A note on the Moser-Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one

S. Iula

Departement Mathematik und Informatik
Fachbereich Mathematik
Universität Basel
CH-4051 Basel

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A note on the Moser-Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one

Stefano Iula*
Universität Basel
stefano.iula@unibas.ch

Abstract

We discuss some recent results by Parini and Ruf on a Moser-Trudinger type inequality in the setting of Sobolev-Slobodeckij spaces in dimension one. We push further their analysis considering the inequality on the whole $\mathbb{R}$ and we give an answer to one of their open questions.

1 Introduction

A classical result in analysis states that, if $\Omega \subset \mathbb{R}^n$ is an open set with finite measure $|\Omega|$ and Lipschitz boundary, k is a positive integer with $k < n$, and $p \in [1, \frac{k}{n})$, then the Sobolev space $W_0^{k,p}(\Omega)$ embeds continuously in $L^{\frac{np}{n-kp}}(\Omega)$. This results doesn't hold for the critical case $p = \frac{n}{k}$, that is $W_0^{k,\frac{n}{k}}(\Omega)$ doesn't embed in $L^\infty(\Omega)$. On the other hand Trudinger [14], Pohozaev [12], Yudovich [6] and others found that, at least in the case $k = 1$, functions in $W_0^{1,n}(\Omega)$ enjoy summability of exponential type. Namely

$$W_0^{1,n}(\Omega) \subset \left\{ u \in L^1(\Omega) \colon \int_\Omega e^{\beta |u|^{\frac{n}{n-1}}}\, dx < +\infty \right\}$$

for any $\beta < +\infty$. Moser [9] sharpened this embedding and determined the optimal exponent α_n such that

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx < C|\Omega|, \quad \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}. \tag{1}$$

Here, ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. In particular the exponent α_n is sharp in the sense that

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx = +\infty$$

for any $\alpha > \alpha_n$. Moreover, the supremum in (1) becomes infinite as soon as we slightly modify the integrand, namely

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega f(|u|) e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx = +\infty \tag{2}$$

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for any measurable function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to+\infty} f(t) = \infty$. This can be proved, for instance, using the same test functions defined in [9]. In [1] Adams, exploiting Riesz potentials, extended Moser's result to higher order Sobolev spaces $W_0^{k,p}(\Omega)$, $k > 1$, $p = \frac{n}{k}$.

In the present work, we are interested in generalizations of (1) that concern Sobolev spaces of fractional orders. The usual approach is to consider Bessel potential spaces $H^{s,p}$. In this setting, sharp versions of (1) are proven both in the cases of bounded and unbounded domains of $\mathbb{R}^n$, $n \geq 1$ (see [5], [8] and [4]).

Here, we focus our attention on the case (in general different from the one of Bessel potential spaces) of Sobolev Slobodeckij spaces (see definitions below), which has been recently proposed, together with some open questions, by Parini and Ruf. In [10] they considered $\Omega \subset \mathbb{R}^n$ to be a bounded and open domain, $n \geq 2$ and $sp = n$ and they were able to prove the existence of $\alpha_* > 0$ such that the corresponding version of inequality (1) is satisfied for any $\alpha \in (0, \alpha_*)$ (see also [11]). Even though the result is not sharp, in the sense that the value of the optimal exponent is not yet known, an explicit upper bound for the optimal exponent α^* is given.

As a first step, we extend the results in [10] to the case $n = 1$. For any $s \in (0, 1)$ and $p > 1$, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R})$ is defined as

$$W^{s,p}(\mathbb{R}) := \left\{u \in L^p(\mathbb{R})\colon [u]_{W^{s,p}(\mathbb{R})} < +\infty\right\}$$

where $[u]_{W^{s,p}(\mathbb{R})}$ is the Gagliardo seminorm defined by

$$[u]_{W^{s,p}(\mathbb{R})} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2}\, dx\, dy\right)^{\frac{1}{p}}. \tag{3}$$

We will often write $[\cdot] := [\cdot]_{W^{s,p}(\mathbb{R})}$. The space $W^{s,p}(\mathbb{R})$ is a Banach space with respect to the norm

$$||u||_{W^{s,p}(\mathbb{R})} := \left(||u||^p_{L^p(\mathbb{R})} + [u]^p_{W^{s,p}(\mathbb{R})}\right)^{\frac{1}{p}}. \tag{4}$$

Let I be an open interval in $\mathbb{R}$. We define the space $\tilde{W}_0^{s,p}(I)$ as the closure of $\left(C_0^\infty(I), \|u\|_{W^{s,p}(\mathbb{R})}\right)$. An equivalent definition for $\tilde{W}_0^{s,p}(I)$ can be obtained taking the completion of $C_0^\infty(I)$ with respect to the seminorm $[u]_{W^{s,p}(\mathbb{R})}$ (see [3, Remark 2.5]).

With a mild adaptation of the techniques used in [10], we are able to prove that their result holds also in dimension one.

Theorem 1.1. *Let $s \in (0, 1)$ and $p > 1$ be such that $sp = 1$. There exists $\alpha_* = \alpha_*(s) > 0$ such that for all $\alpha \in [0, \alpha_*)$ it holds*

$$\sup_{u\in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})}\leq 1} \int_I e^{\alpha|u|^{\frac{1}{1-s}}}\, dx < \infty. \tag{5}$$

Moreover, there exists $\alpha^* = \alpha^*(s) := \gamma_s^{\frac{s}{1-s}}$ such that the supremum in (5) is infinite for any $\alpha \in (\alpha^*, +\infty)$.

It is worth to remark that, as already pointed out in [10], the exponent $\alpha^*(\frac{1}{2})$ is equal to $2\pi^2$ and it coincides, up to a normalization constant, with the optimal exponent π determined in [5] in the setting of Bessel potential spaces.

We move now to the case $I = \mathbb{R}$, pushing further the analysis of [10]. An inequality of the form (5) cannot hold if we don't consider the full $W^{s,p}(\mathbb{R})$-norm, i.e. we take into account also the term $\|u\|_{L^p(\mathbb{R})}$. This has been done by Ruf [13] in the case of $H^{1,2}(\mathbb{R}^2)$, see also [5], [4] for the case of Bessel potential spaces. We define

$$\Phi(t) := e^t - \sum_{k=0}^{\lceil p-2 \rceil} \frac{t^k}{k!}, \tag{6}$$

where $\lceil p-2 \rceil$ is the smallest integer greater than, or equal to $p-2$.

Theorem 1.2. *Let $s \in (0,1)$ and $p > 1$ be such that $sp = 1$. There exists $\alpha_* = \alpha_*(s) > 0$ such that for all $\alpha \in [0, \alpha_*)$ it holds*

$$\sup_{u \in W^{s,p}(\mathbb{R}), ||u||_{W^{s,p}(\mathbb{R})} \le 1} \int_{\mathbb{R}} \Phi(\alpha |u|^{\frac{1}{1-s}})\, dx < \infty. \tag{7}$$

Moreover the supremum in (5) *is infinite for any $\alpha \in (\alpha^*, +\infty)$, where α^* is as in Theorem 1.1*

As we shall see, Theorem 1.1 and 1.2 are sharp in the sense of (2). Indeed one of the open questions in [10] was whether an inequality of the type

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{\tilde{W}_0^{s,p}(I)} \le 1} \int_I f(|u|) e^{\alpha |u|^{\frac{1}{1-s}}}\, dx < +\infty,$$

where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is such that $f(t) \to \infty$ as $t \to \infty$ holds true for the same exponents of the standard Moser-Trudinger inequality (see [4],[5]). For $n = 1$ we prove the following

Theorem 1.3. *Let $I \subset \mathbb{R}$ be a bounded interval, $s \in (0,1)$ and $p > 1$ such that $sp = 1$. We have*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \le 1} \int_I f(|u|) e^{\alpha^* |u|^{\frac{1}{1-s}}}\, dx = \infty, \tag{8}$$

$$\sup_{u \in W^{s,p}(\mathbb{R}), \|u\|_{W^{s,p}(\mathbb{R})} \le 1} \int_{\mathbb{R}} f(|u|) \Phi(\alpha^* |u|^{\frac{1}{1-s}})\, dx = \infty, \tag{9}$$

where $f\colon [0,\infty) \to [0,\infty)$ is any Borel measurable function such that $\lim_{t \to +\infty} f(t) = \infty$.

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2 Proof of Theorem 1.1

We start this section proving the validity of the Moser-Trudinger inequality (5). The result for $n \ge 2$ is proved in [10] and the proof in the one dimensional case, which we report here for the sake of completeness, follows by a mild adaptation of the techniques in [10].

Thanks to [11, Theorem 9.1], using Sobolev embeddings and Hölder's inequality we have that there exists a constant $C > 0$ independent of u such that for any $u \in \tilde{W}_0^{s,p}(I)$

$$||u||_{L^q(\mathbb{R})} \le C [u]_{W^{s,p}(\mathbb{R})} q^{1-s} \tag{10}$$

for any $q > 1$. For $[u]_{W^{s,p}(\mathbb{R})} \leq 1$ we write

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}}\,dx = \sum_{k=0}^{\infty}\int_I \frac{\alpha^k}{k!}|u|^{\frac{k}{1-s}}\,dx \leq \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{C}{1-s}\alpha k\right)^k, \tag{11}$$

where in the last inequality we used (10). Thanks to Stirling's formula

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k\left(1 + O(\frac{1}{k})\right) \tag{12}$$

the series in (11) converges for small α and we recover a bound (uniform w.r.t. u) for

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}}\,dx,$$

yielding (5).

As a direct consequence of (5), using the density of $C_c^\infty(I)$ in $\tilde{W}_0^{s,p}(I)$, we have the following corollary (see [10, Proposition 3.2]).

Corollary 2.1. *If $u \in \tilde{W}_0^{s,p}(I)$, for every $\alpha > 0$ it holds*

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}}\,dx < \infty.$$

We now give a useful result on the Gagliardo seminorm of radially symmetric functions (see [10, Proposition 4.3]), which will turn out to be useful later on.

Proposition 2.1. *Let $u \in W^{s,p}(\mathbb{R})$ be radially symmetric and let $sp = 1$. Then*

$$[u]_{W^{s,p}(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy = 4\int_0^{+\infty}\int_0^{+\infty}|u(x)-u(y)|^p\frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy \tag{13}$$

Proof. The proof will follow from a direct computation. We split

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy \\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy + \int_{-\infty}^0\int_{-\infty}^0\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy \\
&+ \int_0^{+\infty}\int_{-\infty}^0\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy + \int_{-\infty}^0\int_0^{+\infty}\frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy.
\end{aligned}$$

Using a straightforward change of variable and the symmetry of u, we obtain the claim. $\square$

To give an upper bound for the optimal exponent $\bar{\alpha}$ such that the supremum in (5) is finite for $\alpha \in [0, \bar{\alpha})$, we define the family of functions

$$u_\varepsilon(x) := \begin{cases} |\log\varepsilon|^{1-s} & \text{if } |x| \leq \varepsilon \\ \frac{|\log|x||}{|\log\varepsilon|^s} & \text{if } \varepsilon < |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{14}$$

Notice that the restrictions of u_ε to I belong to $\tilde{W}_0^{s,p}(I)$.

Proposition 2.2. *Let $sp = 1$ and $(u_\varepsilon) \subset \tilde{W}_0^{s,p}(I)$ be the family of functions defined in (14). Then*

$$\lim_{\varepsilon \to 0} [u_\varepsilon]^p_{W^{s,p}(\mathbb{R})} = \gamma_s := 8\,\Gamma(p+1) \sum_{k=0}^{\infty} \frac{1}{(1+2k)^p}. \tag{15}$$

Proof. We will follow the proof in [10]. Define

$$I(\varepsilon) := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^2} \, dx\, dy. \tag{16}$$

Using Proposition 2.1 and (14) we see that $I(\varepsilon)$ can be decomposed as

$$I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon),$$

where

$$I_1(\varepsilon) = \frac{8}{|\log \varepsilon|} \int_\varepsilon^1 \int_0^\varepsilon |\log x - \log \varepsilon|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} \, dx\, dy,$$

$$I_2(\varepsilon) = \frac{4}{|\log \varepsilon|} \int_\varepsilon^1 \int_\varepsilon^1 |\log x - \log y|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} \, dx\, dy,$$

$$I_3(\varepsilon) = 8 |\log \varepsilon|^{p-1} \int_1^{+\infty} \int_0^\varepsilon \frac{x^2 + y^2}{(x^2 - y^2)^2} \, dx\, dy,$$

$$I_4(\varepsilon) = \frac{8}{|\log \varepsilon|} \int_\varepsilon^1 \int_1^{+\infty} |\log x|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} \, dx\, dy.$$

With an integration by parts, it is easy to check that $\lim_{\varepsilon \to 0} I_i(\varepsilon) = 0$ for $i = 1, 3, 4$. As for $I_2(\varepsilon)$, integrating by parts after a change of variables we have

$$\begin{aligned}
I_2(\varepsilon) = {} & \frac{4}{|\log \varepsilon|} \left\{ \log y \left(\int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} \, dx \right) \right\} \Bigg|_{y=\varepsilon}^{y=1} \\
& + \frac{4}{|\log \varepsilon|} \int_\varepsilon^1 \frac{\log y}{y^2} |\log \frac{1}{y}|^p \frac{\frac{1}{y^2} + 1}{\left(\frac{1}{y^2} - 1\right)^2} \, dy \\
& - \frac{4\varepsilon}{|\log \varepsilon|} \int_\varepsilon^1 \frac{\log y}{y^2} |\log \frac{\varepsilon}{y}|^p \frac{\left(\frac{\varepsilon}{y}\right)^2 + 1}{\left(\left(\frac{\varepsilon}{y}\right)^2 - 1\right)^2} \, dy.
\end{aligned}$$

A direct computation for the first term gives

$$\begin{aligned}
& \frac{4}{|\log \varepsilon|} \left\{ \log y \left(\int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} \, dx \right) \right\} \Bigg|_{y=\varepsilon}^{y=1} \\
& = 4 \int_1^{\frac{1}{\varepsilon}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} \, dx,
\end{aligned}$$

which converges to

$$4\int_1^{+\infty}|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx,$$

as $\varepsilon\to 0$. Moreover, since

$$\int_0^1\frac{\log y}{y^2}|\log\frac{1}{y}|^p\frac{\frac{1}{y^2}+1}{\left(\frac{1}{y^2}-1\right)^2}\,dy<+\infty$$

the second term in the sum converges to 0 as $\varepsilon\to 0$.

After setting $\frac{\varepsilon}{y}=x$, for the last term in the sum we have

$$\begin{aligned}
&-\frac{4\varepsilon}{|\log\varepsilon|}\int_\varepsilon^1\frac{\log y}{y^2}|\log\frac{\varepsilon}{y}|^p\frac{\left(\frac{\varepsilon}{y}\right)^2+1}{\left(\left(\frac{\varepsilon}{y}\right)^2-1\right)^2}\,dy\\
&=-\frac{4}{|\log\varepsilon|}\int_\varepsilon^1\log\left(\frac{\varepsilon}{x}\right)|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx\\
&=4\int_\varepsilon^1|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx-\frac{4}{|\log\varepsilon|}\int_\varepsilon^1|\log x|^{p+1}\frac{x^2+1}{(x^2-1)^2}\,dx
\end{aligned}$$

which converges to

$$4\int_0^1|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx=4\int_1^{+\infty}|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx$$

as $\varepsilon\to 0$. Summing up, we have

$$\lim_{\varepsilon\to 0}[u_\varepsilon]^p_{W^{s,p}(\mathbb{R})}=\lim_{\varepsilon\to 0}I_2(\varepsilon)=8\int_1^{+\infty}|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx. \tag{17}$$

Integrating by parts we obtain

$$\begin{aligned}
&\int_1^{+\infty}|\log x|^p\frac{x^2+1}{(x^2-1)^2}\,dx=p\int_1^{+\infty}\frac{|\log x|^{p-1}}{x^2-1}\,dx\\
&=p\int_0^1\frac{|\log t|^{p-1}}{1-t^2}\,dt,
\end{aligned}$$

where we set $t=\frac{1}{x}$. Recall now

$$\frac{1}{1-x^2}=\sum_{k=0}^\infty x^{2k},\qquad \int_0^1|\log x|^{p-1}x^{2k}\,dx=\frac{\Gamma(p)}{(1+2k)^p}, \tag{18}$$

where $\Gamma(\cdot)$ is the Euler Gamma function. Thanks to (18) we write

$$\int_0^1\frac{|\log t|^{p-1}}{1-t^2}\,dt=\sum_{k=0}^\infty\int_0^1|\log t|^{p-1}t^{2k}\,dt=\Gamma(p)\sum_{k=0}^\infty\frac{1}{(1+2k)^p}, \tag{19}$$

proving (15). $\square$

The upper bound for the optimal exponent follows directly from Proposition 2.2.

Proposition 2.3. *Let $sp = 1$. There exists $\alpha^* := \gamma_s^{\frac{s}{1-s}}$ such that*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I e^{\alpha |u|^{\frac{1}{1-s}}} \, dx = +\infty \quad \text{for } \alpha \in (\alpha^*, +\infty).$$

Proof. Let u_ε be the family of functions in $\tilde{W}_0^{s,p}(I)$ defined in (14). Thanks to Proposition 2.2 we have that $[u_\varepsilon]_{W^{s,p}(\mathbb{R})} \to (\gamma_s)^{\frac{1}{p}}$ as $\varepsilon \to 0$. Fix $\alpha > \gamma_s^{\frac{s}{1-s}}$. For ε small enough, there exists $\beta > 0$ such that $\alpha [u_\varepsilon]^{-\frac{1}{1-s}} \geq \beta > 1$. If we set $v_\varepsilon := \frac{u_\varepsilon}{[u_\varepsilon]}$ we have

$$\int_I e^{\alpha |v_\varepsilon|^{\frac{1}{1-s}}} \, dx \geq \int_{-\varepsilon}^{\varepsilon} e^{\alpha |v_\varepsilon|^{\frac{1}{1-s}}} \, dx \geq \int_{-\varepsilon}^{\varepsilon} e^{-\beta \log \varepsilon} \, dx = 2\varepsilon^{1-\beta} \to +\infty$$

as $\varepsilon \to 0$, since $\beta > 1$. □

3 Proof of Theorem 1.2

We shall adapt a technique by Ruf [13] to our setting.

For a measurable function u we set $|u|^* : \mathbb{R} \to \mathbb{R}_+$ to be its non-increasing symmetric rearrangement, whose definition we shall now recall. For a measurable set $A \subset \mathbb{R}$, we define

$$A^* = (-|A|/2, |A|/2).$$

The set A^* is symmetric (with respect to 0) and $|A^*| = |A|$. For a non-negative measurable function f, such that

$$|\{x \in \mathbb{R} : f(x) > t\}| < \infty \quad \text{for every } t > 0,$$

we define the symmetric non-increasing rearrangement of f by

$$f^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R} : f(y) > t\}^*}(x) dt.$$

Notice that f^* is even, i.e. $f^*(x) = f^*(-x)$ and non-increasing (on $[0, \infty)$).
We will state here the two properties that we shall use in the proof of Proposition 1.2. The following one is proven e.g. in [7, Section 3.3].

Proposition 3.1. *Given a measurable function $F : \mathbb{R} \to \mathbb{R}$ and a non-negative non-decreasing function $f : \mathbb{R} \to \mathbb{R}$, it holds*

$$\int_{\mathbb{R}} F(f) dx = \int_{\mathbb{R}} F(f^*) dx.$$

The following Pólya-Szegő type inequality can be found e.g. in [2, Theorem 9.2].

Theorem 3.1. *Let $0 < s < 1$ and $u \in W^{s,p}(\mathbb{R})$. Then*

$$[|u|^*]_W^{s,p}(\mathbb{R}) \leq [u]_W^{s,p}(\mathbb{R}).$$

Now given $u \in W^{s,p}(\mathbb{R})$, from Proposition 3.1 we get

$$\int_{\mathbb{R}} \Phi(\alpha(|u|)^{\frac{1}{1-s}})\,dx = \int_{\mathbb{R}} \Phi(\alpha(|u|^*)^{\frac{1}{1-s}})\,dx, \quad \||u|^*\|_{L^p} = \|u\|_{L^p},$$

and according to Theorem 3.1

$$\||u|^*\|^p_{W^{s,p}(\mathbb{R})} = \||u|^*\|^p_{L^p(\mathbb{R})} + [|u|^*]^p_{W^{s,p}(\mathbb{R})} \leq \|u\|^p_{L^p(\mathbb{R})} + [u]^p_{W^{s,p}(\mathbb{R})} = \|u\|^p_{W^{s,p}(\mathbb{R})}.$$

Therefore in the rest of the proof of (7) we may assume that $u \in W^{s,p}(\mathbb{R})$ is even, non-increasing on $[0,\infty)$, and $\|u\|_{W^{s,p}(\mathbb{R})} \leq 1$. We will use a technique by Ruf [13] (see also [5]) and write

$$\begin{aligned}
&\int_{\mathbb{R}} \Phi(\alpha(|u|)^{\frac{1}{1-s}})\,dx \\
&= \int_{I^c} \Phi(\alpha(|u|)^{\frac{1}{1-s}})\,dx + \int_{I} \Phi(\alpha(|u|)^{\frac{1}{1-s}})\,dx \\
&=: (I) + (II),
\end{aligned}$$

where $I = (-r_0, r_0)$, with $r_0 > 0$ to be chosen. Notice that since u is even and non-increasing, for $x \neq 0$ and $p > 1$, we have

$$|u(x)|^p \leq \frac{1}{2|x|}\int_{-|x|}^{|x|} |u(y)|^p\,dy \leq \frac{\|u\|^p_{L^p}}{2|x|}. \tag{20}$$

We start by bounding (I). We observe that for $r_0 >> 1$, we have $|u(x)| \leq 1$ on I^c and hence

$$|u|^{\frac{p\lceil p-1\rceil}{p-1}} \leq |u|^p \quad \text{on } I^c,$$

since $\frac{p\lceil p-1\rceil}{p-1} \geq p$. For $k > p-1$ we bound

$$\int_{I^c} (|u|^p)^{\frac{k}{p-1}}\,dx \leq \int_{I^c} \left(\frac{\|u\|^p_{L^p}}{2|x|}\right)^{\frac{k}{p-1}} = \frac{\|u\|^{\frac{pk}{p-1}}_{L^p} r_0^{1-\frac{k}{p-1}}(p-1)}{2^{\frac{k}{p-1}}(k+1-p)}.$$

Hence

$$\begin{aligned}
(I) &= \sum_{k=\lceil p-1\rceil}^{\infty} \int_{I^c} \frac{\alpha^k}{k!}|u|^{\frac{kp}{p-1}}\,dx \\
&= \frac{\alpha^{\lceil p-1\rceil}}{\lceil p-1\rceil!}\int_{I^c} |u|^{\frac{p\lceil p-1\rceil}{p-1}}\,dx + \sum_{k=\lceil p\rceil}^{\infty}\int_{I^c} \alpha^k \frac{|u|^{\frac{kp}{p-1}}}{k!}dx \\
&\leq C(\alpha,p)\|u\|^p_{L^p} + r_0(p-1)\sum_{k=\lceil p\rceil}^{\infty} \frac{\alpha^k\left(\|u\|^p_{L^p}\right)^{\frac{k}{p-1}}}{k!(k+1-p)(2r_0)^{\frac{k}{p-1}}} \\
&\leq C(\alpha,p)\|u\|^p_{L^p} + C\sum_{k=\lceil p\rceil}^{\infty}\left(\frac{\alpha}{(2r_0)^{p-1}}\right)^k \frac{1}{k!(k+1-p)} \leq C.
\end{aligned}$$

As for (II), define $v \in \tilde{W}_0^{s,p}(I)$ as follows

$$v(x) = \begin{cases} u(x) - u(r_0) & |x| \le r_0 \\ 0 & |x| > r_0. \end{cases}$$

Let $x \in I$. We compute using the monotonicity of u

$$\int_0^\infty |v(x) - v(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2}\, dy \le \int_0^\infty |u(x) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2}\, dy. \tag{21}$$

Let $x \in I^c$. We have

$$\begin{aligned} &\int_0^\infty |v(x) - v(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2}\, dy \\ &= \int_I |u(r_0) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2}\, dy \\ &\le \int_I |u(x) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2}\, dy. \end{aligned} \tag{22}$$

Combining (21), (22) and integrating in x, we get

$$[v]^p \le [u]^p. \tag{23}$$

Using the definition of v and the inequality $(a + b)^\sigma \le a^\sigma + \sigma 2^{\sigma-1}(a^{\sigma-1}b + b^\sigma)$ for $a, b \ge 0$ and $\sigma \ge 1$, we have

$$\begin{aligned} u^{\frac{1}{1-s}} &\le v^{\frac{1}{1-s}} + \frac{1}{1-s} 2^{\frac{s}{1-s}} \left(v^{\frac{s}{1-s}} u(r_0) + u(r_0)^{\frac{1}{1-s}}\right) \\ &\le v^{\frac{1}{1-s}} \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right) + 2^{\frac{s}{1-s}} + \frac{2^{\frac{s}{1-s}}}{1-s} r_0 \\ &= v^{\frac{1}{1-s}} \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right) + C(r_0). \end{aligned} \tag{24}$$

This implies

$$\begin{aligned} u(x) &\le v(x) \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{1-s} + C^{1-s}(r_0) \\ &:= w(x) + C^{1-s}(r_0). \end{aligned}$$

From (23) and the definition of w, we get

$$\begin{aligned} [w]^p &= [v]^p \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{\frac{1-s}{s}} \\ &\le \left(1 - ||u||_p^p\right) \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{\frac{1-s}{s}} \end{aligned} \tag{25}$$

Consider now the function $f(t) = (1-t)(1+\tau t)^\sigma$, where $\tau := \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)}$ and $\sigma = \frac{1-s}{s} > 0$. We compute

$$f'(t) = (1+\tau t)^{\sigma-1}\left(\tau t(-\sigma-1) + \tau\sigma - 1\right) \tag{26}$$

which vanishes for $t_1 = -\frac{1}{\tau} < 0$ and $t_2 = \frac{\tau\sigma-1}{\tau(\sigma+1)}$. We choose now $r_0 > 2^{\frac{2s-1}{1-s}}$ so that $t_2 < 0$. This implies that f is decreasing in $(0,1)$ and since $f(0) = 1$ we have that $f(t) < 1$ for $t \in (0,1)$, which implies

$$[w]^p \leq 1. \tag{27}$$

We can apply now Theorem 1.1 on the interval $I = (-r_0, r_0)$ to get that there exists $\alpha_* > 0$ such that

$$\int_I e^{\alpha_* w^{p'}}\, dx \leq C \tag{28}$$

and using (24) we get

$$\int_I e^{\alpha_* u^{\frac{1}{1-s}}}\, dx \leq C \int_I e^{\alpha_* w^{\frac{1}{1-s}}}\, dx \leq C, \tag{29}$$

concluding the proof of (7).

To prove the second part of the claim one can argue as in the previous section, using the sequence of functions u_ε defined in (14) and taking into account that now the norm we are working with is the full $W^{s,p}$-norm. Indeed we have

$$\|u_\varepsilon\|_{L^p}^p = \int_{\mathbb{R}} |u_\varepsilon|^p\, dx = \int_{|x|\leq\varepsilon} \left(|\log\varepsilon|^{p-sp}\right) dx + \int_{\varepsilon<|x|<1} \frac{|\log x|}{|\log\varepsilon|^{sp}}\, dx = O(|\log\varepsilon|^{-1}). \tag{30}$$

Hence from (15), it follows that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}^p = \gamma_s. \tag{31}$$

Choose $M > 0$ large enough so that

$$\Phi(t) \geq \frac{1}{2}e^t, \quad t \geq M.$$

Then one has

$$\begin{aligned}\int_{\mathbb{R}} \Phi\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}}\right) dx &\geq \int_{u_\varepsilon\geq M} \Phi\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}}\right) dx \\ &\geq \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} e^{\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}\right)^{\frac{1}{1-s}}} dx.\end{aligned} \tag{32}$$

for ε small enough. Now, thanks to (31), one can argue as in the proof of Proposition 2.3 to conclude the proof of Theorem 1.2.

4 Proof of Theorem 1.3

We will start by proving (8) since the proof of (9) will follow adapting the reasoning of the previous section.

Let u_ε be as in (14). To prove (8) it is enough to show that there exists a constant $\delta > 0$ such that

$$\int_{-\varepsilon}^{\varepsilon} e^{\alpha^*\left(\frac{u_\varepsilon}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx \geq \delta.$$

Indeed, $u_\varepsilon \to +\infty$ uniformly for $|x| < \varepsilon$ as $\varepsilon \to 0$ and we have

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I f(|u|) e^{\alpha^*\left(\frac{|u|}{[u]}\right)^{\frac{1}{1-s}}} dx \geq \inf_{|x|<\varepsilon} f(|u_\varepsilon|) \int_{-\varepsilon}^{\varepsilon} e^{\alpha^*\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx.$$

From Proposition 2.2, it follows that

$$\lim_{\varepsilon \to 0} \frac{[u_\varepsilon]}{\gamma_s^s} = 1 \tag{33}$$

and in particular

$$\lim_{\varepsilon \to 0} [u_\varepsilon]^p = 8 \int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2} dx = \gamma_s.$$

We compute

$$\lim_{\varepsilon \to 0} \log \frac{1}{\varepsilon} \left([u_\varepsilon]^p - \gamma_s\right) = 8 \lim_{\varepsilon \to 0} \log \frac{1}{\varepsilon} \int_{\frac{1}{\varepsilon}}^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2} dx = 0. \tag{34}$$

Then we can write

$$\frac{[u_\varepsilon]^p}{\gamma_s} \leq 1 + (C \log \frac{1}{\varepsilon})^{-1} \tag{35}$$

and in particular, recalling

$$\lim_{t \to +\infty} \frac{t}{(1+\frac{C}{t})^{\frac{1}{1-s}}} - t = -\frac{1}{1-s},$$

we have

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} e^{\gamma_s^{\frac{s}{1-s}}\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx &= \int_{-\varepsilon}^{\varepsilon} e^{\left(\frac{\gamma_s^s}{[u_\varepsilon]}\right)^{\frac{1}{1-s}} |u_\varepsilon|^{\frac{1}{1-s}}} dx \\
&\geq \int_{-\varepsilon}^{\varepsilon} e^{\frac{\log \frac{1}{\varepsilon}}{(1+C(\log \frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}} dx \\
&= 2\varepsilon e^{\frac{\log \frac{1}{\varepsilon}}{(1+C(\log \frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}} \to e^{-\frac{1}{1-s}}
\end{aligned} \tag{36}$$

as $\varepsilon \to 0$. Therefore

$$\int_I e^{\gamma_s^{\frac{s}{1-s}}\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx \geq \delta \tag{37}$$

for some $\delta > 0$, proving (8). We shall now prove (9). From (30) and (34) it follows that

$$\frac{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}^p}{\gamma_s} \leq 1 + O(|\log \varepsilon|^{-1}). \tag{38}$$

Now using (32) and arguing as in (36) and (37), we conclude the proof.

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