if the mentioned conditions on $\alpha$ hold. Then (3.76) is clear. (3.77) immediately follows from (3.76) [recall that in the case $n = 1$ and $s > \frac{1}{2}$, $G^{(s)}_{v,\omega}(x)$ was continuous and bounded, hence $p = \infty$ also appears in (3.77)].

It remains to prove (iii). Let $n \geqslant 2$. Abbreviate $h(x) = |x|^{-(n-2s)}$, $p = \frac{n}{n-2s}$. Observe

$$d_h(t) := |\{x \in \mathbb{R}^n; |h(x)| > t\}| = \omega_n t^{-\frac{n}{n-2s}} = \omega_n t^{-p}, \quad \text{where } \omega_n = |B_1(0)|.$$

We deduce

$$\|h\|_{L^{p,\infty}} = \sup_{t>0} \left( d_h(t)^{\frac{1}{p}} t \right) = \omega_n^{\frac{1}{p}} < \infty,$$

so that $h \in L^{p,\infty}(\mathbb{R}^n)$.[11] By (3.78), there exists $C > 0$ such that $|G^{(s)}_{v,\omega}(x)| \leqslant C|h(x)|$ for all $x$, which implies

$$\left\{x \in \mathbb{R}^n;\ |G^{(s)}_{v,\omega}(x)| > Ct\right\} \subset \{x \in \mathbb{R}^n;\ |h(x)| > t\} \quad \text{for all } t > 0.$$

Hence

$$d_{G^{(s)}_{v,\omega}}(Ct) \leqslant d_h(t), \quad \text{for all } t > 0,$$

by monotonicity of the measure. It follows that

$$\|G^{(s)}_{v,\omega}\|_{L^{p,\infty}} = \sup_{t>0} \left( d_{G^{(s)}_{v,\omega}}(t)^{\frac{1}{p}} t \right) = \sup_{t>0} \left( d_{G^{(s)}_{v,\omega}}(Ct)^{\frac{1}{p}} Ct \right) \leqslant C \sup_{t>0} \left( d_h(t)^{\frac{1}{p}} t \right) = C\|h\|_{L^{p,\infty}}.$$

The proof of Lemma 3.22 is now complete. □

**Remark 3.23.** In the case $n \geqslant 2$ (i.e. $s < \frac{n}{2}$) Lemma 3.22 shows

$$|G^{(s)}_{v,\omega}(x)| \lesssim \begin{cases} |x|^{-(n-2s)} & \text{if } |x| \leqslant 1, \\ |x|^{-(n+1)} & \text{if } |x| \geqslant 1. \end{cases} \tag{3.79}$$

In particular, this gives $G^{(s)}_{v,\omega} \in L^1(\mathbb{R}^n)$. By explicit formulae for the inverse Fourier transform of $e^{|\xi|^{2s} - v\cdot\xi + \omega}$ and analytic continuation, with techniques as in [BG60], [Pól23], it may be possible to improve the bound (3.79) up to the following optimal one:

$$|G^{(s)}_{v,\omega}(x)| \lesssim \begin{cases} |x|^{-(n-2s)} & \text{if } |x| \leqslant 1, \\ |x|^{-(n+2s)} & \text{if } |x| \geqslant 1. \end{cases} \tag{3.80}$$

**Proposition 3.24** ($Q_v \in L^\infty(\mathbb{R}^n)$ for $L = (-\Delta)^s$)**.** *Let $n \geqslant 1$, $s \in \left[\frac{1}{2}, 1\right)$, and $L = (-\Delta)^s$. Let $v \in \mathbb{R}^n$ be arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Suppose that $Q_v \in H^s(\mathbb{R}^n)$ solves the Euler-Lagrange equation (3.3). Then $Q_v \in L^\infty(\mathbb{R}^n)$. In particular, any minimizer $Q_v \in H^s(\mathbb{R}^n) \setminus \{0\}$ of problem (3.9) is likewise in $L^\infty(\mathbb{R}^n)$.*

[11] This is analogous to the general argument that $|x|^{-\lambda} \in L^{n/\lambda,\infty}(\mathbb{R}^n)$ (cf. also [LL01, p. 106]).

**Remark 3.25.** Recall the convolution (integral) equality (3.55):

$$Q_v = G^{(s)}_{v,\omega} \star (|Q_v|^{2\sigma} Q_v). \tag{3.81}$$

The idea of the proof is to iterate (3.81) finitely many times, using the (weak) $L^p$ estimates on the Green's function $G^{(s)}_{v,\omega}$ from Lemma 3.22 and the (weak[12]) Young inequality. We make essential use of the method in [Caz03, p. 256]; see also [FLS16], [FL13].

Recall also that we have already proved $Q_v \in L^\infty(\mathbb{R}^n)$ in the cases $s > \frac{n}{2}$ and $s = \frac{n}{2}$; see page 81.

*Proof of Proposition 3.24.* As mentioned, it remains to treat the case $s < \frac{n}{2}$ (i.e. $n \geqslant 2$). Let $q := \frac{2\sigma+2}{2\sigma+1} \in (1,\infty)$. Then

$$Q_v \in L^r(\mathbb{R}^n) \quad \text{for any } q < r < \infty \text{ satisfying } \frac{1}{r} \geqslant \frac{1}{q} - \frac{2s}{n}. \tag{3.82}$$

To see this, let $r$ like this be given. Define $\frac{1}{p} := 1 + \frac{1}{r} - \frac{1}{q}$. It is then clear that $\frac{1}{p} \geqslant \frac{n-2s}{n}$ and $\frac{1}{p} < 1$, in other words $1 < p \leqslant \frac{n}{n-2s}$. Since $G^{(s)}_{v,\omega} \in L^{p,\infty}(\mathbb{R}^n)$ by Lemma 3.22, $|Q_v|^{2\sigma} Q_v \in L^q(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 < p,q,r < \infty$, (3.81) and the weak Young inequality imply $Q_v \in L^r(\mathbb{R}^n)$.

Note that the endpoint $r = \infty$ cannot be reached by (3.82). Similarly to [Caz03, p. 256], now define the sequence $(r_j)_{j\in\mathbb{N}_0}$ by

$$\frac{1}{r_j} = (2\sigma+1)^j \left( \frac{1}{2\sigma+2} - \frac{s}{\sigma n} + \frac{s}{\sigma n (2\sigma+1)^j} \right).$$

The $H^s(\mathbb{R}^n)$-subcriticality assumption $\sigma < \sigma_* = \frac{2s}{n-2s}$ ensures that $\frac{2\sigma}{2\sigma+2} - \frac{2s}{n} =: -\delta$ satisfies $\delta > 0$.[13] Keeping this in mind we check that

$$\frac{1}{r_{j+1}} - \frac{1}{r_j} = -(2\sigma+1)^j \delta \leqslant -\delta,$$

therefore $(\frac{1}{r_j})_{j\in\mathbb{N}_0}$ is strictly decreasing with $\frac{1}{r_j} \leqslant -j\delta + \frac{1}{r_0} \xrightarrow{j} -\infty$. Since $\frac{1}{r_0} > 0$, there exists some $k \geqslant 0$ such that

$$\frac{1}{r_j} > 0 \quad \text{for } 0 \leqslant j \leqslant k, \quad \text{but } \frac{1}{r_{k+1}} \leqslant 0.$$

Let us now show that $Q_v \in L^{r_k}(\mathbb{R}^n)$. Note $|Q_v|^{2\sigma} Q_v \in L^q(\mathbb{R}^n) = L^{\frac{r_0}{2\sigma+1}}(\mathbb{R}^n)$. But, if $Q_v \in L^{r_j}(\mathbb{R}^n)$ for some $0 \leqslant j \leqslant k-1$,[14] then also $Q_v \in L^{r_{j+1}}(\mathbb{R}^n)$. Indeed, analogously to the deduction of (3.82), we deduce (weak Young inequality)

$$Q_v \in L^r(\mathbb{R}^n) \quad \text{for any } \frac{r_j}{2\sigma+1} < r < \infty \text{ satisfying } \frac{1}{r} \geqslant \frac{2\sigma+1}{r_j} - \frac{2s}{n} = \frac{1}{r_{j+1}}. \tag{3.83}$$

[12] For the statement of weak Young inequality, see, e.g., [Gra08, Theorem 1.4.24] or also [LL01, p.107, inequality (9)] and [Lie83, p. 351, inequality (1.8)].

[13] This also holds for $s = \frac{n}{2}$, where $\sigma_* = +\infty$.

[14] We can assume $k \geqslant 1$, since in case $k = 0$, we already know $Q_v \in L^{r_0}(\mathbb{R}^n) = L^{2\sigma+2}(\mathbb{R}^n)$.

Taking $r = r_{j+1}$ in (3.83) gives $Q_v \in L^{r_{j+1}}(\mathbb{R}^n)$, as claimed.[15] With this argument, we conclude in finitely many steps that indeed $Q_v \in L^{r_k}(\mathbb{R}^n)$ holds:

$$Q_v \in L^{r_0}(\mathbb{R}^n) \overset{j=0\leqslant k-1}{\Longrightarrow} Q_v \in L^{r_1}(\mathbb{R}^n) \overset{j=1\leqslant k-1}{\Longrightarrow} \cdots \overset{j=k-1\leqslant k-1}{\Longrightarrow} Q_v \in L^{r_k}(\mathbb{R}^n).$$

Now $|Q_v|^{2\sigma}Q_v \in L^{\frac{r_k}{2\sigma+1}}$ with $1 < \frac{r_k}{2\sigma+1} < \infty$, and $G_{v,\omega}^{(s)} \in L^p(\mathbb{R}^n)$ for any $1 < p < \frac{n}{n-2s}$, hence (3.81) and the *usual* Young inequality imply that

$$Q_v \in L^r(\mathbb{R}^n) \quad \text{for any } \frac{r_k}{2\sigma+1} \leqslant r \leqslant \infty \text{ satisfying } \frac{1}{r} > \frac{2\sigma+1}{r_k} - \frac{2s}{n} = \frac{1}{r_{k+1}}. \tag{3.84}$$

We know $\frac{1}{r_{k+1}} \leqslant 0$. If $\frac{1}{r_{k+1}} < 0$, the choice $r = \infty$ is allowed in (3.84) and we are done. Otherwise $\frac{1}{r_{k+1}} = 0$, and we fix $\frac{r_k}{2\sigma+1} \leqslant r < \infty$ in (3.84) so large that $\gamma := \frac{2\sigma+1}{r} - \frac{2s}{n} < 0$ and $1 < \frac{r}{2\sigma+1}$. Since $|Q_v|^{2\sigma}Q_v \in L^{\frac{r}{2\sigma+1}}(\mathbb{R}^n)$ with $1 < \frac{r}{2\sigma+1} < \infty$ and $G_{v,\omega}^{(s)} \in L^p(\mathbb{R}^n)$ for any $1 < p < \frac{n}{n-2s}$, applying the *usual* Young inequality once more yields

$$Q_v \in L^{\tilde{r}}(\mathbb{R}^n) \quad \text{for any } \frac{r}{2\sigma+1} \leqslant \tilde{r} \leqslant \infty \text{ satisfying } \frac{1}{\tilde{r}} > \frac{2\sigma+1}{r} - \frac{2s}{n} = \gamma. \tag{3.85}$$

In (3.85) we can now choose the endpoint $\tilde{r} = \infty$. The proof of Proposition 3.24 is now complete. □

## 3.7 Spatial Decay

The aim of this section is to prove the spatial decay estimate of Theorem 3.3, namely that at infinity solutions $Q_v$ of the Euler-Lagrange equation (3.3) decay polynomially like

$$|Q_v(x)| \lesssim \frac{1}{1+|x|^{n+1}}.$$

### 3.7.1 A Preliminary Convolution Lemma

We exploit the behaviour of the Green's function $G_{v,\omega}^{(s)}$ in the following lemma. Recall

$$(\mathcal{F}G_{v,\omega}^{(s)})(\xi) = \frac{1}{\varphi(\xi)} = \frac{1}{|\xi|^{2s} - v\cdot\xi + \omega}.$$

**Lemma 3.26** (Convolution Lemma, cf. [Len06, Lemma A.9])**.** *Let $n \geqslant 1$, $s \in [\frac{1}{2}, 1)$, $v \in \mathbb{R}^n$ arbitrary for $s \in (\frac{1}{2}, 1)$, and $|v| < 1$ for $s = \frac{1}{2}$. Let $f$ be a measurable function satisfying*

$$|f(x)| \leqslant \frac{c}{1+|x|^{\alpha}}, \quad \textit{for all } x \in \mathbb{R}^n$$

[15] $(\frac{1}{r_j})_{j\in\mathbb{N}_0}$ is decreasing, so $\frac{1}{r_j} \leqslant \frac{1}{r_0}$, and by assumption $\frac{1}{r_j} > 0$ (since $j \leqslant k-1$), hence indeed $1 < \frac{r_0}{2\sigma+1} \leqslant \frac{r_j}{2\sigma+1} < \infty$. Moreover, from $j+1 \leqslant k$ we have $\frac{1}{r_{j+1}} > 0$ and $1 < r_{j+1} < \infty$.

*with some fixed constants $c \geqslant 0$ and $n+1 \geqslant \alpha \geqslant 1$. Then there exists a constant $C > 0$, depending only on $n, s, v, \omega$ and $\alpha$, such that for all $x \in \mathbb{R}^n$*

$$|(G_{v,\omega}^{(s)} \star f)(x)| \leqslant \frac{C}{1+|x|^\alpha}.$$

**Remark 3.27.** The proof of Lemma 3.26 makes use of the following two facts. First, the condition $\alpha \geqslant 1$ guarantees the convexity of the function $\phi : (0,\infty) \to (0,\infty)$ defined by $\phi(t) := |t|^\alpha$. Second, the condition $n+1 \geqslant \alpha$ guarantees the existence of some $\beta \geqslant 0$ satisfying $n+1-\alpha \geqslant \beta > n-\alpha$. Note that when $\alpha > n+2s$ such a $\beta \geqslant 0$ obviously does not exist (since $s \geqslant \frac{1}{2}$). The lemma would hold up to $n+2s \geqslant \alpha$, if one proved the optimal bound (3.80). This would result in a better decay in Theorem 3.3, namely $|Q_v(x)| \lesssim (1+|x|^{n+2s})^{-1}$ instead of $|Q_v(x)| \lesssim (1+|x|^{n+1})^{-1}$.

*Proof of Lemma 3.26 (cf. [Len06]).* From Lemma 3.22 (i), we recall the estimate

$$|G_{v,\omega}^{(s)}(x)| \lesssim |x|^{-(n+1)} \quad \text{for } |x| \geqslant 1. \tag{3.86}$$

Fix any $x \in \mathbb{R}^n$ and consider the disjoint partition $\mathbb{R}^n = S_1^x \cup S_2^x \cup S_3^x$ with the sets

$$S_1^x := \{y \in \mathbb{R}^n;\ |x-y| < 1\}, \quad S_2^x := \{y \in \mathbb{R}^n;\ |x-y| \geqslant 1,\ \frac{1}{2}|x| < |y|\},$$

$$S_3^x := \{y \in \mathbb{R}^n;\ |x-y| \geqslant 1,\ \frac{1}{2}|x| \geqslant |y|\}.$$

We have

$$\frac{1}{1+|y|^\alpha} \lesssim_\alpha \frac{1}{1+|x|^\alpha}, \quad \text{for all } y \in S_1^x. \tag{3.87}$$

Indeed, the triangle inequality and the definition of $S_1^x$ give $|x| \leqslant |x-y|+|y| \leqslant 1+|y|$, thus

$$|x|^\alpha \leqslant (1+|y|)^\alpha = |1+|y||^\alpha =: \varphi(1+|y|).$$

By convexity of $\phi : (0,\infty) \to (0,\infty)$, $\phi(t) = |t|^\alpha$,

$$\phi(1+|y|) = \phi\left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2|y|\right) \leqslant 2^{\alpha-1}\left(1+|y|^\alpha\right).$$

It follows that $1+|x|^\alpha \leqslant 1+2^{\alpha-1}\left(1+|y|^\alpha\right) \leqslant \left(1+2^{\alpha-1}\right)\left(1+|y|^\alpha\right)$, which gives (3.87) with a constant $C_\alpha \geqslant 1+2^{\alpha-1}$.[16] Note that $C_\alpha$ does not depend on $x$.

For $y \in S_2^x$, we have $|x| < 2|y|$, thus

$$1+|x|^\alpha < 1+2^\alpha|y|^\alpha \leqslant 2^\alpha\left(1+|y|^\alpha\right).$$

[16] We could have also simply estimated

$$1+|x|^\alpha \leqslant 1+(1+|y|)^\alpha \leqslant 1+(2\max\{1,|y|\})^\alpha \leqslant 1+2^\alpha(1+|y|^\alpha) \leqslant (1+2^\alpha)(1+|y|^\alpha).$$

This and (3.87) give

$$\frac{1}{1+|y|^\alpha} \lesssim_\alpha \frac{1}{1+|x|^\alpha}, \quad \text{for all } y \in S_1^x \cup S_2^x. \tag{3.88}$$

From (3.88), it follows by using the hypothesis on $f$ that

$$\begin{aligned} &\int_{S_1^x \cup S_2^x} |G_{v,\omega}^{(s)}(x-y)||f(y)|\,\mathrm{d}y \leqslant \int_{S_1^x \cup S_2^x} |G_{v,\omega}^{(s)}(x-y)| \frac{c}{1+|y|^\alpha}\,\mathrm{d}y \\ &\overset{(3.88)}{\leqslant} \frac{C_\alpha}{1+|x|^\alpha} \int_{S_1^x \cup S_2^x} |G_{v,\omega}^{(s)}(x-y)|\,\mathrm{d}y \leqslant \frac{C_\alpha}{1+|x|^\alpha} \int_{\mathbb{R}^n} |G_{v,\omega}^{(s)}(y)|\,\mathrm{d}y \leqslant \frac{C}{1+|x|^\alpha}, \end{aligned} \tag{3.89}$$

because $G_{v,\omega}^{(s)} \in L^1(\mathbb{R}^n)$ by Lemma 3.22 (ii). The constant $C > 0$ only depends on $n, s, v, \omega$ and $\alpha$. Finally, if $y \in S_3^x$, then $\frac{1}{2}|x| \leqslant |x-y| \leqslant \frac{3}{2}|x|$, where the first inequality results from the reversed triangle inequality. This implies

$$\frac{1}{1+|x-y|^{n+1}} \lesssim_n \frac{1}{1+|x|^{n+1-\beta}} \frac{1}{1+|y|^\beta}, \quad \text{for all } y \in S_3^x,\ 0 \leqslant \beta \leqslant n+1. \tag{3.90}$$

Indeed, if $0 \leqslant \beta \leqslant n+1$, we check that

$$\begin{aligned} \left(1+|x|^{n+1-\beta}\right)\left(1+|y|^\beta\right) &= 1+|x|^{n+1-\beta} + |y|^\beta + |x|^{n+1-\beta}|y|^\beta \\ &\lesssim_n 1+|x-y|^{n+1-\beta} + |x-y|^\beta + |x-y|^{n+1-\beta}|x-y|^\beta \\ &\lesssim_n 1+|x-y|^{n+1}, \end{aligned}$$

using $|x| \leqslant 2|x-y|$, $|y| \leqslant \frac{1}{2}|x|$ and $|x-y| \geqslant 1$. By Remark 3.27, we may pick $\beta \geqslant 0$ such that $\alpha + \beta > n$ and $n+1-\beta \geqslant \alpha$. Then from (3.90) and (3.86) it follows that[17]

$$\begin{aligned} &\int_{S_3^x} |G_{v,\omega}^{(s)}(x-y)||f(y)|\,\mathrm{d}y \leqslant \int_{S_3^x} |G_{v,\omega}^{(s)}(x-y)| \frac{c}{1+|y|^\alpha}\,\mathrm{d}y \\ &\overset{(3.86)}{\leqslant} C \int_{S_3^x} \frac{1}{|x-y|^{n+1}} \frac{1}{1+|y|^\alpha}\,\mathrm{d}y \overset{(3.90)}{\leqslant} \frac{C}{1+|x|^{n+1-\beta}} \int_{S_3^x} \frac{1}{1+|y|^\beta} \frac{1}{1+|y|^\alpha}\,\mathrm{d}y \\ &\leqslant \frac{C}{1+|x|^{n+1-\beta}} \int_{\mathbb{R}^n} \frac{1}{1+|y|^{\alpha+\beta}}\,\mathrm{d}y \leqslant \frac{C}{1+|x|^{n+1-\beta}} \leqslant \frac{C}{1+|x|^\alpha}. \end{aligned} \tag{3.91}$$

The last inequality in (3.91) holds by boundedness of $\psi(x) = \frac{1+|x|^\alpha}{1+|x|^{n+1-\beta}}$ on $\mathbb{R}^n$ ($\psi$ is continuous on the compact set $\{|x| \leqslant 1\}$, whereas for $|x| \geqslant 1$ one has that $\psi(x) \leqslant 1$ if and only if $n+1-\beta \geqslant \alpha$). Now, (3.91) and (3.89) imply the claim, since $x$ was arbitrary and $C > 0$ does not depend on $x$. The proof of Lemma 3.26 is now complete. □

[17] Note also that from $|x-y| \geqslant 1$ we have

$$|x-y|^{n+1} = \frac{1}{2}|x-y|^{n+1} + \frac{1}{2}|x-y|^{n+1} \geqslant \frac{1}{2} + \frac{1}{2}|x-y|^{n+1} = \frac{1}{2}\left(1+|x-y|^{n+1}\right)$$

and thus $\frac{1}{|x-y|^{n+1}} \lesssim \frac{1}{1+|x-y|^{n+1}}$.

**Remark 3.28.** Note that the above proof actually shows the stronger estimate

$$(|G^{(s)}_{v,\omega}| \star |f|)(x) \leqslant \frac{C}{1+|x|^\alpha}.$$

### 3.7.2 Proof of Theorem 3.3

Let us now prove Theorem 3.3. We proceed in three steps.

**Step 1: The spectrum of the associated Schrödinger operator.** The statement that $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\} \cap L^\infty(\mathbb{R}^n)$ solves the Euler-Lagrange equation (3.3) is equivalent to saying that $Q_v \in H^s(\mathbb{R}^n)\backslash\{0\}$ is an eigenfunction (with corresponding eigenvalue $-\omega$) of the operator $H : D(H) \to H^{-s}(\mathbb{R}^n)$ defined on the domain $D(H) = H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ by $H := \mathcal{T}_{s,v} + V$, with the Schrödinger operator $\mathcal{T}_{s,v} := (-\Delta)^s + iv \cdot \nabla$ (recall $s \geqslant \frac{1}{2}$) and the nonlinear potential $V = -|Q_v|^{2\sigma} \in L^\infty(\mathbb{R}^n)$,

$$HQ_v = (\mathcal{T}_{s,v} + V)Q_v = -\omega Q_v$$

see also [HS96]. Observe that $\mathcal{T}_{s,v}$ cannot have an eigenvalue. Indeed, suppose $E$ is an eigenvalue of $\mathcal{T}_{s,v}$ and $f \in \ker(\mathcal{T}_{s,v} - E)$, $f \not\equiv 0$ a corresponding eigenfunction, i.e., on the Fourier side, we have the eigenvalue equation

$$\left(|\xi|^{2s} - v \cdot \xi\right) \widehat{f}(\xi) = E\widehat{f}(\xi).$$

If $\widehat{f} \equiv 0$, Fourier inversion gives $f \equiv \mathcal{F}^{-1}\widehat{f} \equiv 0$, a contradiction. So $\widehat{f} \not\equiv 0$, and we consider the set $N := \{\xi \in \mathbb{R}^n; \widehat{f}(\xi) \neq 0\}$. Defining $g(\xi) := |\xi|^{2s} - v \cdot \xi - E$ and $M := g^{-1}(\{0\}) = \{\xi \in \mathbb{R}^n; a(\xi) = 0\}$, we have $N \subset M$. But $M$ is an $n-1$ dimensional submanifold of $\mathbb{R}^n$, hence of measure zero. In particular $N$ has measure zero, which gives $\widehat{f} \equiv 0$, a contradiction. Consequently, the discrete spectrum of $\mathcal{T}_{s,v}$ is empty, so its spectrum equals its essential spectrum, namely

$$\sigma_{\mathrm{ess}}(\mathcal{T}_{s,v}) = \sigma(\mathcal{T}_{s,v})\backslash\sigma_d(\mathcal{T}_{s,v}) = [-\omega_*, \infty)\,.$$

Here the bottom of the essential spectrum is the number

$$-\omega_* = |\xi_*|^{2s} - v \cdot \xi_* \begin{cases} = 0 & \text{if } v = 0, \\ = 0 & \text{if } s = \frac{1}{2} \text{ and } 0 < |v| < 1, \\ < 0 & \text{if } s > \frac{1}{2} \text{ and } v \neq 0, \end{cases}$$

where

$$\xi_* = \begin{cases} 0 & \text{if } v = 0, \\ 0 & \text{if } s = \frac{1}{2} \text{ and } 0 < |v| < 1, \\ \beta v & \text{if } s > \frac{1}{2} \text{ and } v \neq 0, \quad \text{where } \beta = \beta(s, |v|) = \frac{1}{2s}\left(\frac{v^2}{4s^2}\right)^{\frac{1-s}{2s-1}}. \end{cases}$$

See Lemma B.8 for the fact that $|\xi|^{2s} - v\cdot\xi \geqslant -\omega_*$, with equality if and only if $\xi = \xi_*$. Since $-\omega < -\omega_*$, we can rewrite the above equation $(\mathcal{T}_{s,v} - (-\omega))Q_v = -VQ_v$ with the resolvent $R_{\mathcal{T}_{s,v}}(\mu) := (\mathcal{T}_{s,v} - \mu)^{-1}$ as $Q_v = -R_{\mathcal{T}_{s,v}}(-\omega)(VQ_v)$. In other words, $Q_v = G^{(s)}_{v,\omega} \star (|Q_v|^{2\sigma} Q_v)$, i.e.

$$Q_v(x) = -\int_{\mathbb{R}^n} G^{(s)}_{v,\omega}(x-y)V(y)Q_v(y)\,\mathrm{d}y. \tag{3.92}$$

**Step 2: Adapting the Slaggie-Wichmann method, cf. [His00].** We claim there exists some constant $C > 0$ such that

$$|Q_v(x)| \leqslant \frac{C}{1+|x|}. \tag{3.93}$$

Define the functions

$$h(x) := \int_{\mathbb{R}^n} (1+|x-y|)|G^{(s)}_{v,\omega}(x-y)||V(y)|\,\mathrm{d}y,$$

$$m(x) := \sup_{y\in\mathbb{R}^n} \frac{|Q_v(y)|}{1+|x-y|}. \tag{3.94}$$

By (3.92),

$$|Q_v(x)| \leqslant m(x)h(x). \tag{3.95}$$

We claim that $h$ is continuous on $\mathbb{R}^n$ and vanishes at infinity in the strong sense that

$$\forall \varepsilon > 0\ \exists R_\varepsilon > 0\ \forall x \in \mathbb{R}^n \text{ with } |x| > R_\varepsilon : \quad |h(x)| < \varepsilon.$$

To see this, let us observe that $h$ is the convolution of two functions in dual $L^p(\mathbb{R}^n)$ spaces, so that the claim follows from [LL01, Lemma 2.20]. Let $p > 1$ be given. The function $\widetilde{h}(x) := (1+|x|)|G^{(s)}_{v,\omega}(x)|$ satisfies

$$|\widetilde{h}(x)|^p \leqslant 2^{p-1}\left(|G^{(s)}_{v,\omega}(x)|^p + |x|^p|G^{(s)}_{v,\omega}(x)|^p\right).$$

By Lemma 3.22 (ii), we know that $G^{(s)}_{v,\omega}(x)$ and $|x|G^{(s)}_{v,\omega}(x)$ are in $L^p(\mathbb{R}^n)$ for $p > 1$ sufficiently close to 1. Hence $\widetilde{h} \in L^p(\mathbb{R}^n)$ for $p > 1$ sufficiently close to 1. On the other hand, we know that as a member of $H^s(\mathbb{R}^n)$ and simultaneously a solution to the Euler-Lagrange equation, $Q_v$ belongs to $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. By interpolation, $|V| = |Q_v|^{2\sigma} \in L^{\frac{q}{2\sigma}}(\mathbb{R}^n)$ for all $2 \leqslant q \leqslant \infty$. Picking $p > 1$ close enough to 1, we thus guarantee that both $\widetilde{h} \in L^p(\mathbb{R}^n)$ and $|V| \in L^{p'}(\mathbb{R}^n)$ with $p' = \frac{p}{p-1} > 2$. Hence the conclusion follows from $h = \widetilde{h} \star |V|$. Using the integral equation (3.92), we deduce by the same argument that $Q_v$ is continuous on $\mathbb{R}^n$ and vanishes strongly at infinity in the above sense.

It is an obvious consequence of the triangle inequality that

$$\sup_{z\in\mathbb{R}^n} \frac{1}{(1+|x-z|)(1+|y-z|)} \leqslant \frac{1}{1+|x-y|}. \tag{3.96}$$

Thus

$$\begin{aligned}\sup_{z\in\mathbb{R}^n}\frac{m(z)}{1+|x-z|} &\overset{(3.94)}{=} \sup_{z\in\mathbb{R}^n}\left\{\sup_{y\in\mathbb{R}^n}\frac{|Q_v(y)|}{(1+|x-z|)(1+|z-y|)}\right\}\\ &= \sup_{y\in\mathbb{R}^n}\left\{\sup_{z\in\mathbb{R}^n}\frac{|Q_v(y)|}{(1+|x-z|)(1+|y-z|)}\right\}\\ &\overset{(3.96)}{\leqslant} \sup_{y\in\mathbb{R}^n}\frac{|Q_v(y)|}{1+|x-y|} \overset{(3.94)}{=} m(x).\end{aligned} \tag{3.97}$$

Since $h$ vanishes strongly at infinity, (3.95) allows us to choose $R>0$ so large that

$$|Q_v(x)| \leqslant \frac{1}{2}m(x), \quad \text{for all } x\in\mathbb{R}^n \text{ with } |x|>R.$$

Therefore

$$\begin{aligned}\sup_{|y|>R}\frac{|Q_v(y)|}{1+|x-y|} &\leqslant \frac{1}{2}\sup_{|y|>R}\frac{m(y)}{1+|x-y|} < \sup_{|y|>R}\frac{m(y)}{1+|x-y|}\\ &\leqslant \sup_{y\in\mathbb{R}^n}\frac{m(y)}{1+|x-y|} \overset{(3.97)}{\leqslant} m(x),\end{aligned}$$

provided that $M := \sup_{|y|>R}\frac{m(y)}{1+|x-y|} > 0$. [Note that the assumption $M=0$ yields $m(y)=0$ for all $y\in\mathbb{R}^n$ with $|y|>R$, hence the definition of $m(y)$ gives $Q_v\equiv 0$, and there is nothing to prove (claim (3.93) is evident).] Thus we see that $m(x)$ is strictly greater than $\sup_{|y|>R}\frac{|Q_v(y)|}{1+|x-y|}$ and this implies that $m(x)$ is really a supremum over the ball $\{y\in\mathbb{R}^n;\ |y|\leqslant R\}$. It follows that

$$m(x) = \sup_{|y|\leqslant R}\frac{|Q_v(y)|}{1+|x-y|} \leqslant \frac{C}{1+|x|}, \tag{3.98}$$

using continuity of $Q_v$ on the compact set $\{y\in\mathbb{R}^n;\ |y|\leqslant R\}$ and the triangle inequality $1+|x|\leqslant 1+|x-y|+|y|\leqslant(1+R)(1+|x-y|)$ to bound the denominator. Inserting (3.98) in (3.95) and recalling the boundedness of $h$ on $\mathbb{R}^n$ ($h\in C(\mathbb{R}^n)$ and vanishes strongly at infinity) proves the claimed decay (3.93).

**Step 3: A bootstrap argument.** From $|V(x)| = |Q_v(x)|^{2\sigma}$ and (3.93) we get

$$|V(x)| \overset{(3.93)}{\leqslant} \frac{C}{(1+|x|)^{2\sigma}} \leqslant \frac{C}{1+|x|^{2\sigma}}. \tag{3.99}$$

We start a bootstrap argument, using always the potential bound (3.99).

(1) Using the bounds (3.99), (3.93) in the integral equation (3.92) yields

$$|Q_v(x)| \leqslant C\int_{\mathbb{R}^n}|G^{(s)}_{v,\omega}(x-y)|\frac{1}{1+|y|^{1+2\sigma}}\,\mathrm{d}y. \tag{3.100}$$

If $1 + 2\sigma > n + 1$, the theorem is proved. Indeed, since $\varphi_1(y) := \frac{1+|y|^{n+1}}{1+|y|^{1+2\sigma}} \leqslant C$ on all of $\mathbb{R}^n$ (the continuity of $\varphi_1$ yields an upper bound on the closed unit ball, and outside the condition $1 + 2\sigma > n + 1$ yields that $\varphi_1$ is bounded from above by 1), (3.100) implies

$$|Q_v(x)| \leqslant C \int_{\mathbb{R}^n} |G^{(s)}_{v,\omega}(x-y)| \frac{1}{1+|y|^{n+1}} \,\mathrm{d}y. \tag{3.101}$$

Applying Lemma 3.26 (more precisely, Remark 3.28) with $\alpha = n + 1$ to (3.101) proves $|Q_v(x)| \leqslant C/(1 + |x|^{n+1})$.

If $1 + 2\sigma \leqslant n + 1$, applying Lemma 3.26 with $\alpha = 1 + 2\sigma$ to (3.100) gives a new bound

$$|Q_v(x)| \leqslant \frac{C}{1+|x|^{1+2\sigma}}. \tag{3.102}$$

(2) Using the bounds (3.99), (3.102) in the integral equation (3.92) yields

$$|Q_v(x)| \leqslant C \int_{\mathbb{R}^n} |G^{(s)}_{v,\omega}(x-y)| \frac{1}{1+|y|^{1+2\cdot(2\sigma)}} \,\mathrm{d}y.$$

We conclude like in (1) above: if $1 + 2\cdot(2\sigma) > n + 1$, the theorem is proved; if $1 + 2\cdot(2\sigma) \leqslant n + 1$, we deduce a new bound $|Q_v(x)| \leqslant C/(1 + |x|^{1+2\cdot(2\sigma)})$, which initiates a next step.

Continuing like this, we complete the proof with the $\nu_{\min}$-th step, where

$$\nu_{\min} = \min\{\nu \in \mathbb{N};\ 1 + \nu \cdot (2\sigma) > n + 1\}.$$

The proof of Theorem 3.3 is now complete. □

# B Traveling Solitary Waves

## B.1 Used Theorems

**Lemma B.1** (Compactness modulo translations in $\dot{H}^s(\mathbb{R}^n)$; see [BFV14]). *Let $s > 0$, $1 < p < \infty$ and $(u_j)_{j\in\mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ be a sequence with*

$$\sup_{j\in\mathbb{N}} (\|u_j\|_{\dot{H}^s} + \|u_j\|_{L^p}) < \infty,$$

*and, for some $\eta, c > 0$ (with $|\cdot|$ being Lebesgue measure)*

$$\inf_{j\in\mathbb{N}} |\{x \in \mathbb{R}^n;\ |u_j(x)| > \eta\}| \geqslant c.$$

*Then there exists a sequence of vectors $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that the translated sequence $(\tilde{u}_j)_{j\in\mathbb{N}}$, where $\tilde{u}_j(x) := u_j(x + x_j)$, has a subsequence that converges weakly in $\dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to a nonzero function $u \not\equiv 0$.*

*Proof (see [BFV14, Lemma 2.1]).* The proof rests on refined[1] Sobolev inequalities by means of homogeneous Besov spaces $\dot{B}^{-\beta}_{\infty,\infty}$ of negative smoothness in the form (see [BCD13, Theorem 1.43] and its proof; and [GMO97, Théorème 2] for the original)

$$\|u\|_{L^{2^*}} \leqslant \frac{C}{(2^* - 2)^{\frac{1}{2^*}}} \|u\|^{1-\frac{2}{2^*}}_{\dot{B}^{-n/2^*}_{\infty,\infty}} \|u\|^{\frac{2}{2^*}}_{\dot{H}^s} \quad \text{with } 2^* = \tfrac{2n}{n-2s}, \tag{B.1}$$

for $0 < s < \frac{n}{2}$. Herein, $\|\cdot\|_{\dot{B}^{-\beta}_{\infty,\infty}}$ is a homogeneous Besov norm,[2] given for $\beta > 0$ and tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ through the expression ('thermic description')

$$\|u\|_{\dot{B}^{-\beta}_{\infty,\infty}} := \sup_{A>0} A^{n-\beta} \|\theta(A\cdot) \star u\|_{L^\infty}, \tag{B.2}$$

[1] By a refinement of the Sobolev embedding $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$, $2^* = \frac{2n}{n-2s}$, one understands the existence of some Banach space $X$ such that $\dot{H}^s(\mathbb{R}^n)$ embeds continuously in $X$ and, for some $0 < \vartheta < 1$ and $C > 0$, the inequality

$$\|u\|_{L^{2^*}} \leqslant C\|u\|^{\vartheta}_{\dot{H}^s}\|u\|^{1-\vartheta}_X, \quad \forall u \in \dot{H}^s(\mathbb{R}^n) \tag{$*$}$$

is valid. Such an inequality and the continuity of the embedding $\dot{H}^s(\mathbb{R}^n) \hookrightarrow X$ immediately imply the Sobolev embedding

$$\|u\|_{L^{2^*}} \leqslant C\|u\|_{\dot{H}^s}, \quad \forall u \in \dot{H}^s(\mathbb{R}^n). \tag{$**$}$$

and therefore ($*$) is called a refinement of ($**$). See the article [PP14, p. 801].

[2] Changing variables, it is easy to check that the Besov norm $\|\cdot\|_{\dot{B}^{-\beta}_{\infty,\infty}}$ has the following behaviour

where $\theta \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function the Fourier transform of which is compactly supported, $\widehat{\theta} \in C_c^\infty(\mathbb{R}^n)$, such that $\widehat{\theta} \equiv 1$ near the origin $0 \in \mathbb{R}^n$ and $0 \leqslant \widehat{\theta} \leqslant 1$. With estimate (B.1), the proof is established with a technique of low and high frequency localization, see [BFV14] for more details. □

**Lemma B.2** (pqr Lemma; see [FLL86]). *Let $(\Omega, \Sigma, \mu)$ be a measure space. Let $1 \leqslant p < q < r \leqslant \infty$ and let $C_p, C_q, C_r > 0$ be positive constants. Then there exist constants $\eta, c > 0$ such that, for any measurable function $f \in L^p_\mu(\Omega) \cap L^r_\mu(\Omega)$ satisfying*

$$\|f\|^p_{L^p_\mu} \leqslant C_p, \quad \|f\|^q_{L^q_\mu} \geqslant C_q, \quad \|f\|^r_{L^r_\mu} \leqslant C_r,$$

*it holds that*

$$d_f(\eta) := \mu(\{x \in \Omega;\ |f(x)| > \eta\}) \geqslant c.$$

*The constant $\eta > 0$ only depends on $p, q, C_p, C_q$ and the constant $c > 0$ only depends on $p, q, r, C_p, C_q, C_r$.*

*Proof (see [FLL86, Lemma 2.1]).* By monotonicity of the measure $\mu$ the distribution function $d_f$ is monotone non-increasing. Moreover, we have

$$\|f\|^p_{L^p_\mu} = p \int_0^\infty d_f(t) t^{p-1}\,\mathrm{d}t.\text{[3]}$$

These two facts imply

$$C_p \geqslant p \int_0^\infty d_f(t) t^{p-1}\,\mathrm{d}t \geqslant p \int_0^\eta t^{p-1} d_f(t)\,\mathrm{d}t \geqslant d_f(\eta) p \int_0^\eta t^{p-1}\,\mathrm{d}t = \eta^p d_f(\eta),$$

in other words,

$$d_f(\eta) \leqslant \eta^{-p} C_p \quad \text{for all } \eta > 0. \tag{B.3}$$

Similarly,

$$d_f(\eta) \leqslant \eta^{-r} C_r \quad \text{for all } \eta > 0. \tag{B.4}$$

under dilations $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ :

$$\|u_\lambda\|_{\dot{B}^{-\beta}_{\infty,\infty}} = \lambda^{-\beta} \|u\|_{\dot{B}^{-\beta}_{\infty,\infty}}.$$

Since $\|u_\lambda\|_{L^p} = \lambda^{-\frac{n}{p}} \|u\|_{L^p}$, $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}} \|u\|_{\dot{H}^s} = \lambda^{-\frac{n}{2^*}} \|u\|_{\dot{H}^s}$, one observes the invariance of (B.1) under dilations. Moreover though, as stated in [BCD13, p. 33], it can be checked that the homogeneous Besov norm $\|\cdot\|_{\dot{B}^{-\beta}_{\infty,\infty}}$ is invariant under multiplication by a character, $u(x) \mapsto e^{ix\cdot\omega} u(x)$, - in contrast to the homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s}$ (see [BCD13, p. 30] for a counterexample) - and that the whole inequality (B.1) is invariant (up to an irrelevant constant) under multiplication by a character.

[3]Clearly, we have $|f(x)|^p = \int_0^{|f(x)|} \frac{\mathrm{d}}{\mathrm{d}t} t^p\,\mathrm{d}t = p \int_0^{|f(x)|} t^{p-1}\,\mathrm{d}t$. Since $\chi_{\{y \in \Omega; |f(y)| > t\}}(x) = 1$ if and only if $t < |f(x)|$ and $\chi_{\{y\in\Omega;|f(y)|>t\}}(x) = 0$ if and only if $t \geqslant |f(x)|$, we have $|f(x)|^p = p \int_0^\infty \chi_{\{y\in\Omega;|f(y)|>t\}}(x) t^{p-1}\,\mathrm{d}t$. Integration of this identity over $x \in \Omega$ and using Fubini's Theorem gives the claimed formula $\|f\|^p_{L^p_\mu} = p\int_0^\infty d_f(t) t^{p-1}\,\mathrm{d}t$. See also the layer-cake principle (see [LL01, Theorem 1.13]).

Define $S$ and $T$ through the equations

$$\begin{aligned} qC_pS^{q-p} &= \tfrac{1}{4}(q-p)C_q, \\ qC_rT^{q-r} &= \tfrac{1}{4}(r-q)C_q. \end{aligned}$$

From (B.3) it follows that

$$q\int_0^S d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \overset{\text{(B.3)}}{\leqslant} qC_p\int_0^S \eta^{q-p-1}\,\mathrm{d}\eta = \frac{1}{4}C_q. \tag{B.5}$$

Similarly, from (B.4) it follows that

$$q\int_T^\infty d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \overset{\text{(B.4)}}{\leqslant} qC_r\int_T^\infty \eta^{q-r-1}\,\mathrm{d}\eta = \frac{1}{4}C_q. \tag{B.6}$$

(B.5) and (B.6) imply that $S < T$, since, if $S \geqslant T$, we infer

$$\frac{1}{2}C_q \geqslant q\int_0^S d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta + q\int_T^\infty d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \geqslant q\int_0^\infty d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta = \|f\|^q_{L^q_\mu} \geqslant C_q,$$

contradictory to $C_q > 0$. From (B.5), (B.6) and $S < T$ we conclude

$$\begin{aligned} \frac{1}{2}C_q &\geqslant q\int_0^S d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta + q\int_T^\infty d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \\ &= q\int_0^\infty d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta - q\int_S^T d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \\ &= \|f\|^q_{L^q_\mu} - q\int_S^T d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \\ &\geqslant C_q - q\int_S^T d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta, \end{aligned}$$

in other words

$$I := q\int_S^T d_f(\eta)\eta^{q-1}\,\mathrm{d}\eta \geqslant \frac{1}{2}C_q.$$

But $I \leqslant d_f(S)|T^q - S^q|$, again since $d_f$ is monotone non-increasing. This reads $d_f(S) \geqslant \frac{I}{|T^q-S^q|} \geqslant \frac{C_q}{2|T^q-S^q|}$, proving the lemma with the $f$-independent constants $\eta := S$ and $c := \frac{C_q}{2|T^q-S^q|}$, which obviously have the claimed dependencies. □

**Theorem B.3** (Weak convergence implies strong convergence on small sets; see [LL01]). *Assume that $(u_j)_{j\in\mathbb{N}} \subset \dot H^{\frac{1}{2}}(\mathbb{R}^n)$ is a sequence converging weakly to some $u \in \dot H^{\frac{1}{2}}(\mathbb{R}^n)$ in the sense that for every $v \in \dot H^{\frac{1}{2}}(\mathbb{R}^n)$*

$$\begin{aligned} \lim_{j\to\infty} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u_j(x)-u_j(y))(\overline{v(x)}-\overline{v(y)})}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\overline{v(x)}-\overline{v(y)})}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

*Under the assumption*

- $p < \frac{2n}{n-1}$ *when* $n \geqslant 2$,
- $p < \infty$ *and additionally* $u_j \rightharpoonup u$ *weakly in* $L^2(\mathbb{R})$ *when* $n = 1$

*the following conclusion holds: for any set* $A \subset \mathbb{R}^n$ *of finite measure*

$$\chi_A u_j \to \chi_A u \quad \textit{strongly in } L^p(\mathbb{R}^n).$$

*Proof.* See [LL01, Theorem 8.6]. □

**Corollary B.4** (Weak convergence implies a.e. convergence; see [LL01])**.** *Let* $(u_j)_{j\in\mathbb{N}}$ *be any sequence satisfying the assumptions of Theorem B.3. Then there exists a subsequence* $n(j)$*, such that* $(u_{n(j)}(x))_{j\in\mathbb{N}}$ *converges to* $u(x)$ *for almost every* $x \in \mathbb{R}^n$*.*

*Proof (see [LL01, Corollary 8.7]).* Consider the following sequence of sets with finite measure: $B_k := B_k(0)$, balls centered at the origin with radius $k = 1, 2, \ldots$. By Theorem B.3 $\chi_{B_1} u_j \to \chi_{B_1} u$ strongly in $L^p(\mathbb{R}^n)$ for the $p$'s given there. Equivalently $u_j \to u$ strongly in $L^p(B_1)$. Thus (e.g., by [Alt06, Lemma 1.20⟨1⟩, p. 56], or by [LL01, Theorem 2.7, p.52], or by [AF03, Corollary 2.17, p. 30]) for a subsequence $(n_1(j))$ of $(j)_{j\in\mathbb{N}}$ we get $u_{n_1(j)}(x) \to u(x)$ a.e. $x \in B_1$. Applying the same argument again, there is a subsequence $(n_2(j))_{j\in\mathbb{N}}$ of $(n_1(j))_{j\in\mathbb{N}}$ such that $u_{n_2(j)}(x) \to u(x)$ a.e. $x \in B_2$. Continuing inductively, by construction,

$$\forall k \in \mathbb{N}: \quad u_{n_k(j)}(x) \to u(x) \quad \text{a.e. } x \in B_k, \tag{B.7}$$

say, for all $x \in B_k \backslash N_k$ where $N_k = \{x \in B_k; u_{n_k(j)}(x) \nrightarrow u(x)\}$ is a set of measure zero. It is clear that the diagonal sequence $(u_{n_j(j)})_{j\in\mathbb{N}}$ satisfies $u_{n_j(j)}(x) \to u(x)$ a.e. $x \in \mathbb{R}^n$. Indeed, let $x \in \mathbb{R}^n$, say, $x \in B_k$ and suppose $x \notin N = \cup_k N_k$. Then, since $(n_j(\ell))_{\ell\in\mathbb{N}} \subset (n_k(\ell))_{\ell\in\mathbb{N}}$ for $j \geqslant k$, we have $n_j(j) = n_k(\ell)$ for some $\ell = \ell(j)$. Of course, as $j \to \infty$, also $\ell = \ell(j) \to \infty$[4] and we conclude from (B.7) ($x \notin N_k$) that

$$u_{n_j(j)}(x) = u_{n_k(\ell(j))}(x) \overset{(B.7)}{\to} u(x) \quad \forall x \in \mathbb{R}^n \backslash N.$$ □

**Theorem B.5** (Brézis-Lieb improvement of Fatou's lemma; see [BL83])**.** *Let* $(\Omega, \Sigma, \mu)$ *be a measure space and let* $(f_j)_{j\in\mathbb{N}}$ *be a sequence of complex-valued* $\mu$*-measurable functions. Suppose that the* $f_j$*'s converge pointwise* $\mu$*-almost everywhere to some function* $f$*, and that they are uniformly bounded in* $L^p_\mu(\Omega, \Sigma)$ *for some* $0 < p < \infty$*, i.e.*

$$\exists\, 0 < p < \infty: \ \exists\, C > 0: \ \sup_{j\in\mathbb{N}} \|f_j\|_{L^p_\mu} \leqslant C, \quad f_j(x) \to f(x) \ \mu\text{-a.e. } x \in \Omega. \tag{B.8}$$

*Then*

$$\lim_{j\to\infty} \left\{ \|f_j\|^p_{L^p_\mu} - \|f - f_j\|^p_{L^p_\mu} \right\} = \|f\|^p_{L^p_\mu}. \tag{B.9}$$

[4] We have $n_j(j) = n_k(\ell(j)) \leqslant n_j(\ell(j))$ by $j \geqslant k$, and so $\ell(j) \geqslant j$.

**Remark B.6.** Note that (B.9) follows from the stronger claim

$$\lim_{j\to\infty}\int_\Omega \left||f_j(x)|^p - |f(x)-f_j(x)|^p - |f(x)|^p\right| \mathrm{d}\mu(x) = 0. \tag{B.10}$$

*Proof.* See [LL01, Theorem 1.9]. See also the proof of the more general Theorem [BL83, Theorem 2], implying this theorem via example (a) in [BL83, p. 488].

For convenience, let us repeat the proof of (B.10) as in [LL01, Theorem 1.9, p. 21f] here, adding some details. Assume first[5] that for any $\varepsilon > 0$ there is a constant $\tilde{C}_\varepsilon$ such that for all numbers $a, b \in \mathbb{C}$

$$\left||a+b|^p - |b|^p\right| \leqslant \varepsilon|b|^p + \tilde{C}_\varepsilon |a|^p. \tag{B.11}$$

Write $f_j = f + g_j$; then by hypothesis $g_j(x) \to 0$ for $\mu$-a.e. $x \in \Omega$. The quantity[6]

$$G_j^\varepsilon := \left(\left||f+g_j|^p - |g_j|^p - |f|^p\right| - \varepsilon|g_j|^p\right)_+$$

is now shown to satisfy $\lim_{j\to\infty}\int G_j^\varepsilon = 0$. To see this, note first that by the triangle inequality on $\mathbb{R}$ and (B.11)

$$\left||f+g_j|^p - |g_j|^p - |f|^p\right| \leqslant \left||f+g_j|^p - |g_j|^p\right| + |f|^p \overset{\text{(B.11)}}{\leqslant} \varepsilon|g_j|^p + (\tilde{C}_\varepsilon + 1)|f|^p,$$

which immediately gives $G_j^\varepsilon \leqslant (\tilde{C}_\varepsilon + 1)|f|^p$. Also, since $g_j \to 0$ $\mu$-a.e., it is clear that $G_j^\varepsilon \to 0$ $\mu$-a.e. The dominated convergence theorem[7] now implies $\lim_{j\to\infty}\int G_j^\varepsilon = 0$. Observe[8]

$$\int \left||f+g_j|^p - |g_j|^p - |f|^p\right| \leqslant \varepsilon \int |g_j|^p + \int G_j^\varepsilon. \tag{B.13}$$

Let us show that $\int |g_j|^p$ is uniformly bounded. Indeed,

$$\int |g_j|^p = \int |f_j - f|^p \leqslant 2^p \int (|f_j|^p + |f|^p) \leqslant 2^{p+1}C, \tag{B.14}$$

where we used the hypothesis (B.8) as well as (B.12) in the second inequality, and the estimate

$$|z+w|^p \leqslant (|z|+|w|)^p \leqslant (2\max\{|z|,|w|\})^p = 2^p \max\{|z|^p, |w|^p\} \leqslant 2^p(|z|^p + |w|^p)$$

[5] The proof of this is given in a moment.

[6] As usual, $h_+(x) := \max\{h(x), 0\}$ denotes the positive part of a function $h : \Omega \to \mathbb{R}$.

[7] Note that $(\tilde{C}_\varepsilon + 1)|f|^p$ is an integrable (dominating) function, since $f_j \to f$ $\mu$-a.e. implies $|f(x)|^p = \liminf_{j\to\infty}|f_j(x)|^p$ $\mu$-a.e. $x$, and thus Fatou's lemma yields

$$\int_\Omega |f(x)|^p \,\mathrm{d}\mu(x) \leqslant \liminf_{j\to\infty}\int_\Omega |f_j(x)|^p \,\mathrm{d}\mu(x) \overset{\text{(B.8)}}{\leqslant} C. \tag{B.12}$$

[8] Indeed, $h_j^\varepsilon := \left||f+g_j|^p - |g_j|^p - |f|^p\right| - \varepsilon|g_j|^p$ satisfies $h_j^\varepsilon \leqslant (h_j^\varepsilon)_+ = G_j^\varepsilon$, thus $\int h_j^\varepsilon \leqslant \int G_j^\varepsilon$, which is precisely inequality (B.13).

in the first inequality. From (B.13), the subadditivity of lim sup, the fact that $\lim_{j\to\infty}\int G_j^\varepsilon = 0$ and (B.14), we obtain

$$\begin{aligned}\limsup_{j\to\infty}\int \big||f+g_j|^p - |g_j|^p - |f|^p\big| &\overset{(B.13)}{\leqslant} \limsup_{j\to\infty}\left(\varepsilon\int |g_j|^p + \int G_j^\varepsilon\right)\\ &\leqslant \varepsilon \limsup_{j\to\infty}\int |g_j|^p + \limsup_{j\to\infty}\int G_j^\varepsilon\\ &\overset{(B.14)}{\leqslant} \varepsilon 2^{p+1}C.\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get $\limsup_{j\to\infty}\int\big||f+g_j|^p - |g_j|^p - |f|^p\big| = 0$. Since the quantities $a_j := \int\big||f+g_j|^p - |g_j|^p - |f|^p\big|$ are nonnegative, this means $\lim_{j\to\infty} a_j = 0$, which is precisely claim (B.10).

We come to the remaining proof of (B.11): clearly, $g(t) := t^p$, $t > 0$, is a convex function for $p > 1$ (in fact, $g''(t) > 0$ on all of $\mathbb{R}_+$) and therefore

$$\begin{aligned}|a+b|^p \leqslant (|a|+|b|)^p &= g\left((1-\lambda)\frac{|a|}{(1-\lambda)} + \lambda\frac{|b|}{\lambda}\right)\\ &\leqslant (1-\lambda)g\left(\frac{|a|}{1-\lambda}\right) + \lambda g\left(\frac{|b|}{\lambda}\right) = (1-\lambda)^{1-p}|a|^p + \lambda^{1-p}|b|^p\end{aligned} \tag{B.15}$$

for any $0 < \lambda < 1$. In the case $p > 1$, let us set $\lambda = (1+\eta)^{-\frac{1}{p-1}} \in (0,1)$, with $\eta > 0$, which gives

$$|a+b|^p - |b|^p \leqslant \eta|b|^p + C_\eta|a|^p, \quad C_\eta = \left(1-(1+\eta)^{-\frac{1}{p-1}}\right)^{1-p}. \tag{B.16}$$

(B.16) coincides with (B.11) (choosing $\eta = \varepsilon$) provided that $|a+b|^p - |b|^p \geqslant 0$. If otherwise $|a+b|^p - |b|^p < 0$, let us write $a' + b' := b$ and $b' := a + b$ (that is, define $a' := -a, b' := a + b$) and apply (B.15) to get

$$\begin{aligned}|b|^p - |a+b|^p = |a'+b'|^p - |b'|^p &\overset{(B.16)}{\leqslant} \eta|b'|^p + C_\eta|a'|^p\\ &\leqslant \eta(|a|+|b|)^p + C_\eta|a|^p\\ &\overset{(B.15)}{\leqslant} \eta\left(C_\eta|a|^p + (1+\eta)|b|^p\right) + C_\eta|a|^p\\ &= \eta(1+\eta)|b|^p + C_\eta(1+\eta)|a|^p.\end{aligned} \tag{B.17}$$

Combining results (B.17) and (B.16) (increase $\eta$ to $\eta(1+\eta)$ and $C_\eta$ to $C_\eta(1+\eta)$ on the right side of (B.16)!), we have found

$$\big||a+b|^p - |b|^p\big| \leqslant \eta(1+\eta)|b|^p + C_\eta(1+\eta)|a|^p, \quad C_\eta = \left(1-(1+\eta)^{-\frac{1}{p-1}}\right)^{1-p}, \tag{B.18}$$

for any $\eta > 0$. Now, given any $\varepsilon > 0$, pick some $\eta > 0$ such that $\eta(1+\eta) \leqslant \varepsilon$ and set $\tilde{C}_\varepsilon := C_\eta(1+\eta)$ to deduce from (B.18) the validity of (B.11) in the case $p > 1$. In the case $0 < p \leqslant 1$, we have (by inequality (B.22) below)

$$|a+b|^p - |b|^p \overset{(B.22)}{\leqslant} |a|^p \leqslant \varepsilon|b|^p + 1\cdot|a|^p \tag{B.19}$$

for all $a, b \in \mathbb{C}$. On the other hand, defining again $a' := -a, b' := a + b$, we have

$$\begin{aligned}|b|^p - |a+b|^p = |a'+b'|^p - |b'|^p &\overset{(B.19)}{\leqslant} \varepsilon|b'|^p + |a'|^p \\ = \varepsilon|a+b|^p + |a|^p &\overset{(B.22)}{\leqslant} \varepsilon\left(|a|^p + |b|^p\right) + |a|^p \\ = \varepsilon|b|^p + (\varepsilon+1)|a|^p.\end{aligned} \tag{B.20}$$

Combining (B.20) and (B.19) (increase the constant 1 to $\tilde{C}_\varepsilon := (\varepsilon + 1)$ on the right side of (B.19)!), we have proved

$$\left||a+b|^p - |b|^p\right| \leqslant \varepsilon|b|^p + \tilde{C}_\varepsilon|a|^p$$

also in the case $0 < p \leqslant 1$. □

**Lemma B.7** (An elementary inequality)**.** *For $\alpha, \beta \geqslant 0$ and $\sigma \geqslant 0$ we have*

$$(\alpha+\beta)^{\sigma+1} \geqslant \alpha^{\sigma+1} + \beta^{\sigma+1}. \tag{B.21}$$

*Proof.* If $\sigma = 0$ or if $\alpha$ or $\beta$ is zero, the inequality is clear (understanding $0^{\sigma+1} = 0$). Assuming $\sigma > 0$ and $\alpha, \beta > 0$, the inequality is equivalent to

$$\left(\alpha^{\sigma+1} + \beta^{\sigma+1}\right)^{\frac{1}{\sigma+1}} \leqslant \alpha + \beta.$$

The last inequality follows from the elementary inequality

$$|z+w|^p \leqslant |z|^p + |w|^p, \quad 0 < p < 1, \quad z, w \in \mathbb{C}, \tag{B.22}$$

by setting $p = \frac{1}{\sigma+1}, z = \alpha^{\sigma+1}, w = \beta^{\sigma+1}$. We show (B.22). By the triangle inequality $|z+w|^p \leqslant (|z| + |w|)^p$. Set $f(t) := (1+t)^p - 1 - t^p$, $t \geqslant 0$ and note that $f'(t) = p(1+t)^{p-1} - pt^{p-1} < 0$ for $t \in (0, \infty)$, since $p < 1$. Since $f(0) = 0$, this yields $f(t) < 0$ on $t \in (0, \infty)$, because assuming there existed some $t_* > 0$ such that $f(t_*) \geqslant 0$ leads with the mean value theorem of differentiation to the existence of some $\eta \in (0, t_*)$ such that

$$f'(\eta) = \frac{f(t_*) - f(0)}{t_*} = \frac{f(t_*)}{t_*} \geqslant 0,$$

a contradiction. Let $|z|, |w| > 0$ (otherwise, the claim is clear again) and $\widetilde{t} = \frac{|z|}{|w|} > 0$ to obtain

$$0 > f\left(\widetilde{t}\right) = \left(\frac{|z| + |w|}{|w|}\right)^p - 1 - \left(\frac{|z|}{|w|}\right)^p,$$

which is equivalent to $(|z| + |w|)^p < |z|^p + |w|^p$. □

## B.2 The Operator $(-\Delta)^s + iv \cdot \nabla$

**Lemma B.8** (Explicit lower bound $-\omega_*$ in the case of $L = (-\Delta)^s$)**.** *Let $n \geqslant 1$, $s \geqslant \frac{1}{2}$, $v \in \mathbb{R}^n$ arbitrary for $s > \frac{1}{2}$, and $|v| < 1$ for $s = \frac{1}{2}$. Define $g_{s,v}(\xi) := |\xi|^{2s} - v \cdot \xi$ (with $g_{s,v}(0) := 0$) and $-\omega_*(n,s,v) := \inf_{\xi \in \mathbb{R}^n} g_{s,v}(\xi)$. Then*

$$-\omega_*(n,s,v) \begin{cases} = 0 & \text{if } v = 0, \\ = 0 & \text{if } s = \frac{1}{2} \text{ and } 0 < |v| < 1, \\ < 0 & \text{if } s > \frac{1}{2} \text{ and } v \neq 0. \end{cases}$$

*Explicitly, for $s > \frac{1}{2}$ and $v \neq 0$*

$$-\omega_*(n,s,v) = g_{s,v}(\xi_*) = \left\{ \left( \frac{1}{2s} \left( \frac{1}{4s^2} \right)^{\frac{1-s}{2s-1}} \right)^{2s} - \frac{1}{2s} \left( \frac{1}{4s^2} \right)^{\frac{1-s}{2s-1}} \right\} |v|^{\frac{2s}{2s-1}} < 0.$$

*We have $g_{s,v}(\xi) \geqslant -\omega_*(n,s,v)$ with equality if and only if*

$$\xi = \xi_* = \begin{cases} 0 & \text{if } v = 0, \\ 0 & \text{if } s = \frac{1}{2} \text{ and } 0 < |v| < 1, \\ \beta v & \text{if } s > \frac{1}{2} \text{ and } v \neq 0, \quad \text{where } \beta = \beta(s,|v|) = \frac{1}{2s} \left( \frac{v^2}{4s^2} \right)^{\frac{1-s}{2s-1}}. \end{cases}$$

**Remark B.9.** Note the following:

(i) We have $\omega_*(n,s,v) = \sup_{\xi \in \mathbb{R}^n} (v \cdot \xi - |\xi|^{2s})$, that is, $\omega_*$ is precisely the Legendre transform of the function $\xi \mapsto |\xi|^{2s}$, evaluated at $v$. [Note that $\xi \mapsto |\xi|^{2s}$ is always convex for $s \geqslant \frac{1}{2}$.]

(ii) If $s = \frac{1}{2}$ and $|v| > 1$, the boost term becomes dominant over the dispersive term, namely $g_{\frac{1}{2},v}(\xi)$ is neither bounded from below nor from above. Indeed, taking $\xi = \alpha v$ with $\alpha > 0$ leads to $g_{\frac{1}{2},v}(\xi) = -\alpha|v|(|v| - 1) \to -\infty$ as $\alpha \uparrow \infty$, while taking $\xi = \alpha v$ with $\alpha < 0$ leads to $g_{\frac{1}{2},v}(\xi) = -\alpha|v|(1 + |v|) \to +\infty$ as $\alpha \downarrow -\infty$. The latter also shows that $g_{\frac{1}{2},v}$ is not bounded from above when $|v| = 1$. In this case, $g_{\frac{1}{2},v}$ is however bounded from below by 0 and this value is attained (namely if and only if $\xi = 0$ or $\xi \neq 0$ and (by Cauchy-Schwarz) $\xi, v$ are linearly dependent and satisfy $v \cdot \xi > 0$).

(iii) When $s = \frac{1}{2}$, one needs the condition $|v| < 1$ to ensure the positivity [KLR13]

$$\int_{\mathbb{R}^n} \left[ \overline{u}\sqrt{-\Delta}u + \overline{u}(iv \cdot \nabla)u \right] \mathrm{d}x > 0 \quad \text{for } u \not\equiv 0.$$

Indeed, assuming for example $|v| > 1$, and taking $u \not\equiv 0$ such that supp $\widehat{u} \subset C_v$ lies in the cone $C_v := \{\xi \in \mathbb{R}^n; \cos \measuredangle(v,\xi) \geqslant 1/|v|\}$, gives

$$\int_{\mathbb{R}^n} (|\xi| - v \cdot \xi)|\widehat{u}(\xi)|^2 \, \mathrm{d}\xi = \int_{C_v} |\xi|(1 - |v| \cos \measuredangle(v,\xi))|\widehat{u}(\xi)|^2 \, \mathrm{d}\xi \leqslant 0.$$

*Proof of Lemma B.8.* If $v = 0$, we check all the claims immediately. We let $v \neq 0$ in the following. Let first $s = \frac{1}{2}$. Since $|v| < 1$, one has $g_{\frac{1}{2},v}(\xi) := |\xi| - v \cdot \xi \geqslant |\xi|\,(1 - |v|) \to \infty$ as $|\xi| \to \infty$. Thus by continuity the infimum exists and is really a minimum. Since $g_{\frac{1}{2},v}(\xi) \geqslant |\xi|(1 - |v|) \geqslant 0$ and $g_{\frac{1}{2},v}(0) = 0$ it is clear that $-\omega_*(n, \frac{1}{2}, v) = 0$. This also shows that $\xi_* = 0$ satisfies the equality case $g_{\frac{1}{2},v}(\xi_*) = 0$. Conversely, assume there exists $\xi \neq 0$ such that the equality case $g_{\frac{1}{2},v}(\xi) = 0$ holds. Since $g_{\frac{1}{2},v}$ is differentiable at $\xi \neq 0$, necessarily $\nabla g_{\frac{1}{2},v}(\xi) = 0$, which reads $\frac{\xi}{|\xi|} = v$. But then $|v| = 1$, a contradiction.

Let now $s > \frac{1}{2}$. Differentiation of $g_{s,v}$ with respect to $\xi$ gives

$$\nabla g_{s,v}(\xi) = 2s|\xi|^{2s-2}\xi - v,$$

from which we see

$$\nabla g_{s,v}(\xi) = 0 \iff \xi = \beta v, \quad \text{where } \beta = \frac{1}{2s}\left(\frac{v^2}{4s^2}\right)^{\frac{1-s}{2s-1}}.$$

The function $g_{s,v}$ possesses a local minimum at the point $\xi_* = \beta v \in \mathbb{R}^n \backslash \{0\}$ (see below), with value

$$g_{s,v}(\xi_*) = \left\{\left(\frac{1}{2s}\left(\frac{1}{4s^2}\right)^{\frac{1-s}{2s-1}}\right)^{2s} - \frac{1}{2s}\left(\frac{1}{4s^2}\right)^{\frac{1-s}{2s-1}}\right\}|v|^{\frac{2s}{2s-1}} < 0.$$

Since $\xi_* \in \mathbb{R}^n \backslash \{0\}$ is the only point for which the gradient of $g_{s,v}$ vanishes and $g_{s,v}$ is continuous on $\mathbb{R}^n$ with $g_{s,v}(\xi) \to +\infty$ as $|\xi| \to \infty$ and $g_{s,v}(\xi) \to 0$ as $|\xi| \to 0$, we conclude that the local minimum $g_{s,v}(\xi_*)$ is also a global one, that is $g_{s,v}(\xi_*) = -\omega_*(n, s, v)$. Furthermore, since $\xi_*$ is the unique global minimum of $g_{s,v}$,

$$g_{s,v}(\xi) = -\omega_* \iff \xi = \xi_*.$$

It remains to verify that $\xi_* = \beta v$ is a local minimum point. Indeed, the Hessian of $g_{s,v}$ evaluated at $\xi_*$ is positive definite: we have

$$D^2 g_{s,v}(\xi) = \left(2s|\xi|^{2s-4}\left\{(2s-2)\xi_j\xi_k + |\xi|^2\delta_{jk}\right\}\right)_{1 \leqslant j,k \leqslant n}, \quad \text{for } \xi \in \mathbb{R}^n \backslash \{0\},$$

hence for any $\xi \neq 0$ we get $\xi \cdot D^2 g_{s,v}(\xi_*)\xi > 0$. Namely, by $s > \frac{1}{2}$, if $\xi \cdot \xi_* \neq 0$ (i.e. $\xi \not\perp v$), we have

$$\begin{aligned}
\xi \cdot D^2 g_{s,v}(\xi_*)\xi &= 2s(2s-2)|\xi_*|^{2s-4}|\xi \cdot \xi_*|^2 + 2s|\xi_*|^{2s-2}|\xi|^2 \\
&> -2s|\xi_*|^{2s-4}|\xi \cdot \xi_*|^2 + 2s|\xi_*|^{2s-2}|\xi|^2 \\
&\geqslant -2s|\xi_*|^{2s-4}|\xi_*|^2|\xi|^2 + 2s|\xi_*|^{2s-2}|\xi|^2 \\
&= 0,
\end{aligned}$$

whereas if $\xi \cdot \xi_* = 0$ (i.e. $\xi \perp v$) we also have

$$\begin{aligned}
\xi \cdot D^2 g_{s,v}(\xi_*)\xi &= 2s(2s-2)|\xi_*|^{2s-4}|\xi \cdot \xi_*|^2 + 2s|\xi_*|^{2s-2}|\xi|^2 \\
&= 2s|\xi_*|^{2s-2}|\xi|^2 > 0
\end{aligned}$$

because $\xi, \xi_* \neq 0$. □

# 4 Scattering for Fractional NLS

## 4.1 Introduction and Main Result

This chapter deals with the scattering problem for the fractional nonlinear Schrödinger equation in the defocusing case. That is, we consider the initial value problem

$$\begin{cases} i\partial_t u &= (-\Delta)^s u + F(u), \\ u(0) &= u_0 \in H^s(\mathbb{R}^n), \quad u : [0,T) \times \mathbb{R}^n \to \mathbb{C} \end{cases} \tag{fNLS}$$

with defocusing nonlinearity $F(u) = +|u|^{2\sigma}u$. To clarify the exposition, we restrict ourselves to the case of cubic nonlinearity $\sigma = 1$ and three spatial dimensions $n = 3$. Supposing $\frac{3}{4} < s < 1$, we guarantee that the nonlinearity $\sigma = 1$ is $H^s(\mathbb{R}^3)$-subcritical. By Sobolev's embedding we have $H^s(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ with $2_s^* = \frac{6}{3-2s}$, in particular $H^s(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \cap L^{2_s^*}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$. We then recall the conservation laws for the mass $M[u]$ and energy $E[u]$, that is

$$\begin{aligned} M[u(t)] &= \|u(t)\|_{L^2}^2 \equiv M[u_0], \\ E[u(t)] &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^4}^4 \equiv E[u_0], \end{aligned}$$

and therefore notice an apriori bound on the $H^s(\mathbb{R}^3)$-norm of the solution of (fNLS) thanks to the defocusing sign of the equation:

$$\begin{aligned} \|u(t)\|_{H^s} &\lesssim \|u(t)\|_{L^2} + \||\nabla|^s u(t)\|_{L^2} \\ &\leqslant \|u(t)\|_{L^2} + \sqrt{2\left(\frac{1}{2}\||\nabla|^s u(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^4}^4\right)} \\ &= \sqrt{M[u_0]} + \sqrt{2E[u_0]} \lesssim_{\|u_0\|_{H^s}} 1, \end{aligned}$$

where the last step follows from the Gagliardo-Nirenberg inequality for this $H^s(\mathbb{R}^3)$-subcritical equation. The solution $u(t)$ is thus global.

We are interested in the asymptotic behaviour of $u(t)$ as $t \to \pm\infty$, and will show that the nonlinearity becomes asymptotically negligible, meaning that as $t \to \pm\infty$, the solution $u(t)$ behaves like a solution to the linear (free) fractional Schrödinger equation. The corresponding linear problem for a given initial datum $u_+ \in H^s(\mathbb{R}^3)$ reads

$$\begin{cases} i\partial_t u &= (-\Delta)^s u, \\ u(0) &= u_+ \in H^s(\mathbb{R}^3), \quad u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}. \end{cases} \tag{4.1}$$

Using the Fourier transform, we see that the solution to (4.1) is determined by the propagator $e^{-it(-\Delta)^s}$, namely

$$u(t,x) = \mathcal{F}^{-1}(e^{-it|\cdot|^{2s}}\widehat{u_+}(\cdot))(x) =: e^{-it(-\Delta)^s}u_+(x).$$

Supposing the data $u_+$ is Schwartz (cf. also [OR13, p. 166]), we can write this as

$$u(t,x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^{2s})}\widehat{u_+}(\xi)\,\mathrm{d}\xi.$$

Let us introduce some terminology to state our problem and result. We formulate everything for the asymptotics $t \to +\infty$ (the case $t \to -\infty$ is analogous thanks to time reversal symmetry).

## Scattering States and Wave Operators

**Definition** (cf. [Tao06, p. 163])**.** Let $u \in C(\mathbb{R}_t; H^s_x(\mathbb{R}^3))$ be the global solution of defocusing (fNLS) with initial condition $u_0 \in H^s_x(\mathbb{R}^3)$. We say that $u$ scatters in $H^s_x(\mathbb{R}^3)$ to the solution $e^{-it(-\Delta)^s}u_+$ of the linear equation (4.1) (with initial condition $u_+ \in H^s_x(\mathbb{R}^3)$) as $t \to +\infty$ provided that

$$\|u(t) - e^{-it(-\Delta)^s}u_+\|_{H^s} = \|e^{it(-\Delta)^s}u(t) - u_+\|_{H^s} \to 0, \qquad \text{as } t \to +\infty. \tag{4.2}$$

In this case, we call $u_+$ the **scattering state of** $u_0$ **at** $+\infty$ (cf. [Caz03, p. 211]).[1]

Equality in (4.2) holds because $e^{-it(-\Delta)^s}$ is a unitary operator on $H^s_x(\mathbb{R}^3)$.[2] Indeed, one can shift the multiplier on the Fourier side and obtain

$$\begin{aligned}\langle e^{-it(-\Delta)^s}u, v\rangle_{H^s} &= \int_{\mathbb{R}^n} \overline{\mathcal{F}(e^{-it(-\Delta)^s}u)(\xi)}\mathcal{F}v(\xi)(1+|\xi|^2)^s\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^n} \overline{\mathcal{F}u(\xi)}e^{it|\xi|^{2s}}\mathcal{F}v(\xi)(1+|\xi|^2)^s\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^n} \overline{\mathcal{F}u(\xi)}\mathcal{F}(e^{it(-\Delta)^s}v)(\xi)(1+|\xi|^2)^s\,\mathrm{d}\xi\\ &= \langle u, e^{+it(-\Delta)^s}v\rangle_{H^s},\end{aligned}$$

so that $(e^{-it(-\Delta)^s})^* = e^{+it(-\Delta)^s}$, but $e^{+it(-\Delta)^s} = (e^{-it(-\Delta)^s})^{-1}$. In particular, $e^{-it(-\Delta)^s}$ defines an isometry on $H^s_x(\mathbb{R}^3)$.

By (4.2), a scattering state $u_+ \in H^s_x(\mathbb{R}^3)$ of $u_0 \in H^s_x(\mathbb{R}^3)$ at $+\infty$ is necessarily unique (hence we speak of the scattering state). However, it is not clear whether

[1]Compare this to the notion in [RS79, p. 3].

[2]In fact, $\{e^{-it(-\Delta)^s}\}_{t\in\mathbb{R}}$ is a continuous one-parameter group on $H^s_x(\mathbb{R}^3)$, in particular (by continuity), given $\varphi \in H^s_x(\mathbb{R}^3)$, $u$ defined by $u(t) := e^{-it(-\Delta)^s}\varphi$ satisfies $u \in C(\mathbb{R}; H^s_x(\mathbb{R}^3))$.

to a given initial state $u_0$ there *exists* an associated scattering state $u_+$ in the first place. In other words, it is not clear whether $u_0$ belongs to the set

$$\mathcal{R}_+ := \{u_0 \in H^s_x(\mathbb{R}^3);\ \exists \text{ scattering state } u_+ \in H^s_x(\mathbb{R}^3) \text{ of } u_0 \text{ at } +\infty\ \}, \tag{4.3}$$

to which we associate the (by uniqueness of scattering states well-defined) mapping

$$U_+ :\ \mathcal{R}_+ \to H^s_x(\mathbb{R}^3), \quad u_0 \mapsto U_+u_0 := u_+. \tag{4.4}$$

Moreover, even if to given initial state $u_0$ there exists an associated scattering state $u_+$ (that is, $u_0 \in \mathcal{R}_+$ and $U_+u_0 = u_+$), it is not clear whether $u_0$ is *unique* among all initial states having the scattering state $u_+$; there may be more than one initial datum $u_0$ such that the corresponding solution $u$ to (fNLS) scatters to $e^{-it(-\Delta)^s}u_+$ as $t \to \infty$. In other words, it is not clear whether $U_+$ given by (4.4) is *injective*. [This issue is different from the previous assertion that to given initial datum $u_0$, there can be at most one $u_+$ such that the nonlinear solution $u$ scatters to $e^{-it(-\Delta)^s}u_+$.] If this was the case however, one may invert $U_+$ on the image $U_+(\mathcal{R}_+)$, leading to the following definition.

**Definition** (Wave Operator)**.** If for each $u_+ \in U_+(\mathcal{R}_+)$ there exists a unique initial datum $u_0 \in \mathcal{R}_+$ such that the corresponding solution of (fNLS) scatters to $e^{-it(-\Delta)^s}u_+$ as $t \to \infty$ (in other words, if each state $u_+ \in U_+(\mathcal{R}_+)$ is the scattering state of one and only one $u_0 \in \mathcal{R}_+$ at $+\infty$), we define the **wave operator**:

$$\Omega_+ = U_+^{-1} : U_+(\mathcal{R}_+) \to \mathcal{R}_+ \subset H^s_x(\mathbb{R}^n), \quad u_+ \mapsto \Omega_+u_+ := u_0, \tag{4.5}$$

where $u_0 \in \mathcal{R}_+$ is the unique initial datum such that the corresponding solution $u$ of (fNLS) satisfies (4.2).

Scattering theory is concerned with two fundamental tasks:

1. Proof of *existence of the wave operator* $\Omega_+$ (i.e., injectivity of the map $U_+$). In case of existence, the wave operator $\Omega_+ : U_+(\mathcal{R}_+) \to H^s_x(\mathbb{R}^n)$ is assured to be injective by the well-posedness theory.
2. Proof of surjectivity (hence bijectivity) of the wave operator. This means that $\mathcal{R}_+ = H^s_x(\mathbb{R}^n)$, thus every $u_0 \in H^s_x(\mathbb{R}^n)$ is also in $\mathcal{R}_+$, hence for any initial condition $u_0$ the associated global solution $u$ to (fNLS) scatters in $H^s_x(\mathbb{R}^n)$. This property is called *asymptotic completeness*.

## Our Theorem: Scattering on $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$

We shall solve the problem above in the radial subclass of $H^s_x(\mathbb{R}^3)$. More precisely, we prove the existence of the wave operator $\Omega_+$ on the subspace $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ of radial $H^s_x(\mathbb{R}^3)$ functions and show that it is a continuous bijection $H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$.

**Theorem 4.1** (Scattering on $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$)**.** *Let $n = 3$ and $\sigma = 1$. Let $s \in [s_0, 1)$, where $s_0 = \frac{1}{4}(7-\sqrt{13}) \approx 0.849$. Then for every $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a unique $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that the global solution $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ of defocusing* (fNLS) *with initial value $u_0$ satisfies*

$$\|u(t) - e^{-it(-\Delta)^s}u_+\|_{H^s} = \|e^{it(-\Delta)^s}u(t) - u_+\|_{H^s} \to 0, \qquad \text{as } t \to +\infty. \tag{4.6}$$

*Consequently, there exists an operator $\Omega_+ : H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$, $u_+ \mapsto u_0$. Furthermore, $\Omega_+$ is a continuous bijection $H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$; in particular, conversely for every $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a unique $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that the global solution $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ of defocusing* (fNLS) *with initial value $u_0$ satisfies* (4.6).

The further restriction of $s > \frac{3}{4}$ to $s \geqslant s_0$ with $s_0$ as above is for technical reasons, in order to avoid a continuity argument in the proof of the strong space-time bound below; see also the proof of the weak space-time bound below.

Due to the existence of solitary wave solutions for the focusing problem, which do not scatter to a linear solution, the question of asymptotic completeness is only relevant in the defocusing case. [Solitons represent a perfect balance between the focusing forces of the nonlinearity and the dispersive forces of the linear component [HR08].] The answer relies on the validity of certain decay estimates for the solution, giving a decay in $L^p_x$ spaces as $t \to +\infty$. This is in the spirit of Morawetz-Lin-Strauss estimates; the Morawetz inequality (see Proposition C.6) expresses the decay in a time-averaged sense. We will always use the mixed space-time Lebesgue and Sobolev norms $\|\cdot\|_{L^q_t L^r_x(I\times\mathbb{R}^3)}$, $\|\cdot\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^3)}$ for certain admissible pairs $(q, r)$.

Let us mention that in the focusing case, Guo and Zhu [GZ17] have obtained the sharp threshold of scattering for the general power-type fractional NLS in the $L^2_x$ supercritical and $H^s_x$ subcritical range, in the sense that if $0 < s_c < s$, and

$$\begin{cases} E[u_0]^{s_c}M[u_0]^{s-s_c} < E[Q]^{s_c}M[Q]^{s-s_c}, \\ \|(-\Delta)^{\frac{s}{2}}u_0\|^{s_c}_{L^2}\|u_0\|^{s-s_c}_{L^2} < \|(-\Delta)^{\frac{s}{2}}Q\|^{s_c}_{L^2}\|Q\|^{s-s_c}_{L^2}, \end{cases}$$

then the radial solution $u(t)$ is global and scatters in $H^s_x$, while if

$$\begin{cases} E[u_0]^{s_c}M[u_0]^{s-s_c} < E[Q]^{s_c}M[Q]^{s-s_c}, \\ \|(-\Delta)^{\frac{s}{2}}u_0\|^{s_c}_{L^2}\|u_0\|^{s-s_c}_{L^2} > \|(-\Delta)^{\frac{s}{2}}Q\|^{s_c}_{L^2}\|Q\|^{s-s_c}_{L^2}, \end{cases}$$

we know from chapter 2 that $u(t)$ blows up in finite time. See also the work of Sun, Wang, Yao and Zheng [SWYZ17] who also show the scattering below this threshold. Building on the fractional virial identities developed in the blowup chapter, they are able to use a method of Dodson and Murphy [DM17] in order to fulfill a sufficient scattering criterion due to Tao [Tao04, Theorem 1.1]. They also prove scattering for the defocusing problem based on the virial identity.

### Outline of Chapter 4

The strategy presented here is an elaboration of the arguments in the book of Tao [Tao06]. In section 4.2, we provide a weak space-time bound, which follows from the radial Sobolev inequality in conjunction with the decay estimates. The key to the latter is provided by the Morawetz inequality, proved in Proposition C.6 of Appendix C, which indeed gives decay of the solution to defocusing (fNLS) in a time-averaged sense (the solution cannot concentrate at the origin for a long period of time [Tao06]). The weak space-time bound implies a strong space-time bound in an appropriate Strichartz norm, which in turn will be sufficient to get asymptotic completeness.

Relying on a backwards-in-time fixed-point method, we prove the existence of the wave operator $\Omega_+$ in section 4.3, and show its continuity. The combination of the fixed-point scheme with Strichartz estimates is responsible for the argument being reminiscent of techniques used in the local existence theory. After the construction of $\Omega_+$, we turn to the asymptotic completeness, and thus finish the proof of Theorem 4.1. Finally, in section 4.4 we construct the inverse $U_+ = \Omega_+^{-1}$.

We refer to Appendix C for the definition of the relevant Strichartz spaces, the Strichartz estimates and the Morawetz estimate.

## 4.2 Weak and Strong Space-Time Bounds

**Lemma 4.2** (Weak space-time bound)**.** *Let $n = 3$ and $\sigma = 1$. Let $s \in [s_0, 1)$, where $s_0 = \frac{1}{4}(7 - \sqrt{13}) \approx 0.849$. Then for every radial initial datum $u_0 \in H^s_{x,rad}(\mathbb{R}^3)$ the associated global solution $u \in C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ to defocusing (fNLS) obeys the weak space-time bound*

$$\|u\|^{\alpha}_{L^{\alpha}_{t,x}} = \int_{-\infty}^{+\infty} \|u(t)\|^{\alpha}_{L^{\alpha}_x} \,\mathrm{d}t < +\infty, \qquad \textit{where } \alpha := \tfrac{3+2s}{s}. \tag{4.7}$$

*Proof.* This is a combination of the radial Sobolev inequality and Morawetz's inequality. Indeed, for any $\widetilde{s} \in \left(\frac{1}{2}, \frac{3}{2}\right) \cap [0, s]$, Proposition A.4 gives

$$\begin{aligned} \||\cdot|^{\frac{3}{2}-\widetilde{s}} u(t,\cdot)\|_{L^\infty_x} &\lesssim \|(-\Delta)^{\frac{\widetilde{s}}{2}} u(t)\|_{L^2_x} \lesssim \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2_x} \\ &\leqslant \sqrt{2\left(\frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2_x} + \frac{1}{4}\|u(t)\|^4_{L^4_x}\right)} \\ &= \sqrt{2E[u(t)]} = \sqrt{2E[u_0]}, \end{aligned} \tag{4.8}$$

where we used $\widetilde{s} \leqslant s$ and then the defocusing sign and conservation of energy. (4.8) bounds the quantity $\||\cdot|^{\frac{3}{2}-\widetilde{s}} u(t,\cdot)\|_{L^\infty_x}$ uniformly in time $t$. Writing $\alpha = 4 + \frac{2}{3-2\widetilde{s}}$

with $\alpha = \frac{3+2s}{s}$, it follows that

$$
\begin{aligned}
\|u\|_{L^\alpha_{t,x}}^\alpha &= \int_{-\infty}^{+\infty}\int_{\mathbb{R}^3} |u(t,x)|^{4+\frac{2}{3-2\tilde{s}}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty}\int_{\mathbb{R}^3} \frac{|u(t,x)|^4}{|x|}\left(|x|^{\frac{3}{2}-\tilde{s}}|u(t,x)|\right)^{\frac{2}{3-2\tilde{s}}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_{-\infty}^{+\infty}\left\|\frac{|u(t,\cdot)|^4}{|\cdot|}\right\|_{L^1_x}\left\||\cdot|^{\frac{3}{2}-\tilde{s}}u(t,\cdot)\right\|_{L^\infty_x}^{\frac{2}{3-2\tilde{s}}}\,\mathrm{d}t \\
&\overset{(4.8)}{\lesssim_{s,E[u_0]}} \int_{-\infty}^{+\infty}\left\|\frac{|u(t,\cdot)|^4}{|\cdot|}\right\|_{L^1_x}\,\mathrm{d}t \lesssim_{s,\|u_0\|_{L^2_x},E[u_0]} 1,
\end{aligned}
\tag{4.9}
$$

where we used Hölder's inequality in space $x$ and finally Proposition C.6.

However, this argument used that

$$
\tilde{s} = \frac{3}{2} - \frac{1}{\alpha-4} = \frac{3}{2} - \frac{s}{3-2s} \in \left(\frac{1}{2},\frac{3}{2}\right) \cap [0,s]\,.
$$

The first condition $\tilde{s} \in \left(\frac{1}{2},\frac{3}{2}\right)$ is true for all $\frac{3}{4} < s < 1$, while the second condition $\tilde{s} \leqslant s$ reads

$$
\left(s - \frac{1}{4}(7-\sqrt{13})\right)\left(s - \frac{1}{4}(7+\sqrt{13})\right) \leqslant 0,
$$

which is true if and only if $\frac{1}{4}(7-\sqrt{13}) \leqslant s \leqslant \frac{1}{4}(7+\sqrt{13})$. This explains the restriction $s_0 \leqslant s < 1$; see also Figure 4.1. ☐

**Lemma 4.3** (Strong space-time bound). *Given the hypotheses of Lemma 4.2, the following strong space-time bound holds:*

$$
\|u\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \lesssim 1. \tag{4.10}
$$

*Proof.* **Step 1: Partitioning the time axis.** Let $\varepsilon = \varepsilon(M[u_0], E[u_0]) > 0$ to be fixed later. We claim there exists some partition $\mathbb{R} = \cup_{j=1}^N I_j$ of finitely many ($N \in \mathbb{N}$) intervals $I_j \subset \mathbb{R}$ ($I_j \cap I_k = \varnothing$ for $j \neq k$) such that

$$
\|u\|_{L^\alpha_{t,x}(I_j\times\mathbb{R}^3)}^\alpha \leqslant \varepsilon, \quad j = 1,\dots,N. \tag{4.11}
$$

This is a direct consequence of the monotone convergence theorem (see e.g., [SS05, p. 65] and also [Alt06, p. 88, Lemma 1.17, ⟨2⟩]). That is, given $\varepsilon > 0$, by the weak space-time bound (4.7) there exists a set of finite measure - for example a ball $(-R,R)$ for $R > 0$ sufficiently big - such that

$$
\int_{-\infty}^{-R} \|u(t)\|_{L^\alpha_x}^\alpha\,\mathrm{d}t \leqslant \varepsilon, \qquad \int_{R}^{+\infty} \|u(t)\|_{L^\alpha_x}^\alpha\,\mathrm{d}t \leqslant \varepsilon.
$$

Moreover, by (4.7) there exists $\delta > 0$ such that $\int_E \|u(t)\|_{L^\alpha_x}^\alpha\,\mathrm{d}t \leqslant \varepsilon$ whenenver $|E| < \delta$. Therefore, let us partition the remaining set $(-R,R)$ by a finite number of

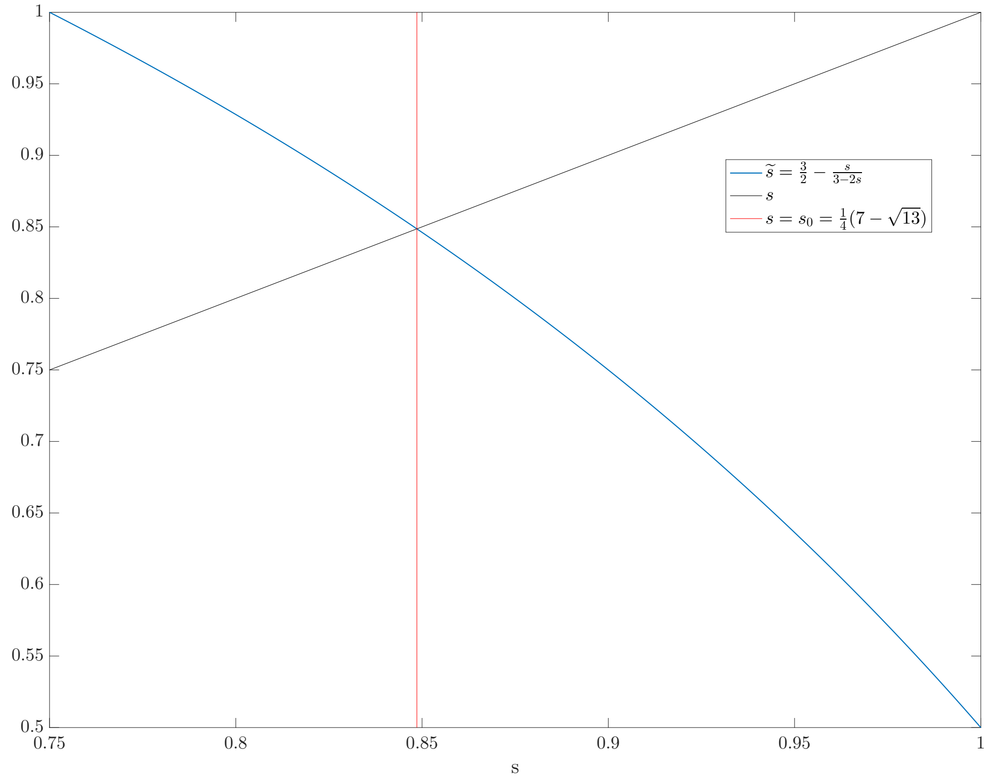


Figure 4.1: Restriction $s \geqslant s_0$ for the weak space-time bound

disjoint intervals $I_j$, $j = 1, \dots, N$, such that $|I_j| < \delta$ for all $j$. The collection $(I_1, \dots, I_N, (-\infty, R], [R, \infty))$ gives the required partition.

**Step 2: Estimating the nonlinearity in the dual Strichartz space.** Now fix one of the intervals from the preceding step, w.l.o.g.[3] $I = [t_0, t_1]$. Since $u \in C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ solves (fNLS) with initial condition $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ at time 0, by global well-posedness it also solves (fNLS) with initial condition $u(t_0) \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ at time $t_0$. By Duhamel's principle

$$u(t) = e^{-i(t-t_0)(-\Delta)^s} u(t_0) - i \int_{t_0}^{t} e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau. \tag{4.12}$$

We estimate the Strichartz norm $\|u\|_{S^s(I\times\mathbb{R}^3)} = \|u\|_{S^0(I\times\mathbb{R}^3)} + \||\nabla|^s u\|_{S^0(I\times\mathbb{R}^3)}$ as follows. For any $(q, r) \in SA$ we have by the homogeneous Strichartz estimate (C.8)

$$\left\|e^{-i(t-t_0)(-\Delta)^s} u(t_0)\right\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} \overset{\text{(C.8)}}{\lesssim} \|u(t_0)\|_{L^2_x}.$$

Since the fractional derivative $|\nabla|^s$ commutes[4] with the semigroup $\{e^{-it(-\Delta)^s}\}_{t\in\mathbb{R}}$,

$$\left\||\nabla|^s e^{-i(t-t_0)(-\Delta)^s} u(t_0)\right\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} \overset{\text{(C.8)}}{\lesssim} \||\nabla|^s u(t_0)\|_{L^2_x}.$$

Thus

$$\left\|e^{-i(t-t_0)(-\Delta)^s} u(t_0)\right\|_{S^s(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{L^2_x} + \||\nabla|^s u(t_0)\|_{L^2_x} \lesssim \|u(t_0)\|_{H^s}. \tag{4.13}$$

Similarly, by the inhomogeneous Strichartz estimates (see [Tao06, estimate (3.26) on p. 135])

$$\left\|\int_{t_0}^{t} e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau\right\|_{S^s(I\times\mathbb{R}^3)} \lesssim \|F(u)\|_{N^s(I\times\mathbb{R}^3)}. \tag{4.14}$$

Consequently, from (4.12), (4.13) and (4.14) there results a *unified Strichartz estimate*

$$\|u\|_{S^s(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{H^s} + \|F(u)\|_{N^s(I\times\mathbb{R}^3)}. \tag{4.15}$$

By construction (see (C.13)), the $N^s$ norm is controlled by $L^{q'}_t W^{s,r'}_x$ norm, for any $(q, r) \in SA$. Notice that $(q, q) \in SA$ implies $q = \frac{2(3+2s)}{3}$, equivalently $q' = \frac{2(3+2s)}{3+4s}$. Note $2 < q < \infty$, equivalently $1 < q' < 2$. As $F(u) = |u|^2 u$, we have

$$\left\||u|^2 u\right\|_{N^s(I\times\mathbb{R}^3)} = \sum_{k\in\{0,s\}} \left\||\nabla|^k (|u|^2 u)\right\|_{N^0(I\times\mathbb{R}^3)} \leqslant \sum_{k\in\{0,s\}} \left\||\nabla|^k (|u|^2 u)\right\|_{L^{q'}_{t,x}(I\times\mathbb{R}^3)}. \tag{4.16}$$

[3] If $I$ is an interval of the form $(-\infty, t_0]$, $(-\infty, t_0)$, $[t_0, \infty)$, $(t_0, \infty)$ or an interval of the form $(t_0, t_1)$, $[t_0, t_1)$, $(t_0, t_1]$ with $t_0, t_1 \in \mathbb{R}$, this does not change the following argument.

[4] See also [Tao06, p. 75].

The elementary formula

$$\left\||f|^\beta\right\|_{L^p_x(\Omega)} = \left(\int_\Omega |f(x)|^{\beta p}\,\mathrm{d}x\right)^{\frac{1}{p}} = \|f\|^\beta_{L^{\beta p}_x(\Omega)}$$

gives

$$\left\||f|^\beta\right\|_{L^p_{t,x}(I\times\Omega)} = \left(\int_I \left\||f(t)|^\beta\right\|^p_{L^p_x(\Omega)}\,\mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_I \|f(t)\|^{\beta p}_{L^{\beta p}_x(\Omega)}\,\mathrm{d}t\right)^{\frac{1}{p}} = \|f\|^\beta_{L^{\beta p}_{t,x}(I\times\Omega)}.$$

Define $p$ by $1/p = 1/q' - 1/q$. Note that from $1 < q' < 2 < q < \infty$ we have $1 < p < \infty$; in fact $p = \left(\frac{1}{q'} - \frac{1}{q}\right)^{-1} = \frac{3+2s}{2s}$. By Hölder, first in space $x$, then in time $t$, we check

$$\begin{aligned}
\left\||u|^2 u\right\|_{L^{q'}_{t,x}(I\times\mathbb{R}^3)} &= \left(\int_I \||u(t)|^2 u(t)\|^{q'}_{L^{q'}_x}\,\mathrm{d}t\right)^{\frac{1}{q'}} \\
&\leqslant \left(\int_I \left(\||u(t)|^2\|_{L^p_x}\|u(t)\|_{L^q_x}\right)^{q'}\,\mathrm{d}t\right)^{\frac{1}{q'}} \\
&\leqslant \left(\int_I \||u(t)|^2\|^p_{L^p_x}\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_I \|u(t)\|^q_{L^q_x}\,\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \left\||u|^2\right\|_{L^p_{t,x}(I\times\mathbb{R}^3)} \|u\|_{L^q_{t,x}(I\times\mathbb{R}^3)} \\
&= \|u\|^2_{L^{2p}_{t,x}(I\times\mathbb{R}^3)} \|u\|_{L^q_{t,x}(I\times\mathbb{R}^3)}.
\end{aligned} \tag{4.17}$$

This gives the estimate for the first ($k = 0$) summand in (4.16). To estimate the second ($k = s$) summand, we use the fractional chain rule,[5] namely

**Lemma 4.4** (Fractional Chain Rule; see [GW11, p. 33])**.** *Let $F \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then*

$$\||\nabla|^s F(u)\|_{L^r_x} \lesssim \|F'(u)\|_{L^{r_1}_x} \||\nabla|^s u\|_{L^{r_2}_x}.$$

By applying the fractional chain rule with $\frac{1}{q'} = \frac{1}{p} + \frac{1}{q}$, where $q = \frac{2(3+2s)}{3}$, $p = \frac{3+2s}{2s}$, $q' = \frac{2(3+2s)}{3+4s}$ are the numbers above and by using $\|F'(u)\| \lesssim |u|^2$, we get the estimate

$$\begin{aligned}
\left\||\nabla|^s(|u|^2 u)\right\|_{L^{q'}_{t,x}(I\times\mathbb{R}^3)} &= \left(\int_I \||\nabla|^s(|u(t)|^2 u(t))\|^{q'}_{L^{q'}_x}\,\mathrm{d}t\right)^{\frac{1}{q'}} \\
&\lesssim \left(\int_I \left(\||u(t)|^2\|_{L^p_x}\||\nabla|^s u(t)\|_{L^q_x}\right)^{q'}\,\mathrm{d}t\right)^{\frac{1}{q'}} \\
&\leqslant \left(\int_I \||u(t)|^2\|^p_{L^p_x}\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_I \||\nabla|^s u(t)\|^q_{L^q_x}\,\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \left\||u|^2\right\|_{L^p_{t,x}(I\times\mathbb{R}^3)} \||\nabla|^s u\|_{L^q_{t,x}(I\times\mathbb{R}^3)} \\
&= \|u\|^2_{L^{2p}_{t,x}(I\times\mathbb{R}^3)} \||\nabla|^s u\|_{L^q_{t,x}(I\times\mathbb{R}^3)}.
\end{aligned} \tag{4.18}$$

[5] See also [CW91, p. 91] for the original and proof.

Inserting (4.18) and (4.17) back into (4.16) yields

$$\begin{aligned}\left\||u|^2u\right\|_{N^s(I\times\mathbb{R}^3)} &\lesssim \|u\|^2_{L^{2p}_{t,x}(I\times\mathbb{R}^3)}\left(\|u\|_{L^q_{t,x}(I\times\mathbb{R}^3)} + \||\nabla|^s u\|_{L^q_{t,x}(I\times\mathbb{R}^3)}\right)\\ &\lesssim \|u\|^2_{L^{2p}_{t,x}(I\times\mathbb{R}^3)}\|u\|_{S^s(I\times\mathbb{R}^3)},\end{aligned} \tag{4.19}$$

where the last inequality follows from the admissibility $(q, q) \in SA$. Then, putting (4.19) into (4.15), we obtain

$$\|u\|_{S^s(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{H^s} + \|u\|^2_{L^{2p}_{t,x}(I\times\mathbb{R}^3)}\|u\|_{S^s(I\times\mathbb{R}^3)}. \tag{4.20}$$

However, since precisely $2p = \alpha$, in view of (4.11) this reads

$$\|u\|_{S^s(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{H^s} + \varepsilon^{\frac{2}{\alpha}}\|u\|_{S^s(I\times\mathbb{R}^3)}.$$

Choosing $\varepsilon > 0$ sufficiently small, we conclude the proof of Lemma 4.3. □

## 4.3 The Wave Operator $\Omega_+$ and its Continuity

### 4.3.1 Existence of the Wave Operator $\Omega_+$

Applying $e^{it(-\Delta)^s}$ to $u(t)$ represented by Duhamel's principle (C.5) gives

$$e^{it(-\Delta)^s}u(t) = u_0 - i\int_0^t e^{i\tau(-\Delta)^s}F(u(\tau))\,\mathrm{d}\tau. \tag{4.21}$$

Then by definition (4.2), the solution $u(t)$ of (fNLS) with initial condition $u_0 \in H^s_x(\mathbb{R}^3)$ scatters in $H^s_x(\mathbb{R}^3)$ to the solution $e^{-it(-\Delta)^s}u_+$ of the free equation (4.1) (with initial condition $u_+ \in H^s_x(\mathbb{R}^3)$) if and only if

$$\|(u_0 - u_+) - i\int_0^t e^{i\tau(-\Delta)^s}F(u(\tau))\,\mathrm{d}\tau\|_{H^s} \to 0, \qquad \text{as } t \to \infty,$$

i.e., if and only if the improper integral $i\int_0^t e^{i\tau(-\Delta)^s}F(u(\tau))\,\mathrm{d}\tau$ is convergent in $H^s_x(\mathbb{R}^3)$ as $t \to \infty$ with the $H^s_x(\mathbb{R}^3)$ limit $u_0 - u_+$:

$$i\lim_{t\to\infty}\int_0^t e^{i\tau(-\Delta)^s}F(u(\tau))\,\mathrm{d}\tau = u_0 - u_+, \qquad \text{in the } H^s_x(\mathbb{R}^3) \text{ sense.}$$

The *scattering state* can thus be written

$$\boxed{u_+ = u_0 - i\int_0^\infty e^{i\tau(-\Delta)^s}F(u(\tau))\,\mathrm{d}\tau.} \tag{4.22}$$

Provided the mentioned improper integral converges in $H^s_x(\mathbb{R}^3)$, let us define $u_+ \in H^s_x(\mathbb{R}^3)$ by the formula (4.22). One can view the solution $u(t)$ as the backwards-in-time evolution of the scattering state $u_+$ imposed as an initial condition at infinite time $t = +\infty$: indeed, subtracting (4.22) from (4.21) in order to eliminate $u_0$ and thereafter applying the linear propagator $e^{-it(-\Delta)^s}$ yields

$$\begin{aligned} u(t) &= e^{-it(-\Delta)^s}u_+ + i\int_t^\infty e^{-i(t-\tau)(-\Delta)^s}F(u(\tau))d\tau. \\ &= e^{-it(-\Delta)^s}u_+ - i\int_\infty^t e^{-i(t-\tau)(-\Delta)^s}F(u(\tau))d\tau. \end{aligned} \tag{4.23}$$

[Indeed, this can be interpreted as the Duhamel representation of the solution $u(t)$ to the (fNLS) problem with intial value $u(+\infty) = u_+$ at $t = +\infty$.]

Let us define an operator $\Gamma_{u_+}$ by

$$\Gamma_{u_+}u(t) := e^{-it(-\Delta)^s}u_+ + i\underbrace{\int_t^\infty e^{-i(t-\tau)(-\Delta)^s}F(u(\tau))d\tau}_{=:\Phi_{F(u)}(t)}. \tag{4.24}$$

Our goal is to show that this operator can be defined on a suitable complete metric space $(X, d)$, mapping $X$ into itself and having the contraction property. The unique fixed point $u \in X$ is then by construction a solution of this backwards-in-time evolution problem.[6]

### Solving the Asymptotic Problem 'From $t = +\infty$ to Some Finite $T$'

Let us use (4.23) instead of the usual Duhamel formula (C.5). Fix some state $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ with the a priori estimate $\|u_+\|_{H^s} \leqslant A$ for some $A > 0$. Similarly as before, one has a unified Strichartz estimate (4.15) and concludes with continuity arguments as before that[7]

$$\left\|e^{-it(-\Delta)^s}u_+\right\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \lesssim_A 1. \tag{4.25}$$

Hence the linear term in (4.24) is bounded with respect to $\|\cdot\|_{S^s(\mathbb{R}\times\mathbb{R}^3)}$ norm. We want to make this norm not only bounded, but small, by restricting time $t$. However, $S^s$ norm contains type $L^\infty_t$ components, which do not necessarily shrink when restricting time to some interval $[T, \infty)$; thus we must proceed more carefully (cf. [Tao06, p. 163]). Namely, let us introduce the following smaller controlling norm

$$\begin{aligned} \|u\|_{\mathfrak{S}^s(I\times\mathbb{R}^3)} &:= \|u\|_{L^{2p}_{t,x}(I\times\mathbb{R}^3)} + \|u\|_{L_t^{\frac{2(3+2s)}{3}} W_x^{s,\frac{2(3+2s)}{3}}(I\times\mathbb{R}^3)} \\ &\leqslant \|u\|_{L^{2p}_{t,x}(I\times\mathbb{R}^3)} + \|u\|_{S^s(I\times\mathbb{R}^3)}, \quad p = \frac{3+2s}{2s}, \end{aligned} \tag{4.26}$$

[6] In other words, $u \in X$ solves the problem

$$\begin{cases} iw_t & = (-\Delta)^s w + F(w), \\ \text{"}w(+\infty)\text{"} & = u_+ \in H^s_x(\mathbb{R}^3). \end{cases}$$

[7] Indeed, $v(t) := e^{-it(-\Delta)^s}u_+$ solves the free equation (4.1), i.e. $F \equiv 0$, with initial value $u_+$ at time 0, thus by unified Strichartz (4.15) one has $\|v\|_{S^s([t_0,t_1]\times\mathbb{R}^3)} \lesssim \|v(t_0)\|_{H^s} = \|u_+\|_{H^s}$.

respecting the admissibility $(q,q) \in SA$, for $q = \frac{2(3+2s)}{3}$ in the last line. The requirement $(2p, \bar{r}) \in SA$[8] implies $\bar{r} = \bar{r}(p,s) = \frac{3\cdot 2p}{3p-2s} \left(= \frac{3\cdot 2(3+2s)}{3\cdot(3+2s)-4s^2}\right)$. Notice that $2 < \bar{r} < +\infty$.[9]
We compute the critical Sobolev exponent $\bar{r}_s^* = \frac{3\bar{r}}{3-s\bar{r}} = \frac{3\cdot 2p}{p(3-2s)-2s}$. Sobolev's embedding states that[10]

$$\|u\|_{L_x^{\bar{r}_s^*}(\mathbb{R}^3)} \lesssim \|u\|_{W_x^{s,\bar{r}}(\mathbb{R}^3)}, \quad \text{provided that } 1 < \bar{r} < \infty \text{ and } 0 < s < \tfrac{3}{\bar{r}}. \tag{4.27}$$

We interpolate the bound (4.27) with the bound $\|u\|_{L_x^{\bar{r}}(\mathbb{R}^3)} \lesssim \|u\|_{W_x^{s,\bar{r}}(\mathbb{R}^3)}$ to obtain

$$\|u\|_{L_x^{\gamma}(\mathbb{R}^3)} \lesssim \|u\|_{W_x^{s,\bar{r}}(\mathbb{R}^3)}, \quad \forall \gamma \in [\bar{r}, \bar{r}_s^*]. \tag{4.28}$$

We use (4.28) for $\gamma = 2p$; see figure 4.2.[11] Thus

$$\begin{aligned} \left\|e^{-it(-\Delta)^s}u_+\right\|_{\mathfrak{S}^s(\mathbb{R}\times\mathbb{R}^3)} &\overset{(4.26)}{\leqslant} \left\|e^{-it(-\Delta)^s}u_+\right\|_{L_{t,x}^{2p}(\mathbb{R}\times\mathbb{R}^3)} + \left\|e^{-it(-\Delta)^s}u_+\right\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \\ &\overset{(4.28)}{\lesssim} \left\|e^{-it(-\Delta)^s}u_+\right\|_{L_t^{2p}W_x^{s,\bar{r}}(\mathbb{R}\times\mathbb{R}^3)} + \left\|e^{-it(-\Delta)^s}u_+\right\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \\ &\overset{(2p,\bar{r})\in SA}{\leqslant} 2\left\|e^{-it(-\Delta)^s}u_+\right\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \overset{(4.25)}{\lesssim_A} 1. \end{aligned} \tag{4.29}$$

Hence the linear term in (4.24) is also bounded with respect to $\|\cdot\|_{\mathfrak{S}^s(\mathbb{R}\times\mathbb{R}^3)}$ norm. By this estimate $\|e^{-it(-\Delta)^s}u_+\|_{\mathfrak{S}^s(\mathbb{R}\times\mathbb{R}^3)} \lesssim 1$, and the definition of the norm $\|\cdot\|_{\mathfrak{S}^s(I\times\mathbb{R}^3)}$ (which is a sum of Lebesgue-Sobolev space-time norms with exponents strictly $< \infty$), it follows from elementary integration theory that for given $\varepsilon > 0$, there exists $T = T(u_+, \varepsilon) > 0$ appropriately large such that

$$K_{T'}^{u_+} := \left\|e^{-it(-\Delta)^s}u_+\right\|_{\mathfrak{S}^s([T',\infty)\times\mathbb{R}^3)} \leqslant \frac{\varepsilon}{2}, \quad \forall T' \geqslant T(u_+,\varepsilon). \tag{4.30}$$

That is, $K_{T'}^{u_+} \to 0$ as $T' \to \infty$. Observe at this point that when $T$ is chosen such that (4.30) holds, then in a full $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-neighborhood $B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ of $u_+$ we still get

$$K_{T'}^{v_+} \leqslant \varepsilon, \quad \forall v_+ \in B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad \forall T' \geqslant T. \tag{4.31}$$

[8] Note that $2p > 2$.
[9] Indeed, from $p > 1$ and $3 > 2s$ we have $3p - 2s > 3 - 2s > 0$, so that $\bar{r} > 2$ is equivalent to $s > 0$.
[10] Sobolev's embedding theorem is applicable because $s < \frac{3}{\bar{r}}$ follows from

$$3 > s\bar{r} \iff 3 > s\frac{3\cdot 2p}{3p-2s} \overset{3p-2s>0}{\iff} 3p - 2s > 2ps \overset{\text{def. of } p}{\iff} 8s^2 < 9$$

and the latter is clear when $s < 1$.
[11] By definition of $\bar{r}$ and $p$, the inequality $2p \geqslant \bar{r}$ holds if and only if $1 \geqslant \frac{2s}{3}$, which is precisely the $L_x^2$ (super-)criticality condition for cubic nonlinearity in three dimensions. From $s < 1$ it's clear that this holds. On the other hand, $p(3-2s) - 2s > 0$ (since by definition of $p$, this is equivalent to $8s^2 < 9$ again, which is clear when $s < 1$) and this implies that the other inequality $2p \leqslant \bar{r}_s^*$ holds if and only if $1 \leqslant \frac{2s}{3-2s}$, which is precisely the $H_x^s$ (sub-)criticality condition for cubic nonlinearity in three dimensions. By $s > \frac{3}{4}$, it's clear that this also holds.

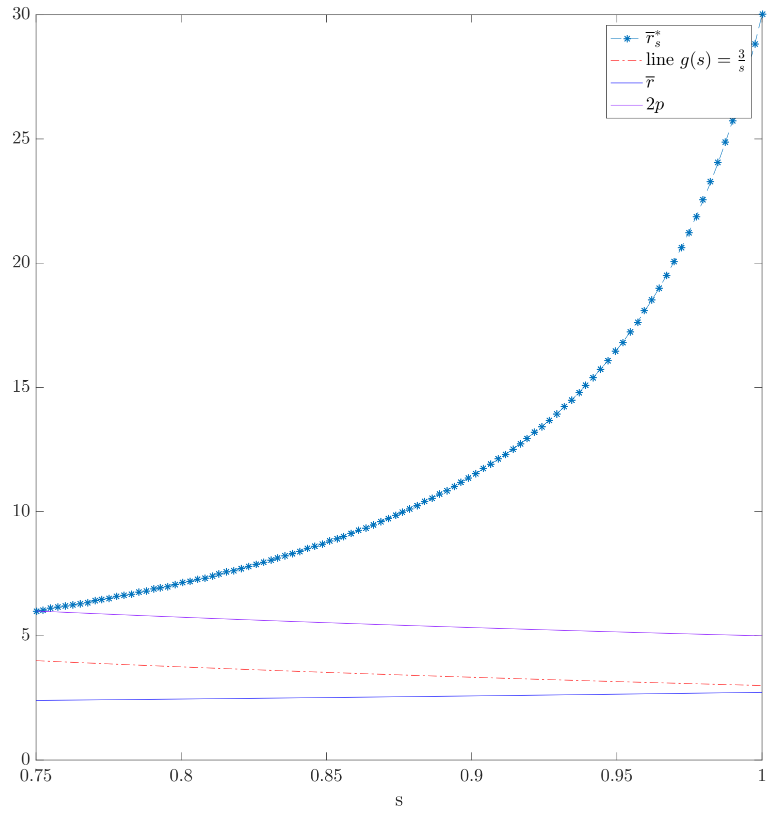


Figure 4.2: Applying Sobolev's embedding

To see this, note first that by Sobolev's embedding and Strichartz estimates arguing as in (4.29) we have for $f \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$

$$\left\|e^{-it(-\Delta)^s} f\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \leqslant 2C \left\|e^{-it(-\Delta)^s} f\right\|_{S^s([T,\infty)\times\mathbb{R}^3)}, \tag{4.32}$$

where $C > 0$ only depends on the exponents involved in Sobolev's embedding and Strichartz estimates, but not on the concrete interval $[T, \infty)$. However, the function $g(t) := e^{-it(-\Delta)^s} f$ obviously solves the linear problem

$$\begin{cases} iw_t &= (-\Delta)^s w, \\ w(T) &= e^{-iT(-\Delta)^s} f \in H^s_x(\mathbb{R}^3), \quad w : [T,\infty) \times \mathbb{R}^3 \to \mathbb{C}. \end{cases}$$

Applying the unified Strichartz estimate (with inhomogeneity $F$ identically zero) (4.15) on the right side of (4.32) and then using the unitary group property gives

$$\begin{aligned} \left\|e^{-it(-\Delta)^s} f\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} &\leqslant 2C \left\|e^{-it(-\Delta)^s} f\right\|_{S^s([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant C\|g(T)\|_{H^s} = C\|f\|_{H^s}, \quad C > 0. \end{aligned} \tag{4.33}$$

Let now $\delta = \frac{\varepsilon}{2C}$ and let $v_+ \in B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. Then

$$\begin{aligned} |K_T^{v_+} - K_T^{u_+}| &= \left| \left\|e^{-it(-\Delta)^s} v_+\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} - \left\|e^{-it(-\Delta)^s} u_+\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \right| \\ &\leqslant \left\|e^{-it(-\Delta)^s} (v_+ - u_+)\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \overset{(4.33)}{\leqslant} C\|v_+ - u_+\|_{H^s} \\ &\leqslant C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}, \end{aligned}$$

hence $K_T^{v_+} \leqslant \frac{\varepsilon}{2} + K_T^{u_+} \overset{(4.30)}{\leqslant} \varepsilon$. Since $K_T^{v_+}$ is decreasing in $T$, this completes the proof of (4.31).[12]

### A Fixed-Point Argument

Recall the fixed numbers $p = \frac{3+2s}{2s}$, $q = \frac{2(3+2s)}{3}$. The state $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ was fixed and it was our goal to show that the operator $\Gamma_{u_+}$ defines a contraction on some complete metric space. For $T > 0$, we consider the Banach spaces

$$\begin{aligned} (X_T, \|\cdot\|_{X_T}) &:= \left(L^q_t W^{s,q}_{x,\mathrm{rad}}([T,\infty) \times \mathbb{R}^3), \|\cdot\|_{L^q_t W^{s,q}_x([T,\infty)\times\mathbb{R}^3)}\right), \\ (Y_T, \|\cdot\|_{Y_T}) &:= \left(L^{2p}_{t,x}([T,\infty) \times \mathbb{R}^3), \|\cdot\|_{L^{2p}_{t,x}([T,\infty)\times\mathbb{R}^3)}\right). \end{aligned}$$

Let $\varepsilon > 0$ be a number to be fixed later. Let then $T = T(u_+, \varepsilon) > 0$ be a fixed positive number so large that (4.30) is true. Introduce

$$B_{\varepsilon,T} := \left\{ u \in X_T \cap Y_T; \quad \|u\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \leqslant 2\varepsilon \right\},$$

[12] We refer to (4.31) shortly by saying that the time $T$ can be chosen to be uniform under small $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-perturbations in $u_+$ thanks to the Strichartz estimates (cf. [Tao06, p. 165]).

equipped with the metric

$$d(u,v) := \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} .$$

Note the nesting $B_{\varepsilon,T} \subset B_{\varepsilon,\tilde{T}}$ for all $\tilde{T} \geqslant T$. Let us remark that the pair $(B_{\varepsilon,T}, d)$ defines a complete metric space.

*Proof of completeness.* According to [BL76, p. 24]), the pair $(X_T \cap Y_T, \|\cdot\|_{X_T\cap Y_T})$ with $\|u\|_{X_T\cap Y_T} := \max\{\|u\|_{X_T}, \|u\|_{Y_T}\}$ is a Banach space. Since

$$\|u\|_{X_T\cap Y_T} \leqslant \|u\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} = \|u\|_{X_T} + \|u\|_{Y_T} \leqslant 2\,\|u\|_{X_T\cap Y_T}$$

for all $u \in X_T \cap Y_T$, the norms $\|\cdot\|_{X_T\cap Y_T}$ and $\|\cdot\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)}$ are equivalent on $X_T \cap Y_T$. By continuity of the norm $\|\cdot\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} : X_T \cap Y_T \to \mathbb{R}$, we get that $B_{\varepsilon,T}$ is a closed subset of $X_T \cap Y_T$. Since $(X_T \cap Y_T, d)$ is complete, so is $(B_{\varepsilon,T}, d)$. □

Now we claim that

$$\Gamma_{u_+} : (B_{\varepsilon,T}, d) \to (B_{\varepsilon,T}, d)$$

defined by

$$\Gamma_{u_+}u(t) := e^{-it(-\Delta)^s}u_+ + i\Phi_{F(u)}(t)$$

is a contraction for $\varepsilon > 0$ chosen appropriately.

*Proof of contraction.* **Step 1 (self-mapping):** Let $u \in B_{\varepsilon,T}$ be arbitrary. Then

$$\begin{aligned}
\|\Gamma_{u_+}u\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} &\leqslant \left\|e^{-it(-\Delta)^s}u_+\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} + \left\|\Phi_{F(u)}\right\|_{X_T} + \left\|\Phi_{F(u)}\right\|_{Y_T} \\
&\leqslant K_T^{u_+} + \left\|\Phi_{F(u)}\right\|_{X_T} + \left\|\Phi_{F(u)}\right\|_{Y_T} \\
&\leqslant \frac{\varepsilon}{2} + \left\|\Phi_{F(u)}\right\|_{X_T} + \left\|\Phi_{F(u)}\right\|_{Y_T} .
\end{aligned}$$

But using Strichartz and then Hölder combined with the fractional chain rule as in (4.18), we see

$$\begin{aligned}
\left\|\Phi_{F(u)}\right\|_{X_T} &= \left\|\Phi_{F(u)}\right\|_{L_t^q W_x^{s,q}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C\,\|F(u)\|_{L_t^{q'} W_x^{s,q'}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C\,\|u\|^2_{L_{t,x}^{2p}([T,\infty)\times\mathbb{R}^3)}\,\|u\|_{L_t^q W_x^{s,q}([T,\infty)\times\mathbb{R}^3)} = C\,\|u\|^2_{Y_T}\,\|u\|_{X_T} \\
&\leqslant C\,\|u\|^3_{X_T\cap Y_T} \leqslant C\,\|u\|^3_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C(2\varepsilon)^3.
\end{aligned}$$

Similarly, using Sobolev embedding, then Strichartz and then estimating as before,

$$\begin{aligned}
\left\|\Phi_{F(u)}\right\|_{Y_T} &= \left\|\Phi_{F(u)}\right\|_{L_{t,x}^{2p}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C\left\|\Phi_{F(u)}\right\|_{L_t^{2p} W_x^{s,\bar{r}}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C\,\|F(u)\|_{L_t^{q'} W_x^{s,q'}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C(2\varepsilon)^3.
\end{aligned}$$

Thus, choosing a posteriori $\varepsilon > 0$ small enough to ensure $\boxed{2C(2\varepsilon)^3 \leqslant \varepsilon}$ (and then $T = T(u_+, \varepsilon) > 0$ as to guarantee (4.30)), we get $\left\|\Gamma_{u_+} u\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \leqslant \frac{3}{2}\varepsilon$, which is less than $2\varepsilon$. Hence the operator $\Gamma_{u_+}$ maps the set $B_{\varepsilon,T}$ to itself.

**Step 2 (contraction property):** Let $u, v \in B_{\varepsilon,T}$ be arbitrary. Then, using Strichartz, the fractional chain rule and the pointwise bound from Lemma C.7 for the nonlinearity, we get

$$\begin{aligned}
\left\|\Gamma_{u_+} u - \Gamma_{u_+} v\right\|_{X_T} &= \left\|\Phi_{F(u)} - \Phi_{F(v)}\right\|_{X_T} = \left\|\Phi_{F(u)-F(v)}\right\|_{X_T} \\
&= \left\|\Phi_{F(u)-F(v)}\right\|_{L_t^q W_x^{s,q}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C \left\|F(u) - F(v)\right\|_{L_t^{q'} W_x^{s,q'}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C \left\{ \|u\|^2_{L^{2p}_{t,x}([T,\infty)\times\mathbb{R}^3)} + \|v\|^2_{L^{2p}_{t,x}([T,\infty)\times\mathbb{R}^3)} \right\} \|u - v\|_{L_t^q W_x^{s,q}([T,\infty)\times\mathbb{R}^3)} \\
&= C \left\{ \|u\|^2_{Y_T} + \|v\|^2_{Y_T} \right\} \|u - v\|_{X_T} \\
&\leqslant C \left\{ \|u\|^2_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} + \|v\|^2_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \right\} \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant 2C(2\varepsilon)^2 \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)},
\end{aligned}$$

where we used $u, v \in B_{\varepsilon,T}$ in the last estimate. Similarly, using Sobolev, Strichartz and then proceeding as before, we get

$$\begin{aligned}
\left\|\Gamma_{u_+} u - \Gamma_{u_+} v\right\|_{Y_T} &= \left\|\Phi_{F(u)} - \Phi_{F(v)}\right\|_{Y_T} = \left\|\Phi_{F(u)-F(v)}\right\|_{Y_T} \\
&\leqslant C \left\|\Phi_{F(u)-F(v)}\right\|_{L_t^{2p} W_x^{s,\tilde{r}}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant C \left\|F(u) - F(v)\right\|_{L_t^{q'} W_x^{s,q'}([T,\infty)\times\mathbb{R}^3)} \\
&\leqslant 2C(2\varepsilon)^2 \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)}.
\end{aligned}$$

Consequently, choosing a posteriori $\varepsilon > 0$ small enough to ensure $\boxed{4C(2\varepsilon)^2 \leqslant \frac{1}{4}}$, we get

$$\left\|\Gamma_{u_+} u - \Gamma_{u_+} v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \leqslant \frac{1}{4} \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)},$$

in other words

$$d(\Gamma_{u_+} u, \Gamma_{u_+} v) \leqslant \frac{1}{4} d(u, v).$$

It follows that $\Gamma_{u_+} : (B_{\varepsilon,T}, d) \to (B_{\varepsilon,T}, d)$ defines a contractive self-mapping provided that first some $\varepsilon > 0$ is chosen small enough to ensure the above two boxed conditions and then some $T = T(u_+, \varepsilon) > 0$ is chosen as to guarantee (4.30). □

Note that the above time $T = T(u_+, \varepsilon) > 0$ is stable under small $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-perturbations in $u_+$, that is, taking any $v_+$ in a sufficiently small neighborhood $B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ of $u_+$ ($\delta > 0$ so small as to guarantee the validity of (4.31)), the above contraction proof still goes through for the operator $\Gamma_{v_+} : B_{\varepsilon,T} \to B_{\varepsilon,T}$ with the same $T = T(u_+, \varepsilon)$. [Because we still had $\frac{1}{2}\varepsilon$ "room" in the above self-mapping proof.]

By Banach's Fixed Point Theorem, there exists a unique $u \in B_{\varepsilon,T}$ such that $\Gamma_{u_+} u = u$, i.e.:

$$u(t) = e^{-it(-\Delta)^s} u_+ + i\Phi_{F(u)}(t) = e^{-it(-\Delta)^s} u_+ + i\int_t^\infty e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))d\tau. \quad (4.34)$$

Recall first that

$$e^{-it(-\Delta)^s} u_+ \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3)) \cap L^\gamma_t W^{s,\rho}_{x,\mathrm{rad}}([T,\infty) \times \mathbb{R}^3), \quad \forall(\gamma,\rho) \in SA.$$

Indeed, continuity follows from the fact that $\{e^{-it(-\Delta)^s}\}_{t\in\mathbb{R}}$ defines a continuous one-parameter unitary group on $H^s_x(\mathbb{R}^3)$, and the $L^\gamma_t W^{s,\rho}_x$ regularity follows from homogeneous Strichartz estimate combined with the commutation of the Fourier multiplier $|\nabla|^s$ with the semigroup operators $e^{-it(-\Delta)^s}$, namely

$$\left\| e^{-it(-\Delta)^s} u_+ \right\|_{L^\gamma_t L^\rho_x([T,\infty)\times\mathbb{R}^3)} \overset{(C.8)}{\lesssim} \|u_+\|_{L^2_x},$$

$$\left\| |\nabla|^s e^{-it(-\Delta)^s} u_+ \right\|_{L^\gamma_t L^\rho_x([T,\infty)\times\mathbb{R}^3)} = \left\| e^{-it(-\Delta)^s} |\nabla|^s u_+ \right\|_{L^\gamma_t L^\rho_x([T,\infty)\times\mathbb{R}^3)} \overset{(C.8)}{\lesssim} \||\nabla|^s u_+\|_{L^2_x},$$

hence

$$\left\| e^{-it(-\Delta)^s} u_+ \right\|_{L^\gamma_t W^{s,\rho}_x([T,\infty)\times\mathbb{R}^3)} \lesssim \|u_+\|_{H^s} < \infty.$$

Recall second that the map $t \mapsto \Phi_{F(u)}(t)$ satisfies

$$\Phi_{F(u)} \in C([T,\infty); H^s_{x,\mathrm{rad}}(\mathbb{R}^3)) \cap L^\gamma_t W^{s,\rho}_{x,\mathrm{rad}}([T,\infty) \times \mathbb{R}^3), \quad \forall(\gamma,\rho) \in SA. \quad (4.35)$$

We conclude that the fixed point $u$ from (4.34) satisfies

$$u \in C([T,\infty); H^s_{x,\mathrm{rad}}(\mathbb{R}^3)) \cap L^\gamma_t W^{s,\rho}_{x,\mathrm{rad}}([T,\infty) \times \mathbb{R}^3), \quad \forall(\gamma,\rho) \in SA.$$

In particular $\psi := u(T) \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ makes good sense.

### Solving the Local Problem 'From Finite $T$ to $0$'

We have found a unique solution $u \in C([T,\infty); H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ of the asymptotic problem (4.34), in particular $\psi = u(T) \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. By semigroup properties, we now check that

$$\begin{aligned}
u(t+T) &\overset{(4.34)}{=} e^{-i(t+T)(-\Delta)^s} u_+ + i\int_{t+T}^\infty e^{-i(t+T-\tau)(-\Delta)^s} F(u(\tau))d\tau \\
&= e^{-it(-\Delta)^s}\left( e^{-iT(-\Delta)^s} u_+ + i\int_T^{+\infty} e^{-i(T-\tau)(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right) \\
&\quad - i\left( \int_T^{+\infty} e^{-i(t+T-\tau)(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau - \int_{t+T}^{+\infty} e^{-i(t+T-\tau)(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right) \\
&= e^{-it(-\Delta)^s} u(T) - i\int_0^t e^{-i(t-\tau')(-\Delta)^s} F(u(\tau'+T))\,\mathrm{d}\tau' \\
&= e^{-it(-\Delta)^s}\psi - i\int_0^t e^{-i(t-\tau)(-\Delta)^s} F(u(\tau+T))\,\mathrm{d}\tau,
\end{aligned}$$

where in the third equality, we changed variable $\tau' = \tau - T$ and then used $\int_0^\infty - \int_t^\infty = \int_0^t$. Therefore, by Duhamel's principle, $u$ solves the problem

$$\begin{cases} iw_t &= (-\Delta)^s w + F(w), \\ w(T) &= \psi \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad w : [T, +\infty) \times \mathbb{R}^3 \to \mathbb{C}. \end{cases} \tag{4.36}$$

But by global well-posedness, the solution $w \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ to this problem is unique and global; since $w(T) = \psi = u(T)$, we obtain $w \equiv u$ and thus $u(0) \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ is well-defined.

### The Global Nonlinear Solution $u(t)$ Scatters to the Given Linear Solution

Next, we prove that the global solution $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ constructed above actually *scatters* in $H^s_x(\mathbb{R}^3)$ to the solution $e^{-it(-\Delta)^s}u_+$ of the free equation as $t \to \infty$. Recall from (4.2) that this means

$$\|u(t) - e^{-it(-\Delta)^s}u_+\|_{H^s} \overset{(4.34)}{=} \|\Phi_{F(u)}(t)\|_{H^s} \to 0, \quad \text{as } t \to \infty. \tag{4.37}$$

*Proof of* (4.37). We see from (4.35) that

$$\Phi_{F(u)} \in C([\tilde{T}, \infty); H^s_{x,\mathrm{rad}}(\mathbb{R}^3)) \cap L^\gamma_t W^{s,\rho}_{x,\mathrm{rad}}([\tilde{T}, \infty) \times \mathbb{R}^3), \quad \forall \tilde{T} \geqslant T, \forall (\gamma, \rho) \in SA.$$

By Strichartz, there exists a constant $C = C(\gamma, \rho, \tilde{\gamma}, \tilde{\rho}, s)$ (independent of time $\tilde{T}$) such that (see estimate (C.16) in Corollary C.5 or analogously the last estimate in [Caz03, Corollary, 2.3.6, p. 37])

$$\left\|\Phi_{F(u)}\right\|_{L^\gamma_t W^{s,\rho}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \leqslant C \left\|F(u)\right\|_{L^{\tilde{\gamma}'}_t W^{s,\tilde{\rho}'}_x([\tilde{T},\infty)\times\mathbb{R}^3)}.$$

In particular, there exists $C = C(\infty, 2, q, q, s) = C(s)$ such that for $t \geqslant \tilde{T}$

$$\begin{aligned} \|\Phi_{F(u)}(t)\|_{H^s} &\leqslant \|\Phi_{F(u)}\|_{C([\tilde{T},\infty);H^s_x(\mathbb{R}^3))} = \left\|\Phi_{F(u)}\right\|_{L^\infty_t W^{s,2}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \\ &\leqslant C \left\|F(u)\right\|_{L^{q'}_t W^{s,q'}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \\ &\leqslant C \left\|u\right\|^2_{L^{2p}_{t,x}([\tilde{T},\infty)\times\mathbb{R}^3)} \left\|u\right\|_{L^q_t W^{s,q}_x([\tilde{T},\infty)\times\mathbb{R}^3)}, \quad t \geqslant \tilde{T}. \end{aligned} \tag{4.38}$$

Since $u \in B_{\varepsilon,T} \subset \cap_{\tilde{T} \geqslant T} B_{\varepsilon,\tilde{T}}$, we have the uniform bounds

$$\begin{aligned} \|u\|_{L^{2p}_{t,x}([\tilde{T},\infty)\times\mathbb{R}^3)} &= \|u\|_{Y_{\tilde{T}}} \leqslant \|u\|_{\mathfrak{S}^s([\tilde{T},\infty)\times\mathbb{R}^3)} \leqslant 2\varepsilon \\ \|u\|_{L^q_t W^{s,q}_x([\tilde{T},\infty)\times\mathbb{R}^3)} &= \|u\|_{X_{\tilde{T}}} \leqslant \|u\|_{\mathfrak{S}^s([\tilde{T},\infty)\times\mathbb{R}^3)} \leqslant 2\varepsilon. \end{aligned}$$

Hence both $\|u\|_{L^{2p}_{t,x}([\tilde{T},\infty)\times\mathbb{R}^3)} \to 0$, $\|u\|_{L^q_t W^{s,q}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \to 0$ as $\tilde{T} \to \infty$ and the result follows from (4.38). □

### Uniqueness

We have deduced that for every $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that the corresponding global solution $u(t)$ scatters to $e^{-it(-\Delta)^s}u_+$ in $H^s_x(\mathbb{R}^3)$ as $t \to \infty$. It remains to show uniqueness of $u_0$. Then the **wave operator** exists:

$$\Omega_+ : H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad u_+ \mapsto u_0, \tag{4.39}$$

where $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ is the unique initial datum whose corresponding global solution $u \in C(\mathbb{R}_t; H^s_x(\mathbb{R}^3))$ scatters to $e^{-it(-\Delta)^s}u_+$ in $H^s_x(\mathbb{R}^3)$ as $t \to \infty$.

*Proof of uniqueness.* Let $\tilde{u} \in C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ be another global solution such that $\tilde{u}(t)$ scatters to $e^{-it(-\Delta)^s}u_+$ in $H^s_x(\mathbb{R}^3)$ as $t \to \infty$. By the characterization of the beginning of Section 4.3 (see (4.23)), we necessarily have

$$\tilde{u}(t) = e^{-it(-\Delta)^s}u_+ + i\Phi_{F(\tilde{u})}(t), \quad \forall t \in \mathbb{R}.$$

Since also $u(t)$ satisfies

$$u(t) = e^{-it(-\Delta)^s}u_+ + i\Phi_{F(u)}(t), \quad \forall t \in [T, \infty),$$

we have

$$u(t) - \tilde{u}(t) = i\Phi_{F(u)-F(\tilde{u})}(t), \quad \forall t \in [T, \infty).$$

Similarly to the contraction proof above, we estimate

$$\begin{aligned}
\|u - \tilde{u}\|_{X_{\tilde{T}}} &= \left\|\Phi_{F(u)-F(\tilde{u})}\right\|_{X_{\tilde{T}}} = \left\|\Phi_{F(u)-F(\tilde{u})}\right\|_{L^q_t W^{s,q}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \\
&\leqslant C\left\|F(u) - F(\tilde{u})\right\|_{L^{q'}_t W^{s,q'}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \\
&\leqslant C\left\{\|u\|^2_{L^{2p}_{t,x}([\tilde{T},\infty)\times\mathbb{R}^3)} + \|\tilde{u}\|^2_{L^{2p}_{t,x}([\tilde{T},\infty)\times\mathbb{R}^3)}\right\}\|u - \tilde{u}\|_{L^q_t W^{s,q}_x([\tilde{T},\infty)\times\mathbb{R}^3)} \\
&= C\left\{\|u\|^2_{Y_{\tilde{T}}} + \|\tilde{u}\|^2_{Y_{\tilde{T}}}\right\}\|u - \tilde{u}\|_{X_{\tilde{T}}} \leqslant \frac{1}{4}\|u - \tilde{u}\|_{X_{\tilde{T}}}
\end{aligned}$$

and

$$\|u - \tilde{u}\|_{Y_{\tilde{T}}} \leqslant \frac{1}{4}\|u - \tilde{u}\|_{\mathfrak{S}^s([\tilde{T},\infty)\times\mathbb{R}^3)},$$

provided that $\tilde{T}$ is large enough. Thus $u = \tilde{u}$ in $X_{\tilde{T}} \cap Y_{\tilde{T}}$ for $\tilde{T}$ large enough. It follows that $u(t,x) = \tilde{u}(t,x)$ for a.e. $x \in \mathbb{R}^3$ and some large $t$. Therefore $u(t) = \tilde{u}(t)$ in $H^s_x(\mathbb{R}^3)$ for that large $t$. By uniqueness of the Cauchy problem at finite time $t$, we conclude $u \equiv \tilde{u}$, hence $u(0) = \tilde{u}(0)$. □

The wave operator $\Omega_+ : H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ is necessarily injective. Indeed, let $u_+, \widetilde{u_+} \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that $\Omega_+ u_+ = \Omega_+\widetilde{u_+}$. Let $u(t)$ be the global solution corresponding to $u_0 := \Omega_+ u_+ = \Omega_+\widetilde{u_+}$. Then, by definition of $\Omega_+$, we have

$$\|u_+ - \widetilde{u_+}\|_{H^s} \leqslant \|u_+ - e^{it(-\Delta)^s}u(t)\|_{H^s} + \|e^{it(-\Delta)^s}u(t) - \widetilde{u_+}\|_{H^s} \to 0, \quad \text{as } t \to \infty,$$

so that $u_+ = \widetilde{u_+}$.

### 4.3.2 Continuity of the Wave Operator $\Omega_+$

In this subsection, we show that $\Omega_+$ is a continuous operator $H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. Let $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ be a given state. We prove the continuity of $\Omega_+$ in $u_+$ by showing that in a $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-neighborhood of $u_+$ the operator $\Omega_+$ can be written as the composition of two continuous operators, $\Omega_+ = \Omega_+^{0,T} \circ \Omega_+^{T,\infty}$.

Indeed, fix some $\varepsilon > 0$ which satisfies the (a posteriori) conditions of the above contraction proof[13] and fix some $T = T(u_+, \varepsilon) > 0$ such that (4.30) holds. Furthermore, fix some $\delta > 0$ such that still (4.31) holds.

**Step 1 (evolving continuously from $\infty$ to $T$):** Let $\varepsilon^* > 0$ be given. We show there exists $\delta^* > 0$ such that if $v_+ \in B_{\delta^*}(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ then $\|u(T) - v(T)\|_{H^s} \leqslant \varepsilon^*$.

Let $v_+ \in B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. Let $u, v \in B_{\varepsilon,T}$ be the corresponding fixed points of the operators[14] $\Gamma_{u_+}, \Gamma_{v_+}$ mapping the set $B_{\varepsilon,T}$ to itself, respectively (they are global solutions of (fNLS) with initial values $u(0), v(0) \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$). Then we have the estimate

$$\begin{aligned} \|u(T) - v(T)\|_{H^s} &= \|e^{-iT(-\Delta)^s} u_+ + i\Phi_{F(u)}(T) - e^{-iT(-\Delta)^s} v_+ - i\Phi_{F(v)}(T)\|_{H^s} \\ &\leqslant \|u_+ - v_+\|_{H^s} + \|\Phi_{F(u)-F(v)}(T)\|_{H^s} \\ &\leqslant \|u_+ - v_+\|_{H^s} + \left\|\Phi_{F(u)-F(v)}\right\|_{L^\infty_t W^{s,2}_x([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant \|u_+ - v_+\|_{H^s} + C\left\|F(u) - F(v)\right\|_{L^{q'}_t W^{s,q'}_x([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant \|u_+ - v_+\|_{H^s} + 2C(2\varepsilon)^2 \left\|u - v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant \|u_+ - v_+\|_{H^s} + \frac{1}{4}\left\|u - v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)}, \end{aligned} \tag{4.40}$$

where from the second to last line on, we estimated again as in step 2 of the above contraction proof. However, using the definition $u = \Gamma_{u_+}u$, $v = \Gamma_{v_+}v$, then arguing as in (4.33) for the homogeneous term, and finally using the estimates in step 2 of the above contraction proof again, we obtain

$$\begin{aligned} \|u - v\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} &\leqslant \left\|e^{-it(-\Delta)^s}(u_+ - v_+)\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} + \left\|\Phi_{F(u)} - \Phi_{F(v)}\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant C\|u_+ - v_+\|_{H^s} + \left\|\Phi_{F(u)-F(v)}\right\|_{X_T} + \left\|\Phi_{F(u)-F(v)}\right\|_{Y_T} \\ &\leqslant C\|u_+ - v_+\|_{H^s} + 4C(2\varepsilon)^2 \left\|u - v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \\ &\leqslant C\|u_+ - v_+\|_{H^s} + \frac{1}{4}\left\|u - v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)}, \end{aligned}$$

which implies

$$\frac{1}{4}\left\|u - v\right\|_{\mathfrak{S}^s([T,\infty)\times\mathbb{R}^3)} \leqslant \frac{C}{3}\|u_+ - v_+\|_{H^s}. \tag{4.41}$$

Inserting (4.41) into (4.40) there results an estimate of the form

$$\|u(T) - v(T)\|_{H^s} \leqslant C\|u_+ - v_+\|_{H^s},$$

[13] That is, $\varepsilon > 0$ must be so small that $2C(2\varepsilon)^{2+1} \leqslant \varepsilon$ and $4C(2\varepsilon)^2 \leqslant \frac{1}{4}$.

[14] That is, $u, v \in B_{\varepsilon,T}$ with $\Gamma_{u_+}u = u$, $\Gamma_{v_+}v = v$.

which holds for any $v_+ \in B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. Picking now $0 < \delta^* < \min\{\delta, \frac{\varepsilon^*}{C}\}$ we have proved that

$$v_+ \in B_{\delta^*}(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \Longrightarrow v(T) \in B_{\varepsilon^*}(u(T)).$$

In a full $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-neighborhood $B_\delta(u_+)$ of $u_+$ we can therefore define a map

$$\begin{cases} \Omega_+^{T,\infty} : B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad v_+ \mapsto v(T), \\ v \text{ is defined as the unique element of } B_{\varepsilon,T} \text{ such that } \Gamma_{v_+} v = v \\ (\text{such } v \text{ is a global solution to (fNLS)}), \end{cases}$$

and we have just proved that it is continuous in $u_+$.

**Step 2 (evolving continuously from $T$ to 0):** This is a consequence of the continuous dependence of strong $H^s_x(\mathbb{R}^3)$-solutions to (fNLS) on the initial data,[15] which says: if $\varphi_T^n \to \varphi_T$ in $H^s_x(\mathbb{R}^3)$ and if $u^n, u \in C(\mathbb{R}; H^s_x(\mathbb{R}^3))$ denote the global solutions of

$$\begin{cases} iw_t &= (-\Delta)^s w + F(w), \\ w(T) &= w_T \in H^s(\mathbb{R}^3), \quad w : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}. \end{cases} \tag{4.42}$$

with initial value $w_T \in \{\varphi_T^n, \varphi_T\}$ at some fixed finite time $T \in \mathbb{R}$, respectively, then necessarily $u^n \to u$ in $L^\infty(\mathbb{R}_t; H^s_x(\mathbb{R}^3))$. In particular, for any $\varepsilon^* > 0$ there exists $\delta^* > 0$ such that $\|u(0) - u^n(0)\|_{H^s} < \varepsilon^*$ whenever $\|\varphi_T - \varphi_T^n\|_{H^s} < \delta^*$. There exists therefore another map

$$\begin{cases} \Omega_+^{0,T} : H^s_x(\mathbb{R}^3) \to H^s_x(\mathbb{R}^3), \quad w_T \mapsto w(0), \\ w \text{ is the unique global solution to (4.42) with initial value } w_T \text{ at time } T, \end{cases}$$

which is continuous in $u(T) := \Omega_+^{T,\infty}(u_+) \in H^s_x(\mathbb{R}^3)$.

**Step 3 (Conclusion):** We glue together step 1 and step 2 to obtain that

$$\begin{cases} \Omega_+ = \Omega_+^{0,T} \circ \Omega_+^{T,\infty} : B_\delta(u_+) \cap H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad v_+ \mapsto v(0), \\ v \text{ is defined as the unique element of } B_{\varepsilon,T} \text{ such that } \Gamma_{v_+} v = v \\ (\text{such } v \text{ is a global solution to (fNLS)}), \end{cases}$$

is continuous in $u_+$. □

### 4.3.3 Asymptotic Completeness

We need to show that $\Omega_+$ is surjective (hence bijective), i.e., $\mathcal{R}_+ = H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$, i.e., for any $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ the associated global solution $u \in C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ scatters

[15] A strong solution to defocusing (fNLS) is global and it is irrelevant if we impose initial data at time 0 or at any other finite time $T \in \mathbb{R}$.

in $H^s_x(\mathbb{R}^3)$. It is sufficient to prove that for given $u_0$ and associated global solution $u$, the integral $\int_0^{+\infty} e^{i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau$ converges in $H^s_x(\mathbb{R}^3)$; indeed, once this is proved, one can then define $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ by the formula (4.22) and it is clear that $u$ scatters in $H^s_x(\mathbb{R}^3)$ to the solution $e^{-it(-\Delta)^s}u_+$ (with initial condition $u_+$) of the free equation (4.1). [Recall the equivalences in the derivation of (4.22)!] Asymptotic completeness is implied by the strong space time bound (4.10) as follows (cf. [Tao06, p. 166]). We have

$$\begin{aligned}
\left\| \int_0^{+\infty} e^{+i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} &\overset{(4.14)}{\lesssim} \|F(u)\|_{N^s(\mathbb{R}\times\mathbb{R}^3)} = \left\| |u|^2 u \right\|_{N^s(\mathbb{R}\times\mathbb{R}^3)} \\
&\overset{(4.19)}{\lesssim} \|u\|^2_{L^{2p}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \|u\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \overset{\text{Sobolev}}{\lesssim} \|u\|^2_{L^{2p}_t W^{s,\bar r}_x(\mathbb{R}\times\mathbb{R}^3)} \|u\|_{S^s(\mathbb{R}\times\mathbb{R}^3)} \\
&\overset{(2p,\bar r)\in SA}{\lesssim} \|u\|^3_{S^s(\mathbb{R}\times\mathbb{R}^3)} \overset{(4.10)}{\lesssim} 1,
\end{aligned} \tag{4.43}$$

using successively an inhomogeneous Strichartz-type estimate analogous to (4.14), the bound on the power-nonlinearity in the dual Strichartz norm (4.19), the Sobolev embedding $L^{2p}_t W^{s,\bar r}_x(\mathbb{R}\times\mathbb{R}^3) \hookrightarrow L^{2p}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$, the admissibility $(2p,\bar r) \in SA$ and finally the crucial strong space-time bound (4.10). Estimate (4.43) shows that the nonlinearity is bounded in the dual Strichartz norm $\|\cdot\|_{N^s(\mathbb{R}\times\mathbb{R}^3)}$. However, this is sufficient, since this norm also controls the $H^s_x(\mathbb{R}^3)$ norm, namely via the dual homogeneous Strichartz estimate (cf. [Tao06, p. 74], estimate (2.25) there)

$$\left\| \int_{\mathbb{R}} e^{+i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right\|_{L^2_x(\mathbb{R}^3)} \lesssim \|F(u)\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^3)}, \quad \forall (q,r) \in SA. \tag{4.44}$$

Postponing for a moment the proof, we see that this implies (using a uniformed boundedness of the appearing constants $C(q,r)$ and going to the infimum over all $(q,r) \in SA$)[16]

$$\left\| \int_0^{+\infty} e^{+i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right\|_{L^2_x(\mathbb{R}^3)} \lesssim \|F(u)\|_{N^0(\mathbb{R}\times\mathbb{R}^3)},$$

and

$$\begin{aligned}
\left\| |\nabla|^s \int_0^{+\infty} e^{+i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \right\|_{L^2_x(\mathbb{R}^3)} &= \left\| \int_0^{+\infty} e^{+i\tau(-\Delta)^s} |\nabla|^s F(u(\tau))\,\mathrm{d}\tau \right\|_{L^2_x(\mathbb{R}^3)} \\
&\lesssim \left\| |\nabla|^s F(u) \right\|_{N^0(\mathbb{R}\times\mathbb{R}^3)},
\end{aligned}$$

and thus

$$\begin{aligned}
\| \int_0^{+\infty} e^{+i\tau(-\Delta)^s} F(u(\tau))\,\mathrm{d}\tau \|_{H^s} &\lesssim \|F(u)\|_{N^0(\mathbb{R}\times\mathbb{R}^3)} + \left\| |\nabla|^s F(u) \right\|_{N^0(\mathbb{R}\times\mathbb{R}^3)} \\
&\lesssim \|F(u)\|_{N^s(\mathbb{R}\times\mathbb{R}^3)}.
\end{aligned}$$

Hence indeed the bound (4.43) is sufficient for asymptotic completeness. It remains to show the dual homogeneous Strichartz estimate:

[16] $(1/q, 1/r)$ varies in a compact set for $n \geqslant 3$.

*Proof of* (4.44). Using $TT^*$ argument, let us write

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \right\|^2_{L^2_x(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} \overline{\left( \int_{\mathbb{R}} e^{i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \right)} \left( \int_{\mathbb{R}} e^{i\tilde{\tau}(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}} e^{-i\tau(-\Delta)^s} \overline{F(u(\tau))}\, \mathrm{d}\tau \right) \left( \int_{\mathbb{R}} e^{i\tilde{\tau}(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}} e^{-i\tau(-\Delta)^s} \overline{F(u(\tau))} \left\{ \int_{\mathbb{R}} e^{i\tilde{\tau}(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\} \mathrm{d}\tau\, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left\langle e^{i\tau(-\Delta)^s} F(u(\tau)), \int_{\mathbb{R}} e^{i\tilde{\tau}(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\rangle_{L^2_\tau(\mathbb{R})} \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left\langle F(u(\tau)), \int_{\mathbb{R}} e^{i(\tilde{\tau}-\tau)(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\rangle_{L^2_\tau(\mathbb{R})} \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}} \overline{F(u(\tau))} \left\{ \int_{\mathbb{R}} e^{i(\tilde{\tau}-\tau)(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\} \mathrm{d}\tau\, \mathrm{d}x \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \overline{F(u(\tau))} \left\{ \int_{\mathbb{R}} e^{i(\tilde{\tau}-\tau)(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\} \mathrm{d}x\, \mathrm{d}\tau \\
&\leqslant \int_{\mathbb{R}} \|F(u(\tau))\|_{L^{r'}_x(\mathbb{R}^3)} \left\| \int_{\mathbb{R}} e^{i(\tilde{\tau}-\tau)(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\|_{L^r_x(\mathbb{R}^3)} \mathrm{d}\tau \\
&\leqslant \|F(u)\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^3)} \left\| \int_{\mathbb{R}} e^{-i(\tau-\tilde{\tau})(-\Delta)^s} F(u(\tilde{\tau}))\, \mathrm{d}\tilde{\tau} \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^3)} \\
&\leqslant \|F(u)\|^2_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^3)}.
\end{aligned}$$

We used $\langle T^* f, T^* g\rangle_{L^2_\tau(\mathbb{R})} = \langle f, TT^* g\rangle_{L^2_\tau(\mathbb{R})}$ with operator $T = e^{-i\tau(-\Delta)^s}$. The last estimates used Hölder in $x$, then Hölder in $t$ and finally the inhomogeneous Strichartz estimate (C.9) (with integral taken over $(-\infty, +\infty)$ instead of $(0, t)$). ☐

## 4.4 The Inversion $U_+ = \Omega_+^{-1}$

We have seen that for any $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a unique $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such that $u_+$ is the scattering state of $u_0$ at $+\infty$. This enabled us to *define* the wave operator by $\Omega_+ u_+ := u_0$. We have seen that $\Omega_+$ is continuous and injective. Subsection 4.3.3 showed that $\Omega_+$ is also surjective. By bijectivity of $\Omega_+$ it is clear that conversely for any initial datum $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a scattering state $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ of $u_0$ at $+\infty$.

**Theorem 4.5.** *Every $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ has a unique scattering state $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ at $+\infty$. That is, for every $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ there exists a unique $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ such* *that*

$$\|u(t) - e^{-it(-\Delta)^s} u_+\|_{H^s} = \|e^{it(-\Delta)^s} u(t) - u_+\|_{H^s} \to 0, \qquad \text{as } t \to +\infty,$$

*where* $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ *denotes the global solution to defocusing* (fNLS) *with initial value* $u_0$*.*

*Proof.* The uniqueness is clear by uniqueness of limits in $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. To prove existence, let $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ and denote by $u \in C(\mathbb{R}_t; H^s_{x,\mathrm{rad}}(\mathbb{R}^3))$ the global solution to defocusing (fNLS) with initial value $u_0$. Recall the representation

$$u(t) = e^{-it(-\Delta)^s} u_0 - i \int_0^t e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \quad \text{for } t \in \mathbb{R},$$

in other words (applying the propagator $e^{+it(-\Delta)^s}$ to both sides)

$$e^{it(-\Delta)^s} u(t) = u_0 - i \int_0^t e^{i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \quad \text{for } t \in \mathbb{R}. \tag{4.45}$$

We know already that the limit

$$\int_0^\infty e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau := \lim_{t\to\infty} \int_0^t e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau$$

exists in $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$ - see Subsection 4.3.3 on asymptotic completeness; the radiality follows from the radiality of $u$, which in turn follows from the radiality of $u_0$. We can thus *define* the $H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$-element

$$u_+ := u_0 - i \int_0^\infty e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau.$$

The statement $i \int_0^t e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \to u_0 - u_+$ in $H^s_x(\mathbb{R}^3)$ as $t \to \infty$ means

$$\begin{aligned}\|e^{it(-\Delta)^s} u(t) - u_+\|_{H^s} &\overset{(4.45)}{=} \|(u_0 - u_+) - i \int_0^t e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau\|_{H^s} \\ &\to 0 \quad \text{as } t \to \infty.\end{aligned}$$

□

**Remark 4.6.** We can thus *define* the operator

$$U_+ : H^s_{x,\mathrm{rad}}(\mathbb{R}^3) \to H^s_{x,\mathrm{rad}}(\mathbb{R}^3), \quad u_0 \mapsto u_+,$$

mapping initial states to the associated scattering states, by

$$\begin{cases} u_+ := U_+ u_0 := u_0 - i \int_0^\infty e^{+i\tau(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau, \\ \text{where } u \in C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^3)) \text{ is the global solution to (fNLS)} \\ \text{with initial datum } u_0. \end{cases} \tag{4.46}$$

Observe that $U_+ \circ \Omega_+ = \Omega_+ \circ U_+ = \mathrm{Id}_{H^s_{x,\mathrm{rad}}(\mathbb{R}^3)}$, in other words, $U_+ = \Omega_+^{-1}$. Indeed, the mappings $\Omega_+$ and $U_+$ are now characterized as follows[17]:

$$\forall u_+ \exists! u_0 : \quad \|e^{it(-\Delta)^s} u_{u_0}(t) - u_+\|_{H^s} \to 0; \quad \text{define } \Omega_+ u_+ := u_0. \tag{4.47}$$

$$\forall u_0 \exists! u_+ : \quad \|e^{it(-\Delta)^s} u_{u_0}(t) - u_+\|_{H^s} \to 0; \quad \text{define } U_+ u_0 := u_+. \tag{4.48}$$

Let now $u_+ \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. We show $U_+(\Omega_+ u_+) = u_+$. Take $u_0 := \Omega_+ u_+$ from (4.47), i.e. $\|e^{it(-\Delta)^s} u_{u_0}(t) - u_+\|_{H^s} \to 0$. But by (4.48), there exists only one $v_+ = U_+ u_0$ such that $\|e^{it(-\Delta)^s} u_{u_0}(t) - v_+\|_{H^s} \to 0$. Hence $v_+ = u_+$, that is $u_+ = v_+ = U_+ u_0 = U_+(\Omega_+ u_+)$.

Conversely, let $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^3)$. We show $\Omega_+(U_+ u_0) = u_0$. Take $u_+ = U_+ u_0$ from (4.48), i.e. $\|e^{it(-\Delta)^s} u_{u_0}(t) - u_+\|_{H^s} \to 0$. But by (4.47), there exists only one $v_0 = \Omega_+ u_+$ such that $\|e^{it(-\Delta)^s} u_{v_0}(t) - u_+\|_{H^s} \to 0$. Hence $v_0 = u_0$, that is $u_0 = v_0 = \Omega_+ u_+ = \Omega_+(U_+ u_0)$. This proves $U_+ = \Omega_+^{-1}$.

[17] $u_{u_0}$ denotes the global solution to (fNLS) corresponding to initial value $u_0$.

# C Scattering

## C.1 Strichartz Estimates in the Radial Case

**Definition C.1** (see [GW11, p. 23])**.** Let $n \geqslant 2$. The exponent pair $(q, r)$ is called $n$-D radial Schrödinger admissible provided $(q, r) \in [2, \infty] \times [2, \infty]$ and

$$\begin{gathered}\frac{4n+2}{2n-1} \leqslant q \leqslant \infty \text{ and } \frac{2}{q} + \frac{2n-1}{r} \leqslant n - \frac{1}{2} \\ \text{or} \\ 2 \leqslant q < \frac{4n+2}{2n-1} \text{ and } \frac{2}{q} + \frac{2n-1}{r} < n - \frac{1}{2}.\end{gathered} \tag{C.1}$$

**Proposition C.2** (see [GW11, p. 26])**.** *Let $n \geqslant 2$ and $u, u_0, F$ be spherically symmetric in space and satisfy* (fNLS)*. Then*

$$\|u\|_{L_t^q L_x^r} + \|u\|_{C(\mathbb{R}; \dot{H}^\gamma)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F(u)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{C.2}$$

*if $\gamma \in \mathbb{R}$, both $(q, r)$ and $(\tilde{q}, \tilde{r})$ are $n$-D radial Schrödinger admissible, $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$, and $(q, r, n)$ and $(\tilde{q}, \tilde{r}, n)$ satisfy the 'gap' condition*

$$\frac{2}{q} s = n\left(\frac{1}{2} - \frac{1}{r}\right) - \gamma, \quad \frac{2}{\tilde{q}} s = n\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) + \gamma. \tag{C.3}$$

**Corollary C.3** (Strichartz Estimates Without Loss of Derivatives; see [GW11, p. 26])**.** *Let $n \geqslant 2$, $\frac{n}{2n-1} < s \leqslant 1$, and $u, u_0, F$ be spherically symmetric in space and satisfy* (fNLS)*. Then*

$$\|u\|_{L_t^q L_x^r} + \|u\|_{C(\mathbb{R}; L^2)} \lesssim \|u_0\|_{L_x^2} + \|F(u)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{C.4}$$

*if both $(q, r)$ and $(\tilde{q}, \tilde{r}) \in \{(q, r) \in [2, \infty] \times [2, \infty]; \frac{2}{q} s = n\left(\frac{1}{2} - \frac{1}{r}\right)\}$ and $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$.*

*Proof of Corollary C.3.* $(q, r)$ (resp., $(\tilde{q}, \tilde{r})$) satisfy the gap condition (C.3) with $\gamma = 0$, from which it is clear that $q = \infty$ if and only if $r = 2$; in this case, one has the equality case $2/q + (2n-1)/r = n - 1/2$ in the $n$-D radial admissibility condition (C.1). In the other case ($2 \leqslant q < \infty \overset{(C.3)}{\Longleftrightarrow} 2 < r \leqslant \infty$) one verifies (C.1) by the restriction on the powers $s$ of the fractional Laplacian, namely

$$\frac{2}{q} + \frac{2n-1}{r} \overset{(C.3)}{=} \frac{n}{s}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{2n-1}{r} < n\frac{2n-1}{n}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{2n-1}{r} = n - \frac{1}{2}. \qquad \square$$

## C.2 Duhamel's Principle and Strichartz Estimates

When $u$ solves (fNLS), then Duhamel's principle gives the representation

$$u(t) = e^{-it(-\Delta)^s} u_0 - i \int_0^t e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau. \tag{C.5}$$

The first part $u_g(t) := e^{-it(-\Delta)^s} u_0$ solves the homogeneous problem [free ($F \equiv 0$) fractional Schrödinger equation]

$$\begin{cases} iu_t &= (-\Delta)^s u, \\ u(0) &= u_0, \end{cases} \tag{C.6}$$

and the second part $u_p(t) := -i \int_0^t e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau$ solves the inhomogeneous (fNLS)

$$\begin{cases} iu_t &= (-\Delta)^s u + F(u), \\ u(0) &= 0. \end{cases} \tag{C.7}$$

If the solution $u$ of (fNLS) given by (C.5) together with $u_0$ and $F$ are spherically symmetric in space, then so are the solutions $u_g$ to (C.6) together with $u_0$ and 0 as well as $u_p$ together with 0 and $F$. Applying Corollary (C.3) to $(u_g, u_0, 0)$ and $(u_p, 0, F)$ yields the **homogeneous Strichartz estimates**

$$\|u_g\|_{L_t^q L_x^r} = \left\| e^{-it(-\Delta)^s} u_0 \right\|_{L_t^q L_x^r} \overset{(C.4)}{\lesssim} \|u_0\|_{L_x^2}, \tag{C.8}$$

and the **inhomogeneous Strichartz estimates**

$$\|u_p\|_{L_t^q L_x^r} = \left\| \int_0^t e^{-i(t-\tau)(-\Delta)^s} F(u(\tau))\, \mathrm{d}\tau \right\|_{L_t^q L_x^r} \overset{(C.4)}{\lesssim} \|F(u)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} . \tag{C.9}$$

## C.3 Definitions of Strichartz Spaces

For $(q, r) \in [2, \infty] \times [2, \infty]$ such that $(q, r, n) \neq (2, \infty, 2)$, we say $(q, r) \in SA$ *(Strichartz admissible)* provided that $(q, r)$ satisfies both the $n$-D radial Schrödinger admissibility condition (C.1) and the gap condition (C.3) with $\gamma = 0$.

Note that for $n \geqslant 3$, the set $SA$ is always compact. Indeed, writing $x = \frac{1}{q}$, $y = \frac{1}{r}$, and rewriting the $n$-D radial Schrödinger admissibility condition and the gap condition with $\gamma = 0$ with $x$ and $y$, we see that[1]

$$\left(\frac{1}{x}, \frac{1}{y}\right) \in SA \iff (x, y) = \left(x, \frac{1}{2} - \frac{2s}{n} x\right), \quad x \in \left[0, \frac{1}{2}\right].$$

[1]The restrictions $\frac{n}{2n-1} < s \leqslant 1$, $n \geqslant 2$ guarantee that $y$ lies in the correct range.

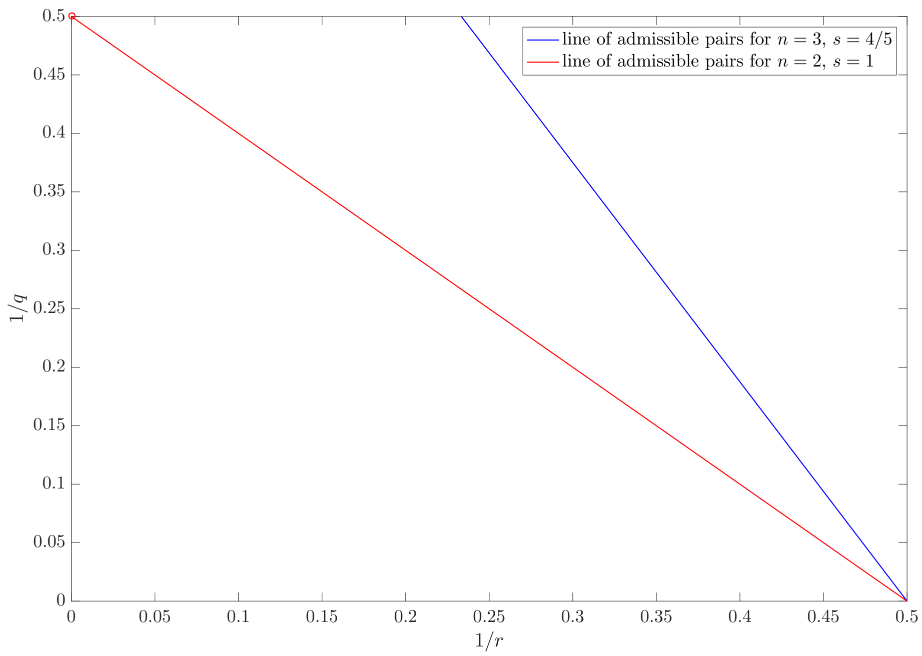


Figure C.1: Line of admissible pairs

As $\varphi : \left[0, \frac{1}{2}\right] \to \mathbb{R}^2$, $\varphi(x) = \left(x, \frac{1}{2} - \frac{2s}{n}x\right)$, is continuous and $\left[0, \frac{1}{2}\right]$ is compact, the set $\varphi\left(\left[0, \frac{1}{2}\right]\right) \subset \mathbb{R}^2$ is compact, and hence $SA$; see figure C.1. The compactness does not hold for $n = 2$ and $s = 1$ (take a sequence $x_j \to \frac{1}{2}, x_j < \frac{1}{2}$, hence $(x_j, y_j) := (x_j, \frac{1}{2} - x_j) \to (\frac{1}{2}, 0)$ and $\left(\frac{1}{x_j}, \frac{1}{y_j}\right) \in SA$, but $(2, \infty) \notin SA$ since Strichartz-admissibility forbids the case $(q, r, n) = (2, \infty, 2)$).

We define (cf. [Tao06, p. 134]) the *Strichartz space* $S^0(I \times \mathbb{R}^n)$ as the completion of the Schwartz functions with respect to the norm[2]

$$\|u\|_{S^0(I\times\mathbb{R}^n)} := \sup_{(q,r)\in SA} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^n)} . \tag{C.10}$$

Similarly, define $S^s(I \times \mathbb{R}^n)$ to be the completion of the Schwartz functions under the norm

$$\|u\|_{S^s(I\times\mathbb{R}^n)} := \|u\|_{S^0(I\times\mathbb{R}^n)} + \||\nabla|^s u\|_{S^0(I\times\mathbb{R}^n)} , \tag{C.11}$$

where $\widehat{|\nabla|^s u}(\xi) := |\xi|^s \widehat{u}(\xi)$. By construction, the space $S^0(I \times \mathbb{R}^n)$ is a Banach space. We consider its topological dual $N^0(I \times \mathbb{R}^n) := S^0(I \times \mathbb{R}^n)^*$. Then by construction

[2] More precisely, we consider space-time Schwartz functions, $\mathscr{S}(\mathbb{R} \times \mathbb{R}^n)|_{I\times\mathbb{R}^n}$, restricted to the space-time domain $I \times \mathbb{R}^n$. It is clear that $\|\cdot\|_{S^0(I\times\mathbb{R}^n)}$ defines a norm on $\mathscr{S}(\mathbb{R} \times \mathbb{R}^n)|_{I\times\mathbb{R}^n}$. The completion of $\mathscr{S}(\mathbb{R} \times \mathbb{R}^n)|_{I\times\mathbb{R}^n}$ with respect to the Strichartz norm $\|\cdot\|_{S^0(I\times\mathbb{R}^n)}$ is then a Banach space, denoted $S^0(I \times \mathbb{R}^n)$.

(cf. [Tao06, p. 135]) the following estimates hold:

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^n)} \leqslant \|u\|_{S^0(I\times\mathbb{R}^n)}, \quad \forall (q,r)\in SA, \tag{C.12}$$

as well as the corresponding dual estimate

$$\|u\|_{N^0(I\times\mathbb{R}^n)} \leqslant \|u\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^n)}, \quad \forall (q,r)\in SA. \tag{C.13}$$

## C.4 Consequences of Strichartz Estimates

We frequently use the following results, which are analogous to [Caz03], Theorem 2.3.3 and Corollary 2.3.6 (see p. 33, p. 37 there).

**Theorem C.4.** *The following properties hold:*

*(i) For every $u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^n)$, the function $t \mapsto e^{-it(-\Delta)^s}u_0$ belongs to*

$$L_t^q W^{s,r}_{x,\mathrm{rad}}(\mathbb{R}\times\mathbb{R}^n) \cap C(\mathbb{R}; H^s_{x,\mathrm{rad}}(\mathbb{R}^n))$$

*for every admissible pair $(q,r)\in SA$. Furthermore, there exists a constant $C$ such that*

$$\|e^{-it(-\Delta)^s}u_0\|_{L_t^q W_x^{s,r}(\mathbb{R}\times\mathbb{R}^n)} \leqslant C\|u_0\|_{H^s} \quad \textit{for every } u_0 \in H^s_{x,\mathrm{rad}}(\mathbb{R}^n).$$

*(ii) Let $I$ be an interval of $\mathbb{R}$ (bounded or not), $J=\overline{I}$, and $t_0\in J$. If $(\gamma,\rho)\in SA$ is admissible and $f\in L_t^{\gamma'}W^{s,\rho'}_{x,\mathrm{rad}}(I\times\mathbb{R}^n)$, then for every admissible pair $(q,r)\in SA$ the function*

$$t \mapsto \varphi_f(t) := \int_{t_0}^t e^{-i(t-\tau)(-\Delta)^s} f(\tau)\,\mathrm{d}\tau \quad \textit{for } t\in I \tag{C.14}$$

*belongs to $L_t^q W^{s,r}_{x,\mathrm{rad}}(I\times\mathbb{R}^n)\cap C(J; H^s_{x,\mathrm{rad}}(\mathbb{R}^n))$. Furthermore, there exists a constant $C$ independent of $I$ such that*

$$\|\varphi_f\|_{L_t^q W_x^{s,r}(I\times\mathbb{R}^n)} \leqslant C\,\|f\|_{L_t^{\gamma'}W_x^{s,\rho'}(I\times\mathbb{R}^n)} \quad \textit{for every } f\in L_t^{\gamma'}W^{s,\rho'}_{x,\mathrm{rad}}(I\times\mathbb{R}^n). \tag{C.15}$$

**Corollary C.5.** *Let $I=(T,\infty)$ for some $T\geqslant -\infty$ and let $J=\overline{I}$. Let $(\gamma,\rho)\in SA$ be an admissible pair, and let $f\in L_t^{\gamma'}W^{s,\rho'}_{x,\mathrm{rad}}(I\times\mathbb{R}^n)$. It follows that the function*

$$t\mapsto \Phi_f(t) := \int_t^\infty e^{-i(t-\tau)(-\Delta)^s} f(\tau)\,\mathrm{d}\tau \quad \textit{for every } t\in J$$

*makes sense as the uniform limit in $H^s_{x,\mathrm{rad}}(\mathbb{R}^n)$, as $m\to\infty$, of the functions*

$$\Phi_f^m(t) := \int_t^m e^{-i(t-\tau)(-\Delta)^s} f(\tau)\,\mathrm{d}\tau \quad \textit{for every } t\in J.$$

*In addition for every admissible pair $(q,r) \in SA$, we have $\Phi_f \in L_t^q W_{x,\mathrm{rad}}^{s,r}(I \times \mathbb{R}^n) \cap C(J; H_{x,\mathrm{rad}}^s(\mathbb{R}^n))$. Furthermore, there exists a constant $C$ such that*

$$\|\Phi_f\|_{L_t^q W_x^{s,r}(I\times\mathbb{R}^n)} \leqslant C\,\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)} \quad \textit{for every } f \in L_t^{\gamma'} W_{x,\mathrm{rad}}^{s,\rho'}(I \times \mathbb{R}^n). \tag{C.16}$$

*Proof of Corollary C.5.* Let $j, m$ be two integers, $T < j < m$. For every $t \in J$,

$$\begin{aligned}\|\Phi_f^m(t) - \Phi_f^j(t)\|_{H^s} &= \|e^{-i(m-t)(-\Delta)^s}(\Phi_f^m(t) - \Phi_f^j(t))\|_{H^s} \\ &= \Big\|\int_j^m e^{-i(m-\tau)(-\Delta)^s} f(\tau)\,\mathrm{d}\tau\Big\|_{H^s}\end{aligned}$$

using the $H_x^s$ isometry and the semigroup property of the propagator. By (C.15), there exists a constant $C$ such that

$$\begin{aligned}\|\Phi_f^m(t) - \Phi_f^j(t)\|_{H^s} \leqslant \big\|\Phi_f^m - \Phi_f^j\big\|_{L_t^\infty W_x^{s,2}((j,\infty)\times\mathbb{R}^n)} &\overset{(C.15)}{\leqslant} C\,\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}((j,\infty)\times\mathbb{R}^n)} \\ &\to 0 \quad \text{as } (m>)j \to \infty.\end{aligned}$$

[We used $\gamma' < \infty$ for $(\gamma,\rho) \in SA$ and the hypothesis $\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)} < \infty$.] Thus $(\Phi_f^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L_t^\infty W_{x,\mathrm{rad}}^{s,2}(I \times \mathbb{R}^n))$, with all members in $C(J; H_{x,\mathrm{rad}}^s(\mathbb{R}^n))$ according to the previous theorem. The uniform limit $\Phi_f$ is therefore also in $C(J; H_{x,\mathrm{rad}}^s(\mathbb{R}^n))$, and we have the estimate

$$\|\Phi_f\|_{L_t^\infty W_x^{s,2}(I\times\mathbb{R}^n)} = \lim_{m\to\infty} \big\|\Phi_f^m\big\|_{L_t^\infty W_x^{s,2}(I\times\mathbb{R}^n)} \leqslant C\,\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)} \tag{C.17}$$

[equality holds since $\Phi_f^m \to \Phi_f$ strongly in $L_t^\infty W_x^{s,2}(I \times \mathbb{R}^n)$, and the inequality follows from the estimates $\big\|\Phi_f^m\big\|_{L_t^\infty W_x^{s,2}(I\times\mathbb{R}^n)} \leqslant C\,\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)}$ given by (C.15) above]. Finally, given any admissible pair $(q,r) \in SA$, it follows from (C.15) that there exists a constant $C$ such that

$$\big\|\Phi_f^m\big\|_{L_t^q W_x^{s,r}(I\times\mathbb{R}^n)} \leqslant C\,\|f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)}\,. \tag{C.18}$$

For $j \in \mathbb{N}$, $j \geqslant T$, define $f_j \in L_t^{\gamma'} W_{x,\mathrm{rad}}^{s,\rho'}(I \times \mathbb{R}^n)$ by

$$f_j(t) := \begin{cases} f(t) & \text{if } t \leqslant j \\ 0 & \text{if } t > j. \end{cases}$$

By dominated convergence (e.g., [CH98, Cor. 1.4.15, p. 8]), we have $f_j \to f$ in $L_t^{\gamma'} W_x^{s,\rho'}(I \times \mathbb{R}^n)$; indeed, abbreviating $X = W_x^{s,\rho'}(\mathbb{R}^n)$ and denoting $\|\cdot\|_X$ its norm, we define $g_j : I \to \mathbb{R}$ by $g_j(t) := \|f_j(t) - f(t)\|_X^{\gamma'}$. We have $(\gamma' \geqslant 1)$

$$|g_j(t)| \leqslant \big(\|f_j(t)\|_X + \|f(t)\|_X\big)^{\gamma'} \lesssim_{\gamma'} \|f_j(t)\|_X^{\gamma'} + \|f(t)\|_X^{\gamma'} =: h(t), \quad \text{a.e. } t \in I,$$

where

$$\int_I h(t)\,\mathrm{d}t \lesssim_{\gamma'} 2\,\|f\|_{L^{\gamma'}(I;X)}^{\gamma'} < +\infty$$

due to $f \in L_t^{\gamma'} W_x^{s,\rho'}(I \times \mathbb{R}^n)$. In particular, $(g_j)_{j\in\mathbb{N}}$ is a sequence of integrable functions $I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ an integrable dominating function. Since by definition of the $f_j$'s we clearly have $g_j(t) \to 0$ for a.e. $t \in I$, we conclude from dominated convergence that $0 = \lim_{j\to\infty} \int_I g_j(t)\,\mathrm{d}t = \lim_{j\to\infty} \|f_j - f\|_{L^{\gamma'}(I;X)}^{\gamma'}$, giving

$$\lim_{j\to\infty} \|f_j - f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)} = 0.$$

From this and (C.17) we deduce that

$$\Phi_{f_j} \to \Phi_f \quad \text{in } H_x^s(\mathbb{R}^n) \text{ uniformly in } t \in J. \tag{C.19}$$

To see this, note that by definition

$$\begin{aligned}\Phi_{f_j}^m(t) - \Phi_f^m(t) &= \int_t^m e^{-i(t-\tau)(-\Delta)^s} f_j(\tau)\,\mathrm{d}\tau - \int_t^m e^{-i(t-\tau)(-\Delta)^s} f(\tau)\,\mathrm{d}\tau \\ &= \int_t^m e^{-i(t-\tau)(-\Delta)^s} (f_j(\tau) - f(\tau))\,\mathrm{d}\tau = \Phi_{f_j-f}^m(t).\end{aligned}$$

That is, $\Phi_{f_j}^m - \Phi_f^m = \Phi_{f_j-f}^m$, where as before $(\Phi_{f_j-f}^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L_t^\infty W_{x,\mathrm{rad}}^{s,2}(I\times\mathbb{R}^n)$ with limit $\Phi_{f_j-f} \in C(J; H_x^s(\mathbb{R}^n))$, and according to (C.17) we have the estimate

$$\left\|\Phi_{f_j-f}\right\|_{L_t^\infty W_x^{s,2}(I\times\mathbb{R}^n)} \leqslant C\,\|f_j - f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)}.$$

Note (abbreviating for a moment $\|\cdot\|_{L_t^\infty W_x^{s,2}(I\times\mathbb{R}^n)}$ by $\|\cdot\|$)

$$\left\|\Phi_{f_j} - \Phi_f\right\| \leqslant \left\|\Phi_{f_j} - \Phi_{f_j}^m\right\| + \left\|\Phi_{f_j-f}^m\right\| + \left\|\Phi_f^m - \Phi_f\right\|,$$

where

$$\begin{aligned}\left\|\Phi_{f_j-f}^m\right\| &\leqslant \left\|\Phi_{f_j-f}^m - \Phi_{f_j-f}\right\| + \left\|\Phi_{f_j-f}\right\| \\ &\leqslant \left\|\Phi_{f_j-f}^m - \Phi_{f_j-f}\right\| + C\,\|f_j - f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)}.\end{aligned}$$

Let now $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that for all $j \geqslant N$ it holds $C\,\|f_j - f\|_{L_t^{\gamma'} W_x^{s,\rho'}(I\times\mathbb{R}^n)} < \frac{\varepsilon}{4}$. Let now $j \geqslant N$ be arbitrary, but fixed. From the known limits

$$\left\|\Phi_{f_j} - \Phi_{f_j}^m\right\| \overset{m}{\to} 0, \quad \left\|\Phi_f^m - \Phi_f\right\| \overset{m}{\to} 0, \quad \left\|\Phi_{f_j-f}^m - \Phi_{f_j-f}\right\| \overset{m}{\to} 0,$$

we see that fixing $m = m(j) \in \mathbb{N}$ large enough, we obtain

$$\left\|\Phi_{f_j} - \Phi_f\right\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $j \geqslant N$ we have $\left\|\Phi_{f_j} - \Phi_f\right\| < \varepsilon$, and (C.19) is proved. By definition of the $f_j$'s, note that for $m \geqslant j$, $\Phi^m_{f_j}$ does not depend on $m$ (in fact, in that case $\Phi^m_{f_j}(t) = \Phi^j_{f_j}(t) = \Phi_{f_j}(t)$). From (C.18),

$$\left\|\Phi_{f_j}\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} \leqslant C \left\|f_j\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(I\times\mathbb{R}^n)} \leqslant C \left\|f\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(I\times\mathbb{R}^n)}, \tag{C.20}$$

we therefore have $\Phi_{f_j} \in L^q_t W^{s,r}_x(I\times\mathbb{R}^n)$. Furthermore, when $T \leqslant j \leqslant k$, we deduce from the definition of the $f_j$'s that $\Phi_{f_j}(t) - \Phi_{f_k}(t) = \Phi^k_{f_j - f_k}(t)$, hence by (C.18)

$$\begin{aligned}\left\|\Phi_{f_j} - \Phi_{f_k}\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} = \left\|\Phi^k_{f_j-f_k}\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} &\overset{(C.18)}{\leqslant} C \left\|f_j - f_k\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(I\times\mathbb{R}^n)} \\ &= C \left\|f\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(( j,k]\times\mathbb{R}^n)}.\end{aligned}$$

But the right side goes to 0 as $(j,k) \to (\infty,\infty)$ by elementary integrability reasons, and so $(\Phi_{f_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^q_t W^{s,r}_x(I\times\mathbb{R}^n)$, which possesses a limit $\Psi \in L^q_t W^{s,r}_x(I\times\mathbb{R}^n)$ such that

$$\left\|\Psi\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} = \lim_{j\to\infty} \left\|\Phi_{f_j}\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} \overset{(C.20)}{\leqslant} C \left\|f\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(I\times\mathbb{R}^n)}. \tag{C.21}$$

In particular, since now $\Phi_{f_j} \to \Psi$ in $L^q_t W^{s,r}_x(I\times\mathbb{R}^n)$, and moreover $\Phi_{f_j} \to \Phi_f$ in $L^\infty_t W^{s,2}_x(I\times\mathbb{R}^n)$ by (C.19), there exists a subsequence (still denoted $(\Phi_{f_j})$) such that for a.e. $t \in I$,

$$\begin{cases}\Phi_{f_j}(t) \to \Psi(t) & \text{in } W^{s,r}_x(\mathbb{R}^n),\\ \Phi_{f_j}(t) \to \Phi_f(t) & \text{in } W^{s,2}_x(\mathbb{R}^n).\end{cases} \tag{$*$}$$

Abbreviate $Y = W^{s,r}_x(\mathbb{R}^n)$, $Z = W^{s,2}_x(\mathbb{R}^n)$ and denote the norms by $\|\cdot\|_Y, \|\cdot\|_Z$, respectively. We claim that $\Psi(t) = \Phi_f(t)$ in $Y \cap Z$ for a.e. $t$. Indeed, denote $N \subset I$ the set of measure zero outside of which ($*$) is valid and let $t \in I\backslash N$. The sequence $(\Phi_{f_j})_{j\in\mathbb{N}}$ is by ($*$) a Cauchy sequence in both $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$, hence also in the space $(Y\cap Z, \|\cdot\|_{Y\cap Z})$, where $\|\cdot\|_{Y\cap Z} = \max\{\|\cdot\|_Y, \|\cdot\|_Z\}$. By completeness of the latter space, there exists $\chi(t) \in Y \cap Z$ such that

$$\left\|\Phi_{f_j}(t) - \chi(t)\right\|_{Y\cap Z} \to 0,$$

but this implies both $\left\|\Phi_{f_j}(t) - \chi(t)\right\|_Y \to 0$, $\left\|\Phi_{f_j}(t) - \chi(t)\right\|_Z \to 0$. By uniqueness of limits in $Y$ and $Z$, we conclude from ($*$) that $\Psi(t) = \chi(t) = \Phi_f(t)$. Since $t \in I\backslash N$ was arbitrary, we have found

$$\Psi(t) = \Phi_f(t) \text{ in } W^{s,r}_x(\mathbb{R}^n) \cap W^{s,2}_x(\mathbb{R}^n) \quad \text{for a.e. } t \in I.$$

Thus (C.21) gives $\left\|\Phi_f\right\|_{L^q_t W^{s,r}_x(I\times\mathbb{R}^n)} \leqslant C \left\|f\right\|_{L^{\gamma'}_t W^{s,\rho'}_x(I\times\mathbb{R}^n)}$. The proof of corollary C.5 is now complete. □

## C.5 Morawetz's Estimate

**Proposition C.6** (Morawetz's Estimate)**.** *Let $n \geqslant 3$ and $\frac{1}{2} \leqslant s < 1$.*[3] *If $u \in C(\mathbb{R}_t; H^s(\mathbb{R}^n))$ solves defocusing (fNLS), then we have the global space-time bound*

$$\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} \frac{|u(t,x)|^{2\sigma+2}}{|x|}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(n,s,\sigma,\|u_0\|_{L^2_x}, E[u_0]) < +\infty. \tag{C.22}$$

There is a classical proof in case $s = 1$; see also [Caz03, Corollary 7.6.6, p. 238]. For $0 < s < 1$, we can use Balakrishnan's formula as in Chapter 2 to treat the dispersive part in (C.23) below. We give an alternative proof via the extension problem for the fractional Laplacian [CS07].

### Proof of Morawetz's Estimate via $s$-Harmonic Extension

As in Chapter 2, let us define the localized virial of the solution $u$ ("Morawetz action" [CKS+04]) by the quantity

$$\mathcal{M}_a[u(t)] := \langle u(t), \Gamma_a u(t)\rangle.$$

Here, $a : \mathbb{R}^n \to \mathbb{R}$ is a given virial weight function which will be chosen adequately, and $\Gamma_a$ is the formally self-adjoint differential operator

$$\Gamma_a := -i(\nabla\cdot\nabla a + \nabla a\cdot\nabla)$$

acting on functions via

$$\Gamma_a f := -i(\nabla a\cdot\nabla f + \mathrm{div}((\nabla a)f)) = -2i\nabla a\cdot\nabla f - i(\Delta a)f.$$

As in (2.24), we compute

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_a[u(t)] = \langle u(t), [(-\Delta)^s, i\Gamma_a]u(t)\rangle + \langle u(t), [|u|^{2\sigma}, i\Gamma_a]u(t)\rangle.} \tag{C.23}$$

We discuss the terms on the right side of (C.23) as follows.

**Step 1 (Dispersive Term):** As $\Gamma_a$ is self-adjoint ($\Gamma_a^* = \Gamma_a$), clearly $i\Gamma_a$ is skew-adjoint ($(i\Gamma_a)^* = -i\Gamma_a$). This plus the self-adjointness of the fractional Laplacian $(-\Delta)^s$ gives

$$\begin{aligned}\langle u, [(-\Delta)^s, i\Gamma_a]u\rangle &= \langle u, (-\Delta)^s(i\Gamma_a u)\rangle - \langle u, i\Gamma_a((-\Delta)^s u)\rangle \\ &= \langle(-\Delta)^s u, i\Gamma_a u\rangle + \langle i\Gamma_a u, (-\Delta)^s u\rangle \\ &= 2\,\mathrm{Re}\langle i\Gamma_a u, (-\Delta)^s u\rangle.\end{aligned} \tag{C.24}$$

[3] In particular, this implies $0 \leqslant s < \frac{n}{2}$, which guarantees the applicability of Hardy's inequality (cf. [BCD13, p. 91]). The cases $n \geqslant 3, s = 1$ are also allowed; use a classical proof of Morawetz inequality, since Balakrishnan's formula holds for $0 < s < 1$.

As in [CS07], we consider the extension problem related to the fractional Laplacian. That is, we consider a function $U(x,y)$ defined on the upper-half space $\mathbb{R}^{n+1}_+ = \{(x,y); x \in \mathbb{R}^n, y > 0\}$ solving the elliptic Dirichlet boundary value problem

$$\begin{cases} \operatorname{div}_{x,y}\left(y^{1-2s}\nabla_{x,y}U\right) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{C.25}$$

The fractional Laplacian of $u$ is then characterized (up to a multiplicative constant) as the Dirichlet-to-Neumann operator in the sense that [FLS16, p. 1684]

$$-d_s \lim_{y\to 0} y^{1-2s}\frac{\partial U}{\partial y} = (-\Delta)^s u, \quad \text{where } d_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)} > 0. \tag{C.26}$$

From (C.25) and integration by parts, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{n+1}_+} i\Gamma_a \overline{U} \operatorname{div}_{x,y}\left(y^{1-2s}\nabla_{x,y}U\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\partial\mathbb{R}^{n+1}_+} i\Gamma_a\overline{U} y^{1-2s}\nabla_{x,y}U\cdot\nu\,\mathrm{d}S(x,y) - \int_{\mathbb{R}^{n+1}_+}\nabla_{x,y}(i\Gamma_a\overline{U})\cdot y^{1-2s}\nabla_{x,y}U\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

As $\nu = -e_y = (0,\dots,0,-1)$ is the outer unit normal to $\partial\mathbb{R}^{n+1}_+$, this implies (when inserting the boundary conditions from (C.25) and the relation (C.26) in the boundary integral) that

$$\frac{1}{d_s}\langle i\Gamma_a u, (-\Delta)^s u\rangle = -\int_{\mathbb{R}^{n+1}_+}\nabla_{x,y}(i\Gamma_a\overline{U})\cdot y^{1-2s}\nabla_{x,y}U\,\mathrm{d}x\,\mathrm{d}y.$$

Thus

$$\begin{aligned} \frac{1}{d_s}\operatorname{Re}\langle i\Gamma_a u, (-\Delta)^s u\rangle &= -\operatorname{Re}\int_{\mathbb{R}^{n+1}_+}\nabla_x(i\Gamma_a\overline{U})\cdot y^{1-2s}\nabla_x U\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \operatorname{Re}\int_{\mathbb{R}^{n+1}_+}\frac{\partial}{\partial y}(i\Gamma_a\overline{U})\left(y^{1-2s}\frac{\partial}{\partial y}U\right)\mathrm{d}x\,\mathrm{d}y =: (I) + (II). \end{aligned}$$

We claim $(II) = 0$ and $(I) \geqslant 0$ (thus the dispersive part has a definite sign). To see this, recall that by skew-adjointness of the operator $i\Gamma_a$ the numbers $\langle -i\Gamma_a f, f\rangle \in i\mathbb{R}$ must be purely imaginary (or zero).[4] Since $\Gamma_a$ does not depend on $y$, we get indeed

$$\begin{aligned} (II) &= -\operatorname{Re}\int_{y>0}\left(\int_{\mathbb{R}^n}\left(i\Gamma_a\frac{\partial\overline{U}}{\partial y}\right)\frac{\partial U}{\partial y}\,\mathrm{d}x\right)y^{1-2s}\,\mathrm{d}y \\ &= -\int_{y>0}\underbrace{\operatorname{Re}\left\langle -i\Gamma_a\frac{\partial U}{\partial y},\frac{\partial U}{\partial y}\right\rangle}_{=0} y^{1-2s}\,\mathrm{d}y = 0. \end{aligned}$$

[4] Clearly, $\overline{\langle i\Gamma_a f, f\rangle} = \overline{\langle f, (i\Gamma_a)^* f\rangle} = \overline{\langle f, -i\Gamma_a f\rangle} = -\langle i\Gamma_a f, f\rangle$, hence $\operatorname{Re}\langle i\Gamma_a f, f\rangle = 0$.

As for $(I)$, we integrate by parts in $x$ to observe (notice again $\nu^k = (-e_y)^k = 0$ for $k = 1, \ldots, n$ for the outer unit normal to $\partial\mathbb{R}^{n+1}_+$) [Einstein summation convention]

$$\begin{aligned}(I) &= -\operatorname{Re}\left\{\int_{\partial\mathbb{R}^{n+1}_+} i\Gamma_a\overline{U}y^{1-2s}\underbrace{\partial_{x_k}U\nu^k}_{=0}\,\mathrm{d}S(x,y) - \int_{\mathbb{R}^{n+1}_+} i\Gamma_a\overline{U}y^{1-2s}\partial^2_{x_k}U\,\mathrm{d}x\,\mathrm{d}y\right\}\\ &= +\operatorname{Re}\int_{y>0}\langle -i\Gamma_a U(\cdot,y), \Delta_x U(\cdot,y)\rangle y^{1-2s}\,\mathrm{d}y\\ &= +\frac{1}{2}\int_{y>0}\langle U(\cdot,y), [-\Delta_x, i\Gamma_a]U(\cdot,y)\rangle y^{1-2s}\,\mathrm{d}y,\end{aligned}$$

where the last equality used

$$\operatorname{Re}\langle -i\Gamma_a U, \Delta_x U\rangle = \frac{1}{2}\left(\langle -i\Gamma_a U, \Delta_x U\rangle + \langle \Delta_x U, -i\Gamma_a U\rangle\right) = \frac{1}{2}\langle U, [-\Delta_x, i\Gamma_a]U\rangle$$

by skew-adjointness of $i\Gamma_a$ and self-adjointness of $\Delta_x$. Collecting the above results and going back to (C.24), we have found for the dispersive part the following expression remarkably only containing local differential operators:

$$\langle u, [(-\Delta)^s, i\Gamma_a]u\rangle = d_s\int_{y>0}\langle U(\cdot,y), [-\Delta_x, i\Gamma_a]U(\cdot,y)\rangle y^{1-2s}\,\mathrm{d}y. \tag{C.27}$$

As in (2.28), we have

$$[-\Delta_x, i\Gamma_a] = -4\partial_{x_k}(\partial^2_{x_kx_l}a)\partial_{x_l} - \Delta_x^2 a.$$

Integrating by parts gives

$$\begin{aligned}\langle U(\cdot,y), [-\Delta_x, i\Gamma_a]U(\cdot,y)\rangle = +4\int_{\mathbb{R}^n}&\overline{\partial_{x_k}U(x,y)}\partial^2_{x_kx_l}a(x)\partial_{x_l}U(x,y)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^n}\Delta_x^2 a(x)|U(x,y)|^2\,\mathrm{d}x =: (I)' + (II)'.\end{aligned}$$

Now we specify to $a(x) = |x|$ (we consider the Morawetz action centered at 0). We easily check $\partial^2_{x_kx_l}a(x) = \left(\delta_{kl} - \frac{x_kx_l}{|x|^2}\right)\frac{1}{|x|}$, and we obtain

$$\begin{aligned}(I)' &= +4\int_{\mathbb{R}^n}\left\{\frac{|\nabla_x U(x,y)|^2}{|x|} - \left(\frac{x}{|x|}\cdot\overline{\nabla_x U}\right)\left(\frac{x}{|x|}\cdot\nabla_x U\right)\frac{1}{|x|}\right\}\mathrm{d}x\\ &= +4\int_{\mathbb{R}^n}\frac{|\not\nabla_x U(x,y)|^2}{|x|}\,\mathrm{d}x.\end{aligned}$$

Herein, for a function $f : \mathbb{R}^n \to \mathbb{C}$, the (by Cauchy-Schwarz on $\mathbb{C}^n$ non-negative) quantity

$$|\not\nabla_x f|^2 := |\nabla_x f|^2 - \left|\frac{x}{|x|}\cdot\nabla_x f\right|^2 \geqslant 0$$

is called *angular part of the gradient of* $f$.[5] Therefore $(I)' \geqslant 0$. As for $(II)'$, first observe that for $a(x) = |x|$ we have $\Delta_x a(x) = \frac{n-1}{|x|}$ and $-\Delta_x^2 a(x) = \frac{(n-1)(n-3)}{|x|^3}$ $(x \neq 0)$. The function

$$\Phi(x) = \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, \quad n \geqslant 3,\ x \neq 0 \tag{C.28}$$

is the fundamental solution to Laplace's equation, i.e., $-\Delta_x \Phi = \delta_0$ in $\mathbb{R}^n$ in the sense of distributions. When $n = 3$ then $\Delta_x a(x) = \frac{2}{|x|} = 6\omega_3 \Phi(x)$ $(x \neq 0)$ and therefore $-\Delta_x^2 a(x) = 6\omega_3 \delta_0$ in $\mathbb{R}^3$ in the sense of distributions. It follows that

$$(II)' = \begin{cases} 6\omega_3 |U(0,y)|^2 \geqslant 0, & n = 3, \\ \int_{\mathbb{R}^n} \frac{(n-1)(n-3)}{|x|^3} |U(x,y)|^2 \,\mathrm{d}x \geqslant 0, & n \geqslant 4. \end{cases} \tag{C.29}$$

[Notice that for $n \geqslant 4$ the function $g(x) = \frac{1}{|x|^3}$ is integrable in a neighborhood around zero, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, so the singularity there poses no problem.] Inserting $(I)', (II)' \geqslant 0$ back into (C.27), we conclude that the dispersive part is nonnegative:

$$\langle u, [(-\Delta)^s, i\Gamma_a] u \rangle \geqslant 0. \tag{C.30}$$

**Step 2 (Nonlinear Term):** Now we turn to the nonlinear term $\langle u, [|u|^{2\sigma}, i\Gamma_a] u \rangle$. By definition of $i\Gamma_a$ and

$$\operatorname{div}(\nabla a(|u|^{2\sigma} u)) = \operatorname{div}(u \nabla a)|u|^{2\sigma} + u \nabla a \cdot \nabla(|u|^{2\sigma})$$

and $\nabla(|u|^{2\sigma} u) = \nabla(|u|^{2\sigma}) u + |u|^{2\sigma} \nabla u$, we obtain by a direct calculation that

$$\begin{aligned} \langle u, [|u|^{2\sigma}, i\Gamma_a] u \rangle &= -2 \int_{\mathbb{R}^n} |u|^2 \nabla a \cdot \nabla(|u|^{2\sigma}) \,\mathrm{d}x \\ &= -\frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^n} \nabla a \cdot \nabla(|u|^{2\sigma+2}) \,\mathrm{d}x \\ &= \underbrace{+\frac{2\sigma}{\sigma+1}(n-1)}_{=:c_{n,\sigma} > 0} \int_{\mathbb{R}^n} \frac{|u|^{2\sigma+2}}{|x|} \,\mathrm{d}x. \end{aligned}$$

In the last equality, an integration by parts was performed and it was used that for $a(x) = |x|$ the Laplacian is given by $\Delta a = \frac{n-1}{|x|}$; the middle equality uses the identity $\nabla(|u|^{2\sigma+2}) = \frac{\sigma+1}{\sigma} \nabla(|u|^{2\sigma})|u|^2$, which one easily verifies. Hence

$$\langle u, [|u|^{2\sigma}, i\Gamma_a] u \rangle = c_{n,\sigma} \int_{\mathbb{R}^n} \frac{|u|^{2\sigma+2}}{|x|} \,\mathrm{d}x, \quad c_{n,\sigma} > 0. \tag{C.31}$$

[5] Note that if $f = f(|x|)$ is a radial function, then $\nabla_x f = (\partial_r f) \frac{x}{|x|}$ and hence the angular part vanishes:

$$|\not\nabla_x f|^2 = \left| (\partial_r f) \frac{x}{|x|} \right|^2 - \left| \frac{x}{|x|} \cdot (\partial_r f) \frac{x}{|x|} \right|^2 = |\partial_r f|^2 - |\partial_r f|^2 = 0.$$

**Step 3 (Conclusion):** Putting (C.31) and (C.30) into (C.23), we conclude the monotonicity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_a[u(t)] &= \langle u(t), [(-\Delta)^s, i\Gamma_a]u(t)\rangle + \langle u(t), [|u|^{2\sigma}, i\Gamma_a]u(t)\rangle \\ &\geqslant c_{n,\sigma}\int_{\mathbb{R}^n}\frac{|u|^{2\sigma+2}}{|x|}\,\mathrm{d}x, \quad c_{n,\sigma}>0.\end{aligned}$$

By time integration[6]

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\frac{|u|^{2\sigma+2}}{|x|}\,\mathrm{d}x\,\mathrm{d}t &\leqslant \frac{1}{c_{n,\sigma}}\Big\{\langle u(T),\Gamma_a u(T)\rangle - \langle u(0),\Gamma_a u(0)\rangle\Big\} \\ &\leqslant \frac{2}{c_{n,\sigma}}\sup_{t\in\{0,T\}}|\langle u(t),\Gamma_a u(t)\rangle| \\ &\lesssim_{n,\sigma}\sup_{t\in[0,\infty)}|\langle u(t),\Gamma_a u(t)\rangle| \\ &\overset{(*)}{\lesssim_{n,\sigma}}\sup_{t\in[0,\infty)}\|u(t)\|^2_{\dot H^{\frac12}} \\ &\lesssim_{n,\sigma}\sup_{t\in[0,\infty)}\|u(t)\|_{L^2}^{2-\frac1s}\||\nabla|^s u(t)\|_{L^2}^{\frac1s} \\ &\leqslant C(n,\sigma,s,\|u_0\|_{L^2},E[u_0])\end{aligned} \tag{C.32}$$

for any $T\geqslant 0$. In the second to last step, we have once again used the exact interpolation inequality [BCD13, Proposition 1.32]

$$\|u\|_{\dot H^{\tilde s}}\leqslant\|u\|^{1-\theta}_{\dot H^{s_0}}\|u\|^{\theta}_{\dot H^{s_1}},\quad \tilde s=(1-\theta)s_0+\theta s_1$$

with $\tilde s=\frac12$, $s_0=0$, $s_1=s$. In the last step, we recalled conservation of $L^2$ mass and total energy and noted that in the defocusing case, the kinetic energy is a priori bounded by the total energy as seen in

$$\||\nabla|^s u(t)\|^2_{L^2}\leqslant 2\left(\frac12\||\nabla|^s u(t)\|^2_{L^2}+\frac{1}{2\sigma+2}\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}}\right)=2E[u(t)]\lesssim E[u_0]. \tag{C.33}$$

It remains to show $(*)$, which follows from a Hardy inequality plus some interpolation argument. Indeed, from

$$\Gamma_a u = -2i\nabla a\cdot\nabla u - i(\Delta a)u = -2i\left\{\frac{x}{|x|}\cdot\nabla u+\frac{n-1}{2|x|}u\right\}$$

we get

$$|\langle u(t),\Gamma_a u(t)\rangle|\lesssim_n\left|\left\langle u(t),\frac{x}{|x|}\cdot\nabla u\right\rangle\right|+\left|\left\langle u(t),\frac{u(t)}{|x|}\right\rangle\right|. \tag{C.34}$$

[6]We hypothesized global wellposedness, i.e. $u\in C(\mathbb{R};H^s_x(\mathbb{R}^n))$ for the solution.

For the second term, Hardy's inequality[7] (see e.g. [BCD13, Theorem 2.57] or [Tao06, Lemma A.2]) states

$$\left|\left\langle u(t), \frac{u(t)}{|x|}\right\rangle\right| = \int_{\mathbb{R}^n} \frac{|u(t,x)|^2}{|x|}\,\mathrm{d}x \lesssim_n \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{C.35}$$

For the first term, we conclude from the momentum estimate [Tao06, Lemma A.10], taking $v = u$ there, that

$$\left|\left\langle u(t), \frac{x}{|x|}\cdot \nabla u(t)\right\rangle\right| \leqslant \sum_{j=1}^{n} \left|\left\langle u(t), \frac{x_j}{|x|}\partial_j u(t)\right\rangle\right| \lesssim_n \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{C.36}$$

Putting (C.36) and (C.35) back into (C.34), we see that $(*)$ holds. Thus (C.32) is now legitimate. By arbitrariness of $T \geqslant 0$ and by a similar argument for $T < 0$, we prove Morawetz's estimate

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{|u(t,x)|^{2\sigma+2}}{|x|}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(n, s, \sigma, \|u_0\|_{L^2_x}, E[u_0]).$$

The proof of Proposition C.6 is now complete. □

## C.6 Elementary Pointwise Bound for the Power-Nonlinearity

**Lemma C.7** (Elementary Pointwise Bound for the Power-Nonlinearity.)**.** *For $\alpha > 0$, we have the pointwise bound*

$$||z|^\alpha z - |w|^\alpha w| \leqslant C\left(|z|^\alpha + |w|^\alpha\right)|z - w|, \quad z, w \in \mathbb{C}, \tag{C.37}$$

*with a constant $C$ depending only on $\alpha$.*[8] *In particular, the function $\tilde{g} : \mathbb{C} \to \mathbb{C}$,*

$$\tilde{g}(z) := \begin{cases} |z|^\alpha z, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

*is Lipschitz-continuous on bounded subsets of $\mathbb{C}$.*[9]

[7] Since $0 \leqslant s < \frac{n}{2}$ for $0 < s < 1$ and $n \geqslant 3$, this is applicable.

[8] For instance, $C = 3(\alpha + 1)$ is sufficient, as shown in the proof.

[9] As in [CH98, p. 55], a function $f : X \to X$, $(X, \|\cdot\|)$ an underlying Banach space, is called Lipschitz-continuous on bounded subsets of $X$, if for any $M > 0$ there exists some constant $L(M)$ such that

$$\|f(x) - f(y)\| \leqslant L(M)\|x - y\|, \quad \text{for all } x, y \in B_M(0), \tag{C.38}$$

where $B_M(0) := \{x \in X;\ \|x\| \leqslant M\}$ is the (closed) ball of radius $M$ centered at the origin.

*Proof (cf. [Caz03, p. 60]).* Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) := \begin{cases} |x|^\alpha x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It satisfies

$$g(0) = 0. \tag{C.39}$$

Since $\alpha > 0$, we have

$$\lim_{h\to 0,\ h\neq 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0, h\neq 0} \frac{|h|^\alpha h}{h} = \lim_{h\to 0, h\neq 0} |h|^\alpha \overset{(\alpha>0)}{=} 0,$$

hence the derivative of $g$ at 0 exists with $g'(0) = 0$. Thus $g$ is differentiable (in particular continuous) on all of $\mathbb{R}$ with derivative

$$g'(x) = \begin{cases} (\alpha+1)|x|^\alpha, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Note that $g$ satisfies

$$|g(x) - g(y)| \leqslant (\alpha+1)(|x|^\alpha + |y|^\alpha)|x-y|, \quad x, y \in \mathbb{R}. \tag{C.40}$$

Indeed, given $x, y \in \mathbb{R}$, $x \neq y$, by the mean value theorem of differentiation there exists $\xi \in (\min\{x,y\}, \max\{x,y\})$ such that

$$\left| \frac{g(x) - g(y)}{x - y} \right| = |g'(\xi)| \leqslant (\alpha+1)|\xi|^\alpha \leqslant (\alpha+1)(|x|^\alpha + |y|^\alpha),$$

where we used $|\xi| \leqslant \max\{|x|, |y|\}$ to get the last inequality.[10] Now we extend $g$ to the complex plane $\mathbb{C}$ by defining the function $\tilde{g} : \mathbb{C} \to \mathbb{C}$,

$$\tilde{g}(z) := \begin{cases} \frac{z}{|z|} g(|z|), & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{C.41}$$

Note that $\tilde{g}(z) = |z|^\alpha z$ for $z \in \mathbb{C}\backslash\{0\}$. From (C.41), (C.39) and (C.40) we have

$$|\tilde{g}(w)| \overset{(C.41)}{=} |g(|w|)| \overset{(C.39)}{=} |g(|w|) - g(0)| \overset{(C.40)}{\leqslant} (\alpha+1)|w|^\alpha |w|, \quad w \in \mathbb{C}, w \neq 0. \tag{C.42}$$

Let now $z, w \in \mathbb{C}\backslash\{0\}$ (for $z = 0$ or $w = 0$, the claim (C.37) is clear by (C.41) and (C.42)). From (C.41) one computes

$$\begin{aligned} |z||w|(\tilde{g}(z) - \tilde{g}(w)) &\overset{(C.41)}{=} |w| z g(|z|) - |z| w g(|w|) \\ &= |w| z (g(|z|) - g(|w|)) + (|w| z - |z| w) g(|w|) \\ &= |w| z (g(|z|) - g(|w|)) + \{(|w| - |z|) z + |z|(z - w)\} g(|w|), \end{aligned}$$

[10] From $|\xi| \leqslant \max\{|x|, |y|\}$ we obtain $|\xi|^\alpha \leqslant \max\{|x|^\alpha, |y|^\alpha\} \leqslant |x|^\alpha + |y|^\alpha$.

so taking the modulus gives

$$\begin{aligned}
|z||w||\widetilde{g}(z)-\widetilde{g}(w)| &\leqslant |w||z||g(|z|)-g(|w|)| + \big(\big||w|-|z|\big||z| + |z||z-w|\big)\,|g(|w|)| \\
&\leqslant |w||z||g(|z|)-g(|w|)| + 2|z-w||z||g(|w|)| \\
&\overset{\text{(C.42)}}{\leqslant} |w||z||g(|z|)-g(|w|)| + 2|z-w||z|(\alpha+1)|w|^{\alpha}|w| \\
&\overset{\text{(C.40)}}{\leqslant} |w||z|(\alpha+1)(|z|^{\alpha}+|w|^{\alpha})\big||z|-|w|\big| + 2|z-w||z|(\alpha+1)|w|^{\alpha}|w| \\
&\leqslant 3|w||z||z-w|(\alpha+1)\left(|z|^{\alpha}+|w|^{\alpha}\right).
\end{aligned}$$

Dividing by $|z||w|$ proves the claim (C.37) with constant $C = 3(\alpha+1)$. In particular, $\widetilde{g}$ is Lipschitz-continuous on bounded subsets of $\mathbb{C}$, since for given $M > 0$, one can choose $L(M) := 2CM^{\alpha}$ (where $C$ is the constant from (C.37)) and verify that (C.38) holds. □

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