Diophantine approximations on definable sets

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Abstract. Consider the vanishing locus of a real analytic function on $\mathbb{R}^n$ restricted to $[0,1]^n$. We bound the number of rational points of bounded height that approximate this set very well. Our result is formulated and proved in the context of o-minimal structure which give a general framework to work with sets mentioned above. It complements the theorem of Pila-Wilkie that yields a bound of the same quality for the number of rational points of bounded height that lie on a definable set. We focus our attention on polynomially bounded o-minimal structures, allow algebraic points of bounded degree, and provide an estimate that is uniform over some families of definable sets. We apply these results to study fixed length sums of roots of unity that are small in modulus.

Contents

1. Introduction

The starting point of our investigation is the Counting Theorem [17] of Pila and Wilkie in a fixed o-minimal structure. In Section 2 we recall the definition of an o-minimal structure. If not stated otherwise, sets and functions are called definable if they are definable in this o-minimal structure. The height of a/b where a and b are coprime integers with $b \geq 1$ is $H(a/b) = \max\{|a|, b\}$. The height of $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ is $\max\{H(q_1), \ldots, H(q_n)\}$ for an integer $n \geq 1$. For any subset $X \subseteq \mathbb{R}^n$ we write X^{alg} for the algebraic locus of X, i.e. the union of all connected real semi-algebraic sets of positive dimension that are contained in X. Roughly speaking, Pila and Wilkie show that rational points of bounded height on a definable set are concentrated on its algebraic locus.

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Theorem 1 (Pila-Wilkie, Theorem 1.8 [17]). *Let $X \subseteq \mathbb{R}^n$ be a definable set and let $\epsilon > 0$. There exists a constant $c = c(X, \epsilon) > 0$ such that*

$$\#\left\{q \in (X \setminus X^{\mathrm{alg}}) \cap \mathbb{Q}^n : H(q) \leq T\right\} \leq cT^{\epsilon}$$

for all $T \geq 1$.

This counting result comes after a long series of work including papers of Jarník [10] and Bombieri-Pila [5] in the one-dimensional setting and Pila [15] for certain surfaces. The counting result was further developed by Pila [16] to algebraic points of bounded height and with a more precise substitute for X^{alg}. This led to striking applications towards the André-Oort Conjecture.

The purpose of this paper is to investigate whether one can find similar bounds on the number of rational points that *approximate* a definable set.

Before we come to our first result, let us introduce some notation. We will use $|\cdot|$ to denote the maximum-norm on $\mathbb{R}^n$. For $\epsilon > 0$ we set

$$\mathcal{N}(X, \epsilon) = \{y \in \mathbb{R}^n : \text{there is } x \in X \text{ with } |y - x| \leq \epsilon\}$$

to be the ϵ-tube around the subset $X \subseteq \mathbb{R}^n$.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The absolute Weil height $H : \overline{\mathbb{Q}} \to [1, +\infty)$ extends the height defined above from $\mathbb{Q}$ to $\overline{\mathbb{Q}}$; we give a precise definition and some basic facts in Section 2. The height $H(q)$ of $q = (q_1, \ldots, q_n) \in \overline{\mathbb{Q}}^n$ is $\max\{H(q_1), \ldots, H(q_n)\}$.

Let $T \geq 1$ be a real number and $e \geq 1$ an integer. We disregard algebraic numbers that are not real and set

$$\mathbb{Q}^n(T, e) = \left\{q \in (\overline{\mathbb{Q}} \cap \mathbb{R})^n : H(q) \leq T \text{ and } [\mathbb{Q}(q) : \mathbb{Q}] \leq e\right\}.$$

This is a finite set by Northcott's Theorem, see Theorem 1.6.8 [4].

An o-minimal structure is called *polynomially bounded* if any definable function $\mathbb{R} \to \mathbb{R}$ is bounded from above by a polynomial for all sufficiently large positive arguments, cf. Section 4 [21].

Let $\lambda > 0$. A function $\psi : [1, +\infty) \to [0, 1]$ is said to have order at most $-\lambda$ if $\psi(x) \leq x^{-\lambda}$ for all $x \geq 1$.

Many results in this paper are restricted to polynomially bounded o-minimal structures for reasons that will be explained in Example 2. Our first result shows that rational approximations to a definable set cluster near the algebraic locus.

Theorem 2. *Let $X \subseteq \mathbb{R}^n$ be closed and definable in a polynomially bounded o-minimal structure. Let $e \geq 1$ be an integer and let $\epsilon > 0$. There exist $c = c(X, e, \epsilon) \geq 1$ and $\theta = \theta(X, e, \epsilon) \in (0, 1]$ such that if $\psi : [1, +\infty) \to [0, 1]$ has order at most $-\theta^{-1}$, then*

$$\#\left\{q \in \mathbb{Q}^n(T, e) \setminus \mathcal{N}(X^{\mathrm{alg}}, \psi(T)^{\theta}) : \text{there is } x \in X \text{ with } |x - q| \leq c^{-1}\psi(T)\right\} \leq cT^{\epsilon} \quad (1)$$

for all $T \geq 1$.

Van den Dries [19] recognized that $\mathbb{R}_{\mathrm{an}}$, the structure of restricted real analytic functions, is o-minimal and even polynomially bounded using older work of Gabrielov.

The largest possible choice $\psi(x) = x^{-1/\theta}$ is natural in context of Theorem 2. However, it may prove useful to also allow functions that decrease quicker as this leads to smaller tubes $\mathcal{N}(X^{\mathrm{alg}}, \psi(T)^{\theta})$. Of course $\psi(x) = 0$ for all $x \geq 1$ is also a valid choice. Then

$\mathcal{N}(X^{\mathrm{alg}}, \psi(T)^\theta) = X^{\mathrm{alg}}$ and Theorem 2 reduces to the Pila-Wilkie Theorem for definable closed subsets in a polynomially bounded o-minimal structure.

The statement of the theorem simplifies if X does not contain a connected real semi-algebraic set of positive dimension. The corollary below is a consequence of the previous theorem with the choice $\psi(x) = x^{-\lambda}$.

Corollary 3. *Let $X \subseteq \mathbb{R}^n$ be closed and definable in a polynomially bounded o-minimal structure such that $X^{\mathrm{alg}} = \emptyset$. Let $e \geq 1$ be an integer and let $\epsilon > 0$. There exist $c = c(X, e, \epsilon) > 0$ and $\lambda = \lambda(X, e, \epsilon) > 0$ such that*

$$\#\left\{q \in \mathbb{Q}^n(T, e) : \text{there is } x \in X \text{ with } |x - q| \leq c^{-1}T^{-\lambda}\right\} \leq cT^\epsilon$$

for all $T \geq 1$.

Huxley [8] obtained powerful bounds for the number of rational approximations to the graph of a function $\mathbb{R} \to \mathbb{R}$ that is twice continuously differentiable. Here the second derivative is not allowed to vanish. He made further contributions [9] for trice continuously differential functions. Huxley's result also covers many algebraic functions and does not distinguish between the algebraic and transcendental case. Applying his result to a graph whose algebraic locus is empty does not seem to lead to a T^ϵ bound as in Corollary 3.

We exhibit examples which show that some of the assumptions in our results cannot be dropped.

First, let us see why we cannot drop the hypothesis that X is closed in Corollary 3.

Example 1. *We work in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ in which*

$$X = \left\{(x, y, (e^x - 1)(e^y - 1)) : x, y \in (0, 1)\right\} \subseteq \mathbb{R}^3.$$

is definable. It is a 2-dimensional cell that is not closed. There are $3T^2/\pi^2 + o(T^2)$ rational points $q = (x, 0, 0) \in [0, 1] \times \mathbb{R}^2$ of height at most T, see Theorem 330 [7]*. Each such point lies in the closure of X in $\mathbb{R}^3$. However, using Ax's Theorem* [1] *one can show $X^{\mathrm{alg}} = \emptyset$. So the real semi-algebraic curves in the boundary of a definable set can lead to many good rational approximations.*

If $x \in \mathbb{R}^n$ and if X is any non-empty subset of $\mathbb{R}^n$ then we define

$$\mathrm{dist}(x, X) = \inf\{|x - x'| : x' \in X\}$$

and

$$\mathrm{dist}^*(x, X) = \min\{1, \mathrm{dist}(x, X)\}.$$

It is convenient to define $\mathrm{dist}^*(x, \emptyset) = 1$ for all $x \in \mathbb{R}^n$. The function $x \mapsto \mathrm{dist}^*(x, X)$ is continuous and it is definable if X is.

Second, we construct an example which shows that Corollary 3 is false if we drop the hypothesis that the o-minimal structure in question is not polynomially bounded.

Example 2. *Set*

$$X = \left\{(x, e^{-1/x}) : x \in (0, 1]\right\} \cup \{(0, 0)\}$$

which is definable in $\mathbb{R}_{\exp}$, the structure generated by the exponential function on the reals, which was proved to be o-minimal by Wilkie. Observe that X is compact and $X^{\mathrm{alg}} = \emptyset$ as $x \mapsto e^x$ is not semi-algebraic. For given $\lambda > 0$ there is $x_0 = x_0(\lambda) \geq 1$ such *that we have* $e^{-x/2} \leq x^{-\lambda}$ *if* $x \geq x_0$*. Now let* $n \geq 1$ *be an integer and suppose* $T \geq x_0(\lambda)$*. If* $T/2 \leq n$*, then*

$$\left|(1/n, 0) - (1/n, e^{-n})\right| = e^{-n} \leq e^{-T/2} \leq T^{-\lambda}.$$

Thus $(1/n, e^{-n}) \in X$ *approximates the rational point* $(1/n, 0)$*. Considering all* n *with* $T/2 \leq n \leq T$ *we find*

$$\#\left\{q \in \mathbb{Q}^2 : H(q) \leq T \text{ and } \operatorname{dist}^*(q, X) \leq T^{-\lambda}\right\} \geq \frac{T}{2} - 1$$

for all sufficiently large T*.*

The multiplicative constant in Pila and Wilkie's Theorem is uniform over families of definable sets. Somewhat surprisingly, the constant c in Theorem 2 is not uniform over a definable family, as we now demonstrate.

Example 3. *We take*

$$Z = \{(y, x, e^{xy} - 1) : x, y \in [0, 1]\} \subseteq \mathbb{R} \times \mathbb{R}^2$$

and we consider Z *as a definable family parametrized by* y *with fibers* Z_y*. It is compact and definable in* $\mathbb{R}_{\mathrm{an}}$*.*

Observe that $(Z_y)^{\mathrm{alg}} = \emptyset$ *if* $y \in (0, 1]$ *and* $(Z_0)^{\mathrm{alg}} = Z_0$*. In other words, the family* Z *has transcendental fibers away from* 0 *which "degenerate" to a real semi-algebraic curve above* $y = 0$*. This will affect approximation properties of the transcendental fibers.*

Let $\lambda > 0$*, let* $y \in [0, 1]$*, and suppose* $T \geq 1$*. For small* y *there are many "obvious" rational points close to* X_y *of bounded height. Indeed, say* $\eta \in \mathbb{Q} \cap [0, 1]$ *with* $H(\eta) \leq T$ *then*

$$|(\eta, 0) - (\eta, e^{\eta y} - 1)| = e^{\eta y} - 1 \leq 2\eta y \leq 2y$$

as $e^t - 1 \leq 2t$ *for all* $t \in [0, 1]$*. So if* $y \leq T^{-\lambda}/2$*, then as in Example 1 we find*

$$\#\{q \in \mathbb{Q}^2 : H(q) \leq T \text{ and there is } x \in Z_y \text{ with } |x - q| \leq T^{-\lambda}\} \geq \frac{3}{\pi^2}T^2 + o(T^2)$$

where the constant in $o(\cdot)$ *is independent of* y*. In particular, there cannot exist constants* $c > 0$ *and* $\lambda > 0$ *such that*

$$\#\{q \in \mathbb{Q}^2 : H(q) \leq T \text{ and } \operatorname{dist}^*(q, Z_y) \leq T^{-\lambda}\} \leq cT$$

holds for all $T \geq 1$ *and all* $y \in [0, 1]$ *with* $(Z_y)^{\mathrm{alg}} = \emptyset$*.*

Example 4. *Here is a variation of the last example. We set*

$$Z = \left\{\left(y, x, y^{-1}x^{\sqrt{2}}\right) : y \in [1, +\infty) \text{ and } x \in [0, 1]\right\} \subseteq \mathbb{R} \times \mathbb{R}^2.$$

Then Z *is definable in the structure generated by* $\mathbb{R}_{\mathrm{an}}$ *and taking real powers, cf. the paragraph before Section 3 [21] and Miller's paper [12] for the fact that this structure is o-minimal and polynomially bounded. This time* Z *is closed and* $(Z_y)^{\mathrm{alg}} = \emptyset$ *for all* y*.*

Say $\lambda > 0$ *is arbitrary. Let* $x \in \mathbb{Q} \cap [0, 1]$ *with* $H(x) \leq T$ *and* $y \geq 1$*, then*

$$\left|(x, 0) - \left(x, y^{-1}x^{\sqrt{2}}\right)\right| = y^{-1}x^{\sqrt{2}} \leq y^{-1}.$$

If $y \geq T^{\lambda}$*, then as in Example 3*

$$\#\left\{q \in \mathbb{Q}^2 : H(q) \leq T \text{ and } \operatorname{dist}^*(q, Z_y) \leq T^{-\lambda}\right\} \geq \frac{3}{\pi^2}T^2 + o(T^2).$$

So the constant c in Corollary 3 is not uniformly bounded for families of definable sets.

In this example, the transcendenal fibers Z_y degenerate to the line segment $[0,1]\times\{0\}$ as $y\to+\infty$.

In order to generalize Corollary 3 to a definable family. we must make sure that the family contains no fibers with a non-trivial algebraic locus and that the fibers do not degenerate into something algebraic at infinity. We make these assumptions precise in the next theorem. Let $m\geq 0$ be an integer.

Theorem 4. *Let $Z\subseteq\mathbb{R}^m\times\mathbb{R}^n$ be closed and definable in a polynomially bounded o-minimal structure such that the projection of Z to $\mathbb{R}^m$ is bounded and such that $(Z_y)^{\text{alg}}=\emptyset$ for all $y\in\mathbb{R}^m$. Let $e\geq 1$ be an integer and let $\epsilon>0$. There exist $c=c(Z,e,\epsilon)\geq 1$ and $\lambda=\lambda(Z,e,\epsilon)>0$ such that*

$$\#\left\{q\in\mathbb{Q}^n(T,e): \textit{there is } x\in X_y \textit{ with } |x-q|\leq c^{-1}T^{-\lambda}\right\}\leq cT^{\epsilon}$$

for all $T\geq 1$ and all $y\in\mathbb{R}^m$.

In view of Example 4 we cannot drop the hypothesis that the projection of Z to $\mathbb{R}^m$ is bounded in this last theorem.

Can one replace $\mathcal{N}(X^{\text{alg}},\psi(T)^{\theta})$ by X^{alg} in Theorem 2. The answer is no, as the following example shows.

Example 5. *Let*

$$\xi=\sum_{n=1}^{\infty}10^{-n!}$$

be Liouville's constant and set

$$X=[0,1]\times\{\xi\}.$$

Then X is semi-algebraic, hence definable in any o-minimal structure and $X=X^{\text{alg}}$. We claim that there cannot exist constants $\epsilon\in(0,2), c>0,$ and $\lambda>0$ such that

$$\#\left\{q\in\mathbb{Q}^2\smallsetminus X^{\text{alg}}: H(q)\leq T \textit{ and } \operatorname{dist}^*(q,X)\leq T^{-\lambda}\right\}\leq cT^{\epsilon} \tag{2}$$

for all $T\geq 2$.

Indeed, say $\xi_m=\sum_{n=1}^m 10^{-n!}$ for $m\geq 1$. Then $\xi_m\neq\xi$ and $|\xi_m-\xi|\leq 2\cdot 10^{-(m+1)!}$. Moveover, each ξ_m is rational with height $H(\xi_m)=T$ where $T=10^{m!}$. For all $x\in\mathbb{R}$ we have

$$|(x,\xi_m)-(x,\xi)|\leq 2\cdot 10^{-(m+1)!}=2T^{-(m+1)}\leq T^{-m}$$

as $T\geq 2$. Say $m\geq\lambda$, then $|(x,\xi_m)-(x,\xi)|\leq T^{-\lambda}$. As there are $3T^2/\pi^2+o(T^2)$ rational $x\in[0,1]$ with $H(x)\leq T$, the bound (2) fails for m sufficiently large.

We now give a variant of Theorem 2 which emphasizes points on X that admit a good rational approximation. We will deduce all results above using this point of view.

Theorem 5. *Let $X\subseteq\mathbb{R}^n$ be closed and definable in a polynomially bounded o-minimal structure. Let $e\geq 1$ be an integer and let $\epsilon>0$. There exist $c=c(X,e,\epsilon)>0$ and $\theta=\theta(X,e,\epsilon)\in(0,1]$ with the following property. If $\psi:[1,+\infty)\to[0,1]$ has order at* *most* $-\theta^{-1}$ *and* $T \geq 1$ *there exist an integer* $N \geq 0$ *with* $N \leq cT^\epsilon$ *and* $x_1, \ldots, x_N \in X$ *such that*

(3)
$$\left\{x \in X \smallsetminus \mathcal{N}(X^{\mathrm{alg}}, \psi(T)^\theta) : \textit{there is } q \in \mathbb{Q}^n(T,e) \textit{ with } |x-q| \leq c^{-1}\psi(T)\right\} \subseteq \bigcup_{i=1}^N \mathcal{N}(\{x_i\}, \psi(T)^\theta).$$

Theorems 5 and 2 are both special cases of the next result. As in Pila and Wilkie's Theorem 1.10 [17] we can replace X^{alg}, which need not be definable, by a subset which is for fixed T. Our formulation of the result below is inspired by Pila's concept of blocks, cf. Theorem 3.6 [16]. We refer to Section 5 where some basic definitions involving real algebraic sets are recalled.

Theorem 6. *Let* $X \subseteq \mathbb{R}^n$ *be closed and definable in a polynomially bounded o-minimal structure. Let* $e \geq 1$ *be an integer and let* $\epsilon > 0$*. There exist* $c = c(X, e, \epsilon) \geq 1, \theta = \theta(X, e, \epsilon) \in (0, 1]$*, integers* $l_1, \ldots, l_t \geq 0$ *and definable sets* $D_j \subseteq \mathbb{R}^{l_j} \times \mathbb{R}^n$ *for all* $j \in \{1, \ldots, t\}$ *with the following properties:*

(i) *Say* $D = D_j$ *for some* $j \in \{1, \ldots, t\}$ *and* $z \in \mathbb{R}^{l_j}$*. Then* $D_z \subseteq X$ *and if* $D_z \neq \emptyset$*, then* D_z *is a connected and open subset of the non-singular locus of a real algebraic set of dimension* $\dim D_z$*.*

(ii) *Let* $\psi : [1, +\infty) \to [0, 1]$ *have order at most* $-\theta^{-1}$*. If* $T \geq 1$ *there exists an integer* $N \geq 1$ *with* $N \leq cT^\epsilon$ *and* $(j_i, z_i) \in \{1, \ldots, t\} \times \mathbb{R}^{l_{j_i}}$ *for* $i \in \{1, \ldots, N\}$ *such that if*
$$x \in X \textit{ and } q \in \mathbb{Q}^n(T,e) \textit{ with } |x-q| \leq c^{-1}\psi(T) \tag{4}$$
then there is $i \in \{1, \ldots, N\}$ *and* $x' \in (D_{j_i})_{z_i}$ *with* $|x - x'| \leq \psi(T)^\theta$*.*

In Theorem 10 below we will state a result for definable families which, in view of Example 3, takes some additional care to formulate.

Our argument follows the framework laid out in the proof of Pila and Wilkie of their counting theorem [17]. We use their basic induction scheme, so it is natural to prove the theorem directly for families of definable sets. Moreover, we use their version of the Gromov-Yomdin Reparametrization Theorem in o-minimal structures. In order to treat algebraic points that merely approximate a definable set, we require a suitable Łojasiewicz Inequality. However, even a basic incarnation of this inequality is not uniform over a definable family, cf. Example 6 below. This lack of uniformity is ultimately reflected in Examples 3 and 4. However, to complete the induction step we need uniform control over various quantities attached to fibers of a definable family. We resolve this technical difficulty by introducing a uniform substitute for the Łojasiewicz Inequality, cf. Proposition 12. This inequality is the main new ingredient in this paper. Its proof requires intricate results on o-minimal structures such as the Generic Trivialization Theorem. Another difference to the original work of Pila-Wilkie, as well as to earlier work of Bombieri-Pila [5], is our construction of the auxiliary function. Instead of a Vandermonde Determinant we use an "approximate Thue-Siegel Lemma" to construct the auxiliary function, an idea due to Wilkie [23]. It has the advantage that we can deal directly with algebraic points of bounded degree.

Rational approximations on submanifolds of $\mathbb{R}^n$ are studied in metric diophantine approximation. We mention just a few results and connections to our work here. Mahler's influential problem asked to show that for all $\epsilon > 0$ and all $x \in \mathbb{R}$ outside a Lebesgue zero set,

$$\left\{q \in \mathbb{Z} : q \geq 1 \text{ and there exist } p_1, \ldots, p_n \in \mathbb{Z} \text{ with } \left|x^i - \frac{p_i}{q}\right| \leq \frac{1}{q^{1+1/n+\epsilon}} \text{ for } 1 \leq i \leq n\right\}$$

is finite. Here $(p_1/q, \ldots, p_n/q)$ approximates a point on the curve $\{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$ with error $q^{-\lambda}$ where $\lambda = 1 + 1/n + \epsilon$ is arbitrarily close to the critical value $1 + 1/n$. Sprindzhuk solved Mahler's problem. The more general conjecture of Baker-Sprindzhuk was proved by Kleinbock and Margulis.

In recent work, Beresnevich, Vaughan, Velani, and Zorin [2] obtained upper bounds for the number of sufficiently good rational approximations on certain submanifolds in $\mathbb{R}^n$. As in other work mentioned in this direction, there is a strong emphasis on the quality of the exponent λ such as in Corollary 3.

Our method is of a different nature, it yields little control on this exponent. Indeed, λ produced by Corollary 3 comes out of compacity statements in o-minimality and seems difficult to pin down. The trade-off is that our bounds for the number of rational approximations grows as an arbitrarily small power of the height. This has applications, one of which we present here.

We apply our results to the question of how small a non-vanishing sum of $n + 1 \geq 2$ roots of unity can be. This problem appears in connection with eigenvalues of circulant matrices in work of Graham and Sloane [6]. For an integer $N \geq 1$, Myerson [14] defined $f(n + 1, N)$ to be the least positive value of

$$|1 + \zeta_1 + \cdots + \zeta_n| \quad \text{where} \quad \zeta_1^N = \cdots = \zeta_n^N = 1.$$

He proved asymptotic estimates if $n \in \{1, 2, 3\}$ for N in certain congruence classes and $N \to +\infty$. Here we are interested in lower bounds for $f(n + 1, N)$. Myerson's result *loc.cit.* implies $f(n + 1, N) \geq cN^{-1}$ for some absolute constant $c > 0$ in the cases $n = 1$ and $n = 2$ and $f(4, N) \geq cN^{-2}$. A lower bounds that decreases exponentially in N holds by Konyagin and Lev's Theorem 1 [11]. Using Liouville's Theorem from number theory one finds $f(n+1, N) > (n+1)^{-N}$ in general. Upper bounds for $f(n+1, N)$ are discussed in [11, 14] and they decrease polynomially in N for fixed n and large N. However, it seems to be unknown if a polynomial lower bound holds if $n \geq 4$. The author finds it reasonable to expect the following folklore conjecture. It would follow from a positive answer to the question Myerson [14] asks at the end of his paper.

Conjecture. *For given $n \geq 1$ there exist constants $c(n) > 0$ and $\lambda(n) > 0$ such that $f(n + 1, N) \geq c(n)N^{-\lambda(n)}$ for all $N \geq 1$.*

We use our result on approximations on definable sets to give some give credence to this conjecture. Indeed, we show that set of the prime orders $N = p$ where the conjecture fails is sparse.

Theorem 7. *For $\epsilon > 0, n \geq 1$, and $a_0, \ldots, a_n \in \mathbb{C} \setminus \{0\}$. there exist constants $c = c(a_0, \ldots, a_n, \epsilon) \geq 1$ and $\lambda = \lambda(a_0, \ldots, a_n, \epsilon) > 0$ such that*

$$\begin{aligned}\#\{p \leq T \text{ is a prime } : \quad & \text{there are } \zeta_1, \ldots, \zeta_n \in \mathbb{C} \text{ with } \zeta_1^p = \cdots = \zeta_n^p = 1 \text{ and} \\ & 0 < |a_0 + a_1\zeta_1 + \cdots + a_n\zeta_n| \leq c^{-1}p^{-\lambda}\} \leq cT^\epsilon\end{aligned}$$

for all $T \geq 1$.

We briefly discuss the paper's content. In Section 2 we introduce some common notation. Our Łojasiewicz Inequality is formulated in Section 3, after that we construct the auxiliary function in Section 4. Section 5 is a detour on a class of cells that are locally semi-algebraic and prove useful in the induction step. The induction itself is done in Section 6 and in Section 7 we complete the proofs of the approximation theorems mentioned here in the introduction. Section 8 contains the proof of Theorem 7 on small sums of roots of unity.

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2. General Notation

The natural numbers are $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $n \in \mathbb{N}$. References to a topology are to the Euclidean topology in $\mathbb{R}^n$ if not stated otherwise. Let X be any subset of $\mathbb{R}^n$, the closure of X in $\mathbb{R}^n$ is denoted by $\overline{X}$ and the frontier of X is $\mathrm{fr}(X) = \overline{X} \smallsetminus X$. This should not be confused with the boundary of X, the complement in $\overline{X}$ of the interior of X.

We defined the height of a rational number in the introduction. More generally, if $q \in \overline{\mathbb{Q}}$, then we may proceed as follows. Let $P \in \mathbb{Z}[X]$ be the unique irreducible polynomial with $P(q) = 0$ and positive leading coefficient p_0. Then

$$H(q) = \left(p_0 \prod_{z \in \mathbb{C} : P(z) = 0} \max\{1, |z|\} \right)^{1/\deg P}$$

is the absolute Weil height, or just height, of q. The height of a vector in $\overline{\mathbb{Q}}^n$ is the maximal height of a coordinate. See Bombieri and Gubler's Chapter 1.5 [4] for more details. Examples of basic height properties are

$$H(q + q') \leq 2H(q)H(q') \quad \text{and} \quad H(qq') \leq H(q)H(q'). \tag{5}$$

Our reference for o-minimal structures is van den Dries's book [20]. For this paper we use the following straightforward definition.

A structure $\mathfrak{S}$ is a sequence $(S_1, S_2, \ldots)$ where each S_n is a set of subsets of $\mathbb{R}^n$ such that the following properties hold true for all $n, m \in \mathbb{N}$.

(i) The set S_n is closed under taking finite unions, finite intersections, and passing to the complement.
(ii) If $X \in S_n$ and $Y \in S_m$, then $X \times Y \in S_{n+m}$.
(iii) If $X \in S_n$ and $n \geq 2$, then the projection of X onto the first $n - 1$ coordinates lies in S_{n-1}.

(iv) All real semi-algebraic sets in $\mathbb{R}^n$ lie in S_n.

We call $\mathfrak{S}$ an o-minimal structure if in addition

(v) all elements in S_1 are finite unions of points and open, possibly unbounded, intervals.

A set is called definable in $\mathfrak{S}$ if it is a member of some S_n. A function defined on a subset of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is called definable if its graph is in S_{n+m}. Say $m \in \mathbb{N}_0$. If $m = 0$ we will identify $\mathbb{R}^m$ with a singleton and $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^n$. A definable family, or family parametrized by $\mathbb{R}^m$, is a definable subset $Z \subseteq \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. We think of Z parametrizing fibers $Z_y = \{x \in \mathbb{R}^n : (y, x) \in Z\} \subseteq \mathbb{R}^n$ where $y \in \mathbb{R}^m$. The dimension of a definable set is defined in Chapter 4.1 [20]; we follow the convention $\dim \emptyset = -\infty$.

If there is no ambiguity about the ambient o-minimal structure $\mathfrak{S}$, then we call a set or function definable if it is definable in $\mathfrak{S}$.

Throughout this paper, we will use some basic properties of o-minimal structures without mentioning them explicitly. For example, if X is definable then so are $\overline{X}$ and $\mathrm{fr}(X)$, cf. Lemma 3.4, Chapter 1 [20]. Moreover, the projection of a definable set to any collection of the coordinates is again definable.

Cells are always assumed to definable in the ambient o-minimal structure. They are the "building blocks" of the definable sets, see Chapter 3 of van den Dries's book [20]. Let us recall some of their properties.

(i) Cells are non-empty by definition.
(ii) A cell $C \subseteq \mathbb{R}^n$ is a locally closed subset of $\mathbb{R}^n$, cf. (2.5) in Chapter 3 [20]. So $\mathrm{fr}(C)$ is a closed subset of $\mathbb{R}^n$.
(iii) Let $C \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a cell. If $y \in \mathbb{R}^m$, then the fiber $C_y \subseteq \mathbb{R}^m$ is either empty or a cell, cf. Proposition 3.5(i) in Chapter 3 [20]. Moreover, the dimension $\dim C_y$ does not depend on y if $C_y \neq \emptyset$. We call this value the fiber dimension of C over $\mathbb{R}^m$.
(iv) Suppose $m \geq 1$, and write $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ for the projection onto the first m coordinates of $\mathbb{R}^m \times \mathbb{R}^n$. If $C \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a cell, then so is $\pi(C) \subseteq \mathbb{R}^m$, cf. (2.8) in Chapter 3 [20].

3. Variations on Łojasiewicz

Throughout this section we work in a fixed polynomially bounded o-minimal structure. Here is the prototype of a Łojasiewicz Inequality for definable functions.

Theorem 8 (Łojasiewicz Inequality)**.** *Let $X \subseteq \mathbb{R}^n$ be a compact and definable set. Suppose that $f : X \to \mathbb{R}$ is a continuous and definable function with zero set $Z \subseteq X$. There exist $c > 0$ and a rational number $\delta > 0$ such that*

$$\mathrm{dist}^*(x, Z) \leq c|f(x)|^\delta$$

for all $x \in X$.

Proof. This follows from 4.14(2) [21] applied to f and the continuous and definable function $g(x) = \mathrm{dist}^*(x, Z)$. □

The proof of Pila and Wilkie's Theorem [17] relies on an inductive argument. To make the induction step work it is necessary to work with families of definable sets and to bound various quantities attached to the fibers of the family uniformly. Unfortunately, the constants c and δ in the Łojasiewicz Inequality above cannot be choosen uniformly over a definable family.

Example 6. *We take $X = [-2,2] \times [-2,2]$ and $f(y,x) = y^2 + x^2 - 1$. The zero set Z of f is the unit circle. We consider X and Z as definable families parametrized by the coordinate y. For all $y \in [-2,2]$, the theorem above yields $c_y > 0$ and $\delta_y > 0$ such that*

$$\mathrm{dist}^*(x, Z_y) \le c_y |f(y,x)|^{\delta_y}$$

for all $x \in Z_y$.

If $y < -1$ or $y > 1$, then $Z_y = \emptyset$ and by our convention $x \mapsto \mathrm{dist}^*(x, Z_y)$ is constant with value 1 as a function in $x \in [-2,2]$. So $c_y|f(y,x)|^{\delta_y} \ge 1$ if $|y| > 1$. Now $|f(y,0)| = |y-1||y+1|$ is arbitrarily small as $y \to 1$ from the right. So it is not possible to choose c_y and δ_y independent of y.

Observe that $(y,x) \mapsto \mathrm{dist}^*(x, Z_y)$ is not continuous on X as $\mathrm{dist}^*(0, Z_y)$ jumps from 1 to 0 as $y \to 1$ from the right.

The purpose of this section is to prove a suitable substitute for the Łojasiewicz Inequality above for a definable family.

We begin with several preliminary lemmas. Recall that the frontier $\mathrm{fr}(X)$ of a set $X \subseteq \mathbb{R}^n$ is $\overline{X} \smallsetminus X$.

Lemma 7. *Let $X \subseteq \mathbb{R}^n$ be bounded, locally closed, and definable and suppose $f : X \to \mathbb{R}$ is a continuous, definable function with $f(x) \neq 0$ for all $x \in X$. There exist $c = c(X,f) > 0$ and a rational number $\delta = \delta(X,f) > 0$ such that*

$$\mathrm{dist}^*(x, \mathrm{fr}(X)) \le c|f(x)|^{\delta}$$

for all $x \in X$.

Proof. We set $g(x) = \mathrm{dist}^*(x, \mathrm{fr}(X))$ which yields a continuous, definable function $g : \overline{X} \to \mathbb{R}$. Certainly, $g(x) = 0$ for $x \in \mathrm{fr}(X)$. Conversely, if $x \in \overline{X}$ and $g(x) = 0$ then we may fix a sequence $x_1, x_2, \ldots \in \mathrm{fr}(X)$ with limit x. The frontier $\mathrm{fr}(X)$ is closed in $\mathbb{R}^n$ as X is locally closed, so $x \in \mathrm{fr}(X)$. Therefore, g vanishes precisely on the frontier $\mathrm{fr}(X)$.

We may apply Lemma C.8 [21] to $\overline{X}$ and the functions g and $f_1 = f^{-1} : X \to \mathbb{R}$, which are continuous and definable. We obtain a definable, continuous, odd, increasing, bijective map $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ (as defined on page 512 [21] with $p = 0$) such that for any $y \in \mathrm{fr}(X)$ we have $\phi(g(x))/f(x) \to 0$ if $x \to y$ with $x \in X$. We set $h(x) = \phi(g(x))/f(x)$ if $x \in X$ and $h(x) = 0$ if $x \in \mathrm{fr}(X)$. Thus $h : \overline{X} \to \mathbb{R}$ is continuous.

Now $\overline{X}$ is compact as X is bounded. So there exists $c_1 > 0$ with $|h(x)| \le c_1$ for all $x \in \overline{X}$. Observe that $\phi(g(x)) \ge 0$ since $g(x) \ge 0$ and because ϕ is odd and increasing. Therefore, $\phi(g(x)) \le c_1|f(x)|$ for all $x \in X$.

Finally, as the ambient o-minimal structure is polynomially bounded there are constants $c_2 > 0$ and $\delta > 0$ with $\phi(t) \ge c_2 t^{1/\delta}$ for all $t \in [0,1]$. We may assume that $\delta \in \mathbb{Q}$. The lemma follows with $c = (c_1/c_2)^{\delta}$ since g takes values in $[0,1]$. □

The fact that δ is rational above entails that $t \mapsto t^{\delta}$ is a definable function.

We state an easy consequence of Proposition C.13 [21].

Lemma 8. *Let $X \subseteq \mathbb{R}^n$ be a locally closed, definable set and suppose $f, g : X \to \mathbb{R}$ are continuous and definable functions such that $x \in X$ and $f(x) = 0$ entails $g(x) = 0$*

and such that g is bounded. There exists a rational number $\delta = \delta(X, f, g) > 0$ and a continuous and definable function $h : X \to \mathbb{R}$ with

$$|g(x)| \le |h(x)f(x)|^\delta$$

for all $x \in X$.

Proof. By Proposition C.13 [21] there is $\phi : \mathbb{R} \to \mathbb{R}$ as in the proof of Lemma 7 and a continuous, definable function $h : X \to \mathbb{R}$ with $\phi(g(x)) = h(x)f(x)$ for all $x \in X$. Observe that $|\phi(g(x))| = \phi(|g(x)|)$. The rest of the proof is now much as the end of the proof of Lemma 7. □

We identify polynomials in $\mathbb{R}[X_1, \ldots, X_m]$ of degree bounded by $d \ge 0$ including the zero polynomial with $\mathbb{R}^l$ where $l = \binom{m+d}{m}$. Thus each $f \in \mathbb{R}^l$ corresponds to a polynomial in m variables and we write $\mathcal{Z}(f)$ for its set of zeros in $\mathbb{R}^m$.

Suppose $m \ge 0$ and let $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a definable family parametrized by $\mathbb{R}^m$. Let $Y \subseteq \mathbb{R}^m$ be the projection of Z to $\mathbb{R}^m$. It is a definable set and for $(y, x) \in Y \times \mathbb{R}^n$

$$(y, x) \mapsto \inf\{|x - x'| : x' \in Z_y\}$$

yields a definable function $Y \times \mathbb{R}^n \to \mathbb{R}$. So $(y, x) \mapsto \operatorname{dist}^*(x, Z_y)$ is definable on $Y \times \mathbb{R}^n$ and even on $\mathbb{R}^m \times \mathbb{R}^n$. We cannot expect it to be continuous due to Example 6.

We come to the first variant of the Łojasiewicz Inequality from the beginning of this section.

Lemma 9 (Flexing)**.** *Let $Z \subseteq \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be bounded, definable, and non-empty. There exist $c = c(Z) \in (0, 1]$, a rational number $\delta = \delta(Z) > 0$, and a compact and definable set $Z' \subseteq \overline{Z}$ with $\dim Z' < \dim Z$ such that the following property holds. Suppose $f \in \mathbb{R}^l$, $y \in \mathbb{R}^m$, and $x \in Z_{(f,y)}$ such that $|f(x)| \le c$. Then $\operatorname{dist}^*(x, Z_{(f,y)} \cap \mathcal{Z}(f)) \le |f(x)|^\delta$ or $\operatorname{dist}^*((f, y, x), Z') < |f(x)|^\delta$.*

Proof. Before this lemma we observed that

$$\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \ni (f, y, x) \mapsto \operatorname{dist}^*(x, Z_{(f,y)} \cap \mathcal{Z}(f))$$

yields a definable and bounded function $g : Z \to \mathbb{R}$. Its values are in $[0, 1]$. We partition Z into a finite number of cells $C_1, \ldots, C_N \subseteq Z$ such that $g|_{C_i}$ is continuous for all $1 \le i \le N$.

To prove the lemma it suffices to prove it in the case $Z = C_i$ for some i.

Therefore, Z is locally closed and g is continuous. We apply Lemma 8 to Z, the continuous and definable evaluation map $(f, y, x) \mapsto f(x)$, and g to find

$$\operatorname{dist}^*(x, Z_{(f,y)} \cap \mathcal{Z}(f)) \le |h(f, y, x)f(x)|^{\delta_1} \quad \text{for all} \quad (f, y, x) \in Z \tag{6}$$

where $h : Z \to \mathbb{R}$ is continuous and definable and $\delta_1 > 0$ is rational.

The sets

$$Z_1 = \{(f, y, x) \in Z : |h(f, y, x)||f(x)|^{1/2} \le 1\} \quad \text{and} \quad Z_2 = Z \setminus Z_1$$

are definable.

Let $c \in (0, 1]$ and let $\delta > 0$ be rational, we will determine them in the argument below. Say f and y are as in the hypothesis and suppose $x \in Z_{(f,y)}$ with $|f(x)| \le c \le 1$. There are two cases.

First let us assume $(f,y,x)\in Z_1$; this includes the case $f(x)=0$. Then

$$\mathrm{dist}^*(x,Z_{(f,y)}\cap\mathcal{Z}(f))\le|f(x)|^{\delta_1/2}$$

follows from (6). The first possibility in the assertion holds as we may assume $\delta\le\delta_1/2$ and since $|f(x)|\le 1$.

The second case is $|h(f,y,x)||f(x)|^{1/2}>1$; in particular $f(x)\neq 0$. Recall that Z is bounded by hypothesis. Here we apply Lemma 7 to Z and the continuous function $Z\ni(f',y',x')\mapsto\max\{1,|h(f',y,'x')|\}^{-1}$ which is continuous, definable, and does not attain 0. So there is a $\delta_2\in(0,1]$ and $c_1>0$, both independent of f,y, and x, with

$$\mathrm{dist}^*((f,y,x),\mathrm{fr}(Z))\le c_1\max\{1,|h(f,y,x)|\}^{-\delta_2}.$$

We obtain

$$\mathrm{dist}^*((f,y,x),\mathrm{fr}(Z))\le c_1\max\{1,|f(x)|^{-1/2}\}^{-\delta_2}=c_1|f(x)|^{\delta_2/2}\le c_1c^{\delta_2/4}|f(x)|^{\delta_2/4}.$$

If c is sufficiently small in terms of c_1 and δ_2, then $\mathrm{dist}^*((f,y,x),\mathrm{fr}(Z))<|f(x)|^{\delta_2/4}\le 1$. We may assume $\delta\le\delta_2/4$, so the distance is strictly less than $|f(x)|^{\delta}$.

Now $Z'=\mathrm{fr}(Z)$ is definable and satisfies $\dim Z'<\dim Z$ by Theorem 1.8 in Chapter 4 [20]. Then Z' is closed in $\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^n$ and contained in $\overline{Z}$ as Z is locally closed. Thus Z' is compact and definable; this concludes the proof. □

Next we prove a variant of the Hölder inequality C.15 [21] without a compactness assumption. Suppose $m\ge 1$ and let $\pi:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$ be the projection onto the first m coordinates.

Lemma 10. *Let $A\subseteq\mathbb{R}^m$ be bounded, locally closed, and definable and let $K\subseteq\mathbb{R}^n$ be compact and definable. We suppose that $\chi:A\times K\to\mathbb{R}^k$ is a continuous, bounded, definable function. There exist $c=c(A,\chi)>0$ and rational numbers $\delta_{1,2}=\delta_{1,2}(A,\chi)>0$ such that*

$$\min\{\mathrm{dist}^*(\pi(x),\mathrm{fr}(A)),\mathrm{dist}^*(\pi(y),\mathrm{fr}(A))\}\le c\min\left\{1,\frac{|x-y|^{\delta_1}}{|\chi(x)-\chi(y)|^{\delta_2}}\right\}\tag{7}$$

for all $x,y\in A\times K$ with $\chi(x)\neq\chi(y)$.

Proof. Let us abbreviate $X=A\times K$. This is a locally closed, bounded, and definable subset of $\mathbb{R}^m\times\mathbb{R}^n$.

We will apply Lemma 8 to $X\times X$ and the functions $f(x,y)=|x-y|$ and $g(x,y)=|\chi(x)-\chi(y)|$. Thus there is a continuous definable function $h:X\times X\to[0,+\infty)$ and a rational number $\delta_1=\delta_1(A,\chi)>0$ with

$$|\chi(x)-\chi(y)|\le h(x,y)|x-y|^{\delta_1}\tag{8}$$

for all $x,y\in X$.

Let us apply also Lemma 7 to $X\times X$. This time we take as function $\max\{1,h(x,y)\}^{-1}$, which never vanishes on $X\times X$. We get constants $c=c(A,f)>0$ and a rational number $\delta_2=\delta_2(A,f)>0$ with

$$\mathrm{dist}^*((x,y),\mathrm{fr}(X\times X))\le c\max\{1,h(x,y)\}^{-\delta_2}$$

for all $x,y\in X$. Observe that

$$\mathrm{fr}(X\times X)=(\mathrm{fr}(X)\times\overline{X})\cup(\overline{X}\times\mathrm{fr}(X))=(\mathrm{fr}(A)\times K\times\overline{X})\cup(\overline{X}\times\mathrm{fr}(A)\times K)$$

because K is closed. So

$$\mathrm{dist}^*((x,y),(\mathrm{fr}(A)\times K\times\overline{X})\cup(\overline{X}\times\mathrm{fr}(A)\times K))\le c\max\{1,h(x,y)\}^{-\delta_2}$$

for all $x,y\in X$. The left-hand side of is at least $\min\{\mathrm{dist}^*(\pi(x),\mathrm{fr}(A)),\mathrm{dist}^*(\pi(y),\mathrm{fr}(A))\}$, therefore

$$\min\{\mathrm{dist}^*(\pi(x),\mathrm{fr}(A)),\mathrm{dist}^*(\pi(y),\mathrm{fr}(A))\}\le c\max\{1,h(x,y)\}^{-\delta_2}. \tag{9}$$

If $x\neq y$ we use (8) to bound the right-hand side of (9) from above. Thus

$$\min\{\mathrm{dist}^*(\pi(x),\mathrm{fr}(A)),\mathrm{dist}^*(\pi(y),\mathrm{fr}(A))\}\le c\max\left\{1,\frac{|\chi(x)-\chi(y)|}{|x-y|^{\delta_1}}\right\}^{-\delta_2}$$

and the lemma follows after adjusting δ_1 and δ_2. □

Lemma 11 (Straightening). *Let $Z\subseteq\mathbb{R}^m\times\mathbb{R}^n$ be compact and definable. There exist $c=c(Z)\in(0,1]$ and a rational number $\delta=\delta(Z)>0$ with the following property. If $y_0\in\pi(Z)$ and $0<\epsilon\le c$ there are $y_1,\dots,y_N\in\pi(Z)$ with $N\le c^{-1}$ such that for any $p\in Z$ with $|y_0-\pi(p)|\le\epsilon$ there exist $i\in\{1,\dots,N\}$ and $x\in Z_{y_i}$ with $|(y_i,x)-p|\le\epsilon^\delta$.*

Proof. If Z is a finite set we can take the y_i to be all elements in projection of Z to $\mathbb{R}^m$ and c small enough to ensure that $|y_0-\pi(p)|\le c$ entails $y_0=\pi(p)$. In this case we may take $(y_i,x)=p$.

We now assume $\dim Z\ge 1$. The proof is by induction where we suppose that the lemma is proved in dimensions strictly less than $\dim Z$.

Below, the constants $c_{1,2}$ and $\delta_{1,2,3,4}$ are positive and depend only on Z. The constants $c>0$ and $\delta>0$ from the assertion may depend on them and will be determined below.

By the Generic Trivialization Theorem 1.2, Chapter 9 [20] we can partition each $\pi(Z)$ into finitely many cells $C_1\cup\cdots\cup C_N$ such that $\pi|_Z:Z\to\pi(Z)$ is definably trivializable over each C_i. We let $\chi_i:C_i\times K_i\to\pi|_Z^{-1}(C_i)$ denote the definable homeomorphism coming from a trivialization. As K_i is homeomorphic to a fiber of $Z\to\pi(Z)$ it is compact. We will also use the fact that each C_i is locally closed.

We may assume $c\le 1/16$ and $c\le 1/N$.

Say $y_0\in\pi(Z)$ and $0<\epsilon\le c$. For each $1\le i\le N$ we choose auxiliary points

$$y_i\in C_i \text{ such that } |y_0-y_i|\le\epsilon \tag{10}$$

if such an element exists. After renumbering, the y_i will be the points in the assertion.

Let $p\in Z$ be as in the hypothesis and suppose $\pi(p)\in C_i$. The y_i as described above exists and we will prove that there is $x\in Z_{y_i}$ such that $|(y_i,x)-p|\le\epsilon^\delta$. We are in effect straightening-out the fiber containing the possible p.

Observe that if $y_i=\pi(p)$, then we are allowed to choose x with $p=(y_i,x)$. So let us suppose $y_i\neq\pi(p)$. To simplify notation we write $C=C_i,y=y_i,\chi=\chi_i$, and $K=K_i$.

There is $z(p)\in K$ with

$$\chi(\pi(p),z(p))=p.$$

Recall that $y\neq\pi(p)$, so $\chi(y,z(p))\neq p$. The function $\chi:C\times K\to\mathbb{R}^n$ takes values in the bounded set Z. So we may apply lemma 10 to $C\times K$ and χ. We obtain $c_1>0$ and $\delta_{1,2}>0$ such that

$$\min\{\mathrm{dist}^*(y,\mathrm{fr}(C)),\mathrm{dist}^*(\pi(p),\mathrm{fr}(C))\}\le c_1\min\left\{1,\frac{|y-\pi(p)|^{\delta_1}}{|\chi(y,z(p))-p|^{\delta_2}}\right\};$$

observe that $(y, z(p))$ and $(\pi(p), z(p))$ both lie in $C \times K$ and $\chi(\pi(p), z(p)) = p$.

We set $\delta_3 = \min\{1/2, \delta_1/4\}$ and split-up into 2 cases, the first one being

$$\mathrm{dist}^*(\pi(p), \mathrm{fr}(C)) \geq \epsilon^{\delta_3}. \tag{11}$$

We recall (10) and the hypothesis $|y_0 - \pi(p)| \leq \epsilon$ to bound

$$|y - \pi(p)| \leq |y - y_0| + |y_0 - \pi(p)| \leq 2\epsilon \leq 2c \leq 1.$$

So we have either

$$|\chi(y, z(p)) - p| < |y - \pi(p)|^{\delta_1/(2\delta_2)} \leq (2\epsilon)^{\delta_1/(2\delta_2)} \leq 1 \tag{12}$$

or $|\chi(y, z(p)) - p| \geq |y - \pi(p)|^{\delta_1/(2\delta_2)}$ and thus

$$\min\{\mathrm{dist}^*(y, \mathrm{fr}(C)), \mathrm{dist}^*(\pi(p), \mathrm{fr}(C))\} \leq c_1|y - \pi(p)|^{\delta_1/2} \leq c_1(2\epsilon)^{\delta_1/2}. \tag{13}$$

We can rule out this second possibility. Indeed, if $\mathrm{dist}^*(y, \mathrm{fr}(C)) < \epsilon^{\delta_3}/2$, then

$$\mathrm{dist}^*(\pi(p), \mathrm{fr}(C)) < \epsilon^{\delta_3}/2 + |y - \pi(p)| \leq \epsilon^{\delta_3}/2 + 2\epsilon.$$

By (11) we find $\epsilon^{-(1-\delta_3)} < 4$ which is a contradiction as $\epsilon \leq c \leq 1/16$ and $\delta_3 \leq 1/2$. So we must have $\mathrm{dist}^*(y, \mathrm{fr}(C)) \geq \epsilon^{\delta_3}/2$. By (11) the left-hand side of (13) is at least $\epsilon^{\delta_3}/2$. This is incompatible with $0 < \delta_3 \leq \delta_1/4$ for sufficiently small c.

Therefore, (12) holds true. We may assume $\delta \leq \delta_1/(4\delta_2)$, hence $|\chi(y, z(p)) - p| \leq (2\epsilon)^{2\delta} \leq \epsilon^{\delta}$ because $\epsilon \leq c < 1/4$. We take x as in $\chi(y, z(p)) = (y, x)$ and this yields the lemma.

The second case is

$$\mathrm{dist}^*(\pi(p), \mathrm{fr}(C)) < \epsilon^{\delta_3}.$$

Observe that $\pi|_Z^{-1}(C) = (C \times \mathbb{R}^n) \cap Z$ is locally closed in $\mathbb{R}^m \times \mathbb{R}^n$ because Z is closed. Moreover, this preimage is bounded because Z is. We can thus apply Lemma 7 to $\pi|_Z^{-1}(C)$ and the continuous function $p' \mapsto \mathrm{dist}^*(\pi(p'), \mathrm{fr}(C))$. Observe that this function does not vanishes as $\mathrm{fr}(C)$ is closed in $\mathbb{R}^m$. We set $Z' = \mathrm{fr}(\pi|_Z^{-1}(C))$, so

$$\mathrm{dist}^*(p, Z') \leq c_2 \mathrm{dist}^*(\pi(p), \mathrm{fr}(C))^{\delta_4} < c_2\epsilon^{\delta_3\delta_4}.$$

If c is small enough, the left-hand side is strictly less than 1. So there exists $p' \in Z'$ with $|p - p'| \leq c_2\epsilon^{\delta_3\delta_4}$.

As in the Flexing Lemma the dimension drops $\dim Z' < \dim \pi|_Z^{-1}(C) \leq \dim Z$. The frontier Z' is closed in $\mathbb{R}^m \times \mathbb{R}^n$ and definable as $\pi|_Z^{-1}(C)$ is locally closed and definable. So Z' is compact with $Z' \subseteq Z$ because Z is compact. Therefore, this lemma holds for the compact and definable set Z' by induction on the dimension and if c is small enough. We take as y_0 a fixed projection $\pi(p')$ that occurs in this second case. We obtain a point in Z' near p' that is in a bounded number of fibers. Both the proximity estimate and the bound on the number of fibers are sufficient to conclude the lemma for Z in this first case. □

We combine flexing and straightening to prove a Łojasiewicz Inequality for families.

We will again interpret $\mathbb{R}^l$ as the vector space of tuples of polynomials of degree bounded by d.

Proposition 12 (Łojasiewicz in families)**.** *Let $Z \subseteq \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be compact and definable. There exist $c = c(Z) \in (0,1]$ and a rational number $\delta = \delta(Z) > 0$ with the following property. If $f \in \mathbb{R}^l, y \in \mathbb{R}^m$, and $0 \le \epsilon \le c$, there are $(f_1, y_1), \ldots, (f_N, y_N) \in \mathbb{R}^l \times \mathbb{R}^m$ with $N \le c^{-1}$ such that for all $x \in Z_{(f,y)}$ with $|f(x)| \le \epsilon$ there is $i \in \{1, \ldots, N\}$ and $x' \in Z_{(f_i,y_i)}$ with $f_i(x') = 0$ and $|(f_i, y_i, x') - (f, y, x)| \le \epsilon^\delta$.*

Before we proceed with the proof note that the (f_i, y_i) in the claim may depend on (f, y), but their number is bounded uniformly.

Proof. If $\epsilon = 0$ then the proposition follows on taking $N = 1, f_i = f, y_i = y$, and $x' = x$. So let us assume $\epsilon \in (0,1]$. If Z is finite we take for the (f_i, y_i) the elements in its projection to $\mathbb{R}^l \times \mathbb{R}^m$ and c small enough to ensure that $x \in Z_{(f,y)}$ and $|f(x)| \le c$ entail $f(x) = 0$.

We now assume $\dim Z \ge 1$. We prove the proposition by induction on $\dim Z$ and suppose that it holds in all dimensions that are strictly less than $\dim Z$.

Let $x \in Z_{(f,y)}$ with $|f(x)| \le \epsilon \le c$. The constants $c_{1,2,3}$ and $\delta_{1,2,3,4}$ below are positive and depend only on Z but not on f, y, x, c, δ, or ϵ. We will fix c and δ during the argument below.

Let $c_1 \in (0,1], \delta_1 > 0$ be the constants and $Z' \subseteq \overline{Z} = Z$ the compact and definable set from the Flexing Lemma applied to Z. We may assume $c \le c_1 \le 1/2$. According to the Flexing Lemma there are two cases.

Suppose first $\mathrm{dist}^*(x, Z_{(f,y)} \cap \mathcal{Z}(f)) \le |f(x)|^{\delta_1} \le \epsilon^{\delta_1} \le c^{\delta_1} < 1$. Then $f(x') = 0$ for some $x' \in Z_{(f,y)}$ whose distance to x is at most $\epsilon^\delta > \epsilon^{\delta_1}$ as we may assume $\delta < \delta_1$. In this case the proposition follows since we may take (f, y) to be among the (f_i, y_i).

The second case is when there exits $(f_0, y_0, x_0) \in Z'$ with distance at most $|f(x)|^{\delta_1}$ to $(f, y, x) \in Z$. In particular, $|(f_0, y_0) - (f, y)| \le \epsilon^{\delta_1}$. Let us assume that we have found (f_0, y_0) in the projection of Z' that satisfies this inequality. It will serve as a base point for applying the Straightening Lemma to Z'; we now forget about x_0 and x.

Indeed, suppose $x \in Z_{(f,y)}$ with $|f(x)| \le \epsilon$ is a new point that is not covered by the first case. Thus there is $(f', y', x') \in Z'$ with $|(f', y', x') - (f, y, x)| \le \epsilon^{\delta_1}$. Using the triangle inequality we find

$$|(f', y') - (f_0, y_0)| \le 2\epsilon^{\delta_1} \le 2c^{\delta_1/2}\epsilon^{\delta_1/2} \le \epsilon^{\delta_1/2}$$

for c sufficiently small. After further shrinking c we may apply the Straightening Lemma to $Z', \epsilon^{\delta_1/2}$, and (f_0, y_0) when considering $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{l+m} \times \mathbb{R}^n$ as parametrized by $\mathbb{R}^{l+m}$. So (f', y', x') has distance at most $\epsilon^{\delta_1\delta_2/2}$ to a point (f_i, y_i, x'') in one of finitely many fibers $Z'_{(f_i,y_i)}$ of Z' provided for by Straightening Lemma. We may assume that the number of fibers in question is at most c^{-1}.

By the triangle inequality and the estimates above we get

$$\begin{aligned} |(f_i, y_i, x'') - (f, y, x)| &\le |(f_i, y_i, x'') - (f', y', x')| + |(f', y', x') - (f, y, x)| \\ &\le \epsilon^{\delta_1\delta_2/2} + \epsilon^{\delta_1} \\ &\le 2\epsilon^{\delta_3} \end{aligned} \tag{14}$$

with $\delta_3 = \delta_1 \min\{1, \delta_2/2\}$. Now $f_i(x'') = (f_i - f)(x'') + f(x'')$, so $|f_i(x'')| \le c_2\epsilon^{\delta_3} + |f(x'')|$ since x'' lies in the projection of Z' to $\mathbb{R}^n$, a bounded set. Let $\delta_4 = \min\{1, \delta_3\}$. By developing f in a series around x we find

$$|f(x'')| = |f(x + x'' - x)| \le |f(x)| + c_3|x'' - x| \le \epsilon + 2c_3\epsilon^{\delta_3} \le (1 + 2c_3)\epsilon^{\delta_4}$$

because f lies in the projection of the bounded set Z to $\mathbb{R}^l$ and since $|x'' - x| \le 2\epsilon^{\delta_3}$. Therefore, $|f_i(x'')| \le (1 + c_2 + 2c_3)\epsilon^{\delta_4}$. We may assume $\delta \le \delta_4/2$. If $c > 0$ is sufficiently small, then $\epsilon \le c$ implies $|f_i(x'')| \le \epsilon^{\delta}$.

Recall that $x'' \in Z'_{(f_i,y_i)}$ and $\dim Z' < \dim Z$. The proposition now follows by induction on the dimension combined with (14). $\square$

4. Construction of the Auxiliary Function

Bombieri and Pila [5] and later Pila and Wilkie [17] use the *determinant method* to construct an auxiliary function. Here we use a different approach introduced by Wilkie. He presented it in his lecture course at Manchester in 2013 [23]. It is related to the use of the Thue-Siegel Lemma in transcendence theory. Our tool to construct the auxiliary function is Minkowksi's Lattice Point Theorem.

As in the previous section we let $n \in \mathbb{N}$ and recall that $|\cdot|$ denotes the maximum norm on $\mathbb{R}^n$. Below we also use $|\cdot|$ to denote the maximum norm of the coefficient vector attached to a polynomial in real coefficients and possibly more than one unknown. Moreover, we write $\ell(x) = |x_1| + \cdots + |x_n|$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ we set $x^i = x_1^{i_1} \cdots x_n^{i_n}$ for elements $x_1, \ldots, x_n$ in any given ring where 0^0 is interpreted as 1. Suppose $k \in \mathbb{N}$ and let $\phi : (0,1)^k \to \mathbb{R}$ be a continuous function for which all partial derivatives up-to order b exist. For any $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$ with $\ell(\alpha) \le b$ we define

$$\partial^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial X_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial X_k^{\alpha_k}} \phi.$$

We set $|\phi| = \sup_{x \in (0,1)^k} |\phi(x)|$, which is possibly $+\infty$.

We begin with some elementary estimates.

Lemma 13. (i) *Suppose $x, y \in \mathbb{R}^n$ and $i \in \mathbb{N}_0^n$, then $|x^i - y^i| \le \max\{1, |x - y|\}^{\ell(i)-1}(1 + |x|)^{\ell(i)}|x - y|$.*

(ii) *Let $k \in \mathbb{N}, b \in \mathbb{N}_0$, and suppose $\phi : (0,1)^k \to \mathbb{R}^n$ has coordinate functions that have continuous parital derivatives up-to order b with modulus bounded by a real number $B \ge 1$ on $(0,1)^k$. If $i \in \mathbb{N}_0^k$, then $|\partial^\alpha(\phi^i)| \le B^{\ell(i)}\ell(i)^{\ell(\alpha)}$ for all $\alpha \in \mathbb{N}_0^k$ such that $\ell(\alpha) \le b$.*

Proof. For the proof of (i) we write $h = x - y = (h_1, \ldots, h_n)$ and may assume $i \ne 0$. The Binomial Theorem implies

$$x^i - y^i = x^i - (x - h)^i = -\sum_{\substack{0 \le j_1 \le i_1, \ldots, 0 \le j_n \le i_n \\ j = (j_1, \ldots, j_n) \ne 0}} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n} x^{i-j}(-h)^j$$

where $i = (i_1, \ldots, i_n)$. We observe that $|(-h)^j| = |h_1^{j_1} \cdots h_n^{j_n}| \leq |h|^{\ell(j)} \leq |h| \max\{1, |h|\}^{\ell(j)-1}$ if $j \neq 0$. Say $x = (x_1, \ldots, x_n)$, then the triangle inequality yields

$$\begin{aligned} |x^i - y^i| &\leq |h| \max\{1, |h|\}^{\ell(i)-1} \sum_{\substack{0 \leq j_1 \leq i_1, \ldots, 0 \leq j_n \leq i_n \\ j=(j_1,\ldots,j_n) \neq 0}} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n} |x^{i-j}| \\ &\leq |h| \max\{1, |h|\}^{\ell(i)-1} \prod_{k=1}^{n} \left(\sum_{j_k=0}^{i_k} \binom{i_k}{j_k} |x_k|^{i_k - j_k} \right) \\ &= |h| \max\{1, |h|\}^{\ell(i)-1} (1 + |x_1|)^{i_1} \cdots (1 + |x_n|)^{i_n} \\ &\leq |h| \max\{1, |h|\}^{\ell(i)-1} (1 + |x|)^{\ell(i)} \end{aligned}$$

and thus part (i).

For the proof of (ii) let $\phi_1, \ldots, \phi_d : (0,1)^k \to \mathbb{R}$ be continuous functions for which all partial derivatives up-to order b exist and are bounded in modulus by 1. If $\alpha \in \mathbb{N}_0^k$ with $1 \leq \ell(\alpha) \leq b$, then using the Leibniz rule we find

$$\partial^\alpha (\phi_1 \cdots \phi_d) = \sum_{i=1}^{d} \partial^{\alpha'} \left(\phi_1 \cdots \phi_{i-1} \frac{\partial \phi_i}{\partial X_j} \phi_{i+1} \cdots \phi_d \right)$$

for some $\alpha' \in \mathbb{N}_0^k$ with $\ell(\alpha') = \ell(\alpha) - 1$ if the j-th coefficient of α is non-zero. By induction on $\ell(\alpha)$ we conclude $|\partial^\alpha(\phi_1 \cdots \phi_d)| \leq d^{\ell(\alpha)}$. The lemma follows after scaling ϕ and observing $\phi^i = \phi_1 \cdots \phi_d$ where $d = \ell(i)$ and the $\phi_1, \ldots, \phi_d$ are certain coordinate functions of ϕ. □

The next lemma is a variant of Liouville's Inequality. For $d \in \mathbb{N}$ we write

$$D_n(d) = \binom{n+d}{n}$$

for the number of monomials in n variables and with degree at most d.

Lemma 14. *Let $x \in \mathbb{R}^n$ have algebraic coefficients. If $f \in \mathbb{Z}[X_1, \ldots, X_n] \smallsetminus \{0\}$ has degree d and if $f(x) \neq 0$, then $|f(x)| \geq \left(D_n(d)|f|H(x)^{dn}\right)^{-[\mathbb{Q}(x):\mathbb{Q}]}$.*

Proof. Suppose $f(x) \neq 0$ and set $K = \mathbb{Q}(x)$. Any maximal ideal v of the ring of integers of K defines a non-Archimedean absolute value $|\cdot|_v$ on K with $|p|_v = p^{-1}$ for the prime number p contained in v. We write d_v for the degree of the completion of K with respect to v over the field of p-adic numbers. The product formula, cf. Chapter 1.4 [4], implies

$$\prod_{\sigma : K \to \mathbb{C}} |\sigma(f(x))| \prod_{v} |f(x)|_v^{d_v} = 1$$

where σ runs over all field embeddings and v over all maximal ideals as before.

Let $x = (x_1, \ldots, x_n)$. For a maximal ideal v, the ultrametric triangle inequality and the fact that f has integral coefficients gives

$$|f(x)|_v \leq \max\{1, |x_1|_v, \ldots, |x_n|_v\}^d \leq \max\{1, |x_1|_v\}^d \cdots \max\{1, |x_n|_v\}^d. \tag{15}$$

If $\sigma : K \to \mathbb{C}$ is a field embedding, then

$$|\sigma(f(x))| \le D_n(d)|f| \max\{1, |\sigma(x_1)|, \dots, |\sigma(x_n)|\}^d \le D_n(d)|f| \max\{1, |\sigma(x_1)|\}^d \cdots \max\{1, |\sigma(x_n)|\}^d. \tag{16}$$

We take the product of (15) raised to the d_v-th power over all v and multiply it with the product over all (16) with σ not the identity. On applying the product formula we get

$$1 \le \left(D_n(d)|f|H(x_1)^d \cdots H(x_n)^d\right)^{[K:\mathbb{Q}]} |f(x)|,$$

as desired. □

The key tool for constructing the auxiliary function is the following "approximate Thue-Siegel Lemma" which follows from Minkowski's Lattice Point Theorem. We use $|\cdot|_2$ to denote the Euclidean norm on a power of $\mathbb{R}$.

Lemma 15. *Let $M, N \in \mathbb{N}$ with $M \le N$ and suppose $A \in \mathrm{Mat}_{M,N}(\mathbb{R})$ has rows $a_1, \dots, a_M$ with $|a_i|_2 \ge 1$ for all $1 \le i \le M$. We set $\Delta = |a_1|_2 \cdots |a_M|_2$. If $Q \ge 2\sqrt{N}\Delta^{1/N}$ there exists $f \in \mathbb{Z}^N \smallsetminus \{0\}$ with*

$$|f| \le Q \quad \text{and} \quad |Af| \le (2\sqrt{N})^{N/M} Q^{1-N/M} \Delta^{1/M}.$$

Proof. We set $\epsilon = (2\sqrt{N})^{N/M} Q^{-N/M} \Delta^{1/M}$ which lies in $(0, 1]$ by our hypothesis. The columns of the $(M + N) \times N$ matrix $\overline{A}$ obtained by augmenting A by the $N \times N$ unit matrix scaled by ϵ are a basis of a lattice $\Lambda \subseteq \mathbb{R}^{M+N}$ of rank N. Observe that an orthogonal transformation of Λ lies in $\mathbb{R}^N \times \{0\}$, which we here identify with $\mathbb{R}^N$. By Minkowki's Lattice Point Theorem applied to this transformation there exists $f \in \mathbb{Z}^N \smallsetminus \{0\}$ such that

$$|\overline{A}f|_2 \le 2 \det(\overline{A}^{\mathrm{t}}\overline{A})^{1/(2N)} \nu_N^{-1/N}$$

where $\nu_N > 0$ is the volume of the unit N-ball in $\mathbb{R}^N$ and the determinant is the volume of Λ squared. By the Cauchy-Binet Formula this determinant is the sum of the squares of the determinants of all $N \times N$ submatrices of $\overline{A}$. Hadamard's inequality implies that the absolute value of each determinant is at most $|a_1|_2 \cdots |a_M|_2 \epsilon^{N-M} = \Delta\epsilon^{N-M}$ since $\epsilon \le 1 \le |a_i|_2$ for all $1 \le i \le M$. Thus $\det(\overline{A}^{\mathrm{t}}\overline{A}) \le \binom{M+N}{N}\Delta^2\epsilon^{2(N-M)} \le 4^N\Delta^2\epsilon^{2(N-M)}$ since $M \le N$. Now $\nu_N = \pi^{N/2}/\Gamma(1 + N/2)$ where $\Gamma(\cdot)$ is the gamma function. The inequality $\log\Gamma(x) < (x - 1/2)\log(x) - x + \log(2\pi)/2 + 1$ is well-known for all $x > 1$, cf. Lemma 1 [13]. An elementary calculation yields $\nu_N^{1/N} \ge 2/\sqrt{N}$. We combine this with the estimates above and obtain $|\overline{A}f|_2 \le 2\sqrt{N}\Delta^{1/N}\epsilon^{1-M/N}$. Now $|f| \le |f|_2 \le \epsilon^{-1}|\overline{A}f|_2$ and $|Af| \le |Af|_2 \le |\overline{A}f|_2$, hence

$$|f| \le 2\sqrt{N}\Delta^{1/N}\epsilon^{-M/N} = Q$$

by our choice of ϵ and $|Af| \le 2\sqrt{N}\Delta^{1/N}\epsilon^{1-M/N} = \epsilon Q = (2\sqrt{N})^{N/M}Q^{1-N/M}\Delta^{1/M}$, as desired. □

Proposition 16. *Let $b, d, k, n, e \in \mathbb{N}$, and suppose $D_n(d) \ge (e + 1)D_k(b)$. Let $B \ge 1$. There exists a constant $c = c(b, d, k, n, e, B) \ge 1$ with the following property. Suppose $\phi : (0, 1)^k \to \mathbb{R}^n$ is a map whose coordinate functions have continuous parital derivatives up-to order $b+1$ with modulus bounded by B on $(0, 1)^k$. For any real number $T \ge 1$ there* *exist* $N \in \mathbb{N}$ *with* $N \leq cT^{(k+1)ne\frac{d}{b}}$ *and polynomials* $f_1, \ldots, f_N \in \mathbb{Q}[X_1, \ldots, X_n] \smallsetminus \{0\}$ *with* $\deg f_j \leq d$ *and* $|f_j| = 1$ *for all* $j \in \{1, \ldots, N\}$ *such that the following holds true. If*

$$z \in (0,1)^k \text{ and } q \in \mathbb{Q}^n(T, e) \text{ such that } |\phi(z) - q| \leq c^{-1} T^{-\frac{(k+1)ne}{k}\frac{d(b+1)}{b}},$$

then $f_j(q) = 0$ *and* $|f_j(\phi(z))| \leq c|\phi(z) - q|$ *for some* $j \in \{1, \ldots, N\}$.

Proof. During the proof of this proposition we will increase c several times. This constant shall not depend on T. Below, $c_1, \ldots, c_6$ are positive constants that depend on b, d, k, n, e and B. We will choose c in function of these constants.

For any $i \in \mathbb{N}_0^n$ with $\ell(i) \leq d$ we set $\phi_i(x) = \phi(x)^i$ for all $x \in (0,1)^k$. We thus get a collection of $D = D_n(d)$ functions $\phi_i : (0,1)^k \to \mathbb{R}$ for which all derivatives exist and are continuous up-to order $b + 1$.

Say $T \geq 1$. We take

$$r = c' T^{-\frac{(k+1)ne}{k}\frac{d}{b}} \leq c' \leq 1 \tag{17}$$

where $c' \in (0, 1]$ is small enough in terms of b, d, k, n, e, B, and the c_i and is to be determined. Our choice of c is large enough in terms of c'. The hypercube $(0,1)^k$ is contained in the union of

$$N \leq (1 + r^{-1})^k \leq 2^k r^{-k} = 2^k c'^{-k} T^{(k+1)ne\frac{d}{b}} \tag{18}$$

closed hypercubes of side length r.

Let $V \subseteq \mathbb{R}^k$ be one of these closed hypercubes with $V \cap (0,1)^k \neq \emptyset$. It will eventually lead to a single polynomial $f = f_V$ as in the hypothesis. As we let V vary over the hypercubes covering $(0,1)^k$, we will get N polynomials. After renumbering them, they will be the f_j claimed to exist in the assertion of this lemma. The estimate for N in the assertion will follow from (18) as we may assume $c \geq 2^k c'^{-k}$.

Our approach is to find $D_n(d)$ coefficients $f_i \in \mathbb{Z}$ for a polynomial $f = \sum_{\substack{\ell(i) \leq d \\ i=(i_1,\ldots,i_n)}} f_i X_1^{i_1} \cdots X_n^{i_n}$ using Lemma 15. We develop the Taylor series of $p(z) = f(\phi_1(z), \ldots, \phi_n(z))$ around a fixed auxiliary point $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_k) \in V \cap (0,1)^k$ with Lagrange remainder term. Indeed, for $z \in (0,1)^k$ we have

$$p(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^k \\ \ell(\alpha) \leq b}} \left(\sum_{\substack{i \in \mathbb{N}_0^n \\ \ell(i) \leq d}} f_i \frac{\partial^\alpha \phi^i(\overline{z})}{\alpha!} \right) (z - \overline{z})^\alpha + \sum_{\substack{\alpha \in \mathbb{N}_0^k \\ \ell(\alpha) = b+1}} \left(\sum_{\substack{i \in \mathbb{N}_0^n \\ \ell(i) \leq d}} f_i \frac{\partial^\alpha \phi^i(\xi)}{\alpha!} \right) (z - \overline{z})^\alpha \tag{19}$$

where $(\alpha_1, \ldots, \alpha_k)! = \alpha_1! \cdots \alpha_k!$ and where $\xi \in (0,1)^k$ lies on the line segment connecting $\overline{z}$ and z. We now suppose $z \in V \cap (0,1)^k$, observe that $|z - \overline{z}| \leq r$. We must find f_i such that

$$\frac{r^{-(b-\ell(\alpha))}}{\left| \left(\frac{\partial^\alpha \phi^i(\overline{z})}{\alpha!} \right)_{i \in \mathbb{N}_0^n, \ell(i) \leq d} \right|_2} \sum_{\substack{i \in \mathbb{N}_0^n \\ \ell(i) \leq d}} f_i \frac{\partial^\alpha \phi^i(\overline{z})}{\alpha!} \tag{20}$$

is small in absolute value for all $\alpha \in \mathbb{N}_0^k$ with $\ell(\alpha) \leq b$; the norm in the denominator is, as usual, the Euclidean norm. The Euclidean norm of the coefficient vector in (20) with respect to the f_i is $r^{-(b-\ell(\alpha))} \geq 1$. We thus obtain a matrix with real coefficients,

$D_k(b)$ rows, and $D_n(d)$ columns. In order to apply Lemma 15 we need to estimate the product Δ of the Euclidean norms of the rows of this matrix. This product equals

$$\Delta = r^{-\sum_{\ell(\alpha)\le b}(b-\ell(\alpha))} = r^{-\sum_{j=0}^{b}\binom{k+j-1}{j}(b-j)}$$

as there are $\binom{k+j-1}{j}$ derivatives of precise order j. We have

$$\sum_{j=0}^{b}\binom{k+j-1}{j}(b-j) = \frac{b}{k+1}\binom{b+k}{b} = \frac{b}{k+1}D_k(b)$$

by basic properties of the bionomial coefficients and hence

$$\Delta = r^{-\frac{b}{k+1}D_k(b)}. \tag{21}$$

We define

$$Q = r^{-\frac{b+k+1}{(e+1)(k+1)}} \ge 1. \tag{22}$$

Let us verify that Q satisfies the hypothesis of Lemma 15 applied to the $D_k(b)\times D_n(d)$ matrix constructed above. Indeed, (21) implies the first equality in

$$\Delta^{\frac{1}{D_n(d)}} = r^{-\frac{b}{k+1}\frac{D_k(b)}{D_n(d)}} \le r^{-\frac{b}{(e+1)(k+1)}} = r^{\frac{1}{e+1}-\frac{b+k+1}{(e+1)(k+1)}} = r^{\frac{1}{e+1}}Q$$

the inequality is due to $D_k(b) \le D_n(d)/(e+1)$ and $r \le 1$. As $r \le c'$ we find $2\sqrt{D_n(d)}\Delta^{1/D_n(d)} \le Q$ if $c' \le (2\sqrt{D_n(d)})^{-(e+1)}$, which we may assume.

So there is $f \in \mathbb{Z}^{D_n(d)} \smallsetminus \{0\}$ with $|f| \le Q$ such that (20) is bounded from above in absolute value by $c_1 Q^{1-D_n(d)/D_k(b)}\Delta^{1/D_k(b)} \le c_1 Q^{-e}\Delta^{1/D_k(b)}$, we used $D_k(b) \le D_n(d)/(e+1)$ again.

The terms up-to order b in the Taylor expansion (19) can be bounded as follows. For any $z \in (0,1)^k \cap V$ we have

$$\Bigg|\underbrace{\sum_{\substack{\alpha\in\mathbb{N}_0^k\\ \ell(\alpha)\le b}}\Bigg(\sum_{\substack{i\in\mathbb{N}_0^n\\ \ell(i)\le d}} f_i\frac{\partial^\alpha\phi^i(\overline{z})}{\alpha!}\Bigg)(z-\overline{z})^\alpha}_{p_{\mathrm{main}}}\Bigg| \le \sum_{\substack{\alpha\in\mathbb{N}_0^k\\ \ell(\alpha)\le b}}\Bigg|\sum_{\substack{i\in\mathbb{N}_0^n\\ \ell(i)\le d}} f_i\frac{\partial^\alpha\phi^i(\overline{z})}{\alpha!}\Bigg| r^{\ell(\alpha)}$$

$$\le c_1 Q^{-e}\Delta^{1/D_k(b)}r^b\sum_{\substack{\alpha\in\mathbb{N}_0^k\\ \ell(\alpha)\le b}}\Bigg(\sum_{\substack{i\in\mathbb{N}_0^n\\ \ell(i)\le d}}\left(\frac{\partial^\alpha\phi^i(\overline{z})}{\alpha!}\right)^2\Bigg)^{1/2}, \tag{23}$$

keeping (20) in mind. Each Euclidean norm on the right is at most $D_n(d)^{1/2}B^d d^b$ by Lemma 13(ii). Therefore, $|p_{\mathrm{main}}| \le c_2 Q^{-e}\Delta^{1/D_k(b)}r^b$ where $c_2 = c_1 D_k(b)D_n(d)^{1/2}B^d d^b$. We insert (21) and obtain $|p_{\mathrm{main}}| \le c_2 Q^{-e} r^{b\left(-\frac{1}{k+1}+1\right)} = c_2 Q^{-e} r^{\frac{bk}{k+1}}$. Next we substitute the expression for Q from (22) to get

$$|p_{\mathrm{main}}| \le c_2 r^\sigma \quad \text{with} \quad \sigma = e\frac{b+k+1}{(e+1)(k+1)} + \frac{bk}{k+1}. \tag{24}$$

The remainder in the Taylor expansion (19) can be bounded as follows

$$\begin{aligned}\Bigg|\underbrace{\sum_{\substack{\alpha\in\mathbb{N}_0^k\\ \ell(\alpha)=b+1}}\left(\sum_{\substack{i\in\mathbb{N}_0^n\\ \ell(i)\le d}} f_i\frac{\partial^\alpha\phi^i(\xi)}{\alpha!}\right)(z-\overline{z})^\alpha}_{p_{\mathrm{rem}}}\Bigg| &\le \sum_{\substack{\alpha\in\mathbb{N}_0^k\\ \ell(\alpha)=b+1}}\sum_{\substack{i\in\mathbb{N}_0^n\\ \ell(i)\le d}}\left|f_i\frac{\partial^\alpha\phi^i(\xi)}{\alpha!}\right| r^{b+1}\\ &\le \binom{k+b}{k-1}D_n(d)B^d d^{b+1}|f|r^{b+1}\end{aligned}$$

where we used Lemma 13(ii) again to bound the partial derivatives of ϕ^i at $\xi\in(0,1)^k$. We obtain $|p_{\mathrm{rem}}|\le c_3|f|r^{b+1}$ where $c_3=\binom{k+b}{k-1}D_n(d)B^d d^{b+1}$. Observe that (22) and the choice of σ in (24) imply

$$Qr^{b+1}=r^\sigma. \tag{25}$$

We recall $|f|\le Q$ and find $|p_{\mathrm{rem}}|\le c_3Qr^{b+1}=c_3r^\sigma$ with the same exponent as in (24). Combining both bounds yields

$$|f(\phi(z))|=|p(z)|\le|p_{\mathrm{main}}|+|p_{\mathrm{rem}}|\le c_4r^\sigma \tag{26}$$

with $c_4=c_2+c_3$.

Now suppose that $q\in(\overline{\mathbb{Q}}\cap\mathbb{R})^n$ with $H(q)\le T$ and $[\mathbb{Q}(q):\mathbb{Q}]\le e$ satisfies

$$|\phi(z)-q|\le c^{-1}T^{-\frac{(k+1)ne}{k}\frac{d(b+1)}{b}}=c^{-1}(r/c')^{b+1}$$

where z still lies in $(0,1)^k\cap V$ and where we used (17). We may suppose that $c^{-1}\le c'^{b+1}$, hence $|\phi(z)-q|\le r^{b+1}\le 1$. We note that $f(\phi(z))-f(q)$ is the sum of $D_n(d)$ terms of the form $f_i(\phi(z)^i-q^i)$ where $\ell(i)\le d$. By Lemma 13(i) we find $|f(\phi(z))-f(q)|\le D_n(d)|f|(1+B)^d|\phi(z)-q|\le c_5Qr^{b+1}$ with $c_5=D_n(d)(1+B)^d$ as $|f|\le Q$. Using equality (25) we obtain

$$|f(q)-f(\phi(z))|\le c_5r^\sigma. \tag{27}$$

Together with (26) we get

$$|f(q)|\le|f(q)-f(\phi(z))|+|f(\phi(z))|\le c_6r^\sigma \tag{28}$$

where $c_6=c_4+c_5$.

Suppose $f(q)\ne 0$. Then we obtain $|f(q)|\ge\left(D_n(d)QT^{dn}\right)^{-e}$ from Lemma 14. We compare this inequality with (28) and rearrange to get $D_n(d)^{-e}c_6^{-1}\le r^\sigma Q^eT^{dne}$. Using (17) and (22) we find, after a brief calculation, that $r^\sigma Q^eT^{dne}=c'^{\frac{bk}{k+1}}$ is independent of T. As the exponent $bk/(k+1)$ of c' is positive, we arrive at a contradiction for c' sufficiently small.

So $f(q)=0$. We may replace f by $f/|f|$ to normalize the polynomial. This yields the first claim of the proposition as the number of f is bounded by (18).

For the second and final claim we will bound $|f(\phi(z))|$ from above. Now that we have $f(q)=0$ and $|f|=1$ we find as above (27) that

$$|f(\phi(z))|=|f(\phi(z))-f(q)|\le D_n(d)(1+B)^d|\phi(z)-q|.$$

We may assume $c\ge D_n(d)(1+B)^d$ and from this we conclude the proof. □

5. Quasi-Algebraic Cells

In this section, cells are assumed to be definable in a fixed o-minimal structure which we do not require to be polynomially bounded. Our ambient o-minimal structure contains all semi-algebraic sets which themselves form an o-minimal structure. We also often work with semi-algebraic cells.

Let $n \in \mathbb{N}$. We recall the notion of a non-singular point in real algebraic geometry, our reference is the book of Bochnak, Coste, and Roy [3]. A real algebraic set $A \subseteq \mathbb{R}^n$ is the set of common zeros of a finite number of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$. Let $0 \le r \le n$ be an integer. A point $x \in A$ is called non-singular in dimension r, if there exist polynomials $f_1, \ldots, f_{n-r} \in \mathbb{R}[X_1, \ldots, X_n]$ that vanish on A and satisfy the rank condition

$$\mathrm{Rk}\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{\substack{1\le i\le n-r\\ 1\le j\le n}} = n-r$$

and an open neighborhood U of x in $\mathbb{R}^n$ such that $A \cap U = \mathcal{Z}(f_1, \ldots, f_{n-r}) \cap U$, see Proposition 3.3.10 *loc.cit.* We let $\mathrm{Sing}(A)$ denote the complement in A of all $x \in A$ that are non-singular in dimension $\dim A$. The complement $A \smallsetminus \mathrm{Sing}(A)$ is called the non-singular locus of A. It is open in A with respect to the Euclidean and Zariski topologies. By Proposition 3.3.14 *loc.cit.*, $\mathrm{Sing}(A)$ is a real algebraic set with $\dim \mathrm{Sing}(A) < \dim A$. The dimension of a real semi-algebraic set as in [3] coincides with its dimension as a definable set in an o-minimal structure.

We call a cell of dimension r quasi-algebraic if it is an open subset of the non-singular locus of a r-dimensional real algebraic set.

For example, a 0-dimensional cell is a quasi-algebraic cell. An n-dimensional cell in $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, so it is quasi-algebraic.

Quasi-algebraic cells bare similarities to Pila's definable blocks. Indeed, as all cells are connected, an r-dimensional quasi-algebraic cell is a definable block of dimension of dimension r and degree d for some d in the sense of Definition 3.4 [16]. Working with cells provides advantages in the induction step presented in Section 6 below.

Lemma 17. *Let $C \subseteq \mathbb{R}^n$ be a definable set that is homoeomorphic to an open subset of $\mathbb{R}^r$, e.g. an r-dimensional cell, and contained in a non-empty real algebraic set $A \subseteq \mathcal{Z}(f_1, \ldots, f_M)$ where $f_1, \ldots, f_M \in \mathbb{R}[X_1, \ldots, X_n]$. Suppose $\dim A = r$ and*

$$\mathrm{Rk}\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{\substack{1\le i\le M\\ 1\le j\le n}} \ge n-r \tag{29}$$

for all $x \in C$. Then $C \subseteq A \smallsetminus \mathrm{Sing}(A)$ and C is open in $A \smallsetminus \mathrm{Sing}(A)$. If in addition C is an r-dimensional cell then it is a quasi-algebraic cell.

Proof. We may assume $r \ge 1$ and $C \ne \emptyset$. Say $x \in C$. The jacobian matrix $(\partial f_i/\partial X_j(x))_{i,j}$ contains an invertible $(n-r) \times (n-r)$ submatrix. After permuting coordinates and the f_i we may suppose

$$\det\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{1\le i,j\le n-r} \ne 0.$$

Let us define $B = \mathcal{Z}(f_1, \ldots, f_{n-r})$, it contains A and C. By the implicit function theorem, cf. Corollary 2.9.8 [3], there is an open neighborhood U of x in $\mathbb{R}^n$ such that

$B \cap U$ is homeomorphic to an open subset of $\mathbb{R}^r$. By hypothesis, C is also homeomorphic to an open subset of $\mathbb{R}^r$. Observe that $x \in C \cap U \subseteq B \cap U$. By invariance of domain, $C \cap U$ is open in $B \cap U$, i.e. $C \cap U = B \cap U \cap V$ for an open subset $V \subseteq \mathbb{R}^n$. Recall that $C \subseteq A \subseteq B$, so $C \cap U \cap V = A \cap U \cap V = B \cap U \cap V$. Therefore, $x \in A$ is non-singular in dimension $r = \dim A$ and thus $x \in A \smallsetminus \mathrm{Sing}(A)$. Moreover, x lies in $B \cap U \cap V$ which is open in A and contained in C. We find that C is open in A by taking the union of all $B \cap U \cap V$ as x runs through the points of C. □

If $0 \le r \le n-1$ we write $\mathcal{J}_{n,r}$ for the set of subsets $J \subseteq \{1, \ldots, n\}$ with $\#J = r+1$.

Lemma 18. *Suppose that for each $J \in \mathcal{J}_{n,r}$ we are given an irreducible $f_J \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg_{X_j}(f_J) = 0$ for all $j \in \{1, \ldots, n\} \smallsetminus J$. Then the set of all $x \in \mathcal{Z}(f_J : J \in \mathcal{J}_{n,r})$ with*

$$\mathrm{Rk}\left(\frac{\partial f_J}{\partial X_j}(x)\right)_{\substack{J \in \mathcal{J}_{n,r}\\ 1 \le j \le n}} < n - r$$

is real algebraic of dimesion at most $r-1$.

Proof. Let x be as in the hypothesis and let $J \subseteq \{1, \ldots, n\}$ have cardinality r. For any $i \in \{1, \ldots, n\} \smallsetminus J$ we write $g_i = f_{J \cup \{i\}}$. The $(n-r) \times (n-r)$ diagonal matrix $\left((\partial g_i/\partial X_j)(x)\right)_{i,j}$, where $i, j \in \{1, \ldots, n\} \smallsetminus J$, is singular by hypothesis. So $(\partial g_i/\partial X_i)(x) = 0$ for some i. The polynomial g_i is irreducible by hypothesis. If $\deg_{X_i} g_i \ge 1$, then the resultant of g_i and $\partial g_i/\partial X_i$, taken as polynomials in X_i, is a non-zero polynomial $h \in \mathbb{R}[X_j : j \in J]$. If $\deg_{X_i} g_i = 0$ we set $h = g_i \neq 0$ which only depends on the coordinates in J. Observe that $h(x) = 0$ in both cases.

We have proved that if x is as in the hypothesis, then its projection to any choice of r coordinates of $\mathbb{R}^n$ indexed by J lies in the vanishing locus of a non-zero polynomial in r variables. Therefore, the set of x in question has dimension at most $r-1$. It is clearly a real algebraic set. □

Lemma 19. *Let $D \subseteq \mathbb{R}^n$ be a connected, definable, open subset of a real semi-algebraic set. If $\dim D \ge 1$ then $D^{\mathrm{alg}} = D$.*

Proof. Say $x \in D$, by hypothesis there is an open subset $U \subseteq \mathbb{R}^n$ containing x such that $D \cap U$ is semi-algebraic. All connected components of $D \cap U$ are semi-algebraic and open in $D \cap U$. So we may suppose that $D \cap U$ contains x, is connected, semi-algebraic, and open in D. Now $D \cap U$ cannot be a singleton since D is connected and of positive dimension. So it has positive dimension and $D \cap U \subseteq D^{\mathrm{alg}}$. We conclude $D = D^{\mathrm{alg}}$. □

Let $C \subseteq \mathbb{R}^n$ be an $(i_1, \ldots, i_n)$-cell of dimension $r \ge 0$, cf. Section 3.2 [20] for this terminology. Suppose $1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_r \le n$ are precisely those indices with $i_\lambda = 1$. Let $p : \mathbb{R}^n \to \mathbb{R}^r$ denote the projection onto the coordinates $\lambda_1, \ldots, \lambda_r$. Then $p|_C$ is injective.

Lemma 20. *In the notation above suppose $D \subseteq \mathbb{R}^r$ is a cell with $D \subseteq p(C)$. Then $p|_C^{-1}(D)$ is a cell.*

Proof. The proof is by induction on n. The case $n = 1$ is immediate, so let us assume $n \ge 2$. We may also suppose $r \ge 1$.

We write $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ for the projection onto the first $n-1$ coordinates and $C' = \pi(C)$. We often make use of the fact that C' is an $(i_1, \ldots, i_{n-1})$-cell and use other properties listed in Section 2.

If $i_n = 0$, then C is the graph of a continuous and definable map $f : C' \to \mathbb{R}$. We write $q : \mathbb{R}^{n-1} \to \mathbb{R}^r$ for the projection onto the coordinates $1 \leq \lambda_1 < \cdots < \lambda_r < n$. Then $q \circ \pi = p$, so $q(C') = q(\pi(C)) = p(C)$ and

$$p|_C^{-1}(D) = \{(x', f(x')) : x' \in q|_{C'}^{-1}(D)\}.$$

By induction $q|_{C'}^{-1}(D)$ is a cell. This makes $p|_C^{-1}(D)$ the graph of a continuous and definable function over this cell, hence itself a cell.

Now say $i_n = 1$. Then there are continuous and definable $f, g : C' \to \mathbb{R}$, or $f = -\infty$, or $g = +\infty$, with $f(x') < g(x')$ for all $x' \in C'$ such that

$$C = \{(x', t) \in C' \times \mathbb{R} : f(x') < t < g(x')\}.$$

Say D is a $(j_1, \ldots, j_r)$-cell.

If $r = 1$, then $p : \mathbb{R}^n \to \mathbb{R}$ projects to the final coordinate and $C = \{\text{point}\} \times \text{interval}$, which is easy to handle. Say $r \geq 2$.

Let $p' : \mathbb{R}^{n-1} \to \mathbb{R}^{r-1}$ be the projection on the coordinates $1 \leq \lambda_1 < \cdots < \lambda_{r-1} < n$ and $\pi' : \mathbb{R}^r \to \mathbb{R}^{r-1}$ onto the first $r-1$ coordinates. Then $\pi' \circ p = p' \circ \pi$ and $\pi'(D)$ is a cell in $\mathbb{R}^{r-1}$ contain in $\pi'(p(C)) = p'(\pi(C)) = p'(C')$. By induction we see that

$$C'' = p'|_{C'}{}^{-1}(\pi'(D)) \subseteq \mathbb{R}^{n-1}$$

is a cell.

Say $x' \in C''$. If $t \in \mathbb{R}$ with $(p'(x'), t) \in D$, then there is $\widetilde{x} \in C$ such that $(p'(x'), t) = p(\widetilde{x})$. Observe that $x' \in C'$ and $\pi(\widetilde{x}) \in C'$ and that p' is injective on C'. Therefore, $x' = \pi(\widetilde{x})$ and so $\widetilde{x} = (x', t) \in C$.

In the first subcase we suppose $j_r = 0$. Here D is the graph of a suitable $f_D : \pi'(D) \to \mathbb{R}$. We have

$$p|_C^{-1}(D) = \{(x', t) : x' \in C'', f(x') < t < g(x'), \text{ and } t = f_D(p'(x'))\}.$$

As we saw in the last paragraph, $x' \in C''$ implies $f(x') < f_D(p'(x')) < g(x')$, so

$$p|_C^{-1}(D) = \{(x', t) : x' \in C'' \text{ and } t = f_D(p'(x'))\}$$

is a graph and thus a cell.

The second subcase is $j_r = 1$. Let $f_D, g_D : D' \to \mathbb{R}$ with $f_D < g_D$ on D', or $f_D = -\infty$, or $g_D = +\infty$ describe the boundaries for D. As in the last subcase we find

$$p|_C^{-1}(D) = \{(x', t) : x' \in C'' \text{ and } f_D(p'(x')) < t < g_D(p'(x'))\}.$$

And so $p|_C^{-1}(D)$ is again a cell. □

We require the following result of Wilkie.

Theorem 9 (Wilkie). *A definable, bounded, open subset of $\mathbb{R}^n$ is a finite union of open cells.*

Proof. This is Theorem 1.3 [22]. The open cells may have non-empty intersection. □

This theorem extends to cells in the following way.

Lemma 21. *Suppose $C \subseteq \mathbb{R}^n$ is a cell and let $U \subseteq C$ be a bounded and definable set that is open in C. There exist cells $C_1, \ldots, C_s \subseteq \mathbb{R}^n$, each of dimension $\dim C$, with $U = C_1 \cup \cdots \cup C_s$.*

Proof. Let $r = \dim C$. There is nothing to show if $r = 0$, else say $p : \mathbb{R}^n \to \mathbb{R}^r$ is as before Lemma 20. Then $p|_C : C \to p(C)$ is a homeomorphism and $p(C)$ is open in $\mathbb{R}^r$, cf. 2.7 in Chapter 3 [20]. Therefore, $p(U)$ is open in $\mathbb{R}^r$ and certainly bounded. By Wilkie's Theorem above it is covered by cells that are open in $\mathbb{R}^r$. A cell in such a covering has dimension r and by Lemma 20 its preimage under $p|_C$ is again a cell of dimension r. □

6. Induction Scheme

Here is the main technical result of this paper on diophantine approximation on definable sets. Our theorems mentioned in the introduction are derived from the following statement.

Theorem 10. *Suppose the ambient o-minimal structure is polynomially bounded. Let $m \in \mathbb{N}_0, n, e \in \mathbb{N}, \epsilon > 0$ and suppose $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a closed and definable set whose projection to $\mathbb{R}^m$ is bounded. There exist $c = c(Z, e, \epsilon) \geq 1$, $\theta = \theta(Z, e, \epsilon) \in (0, 1]$, integers $l_1, \ldots, l_t \in \mathbb{N}_0$, and definable sets $D_j \subseteq \mathbb{R}^{l_j} \times \mathbb{R}^m \times \mathbb{R}^n$ for all $j \in \{1, \ldots, t\}$ with the following properties:*

(i) *Say $D = D_j$ for some $j \in \{1, \ldots, t\}, z \in \mathbb{R}^{l_j}$, and $y \in \mathbb{R}^m$. Then $D_{(z,y)} \subseteq Z_y$ and if $D_{(z,y)} \neq \emptyset$, then $D_{(z,y)}$ is a connected and open subset of the non-singular locus of a real algebraic set of dimension $\dim D_{(z,y)}$.*

(ii) *Let $\psi : [1, +\infty) \to [0, 1]$ have order at most $-\theta^{-1}$. If $y \in \mathbb{R}^m$ and $T \geq 1$ there exist an integer $N \geq 1$ with $N \leq cT^\epsilon$ and $(j_p, z_p, y_p) \in \{1, \ldots, j\} \times \mathbb{R}^{l_{j_p}} \times \mathbb{R}^m$ for $p \in \{1, \ldots, N\}$ such that if*

$$x \in Z_y \text{ and } q \in \mathbb{Q}^n(T, e) \text{ with } |x - q| \leq c^{-1}\psi(T)$$

then there exists $p \in \{1, \ldots, N\}$ with $|y - y_p| \leq \psi(T)^\theta$ and $x' \in (D_{j_p})_{(z_p, y_p)}$ with $|x - x'| \leq \psi(T)^\theta$.

In this section we work in a fixed o-minimal structure which is arbitrary at first. The goal is to start the induction step and eventually prove Theorem 10.

For the next lemma we do not need to assume that the ambient o-minimal structure is polynomially bounded as in the theorem above. Let $n \in \mathbb{N}$. For $d \in \mathbb{N}_0$ we define $\mathbb{R}[X_1, \ldots, X_n]_d$ to be the vector space of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ of degree at most d including 0. We will identify this vector space with $\mathbb{R}^{D_n(d)}$. If $f \in \mathbb{R}[X_1, \ldots, X_n]$, then $|f|$ denotes the maximum norm of the coefficient vector of f.

Let $r \in \mathbb{N}_0$ with $r \leq n - 1$. Recall that $\mathcal{J}_{n,r}$ is the set of subsets of $\{1, \ldots, n\}$ with $r + 1$ elements. We define

$$F_{r,d} = \left\{ \sum_{J \in \mathcal{J}_{n,r}} f_J^2 : f_J \in \mathbb{R}[X_j : j \in J]_d \text{ and } |f_J| = 1 \text{ for all } J \in \mathcal{J}_{n,r} \right\}$$

Observe that each f_J depends only on the variables indexed by J. We may identify $F_{r,d}$ with a subset of $\mathbb{R}^{\binom{n+2d}{n}}$. It is the image of

$$\left\{(f_J)_{J\in\mathcal{J}_{n,r}} \in \mathbb{R}[X_1,\dots,X_n]_d^{\binom{n}{r+1}} : f_J \in \mathbb{R}[X_j : j\in J] \text{ and } |f_J| = 1 \text{ for all } J \in \mathcal{J}_{n,r}\right\}, \tag{30}$$

which we may identify with a semi-algebraic subset of $\mathbb{R}^{\binom{n+d}{n}\binom{n}{r+1}}$, under the semi-algebraic map $(f_J)_J \mapsto \sum_J f_J^2$. Thus $F_{r,d}$ is a semi-algebraic set. As this map is continuous and since (30) is compact, we conclude that $F_{r,d}$ is compact.

The zero set $\mathcal{Z}(f) \subseteq \mathbb{R}^n$ of $f = \sum_J f_J^2$ is the intersection of the zero sets of all the f_J. The projection of $\mathcal{Z}(f)$ to the $r+1$ distinct coordinates in a given $J \in \mathcal{J}_{n,r}$ is contained in $\mathcal{Z}(f_J)$, taken as a subset of $\mathbb{R}^{r+1}$. As $f_J \neq 0$, this projection does not contain a non-empty open subset of $\mathbb{R}^{r+1}$. It follows that $\dim \mathcal{Z}(f) \le r$ for all $f \in F_{r,d}$.

For $n = r$ it is convenient to define $F_{n,d} = \{0\}$ and identify 0 with the zero polynomial in $\mathbb{R}[X_1,\dots,X_n]$. This is clearly also a compact and semi-algebraic set with $\dim \mathcal{Z}(f) \le n$ if $f \in F_{n,d}$. Recall that the fiber dimension was introduced near the end of Section 2.

Lemma 22. *Let $m \in \mathbb{N}_0, n, e \in \mathbb{N}$, and $\epsilon \in (0,1]$. Suppose $C \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a cell whose projection to $\mathbb{R}^n$ is bounded and which has fiber dimension r over $\mathbb{R}^m$. There exist constants $c = c(C,e,\epsilon) \ge 1, d = d(n,r,e,\epsilon) \in \mathbb{N}$, and $0 < \lambda \le 4(r+1)^{2r+2}e^{r+1}(e+1)\binom{n}{r+1}^r\epsilon^{-r}$ with the following property. Say $y \in \mathbb{R}^m$. If $T \ge 1$ there exist $N \in \mathbb{N}$ with $N \le cT^\epsilon$ and polynomials $f_1,\dots,f_N \in F_{r,d}$ such that if*

$$x \in C_y \text{ and } q \in \mathbb{Q}^n(T,e) \text{ with } |x-q| \le c^{-1}T^{-\lambda} \tag{31}$$

then $f_j(q) = 0$ and $|f_j(x)| \le |x-q|$ for some $j \in \{1,\dots,N\}$.

Proof. Recall that each fiber $C_y \subseteq \mathbb{R}^n$ is either empty or a cell of dimension r.

The case $r = n$ can be handled easily. Indeed, here we may take $c = d = \lambda = 1$ and one polynomial $f_1 = 0 \in F_{n,1}$ is enough.

Now we assume $r \le n-1$. Let $y \in \mathbb{R}^m$, to prove the lemma we may assume $C_y \neq \emptyset$.

Let Z denote one of the $\binom{n}{r+1}$ projections of C to $\mathbb{R}^m \times \mathbb{R}^{r+1}$. Each such projection corresponds to the choice of $r+1$ variables among $X_1,\dots,X_n$. We let $X_1',\dots,X_{r+1}'$ denote these chosen variables.

We define $k = \dim Z_y$ and note that $k \le \dim C_y = r$ as Z_y is the image of C_y under a projection.

We proceed by proving the following

Intermediate Claim. *There exist $0 < \lambda \le 4(r+1)^{2r+2}e^{r+1}(e+1)\binom{n}{r+1}^r\epsilon^{-r}$, $d \in \mathbb{N}$, and $c_1 \ge 1$ depending only on C, e, and ϵ with the following property. If $T \ge 1$ there exist $N \in \mathbb{N}$ with $N \le c_1 T^{\epsilon/\binom{n}{r+1}}$ and polynomials $f_1',\dots,f_N' \in \mathbb{R}[X_1',\dots,X_{r+1}']_d$ with $|f_1'| = \cdots = |f_N'| = 1$ such that if*

$$x' \in Z_y \text{ and } q' \in \mathbb{Q}^{r+1}(T,e)\cap\mathbb{R}^{r+1} \text{ with } |x'-q'| \le c_1^{-1}T^{-\lambda} \tag{32}$$

then $f_j'(q') = 0$ and $|f_j'(x')| \le c_1|x'-q'|$ for some for $j \in \{1,\dots,N\}$.

We prove the claim in the case $k = 0$ first; here we may take $\lambda = 2e^2$ and $N = 1$. The set Z_y, being the continuous image of a connected space, is a singleton $\{x'\}$. We fix q in the finite set $\mathbb{Q}^{r+1}(T,e)\cap\mathbb{R}^{r+1}$ such that $|x'-q|$ is minimal and take $f'\in\mathbb{R}[X'_1,\ldots,X'_{r+1}]$ to be the normalization of $\widetilde{f}'=(X'_1-q_1)^2+\cdots+(X'_{r+1}-q_{r+1})^2$ where $q=(q_1,\ldots,q_{r+1})$. Observe $|\widetilde{f}'|\geq 1$ for all j. If $q'\in\mathbb{Q}^{r+1}(T,e)\cap\mathbb{R}^{r+1}$ is as in (32) then

$$|f'(x')|\leq|\widetilde{f}'(x')|\leq(r+1)|x'-q|^2\leq(r+1)|x'-q'|^2\leq(r+1)|x'-q'|$$

by minimality of $|x'-q|$ and since $|x'-q'|\leq 1$. As we may assume $c_1\geq r+1$ we find $|f'(x')|\leq c_1|x'-q'|$. It remains to prove that $f'(q')$ vanishes. Note that $|q'-q|\leq|q'-x'|+|x'-q|\leq 2|x'-q'|\leq 2c_1^{-1}T^{-\lambda}$. If $q'\neq q$ then Liouville's Inequality, Theorem 1.5.21 [4], yields $|q'-q|\geq(2H(q')H(q))^{-e^2}\geq 2^{-e^2}T^{-2e^2}$. Combining upper and lower bound yields $c_1\leq 2^{e^2+1}T^{2e^2-\lambda}$ and so $c_1\leq 2^{e^2+1}$ since $\lambda=2e^2$. So if we assume, as we may, that $c_1>2^{e^2+1}$, then $q'=q$. Thus $f'(q')=f'(q)=0$ and this settles our intermediate claim if $k=0$.

Now say $k\geq 1$. Recall that $k\leq r$. Let d be an integer satisfying $d+1\geq(e+1)(r+1)$. We will fix d in terms of ϵ in a moment. But first observe that $(e+1)D_k(1)=(e+1)(k+1)\leq(e+1)(r+1)\leq d+1\leq D_{r+1}(d)$. The binomial coefficient $D_k(b)$ increases strictly in b since $k\geq 1$. So there exists a unique $b\in\mathbb{N}$, depending on d, with

$$(e+1)D_k(b)\leq D_{r+1}(d)<(e+1)D_k(b+1). \tag{33}$$

We obtain

$$e+1>\frac{D_{r+1}(d)}{D_k(b+1)}\geq\frac{D_{k+1}(d)}{D_k(b+1)}=\frac{d+1}{k+1}\left(\frac{d+2}{b+2}\cdots\frac{d+k+1}{b+k+1}\right)\geq\frac{d+1}{r+1}\left(\frac{d+2}{b+2}\cdots\frac{d+k+1}{b+k+1}\right)$$

and thus we must have $d<b$. Hence each one of the k factors in the parentheses on the right is greater than $d/b<1$. Therefore, $e+1>(d/b)^r(d+1)/(r+1)$. We rearrange terms and find

$$\frac{d}{b}<\left(\frac{(e+1)(r+1)}{d+1}\right)^{1/r}. \tag{34}$$

Observe that the right-hand side goes to 0 as d tends to $+\infty$.

We choose d to be the least integer $d\geq(e+1)(r+1)-1\geq 1$ such that

$$(k+1)(r+1)e\frac{d}{b}\leq\frac{\epsilon}{\binom{n}{r+1}} \tag{35}$$

holds. By rearranging and using $\epsilon\in(0,1]$ as well as $k\leq r$ we find, using (34), that d satisfies

$$d\leq(k+1)^r(r+1)^{r+1}e^r(e+1)\binom{n}{r+1}^r\epsilon^{-r}\leq(r+1)^{2r+1}e^r(e+1)\binom{n}{r+1}^r\epsilon^{-r}. \tag{36}$$

The choice of d uniquely determines b, which is bounded from above in terms of n,ϵ, and e only.

We now apply Pila and Wilkie's reparametrization Corollary 5.2 [17]. Thereby, the fiber Z_y can be covered by the images of a finite number of maps $\phi:(0,1)^k\to\mathbb{R}^{r+1}$ for which all derivatives up-to order $b+1$ exist, are continuous, and have modulus bounded by a constant $B\geq 1$. Observe that the number of maps and B are bounded independent of y. Pila and Wilkie assume that the definable set is in $(0,1)^{r+1}$, but this restriction is harmless as the projection of C to $\mathbb{R}^n$ is bounded by hypothesis. So we can recover the desired statement by scaling.

We now apply Proposition 16 with n replaced by $r+1$ to the ϕ, recalling (33) and (35). For given $T \geq 1$ there is an integer $N \leq c_1 T^{\epsilon/\binom{n}{r+1}}$, with $c_1 \geq 1$ as in the said proposition, and polynomials $f'_1, \ldots, f'_N \in \mathbb{Q}[X'_1, \ldots, X'_{r+1}] \setminus \{0\}$ of degree at most d and norm 1 such that the assertion of the claim made above holds true for $\lambda = 4(r+1)ed$ as

$$4(r+1)ed \geq \frac{(k+1)(r+1)e}{k}\frac{d(b+1)}{b}.$$

Observe that in this case c_1 is independent of y. As d is bounded by (36) we retrieve

$$\lambda \leq 4(r+1)^{2r+2}e^{r+1}(e+1)\binom{n}{r+1}^{r}\epsilon^{-r}.$$

This completes the proof of our intermediate claim.

We may treat the constants $\lambda > 0$ and $c_1 \geq 1$ found as independent of the choice of $r+1$ coordinates. The constant in the assertion is $c = \max\{2^n c_1^2, c_1^{\binom{n}{r+1}}\}$. The construction above yields for each choice of $r+1$ coordinates among all n coordinates of $\mathbb{R}^n$, given T, a tuple of at most $c_1 T^{\epsilon/\binom{n}{r+1}}$ normalized polynomials in the corresponding $r+1$ variables and with the stated properties. We take as the f_j all possible sums of squares of the f'_j that appear above where each term corresponds to one of the $\binom{n}{r+1}$ projections. In total there at most $c_1^{\binom{n}{r+1}} T^{\epsilon} \leq cT^{\epsilon}$ possible polynomials by our choice of c, they lie in $F_{r,d}$

Now say $x \in C_y$ and $q \in \mathbb{Q}^n(T, e)$ with $|x - q| \leq c^{-1}T^{-\lambda} \leq (2^n c_1^2)^{-1}$. Then one of the f_j just constructed satisfies $f_j(q) = 0$ and

$$|f_j(x)| \leq \binom{n}{r+1} c_1^2 |x-q|^2 \leq 2^n c_1^2 |x-q||x-q| \leq |x-q|. \qquad \square$$

The coefficients of each polynomial f_j produced by this last lemma are algebraic and have uniformly bounded degree over $\mathbb{Q}$.

The fact that some f_j vanishes at q will play no role in the remaining argument. But from this conclusion we can infer something about algebraic approximations of a bounded cell C without restricting to polynomially bounded sets. Indeed, they lie on at most cT^{ϵ} real algebraic sets of dimension at most $\dim C$ that are cut out by a polynomial of controlled degree.

For the rest of this section we suppose that the ambient o-minimal structure is polynomially bounded.

The following statement is proved by induction on the fiber dimension $r \in \mathbb{N}_0$. In the induction step we need to keep track of additional data, for this reason we work with a prescribed cell partition of our given definable family.

Statement(r). *Let $m \in \mathbb{N}_0, n, e \in \mathbb{N}$ with $r \leq n$, and let $\epsilon \in (0,1], \kappa \in (0,1)$. Suppose we are given $(Z, C_1, \ldots, C_s)$ where $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is compact and definable such that $C_1 \cup \cdots \cup C_s$ is a partition of Z into cells $C_1, \ldots, C_s$. There exist $c = c(C_1, \ldots, C_s, e, \epsilon, \kappa) \geq 1$, $\theta = \theta(C_1, \ldots, C_s, e, \epsilon) \in (0,1]$, integers $l_1, \ldots, l_t \in \mathbb{N}_0$, and bounded cells $D_j \subseteq \mathbb{R}^{l_j} \times \mathbb{R}^m \times \mathbb{R}^n$ for all $j \in \{1, \ldots, t\}$ with the following properties:*

(i) *Say $D = D_j$ for some $j \in \{1, \ldots, t\}, z \in \mathbb{R}^{l_j}$, and $y \in \mathbb{R}^m$. Then $D_{(z,y)} \subseteq Z_y$ and if $D_{(z,y)} \neq \emptyset$, then $\dim D_{(z,y)} \leq r$ and $D_{(z,y)}$ is a quasi-algebraic cell.*

(ii) *Say $C = C_j$ has fiber dimension r over $\mathbb{R}^m$ and suppose $\psi : [1, +\infty) \to [0, 1]$ has order at most $-\theta^{-1}$. If $y \in \mathbb{R}^m$ and $T \geq 1$ there exist an integer $N \geq 1$ with $N \leq cT^\epsilon$ and $(j_p, z_p, y_p) \in \{1, \ldots, t\} \times \mathbb{R}^{l_{j_p}} \times \mathbb{R}^m$ for $p \in \{1, \ldots, N\}$ such that if*

$$x \in C_y \text{ and } q \in \mathbb{Q}^n(T, e) \text{ with } |x - q| \leq c^{-1}\psi(T) \tag{37}$$

then there exists $p \in \{1, \ldots, N\}$ with $|y - y_p| \leq \kappa\psi(T)^\theta$ and $x' \in (D_{j_p})_{(z_p, y_p)}$ with $|x - x'| \leq \kappa\psi(T)^\theta$.

Proof. We prove by induction on r that **Statement(r)** holds true for all r. During the argument we will choose $c \geq 1$ and $\theta > 0$ in terms of the appropiate data.

If $r = 0$ and if C is a cell appearing in (ii) then any non-empty fiber $C_y \neq \emptyset$ consists of a single point. Therefore, **Statement(0)** holds true by taking the D_j to equal the C_j that have fiber dimension 0 over $\mathbb{R}^m$ and $l_j = 0$. Part (ii) follows with $N = \theta = c = 1$ and $y_1 = y$. **Statement(n)** can be handled in a similar fashion. It holds true by adding those C_j to our list in (i) that have fiber dimension n over $\mathbb{R}^m$; indeed, n-dimensional cells are quasi-algebraic.

So let $1 \leq r \leq n - 1$ and suppose that **Statement(r')** holds true for all $r' \leq r - 1$.

Let C, y, and T be as in (ii). We apply Lemma 22 to C and obtain c_1, d, and λ. We may assume that λ attains the upper bound provided by the lemma, so it depends only on n, e, r, and ϵ. We may also suppose $c \geq c_1$ and $\theta \leq \lambda^{-1}$. Hence $c^{-1}\psi(T) \leq c^{-1}T^{-1/\theta} \leq c_1^{-1}T^{-\lambda}$. By the lemma there is a collection $f_1, \ldots, f_U$ of polynomials in $F_{r,d}$ with $U \leq c_1 T^\epsilon$ such that any pair q, x as in (37) satisfies $|f_j(x)| \leq |x - q| \leq c^{-1}\psi(T)$ for some $j \in \{1, \ldots, U\}$.

Recall that $F_{r,d}$ is a compact real semi-algebraic set and that $\overline{C}$ is the closure in $\mathbb{R}^m \times \mathbb{R}^n$ of a bound cell. Therefore, $F_{r,d} \times \overline{C}$ is compact and definable. We will apply Proposition 12 to ϵ replaced by $c^{-1}\psi(T)$ and to $F_{r,d} \times \overline{C}$. After increasing c we can make $c^{-1}\psi(T) \leq c^{-1}T^{-1/\theta} \leq c^{-1}$ smaller than c_2^{-1} where $c_2 = c_2(C, r, d) > 0$ is c from the said proposition. Each f_j from above leads to at most c_2^{-1} new elements in $F_{r,d}$. By abuse of notation let us also call them $f_1, \ldots, f_U$ after renumbering; we have $U \leq cT^\epsilon$ as we may suppose $c \geq c_1 c_2^{-1}$. Observe that these new polynomials approximate the original ones and could now have transcendental coordinates. Being in $F_{r,d}$, each f_j is a sum $\sum_{J \in \mathcal{J}_{n,r}} f_{j,J}^2$ where $f_{j,J}$ depends only on the $r + 1$ variables associated to J. The number of terms is $\binom{n}{r+1}$ and $\deg f_{j,J} \leq d$. We split each $f_{j,J}$ into irreducible factors. So after replacing c by a possibly larger constant we may assume that $U \leq cT^\epsilon$ and that each $f_{j,J}$ is irreducible with $|f_{j,J}| = 1$.

Let $\delta = \delta(C, r, d) > 0$ also come from Proposition 12. This proposition yields $y_1, \ldots, y_U$ with $U \leq cT^\epsilon$ such that the following holds. For any x as above there is j and $x' \in \overline{C}_{y_j} \cap \mathcal{Z}(f_j)$ with

$$\max\{|x' - x|, |y_j - y|\} \leq c^{-\delta}\psi(T)^\delta \leq \frac{\kappa}{2}\psi(T)^\delta \tag{38}$$

as we may assume $c^\delta \geq 2/\kappa$.

The point $\left((f_{j,J})_{J \in \mathcal{J}_{n,r}}, y_j, x'\right)$ is a member of the compact and definable set

(39)
$$Z' = \left\{ \left((f_J)_{J \in \mathcal{J}_{n,r}}, y', x''\right) \in \mathbb{R}[X_1, \ldots, X_n]_d^{\binom{n}{r+1}} \times \overline{C} : f_J(x'') = 0 \text{ and } |f_J| = 1 \text{ for all } J \in \mathcal{J}_{n,r} \right\}.$$

Observe that each fiber $Z'_{((f_J)_J,y')}$ is contained in $\mathcal{Z}((f_J)_{J\in\mathcal{J}_{n,r}})$ which is a real algebraic set of dimension at most r by the remark below (30). To avoid singularities we introduce the subset

$$Z'' = \left\{ ((f_J)_{J\in\mathcal{J}_{n,r}}, y', x'') \in Z' : \mathrm{Rk}\left(\frac{\partial f_J}{\partial x_j}(x'')\right)_{\substack{J\in\mathcal{J}_{n,r}\\ 1\le j\le n}} < n-r \right\}. \tag{40}$$

which is again compact and definable.

We fix a cell partition $D_1 \cup \cdots \cup D_{t''} = Z''$ and a cell partition $D_{t''+1} \cup \cdots \cup D_{t''+t'} = Z' \smallsetminus Z''$. So $D_1 \cup \cdots \cup D_{t''+t'}$ is a partition of Z' into cells. Note that each cell is bounded since Z' is compact.

The point $((f_{j,J})_{J\in\mathcal{J}_{n,r}}, y_j, x')$ from Proposition 12 lies in one of these cells, D, say. As already pointed out above, we have

$$\dim D_{((f_{j,J})_J,y_j)} \le \dim Z'_{((f_{j,J})_J,y_j)} \le \dim \mathcal{Z}((f_{j,J})_J) \le r. \tag{41}$$

We split up into two cases depending on the value of $r' = \dim D_{((f_{j,J})_J,y_j)}$.

First, suppose $r' \le r-1$. In this case, we consider Z' as a definable set parametrized by $\mathbb{R}^{m'}$, where $m' = \binom{n}{r+1}\binom{n+d}{n} + m$. We can thus apply **Statement**(r') to $(Z', D_1, \ldots, D_{t''+t'})$ and $e, \epsilon, \kappa/2$ to obtain c' and θ'. The point x' lies in a fiber of the cell D. Moreover, as $|x-q| \le c^{-1}\psi(T)$, we get

$$|x'-q| \le |x'-x| + |x-q| \le c^{-\delta}\psi(T)^{\delta} + c^{-1}\psi(T) \tag{42}$$

using the first inequality of (38). We are free to increase c and decrease θ to assume $c^{-\delta} + c^{-1} \le c'^{-1}$ and $\theta \le \theta' \min\{1,\delta\}$, respectively. As $\psi(T) \le 1$, the right-hand side of (42) is at most $c'^{-1}\psi(T)^{\min\{1,\delta\}}$. Observe that $\psi(T)^{\min\{1,\delta\}}$ has order at most $-\min\{1,\delta\}/\theta \le -1/\theta'$. By induction we find that x' has distance at most $\frac{\kappa}{2}\psi(T)^{\theta'}$ to the union of at most $c'T^{\epsilon}$ fibers of one of finitely many bounded cells $D'' \subseteq \mathbb{R}^{l'} \times \mathbb{R}^{m'} \times \mathbb{R}^n$. More precisely, we have

$$\max\{|(f'', y'', x'') - ((f_{j,J})_J, y_j, x)|\} \le \frac{\kappa}{2}\psi(T)^{\theta'}$$

for some $x'' \in D''_{(z,f'',y'')}$.

We may also assume $\theta \le \delta$ and have already established $\theta \le \theta'$. Thus by (38) $|x-x''| \le \kappa\psi(T)^{\theta}$. Similarly, $|y''-y| \le |y''-y_j| + |y_j - y| \le \kappa\psi(T)^{\theta}$. This yields (ii). The non-empty fibers $D''_{(z,f'',y'')}$ have dimension at most r' and are quasi-algebraic cells. We are allowed to add the D'' to our collection in (i). Thus **Statement**(r) is established if $r' \le r-1$.

Second, say $r' = r$. In this case we verify that D satisfies the properties from (i). Recall that D is member of a cell partition of Z'. A fiber of D above $\mathbb{R}^{m'}$ is either empty or a cell of dimension r.

We claim that D is not among the cells in the partition of Z''. Indeed, otherwise we would have $D \subseteq Z''$. By (40) the jacobian matrix attached to the $f_{j,J}$ has rank strictly less than $n-r$ on the fibers of D. By construction each $f_{j,J}$ is irreducible as J runs through $\mathcal{J}_{n,r}$ Thus Lemma 18 contradicts the fact that the fiber $D_{((f_{j,J})_J,y_j)}$ has dimension r.

For any $((f_J)_{J\in\mathcal{J}_{n,r}}, y') \in \mathbb{R}^{m'}$ we have $D_{((f_J)_J,y')} \subseteq A = \mathcal{Z}((f_J)_J)$. As the algebraic set on the right has dimension at most r we have $\dim A = r$ if $D_{((f_J)_J,y')} \ne \emptyset$. In this case and since $D \subseteq Z' \setminus Z''$, the jacobian matrix attached to $(f_J)_J$ has rank at least $n-r$ at all points of $D_{((f_J)_J,y')}$. So $D_{((f_J)_J,y')}$ is a quasi-algebraic cell by Lemma 17.

Now

$$D_{((f_J)_J,y')} \subseteq Z'_{((f_J)_J,y')} \subseteq \overline{C}_{y'} \subseteq Z_{y'}$$

by (39) and as the compact set Z contains C and hence its closure $\overline{C}$ in $\mathbb{R}^m \times \mathbb{R}^n$. Thus we can add D to the cells mentioned in (i).

As only many finitely cells appear in the partition of Z', we get at most finitely many cells by this process. We already assumed $\theta \leq \delta$. So x has distance at most $\frac{\kappa}{2}\psi(T)^\theta \leq \kappa\psi(T)^\theta$ to $D_{(f_j,y_j)}$ by (38). Moreover, $|y-y_j| \leq \kappa\psi(T)^\theta$ by the same reasoning. This completes the proof that **Statement**(r) holds true. □

Theorem 11. *Let $m \in \mathbb{N}_0, n, e \in \mathbb{N}, \epsilon > 0, \kappa \in (0,1)$ and suppose $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is compact and definable. There exist $c = c(Z,e,\epsilon,\kappa) \geq 1$, $\theta = \theta(Z,e,\epsilon) \in (0,1]$, integers $l_1,\ldots,l_t \in \mathbb{N}_0$, and bounded cells $D_j \subseteq \mathbb{R}^{l_j} \times \mathbb{R}^m \times \mathbb{R}^n$ for all $j \in \{1,\ldots,t\}$ with the following properties:*

(i) *Say $D = D_j$ for some $j \in \{1,\ldots,t\}, z \in \mathbb{R}^{l_j}$, and $y \in \mathbb{R}^m$. Then $D_{(z,y)} \subseteq Z_y$ and if $D_{(z,y)} \neq \emptyset$, then $D_{(z,y)}$ is a quasi-algebraic cell.*

(ii) *Let $\psi : [1,+\infty) \to [0,1]$ have order at most $-\theta^{-1}$. If $y \in \mathbb{R}^m$ and $T \geq 1$ there exist an integer $N \geq 1$ with $N \leq cT^\epsilon$ and $(j_p, z_p, y_p) \in \{1,\ldots,j\} \times \mathbb{R}^{l_{j_p}} \times \mathbb{R}^m$ for $p \in \{1,\ldots,N\}$ such that if*

$$x \in Z_y \text{ and } q \in \mathbb{Q}^n(T,e) \text{ with } |x-q| \leq c^{-1}\psi(T)$$

then there exists $p \in \{1,\ldots,N\}$ with $|y-y_p| \leq \kappa\psi(T)^\theta$ and $x' \in (D_{j_p})_{(z_p,y_p)}$ with $|x-x'| \leq \kappa\psi(T)^\theta$.

Proof. We may assume $\epsilon \leq 1$. The theorem then follows from **Statement**(r) $(0 \leq r \leq n)$ and since Z admits a partition into a finite number of cells. □

We now extend this theorem to more general families of definable sets. To do this we introduce the semi-algebraic homeomorphism $\varphi : (-1,+\infty)^n \to (-\infty,1)^n$ given by

$$\varphi(x_1,\ldots,x_n) = \left(\frac{x_1}{1+x_1},\ldots,\frac{x_n}{1+x_n}\right)$$

with inverse

$$\varphi^{-1}(x_1,\ldots,x_n) = \left(\frac{x_1}{1-x_1},\ldots,\frac{x_n}{1-x_n}\right).$$

If $x,x' \in [-1/2,+\infty)^n$, then $|\varphi(x)-\varphi(x')| \leq 4|x-x'|$ and if $x,x' = (x'_1,\ldots,x'_n) \in (-\infty,1)$, then

$$|\varphi^{-1}(x)-\varphi^{-1}(x')| \leq \frac{|x-x'|}{\min_{1\leq i\leq n}\{1-x_i\}\min_{1\leq i\leq n}\{1-x'_i\}}. \tag{43}$$

The map is not height-invariant but still satifies

$$H(\varphi(x)) \leq 2H(x)^2$$

for all algebraic $x \in \mathbb{R}^n$ by basic height properties, cf. (5). So φ maps $\mathbb{Q}^n(T,e)$ to $\mathbb{Q}^n(2T^2,e)$.

Lemma 23. *Suppose $D \subseteq \mathbb{R}^n$ is a quasi-algebraic cell of dimension r with $D \subseteq (-\infty,1)^n$. Then $\dim \varphi^{-1}(D) = r$ and $\varphi^{-1}(D)$ is an open subset of the non-singular locus of an r-dimensional real algebraic set.*

Proof. We have $\dim \varphi^{-1}(D) = r$ sind φ is a homeomorphism.

For a non-zero $f \in \mathbb{R}[X_1,\ldots,X_n]$ we set

$$f^* = f\left(\frac{X_1}{1+X_1},\ldots,\frac{X_n}{1+X_n}\right)\prod_{j=1}^n (1+X_j)^{\deg_{X_j}(f)}$$

which is again a polynomial in $\mathbb{R}[X_1,\ldots,X_n]$, we also set $f^*=0$ if $f=0$. If f vanishes on A, then f^* vanishes on B, the Zariski closure of $\varphi^{-1}(A) = \varphi^{-1}(A\cap(-\infty,1)^n)$.

By hypothesis, there exists a real algebraic set $A \subseteq \mathbb{R}^n$ of dimension r such that D is an open subset of $A \smallsetminus \mathrm{Sing}(A)$. So $\dim A \geq \dim \varphi^{-1}(A) \geq \dim \varphi^{-1}(D) = r = \dim A$. Proposition 2.8.2 [3] implies $\dim B = \dim \varphi^{-1}(A)$ and so $\dim B = r$.

We want to apply Lemma 17. First, we observe that $\varphi^{-1}(D)$, being homeomorphic to the cell D, is homeomorphic to an open subset of $\mathbb{R}^r$. Say $x \in \varphi^{-1}(D)$, then $\varphi(x) \in D \subseteq A \smallsetminus \mathrm{Sing}(A)$. There are $f_1,\ldots,f_{n-r} \in \mathbb{R}[X_1,\ldots,X_n]$ that vanish on A such that $(\frac{\partial f_i}{\partial X_j})_{1\leq i\leq n-r, 1\leq j\leq n}$ has rank $n-r$ when evaluated at $\varphi(x)$. By the chain rule $(\frac{\partial f_i^*}{\partial X_j})_{1\leq i\leq n-r,1\leq j\leq n}$ also has rank $n-r$ at x. We apply Lemma 17 to $\varphi^{-1}(D)$, B, and $f_1^*,\ldots,f_{n-r}^*$ to find that $\varphi^{-1}(D)$ lies open in $B\smallsetminus \mathrm{Sing}(B)$, as desired. □

Proof of Theorem 10. After splitting up into the 2^n orthants of $\mathbb{R}^n$ and switching signs we may assume $Z \subseteq \mathbb{R}^m \times [0,+\infty)^n$.

We consider the closure Z' of the image of Z under $\mathrm{id}_{\mathbb{R}^m}\times\varphi$. This is a compact subset of $\mathbb{R}^m\times[0,1]^n$. Since Z is closed we mention

$$\big(\mathbb{R}^m\times[0,1)^n\big)\cap Z' \subseteq (\mathrm{id}_{\mathbb{R}^m}\times\varphi)(Z) \tag{44}$$

for later reference.

Say $y \in \mathbb{R}^m$ and $T \geq 1$ such that there are $q \in \mathbb{Q}^n(T,e)$ and $x \in Z_y$ with $|x-q| \leq c^{-1}\psi(T)$. Here and below $c \geq 2$ is sufficiently large and θ is sufficiently small in terms of the given data. Moreover, we set $\kappa = 2^{-2e-2}$.

We write $q=(q_1,\ldots,q_n)$ and $x=(x_1,\ldots,x_n)$. As $x_i \geq 0$ for all i, we find $q_i \geq x_i - c^{-1}\psi(T) \geq -1/2$. So $|\varphi(x)-\varphi(q)| \leq 4|x-q| \leq 4c^{-1}\psi(T)$.

For large c and small θ, by Theorem 11 applied to Z', e, ϵ, and κ we get $c' \geq 1, \theta' \in (0,1]$ and $z' \in \mathbb{R}^l, y' \in \mathbb{R}^m, x' \in D_{(z',y')}$ with $|y-y'| \leq \kappa\psi(T)^{\theta'}$ and $|\varphi(x)-x'| \leq \kappa\psi(T)^{\theta'}$ and where there are at most $c'T^\epsilon$ possibilities for (z',y'). Here $D \subseteq \mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^n$ is a bounded cell from a finite collection and the fiber $D_{(z',y')}$ is a quasi-algebraic cell.

We want to show that $x' \in [0,1)^n$. Observe that the entries of x' are non-negative. We use Liouville's Inequality to show that for any i we have

$$1-\frac{q_i}{1+q_i} = \frac{1}{1+q_i} \geq \frac{1}{H(1+q_i)^e} \geq \frac{1}{2^eT^e}$$

using again (5) and $q \in \mathbb{Q}^n(T,e)$. Note that $|\varphi(q) - x'| \leq 4c^{-1}\psi(T) + \kappa\psi(T)^{\theta'}$ and hence

$$
\begin{aligned}
x'_i &\leq \frac{q_i}{1+q_i} + \frac{4}{c}\psi(T) + \kappa\psi(T)^{\theta'} \\
&\leq 1 - \frac{1}{2^e T^e} + \frac{4}{cT^{1/\theta}} + \frac{1}{2^{2e+2}T^{\theta'/\theta}} \\
&\leq 1 - \frac{1}{2^{e+1}T^e}
\end{aligned}
\tag{45}
$$

for each coordinate x'_i of x' by our choice of κ, for large c and small θ. So $x' \in [0,1)^n$ as desired. Using a similar argument we find

$$
\frac{x_i}{1+x_i} \leq \frac{q_i}{1+q_i} + \frac{4}{c}\psi(T) \leq 1 - \frac{1}{2^e T^e} + \frac{4}{cT^{1/\theta}} \leq 1 - \frac{1}{2^{e+1}T^e}
\tag{46}
$$

for large c and small θ.

Observe that $D \subseteq \mathbb{R}^l \times \mathbb{R}^m \times [0,1]$. So the intersection $D \cap \mathbb{R}^l \times \mathbb{R}^m \times [0,1)^n$ is open in D. It is a finite union of cells $D_1 \cup \cdots \cup D_s$ with $\dim D_i = \dim D$ for all $1 \leq i \leq s$ by Lemma 21.

Say $1 \leq i \leq s$. The dimension of the cell D_i equals the sum of the dimension of its projection to $\mathbb{R}^l \times \mathbb{R}^m$ and the fiber dimension over $\mathbb{R}^{l+m}$. The same holds for the cell $D \supseteq D_i$. Thus each D_i has the same fiber dimension as D over $\mathbb{R}^{l+m}$. We find that any fiber of D_i above a point in $\mathbb{R}^l \times \mathbb{R}^m$ lies open in the respective fiber of D by Lemma 1.14 in Chapter 4 [20]. Therefore, all non-empty fibers of D_i are quasi-algebraic cells.

The first inequality in

$$
|x - \varphi^{-1}(x')| = |\varphi^{-1}(\varphi(x)) - \varphi^{-1}(x')| \leq 2^{2e+2}T^{2e}|\varphi(x) - x'| \leq 2^{2e+2}\kappa T^{2e}\psi(T)^{\theta'} = T^{2e}\psi(T)^{\theta'}
$$

follows from (45) and (46) applied to (43). The inequalities

$$
\psi(T)^{\theta'-\theta} \leq T^{1-\theta'/\theta} \leq T^{-2e}
$$

holds for all $T \geq 1$ if θ is small enough in terms of θ'. Therefore, $|x - \varphi^{-1}(x')| \leq \psi(T)^{\theta}$.

Recall that $\varphi^{-1}(x')$ lies in the preimage $\varphi^{-1}((D_i)_{(z',y')})$. Observe that $(D_i)_{(z',y')} \subseteq (-\infty,1)^n$ and $\varphi^{-1}((D_i)_{(z',y')})$ is connected as φ is a homeomorphism. By Lemma 23 this preimage satisfies the conditions in (i) of the assertion. By (44) the preimage lies in the respective fiber $Z_{y'}$ of Z. This completes the proof. □

7. Proof of Theorems 2, 4, 5, and 6

Proof of Theorem 6. The theorem follows from Theorem 10 applied to the trivial family $Z = X \subseteq \mathbb{R}^n$ where $m = 0$. □

Proof of Theorem 2. By Northcott's Theorem $\mathbb{Q}^n(2^{1+e^2}, e)$ is finite. So we may assume $T > 2^{1+e^2}$ without loss of generality.

Let c and $\theta' \in (0,1]$ be as in Theorem 6 applied to X, e, and ϵ. We may assume $c \geq 2$, as increasing c makes the conclusion of the said theorem weaker. We fix $\theta = \theta'/(4e^2+2)$, so $0 < \theta \leq \theta'/2$. Say $\psi : [1,+\infty) \to [0,1]$ has order at most $-1/\theta$.

Suppose $q \in \mathbb{Q}^n(T,e)$ such that there is $x \in X$ with $|x - q| \leq c^{-1}\psi(T)$. Then $x' \in D$ and $|x' - x| \leq \psi(T)^{\theta'}$ for one among at most cT^{ϵ} sets $D \subseteq X$ as in (i) of Theorem 6.

Observe that $\psi(T)^{\theta'/2} \le T^{-\theta'/(2\theta)} = T^{-2e^2-1} \le T^{-1} \le 1/2$ by our choice of θ and since $T \ge 2$. Using $c \ge 2$ and $\theta \le \theta'/2 \le 1$ we find using the triangle inequality that

$$|x' - q| \le \psi(T)^{\theta'} + c^{-1}\psi(T) \le \frac{1}{2}\psi(T)^{\theta'/2} + \frac{1}{2}\psi(T)^{\theta'/2} \le \psi(T)^{\theta'/2} \le \psi(T)^{\theta}. \tag{47}$$

If $\dim D \ge 1$, then $D^{\mathrm{alg}} = D$ by Lemma 19 and so $x' \in X^{\mathrm{alg}}$. Hence (47) implies $q \in \mathcal{N}(X^{\mathrm{alg}}, \psi(T)^\theta)$.

If we assume $q \notin \mathcal{N}(X^{\mathrm{alg}}, \psi(T)^\theta)$ as in (1) then $\dim D = 0$ and thus $D = \{x'\}$ as D is connected. In this case our rational point q is close to one of at most cT^ϵ singletons D.

Now suppose a second $q' \in \mathbb{Q}^n(T, e)$ with $q' \ne q$ also satisfies $|x' - q'| \le \psi(T)^{\theta'/2}$, cf. (47). Then $|q' - q| \le 2\psi(T)^{\theta'/2}$. As $q' \ne q$ Liouville's Inequality gives $|q' - q| \ge (2H(q')H(q))^{-e^2} \ge 2^{-e^2}T^{-2e^2}$. Therefore,

$$2^{-e^2}T^{-2e^2} \le 2\psi(T)^{\theta'/2} \le 2T^{-\theta'/(2\theta)} = 2T^{-2e^2-1}$$

and this contradicts $T > 2^{1+e^2}$.

We have shown that at most one algebraic point of height at most T and degree at most e approximates a singleton D. Thus the number of q in question is at most cT^ϵ. □

Proof of Theorem 4. Instead of applying Theorem 6 as before we require Theorem 10, which holds for families, directly. The proof is then very similar to the proof of Theorem 2 with the choice $\psi(x) = x^{-\lambda}$ where $\lambda \ge 1/\theta$. □

Proof of Theorem 5. We use Theorem 6. Indeed, any x as in the set on the left of (3) lies in $\mathcal{N}((D_j)_z, \psi(T)^\theta)$ for one of at most cT^ϵ sets $(D_j)_z$ as in (i) of Theorem 6.

If one particular $(D_j)_z$ has positive dimension, then it equals its algebraic locus by Lemma 19. In particular, $x \in \mathcal{N}(X^{\mathrm{alg}}, \psi(T)^\theta)$, which is impossible. So $(D_j)_z$ has dimension 0 and, being connected, is a singleton. This yields (3) when taking the x_i to be the points appearing in the $(D_j)_z$. □

8. Application to Sums of Roots of Unity

Proof of Theorem 7. Our proof is by induction on n, the statement being elementary if $n = 1$. So say $n \ge 2$ and let

$$X = \left\{(x_1, \ldots, x_n) \in [0,1]^n : a_0 + a_1 e^{2\pi\sqrt{-1}x_1} + \cdots + a_n e^{2\pi\sqrt{-1}x_n} = 0\right\}$$

which is compact and definable in the polynomially bounded o-minimal structure $\mathbb{R}_{\mathrm{an}}$.

We will choose c and λ in the argument below. Say $\zeta_j = e^{2\pi\sqrt{-1}q_j}$ with $q_j \in \frac{1}{p}\mathbb{Z} \cap [0, 1)$ such that $0 < |a_0 + a_1\zeta_1 + \cdots + a_n\zeta_n| \le c^{-1}p^{-\lambda}$ and where $p \le T$ is a prime. We may assume $p \ge T^\epsilon$ as there are at most T^ϵ primes bounded by T^ϵ. So $|a_0 + a_1\zeta_1 + \cdots + a_n\zeta_n| \le c^{-1}T^{-\epsilon\lambda}$.

If c is large enough in terms of $(a_0, \ldots, a_n)$, then at least one among $\zeta_1, \ldots, \zeta_n$ has order p. For large c the Łojasiewicz Inequality from Theorem 8 implies

$$\mathrm{dist}^*(q, X) \le T^{-\epsilon\lambda\delta}$$

where $q = (q_1, \ldots, q_n)$ and where $\delta > 0$ depends only on X.

We suppose $\epsilon\lambda\delta \ge \theta^{-1}$ with θ from Theorem 2 applied to $X, e = 1$, and ϵ; for ψ we take $\psi(x) = x^{-\epsilon\lambda\delta}$. There are two cases.

In the first case q is not in the $\psi(T)^\theta = T^{-\theta\epsilon\lambda\delta}$-tube around X^{alg}. As p is the denominator of q it is among at most cT^ϵ possibilities. We are done in this case.

In the second case there is $x' = (x'_1, \ldots, x'_n) \in X^{\mathrm{alg}}$ with $|q - x'| \le T^{-\theta\epsilon\lambda\delta}$.

The locus X^{alg} plays an important role in Zannier's proof strategy of the Manin-Mumford Conjecture presented in his joint work with Pila [18]. Indeed, it is a well-known consequence of Ax's Theorem, Corollary 2 [1], that a non-trivial subsum

$$a_0 + \sum_{j\in J} a_j e^{2\pi\sqrt{-1}x'_j} = 0$$

vanishes for some non-empty set $J \subsetneq \{1, \ldots, n\}$. The corresponding sum over coordinates of q must be small, i.e.

$$\left| a_0 + \sum_{j\in J} a_j\zeta_j \right| \le 2\pi n \max_{1\le j\le n}\{|a_j|\}|q - x'| \le c' T^{-\theta\epsilon\lambda\delta}$$

where $c' > 0$ depends only on $(a_1, \ldots, a_n)$.

Let λ' be the maximal value of λ for this theorem applied by induction to a sum involving at most $n-1$ roots of unity and a subset of the $a_0, \ldots, a_n$ as coefficients. We may assume $\theta\epsilon\lambda\delta \ge 1 + \lambda'$ and if c'' comes from this theorem applied by induction we may also assume that $T \ge c'c''$. Hence

$$\left| a_0 + \sum_{j\in J} a_j\zeta_j \right| \le c''^{-1} T^{-\lambda'} \le c''^{-1} p^{-\lambda'}.$$

Say $a_0 + \sum_{j\in J} a_j\zeta_j \ne 0$. Then by induction there are at most cT^ϵ possibilites for p, if c is sufficiently large.

Finally, if $a_0 + \sum_{j\in J} a_j\zeta_j = 0$, then $\sum_{j\in I} a_j\zeta_j \ne 0$ where $I = \{1, \ldots, n\} \smallsetminus J$. Say $j_0 \in I$, then

$$0 < \left| a_{j_0} + \sum_{j\in I\smallsetminus\{j_0\}} a_j\zeta_j\zeta_{j_0}^{-1} \right| = \left| a_0 + \sum_{j=1}^{n} a_j\zeta_j \right| \le c^{-1}p^{-\lambda}.$$

then, again by induction on n, we conclude the claim if $\lambda \ge \lambda'$ and if c is large enough. □

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