Minimal energy problems
for strongly singular
Riesz kernels

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Abstract We study minimal energy problems for strongly singular Riesz kernels
$|\mathbf{x}-\mathbf{y}|^{\alpha-n}$, where $n \geq 2$ and $\alpha \in (-1, 1)$, considered for compact $(n-1)$-dimensional
C^∞-manifolds Γ immersed into $\mathbb{R}^n$. Based on the spatial energy of harmonic double
layer potentials, we are motivated to formulate the natural regularization of such
a minimization problem by switching to Hadamard's partie finie integral operator
which defines a strongly elliptic pseudodifferential operator of order $\beta = 1 - \alpha$
on Γ. The measures with finite energy are thus elements from the Sobolev space
$H^{\beta/2}(\Gamma)$, $0 < \beta < 2$, and the corresponding minimal energy problem admits a
unique solution. We relate this approach also to the common approach where for
$\delta > 0$ the set $|\mathbf{x} - \mathbf{y}| \leq \delta$ of $\Gamma \times \Gamma$ is cut out.

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1 Introduction

The classical Gauss problem of minimizing the Coulomb energy to solve the problem of Thomson and its generalization to Riesz potentials together with the discretization is the basic problem of many applications (in [3] are listed coding theory, cubature formulas, tight frames and packing problems). In the works [2], [3], [6], [10], the discretization is obtained by approximating the minimizing charges by nonnegative linear combinations of Dirac measures on the given manifold.

If the number of Dirac points tends to infinity, then the minimizing densities approach distributions in the form of Sobolev space elements. Therefore, in [8], [9], [17], the minimizing measures are considered as distributions in Hilbert spaces of finite Riesz energy. This continuous setting is simpler and more efficient from the numerical point compared to the discrete approach in [2], [3], [6], [10].

For potentials with Riesz kernel $|\mathbf{x} - \mathbf{y}|^{\alpha-n}$, where $1 < \alpha < n$, and Borel measures supported on a given $(n-1)$-dimensional manifold Γ immersed into $\mathbb{R}^n$, a surface potential is generated, which on Γ defines a boundary integral operator with weakly singular kernel. This boundary integral operator is a pseudodifferential operator of *negative* order $\beta = 1 - \alpha$ if $\Gamma \in C^\infty$. The energy space of this pseudodifferential operator on Γ is thus the Sobolev space $H^{\beta/2}(\Gamma)$ of distributions and the minimizing measure of finite energy is an element of this Sobolev space. Hence, the determination of the minimizer is reduced to an optimization problem with a quadratic functional which is defined in terms of the single layer Riesz potential on Γ. The strong ellipticity of the corresponding pseudodifferential operator in $\mathbb{R}^n$ and its trace on Γ then provides the coerciveness of the associated quadratic functional. For $\alpha = 2$, which corresponds to the Newtonian kernel, the Riesz energy of the single layer potential is just its Dirichlet integral over $\mathbb{R}^n \setminus \Gamma$.

In this paper, however, we consider the Riesz kernels with $\alpha \in (-1, 1)$. For $\alpha = 0$, in classical potential theory, the energy of the harmonic double layer potential in $\mathbb{R}^n \setminus \Gamma$ now equals the Riesz energy if we define the latter as to be Hadamard's partie finie integral of the hypersingular potential — which is the natural distributional regularization (see Section 2 where Γ is a $(n-1)$-dimensional planar bounded domain in $\mathbb{R}^n$).

Let $\Gamma = \bigcup_{\in I} \Gamma_i$ where Γ_i, $i \in I$, are finitely many compact, connected $(n-1)$-dimensional C^∞-manifolds immersed into $\mathbb{R}^n$. In Section 3, we then consider the Riesz potential as a pseudodifferential operator just on Γ since we cannot use its extension to $\mathbb{R}^n$ (for $\alpha \neq 0$, the transmission conditions [12, Theorem 8.3.11] are not satisfied). We call the bilinear form with the strongly singular partie finie integral of the Riesz kernel the *energy of the Riesz potential*. The partie finie integral operator with the hypersingular Riesz kernel defines now a strongly elliptic pseudodifferential operator V_β of *positive* order $\beta = 1 - \alpha$ on Γ.

In contrast to the analysis of weakly singular Riesz kernels provided earlier by the authors in [8], [9], in the case under consideration, the trace theorem in $H^{-\beta/2}(\Gamma) = V_\beta H^{\beta/2}(\Gamma)$ is not valid anymore, because of the negativity of the order $-\beta/2$, cf. [1]. Nevertheless, we have succeeded in overcoming this difficulty, and we have shown that all the Borel measures on Γ with finite Riesz energy whose restriction on any Γ_i takes sign either $+1$ or -1 form a certain cone in the Sobolev space $H^{\beta/2}(\Gamma)$, $0 < \beta < 2$. *This is our main result in Section 3, Theorems 3.3 and 3.4.* In this framework, the corresponding Gauss variational problem admits a unique solution which belongs to $H^{\beta/2}(\Gamma)$, which is a compact subspace of $L_2(\Gamma)$.
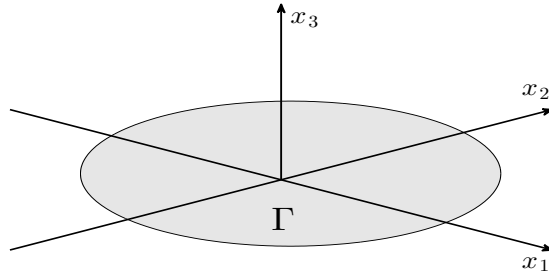
Fig. 1 Illustration of the geometrical setting.

These results have again a potential theoretic meaning in the particular situation $\alpha = 0$ in relation to the harmonic double layer potential as explained in Section 5.

In the papers [3], [4], [6], [10], [11], for the approximation of the minimal energy distribution with nonnegative linear combinations of Dirac distributions, the Riesz energy with hypersingular Riesz potentials is formulated by excluding the set $\mathbf{x} = \mathbf{y}$. In our paper, we analyze also this approach by cutting out the set $|\mathbf{x} - \mathbf{y}| \leq \delta$ of $\Gamma \times \Gamma$ where $\delta > 0$. We first figure out the idea in Section 6 by studying a perturbed Riesz energy problem. Then, in Section 7, we perform the computations in detail for the punched Riesz energy problem and give an asymptotic expansion of the solution in the corresponding family of finite energy spaces for $\delta \to 0$. In particular cases (see Corollary 7.7 for details), the minimizers tend to a constant distribution on Γ while the corresponding minimal energies tend to infinity.

2 Motivation. The energy of the Laplacian's double layer potential

We shall motivate our approach by an example from potential theory where $\alpha = 0$, i.e. $\beta = 1$. To this end, let $\Gamma \subset \mathbb{R}^{n-1}$ be a planar bounded domain in $\mathbb{R}^n$ and $\mathbf{x} = (\mathbf{x}', x_n) \in \mathbb{R}^n$ with $\mathbf{x} \in \Gamma$ when $x_n = 0$, see Figure 1 for an illustration.

The double layer potential of the Laplacian with given dipole charge density $\varphi(\mathbf{y}')$ is given by

$$U_{\boldsymbol{\varphi}}(\mathbf{x}) := -(\mathbf{W}\boldsymbol{\varphi})(\mathbf{x}) := -\int\limits_{y_n=0} \boldsymbol{\varphi}(\mathbf{y}') \frac{\partial E}{\partial y_n}\big((\mathbf{y}',0),\mathbf{x}\big)\,\mathrm{d}\mathbf{y}'$$

$$= -c_n \int\limits_{\mathbf{y}'\in\Gamma} \frac{\big(\mathbf{x}-(\mathbf{y}',0)\big)\cdot\mathbf{e}_n}{|\mathbf{x}-(\mathbf{y}',0)|^n}\boldsymbol{\varphi}(\mathbf{y}')\,\mathrm{d}\mathbf{y}'$$

for $\mathbf{x} \in \mathbb{R}^n$ with $x_n \neq 0$ and $c_n = \frac{1}{\omega_n} = \big(2(n-1)\pi\big)^{-1}$. The fundamental solution for the Laplacian is given by

$$E(\mathbf{x},\mathbf{y}) = c_n|\mathbf{x}-\mathbf{y}|^{2-n}.$$

The vector $\mathbf{e}_n = (0,\ldots,1)^\top$ is the n-th basis vector of $\mathbb{R}^n$ and the unit normal vector on Γ. If $\boldsymbol{\varphi}$ is continuous at $\mathbf{x}'$, then there holds the jump relation

$$\mathbb{R}^n_\pm \ni \mathbf{x} \to (\mathbf{x}',0) :$$

$$U_{\boldsymbol{\varphi}}(\mathbf{x}) \to \mp\frac{1}{2}\varphi(\mathbf{x}') - \int\limits_{\mathbf{y}'\in\Gamma\setminus\{\mathbf{x}'\}} \frac{\big((\mathbf{x}-(\mathbf{y}',0))\cdot\mathbf{e}_n\big)}{|\mathbf{x}-(\mathbf{y}',0)|^n}\varphi(\mathbf{y}')\,\mathrm{d}\mathbf{y}' = \mp\frac{1}{2}\varphi(\mathbf{x}') - 0$$

since for $\mathbf{x}\in\Gamma$ and $\mathbf{y}\in\Gamma\setminus\{\mathbf{x}\}$ the scalar product $(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_n = 0$ and, hence, the integral vanishes. Consequently, the harmonic potential $U_{\boldsymbol{\varphi}}(\mathbf{x})$ solves the transmission problem in $\mathbb{R}^n\setminus\Gamma$

$$[U]_\Gamma := U_{\boldsymbol{\varphi}}(\mathbf{x}',-0) - U_{\boldsymbol{\varphi}}(\mathbf{x}',+0) = \varphi(\mathbf{x}')$$

where $\boldsymbol{\varphi}$ is a given element of $\widetilde{H}^{1/2}(\Gamma)$, the closure of $C_0^\infty(\Gamma)$ in $H^{1/2}(\mathbb{R}^{n-1})$.

The energy of the harmonic field $U_{\boldsymbol{\varphi}}$ is given by its Dirichlet integral, and Green's theorem yields

$$\int\limits_{\mathbb{R}^n\setminus\Gamma} |\nabla U_{\boldsymbol{\varphi}}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}$$

$$= -\int\limits_{\mathbb{R}^{n-1}} U_{\boldsymbol{\varphi}}(\mathbf{x}',+0)\Big(\frac{\partial}{\partial x_n}U_{\boldsymbol{\varphi}}(\mathbf{x}',+0)\Big)\,\mathrm{d}\mathbf{x}' + \int\limits_{\mathbb{R}^{n-1}} U_{\boldsymbol{\varphi}}(\mathbf{x}',-0)\Big(\frac{\partial}{\partial x_n}U_{\boldsymbol{\varphi}}(\mathbf{x}',-0)\Big)\,\mathrm{d}\mathbf{x}'$$

$$= \int\limits_\Gamma \frac{1}{2}\varphi(\mathbf{x}')\Big(\frac{\partial}{\partial x_n}(-\mathbf{W}\varphi)(\mathbf{x}',-0)\Big)\,\mathrm{d}\mathbf{x}' - \int\limits_\Gamma \frac{1}{2}\varphi(\mathbf{x}')\Big(\frac{\partial}{\partial x_n}(-\mathbf{W}\varphi)(\mathbf{x}',+0)\Big)\,\mathrm{d}\mathbf{x}'$$

$$= -c_n\int\limits_\Gamma \left\{ \frac{1}{2}\varphi(\mathbf{x}')\frac{\partial}{\partial x_n}\left(\int\limits_{\mathbb{R}^{n-1}} \frac{(\mathbf{x}'-x_n\mathbf{e}_n-\mathbf{y}')\cdot\mathbf{e}_n}{|\mathbf{x}'-x_n\mathbf{e}_n-\mathbf{y}'|^n}(\varphi(\mathbf{y}')-\varphi(\mathbf{x}'))\,\mathrm{d}\mathbf{y}' \right) \right.$$

$$\left. + \frac{1}{2}\varphi(\mathbf{x}')^2\frac{\partial}{\partial x_n}\left(\int\limits_{\mathbb{R}^{n-1}} \frac{(\mathbf{x}'-x_n\mathbf{e}_n-\mathbf{y}')\cdot\mathbf{e}_n}{|\mathbf{x}'-x_n\mathbf{e}_n-\mathbf{y}'|^n}\,\mathrm{d}\mathbf{y}' \right) \right\}\,\mathrm{d}\mathbf{x}'$$

$$+ c_n\left\{ \frac{1}{2}\varphi(\mathbf{x}')\frac{\partial}{\partial x_n}\left(\int\limits_{\mathbb{R}^{n-1}} \frac{(\mathbf{x}'+x_n\mathbf{e}_n-\mathbf{y}')\cdot\mathbf{e}_n}{|x'+x_n\mathbf{e}_n-\mathbf{y}'|^n}(\varphi(\mathbf{y}')-\varphi(\mathbf{x}'))\,\mathrm{d}\mathbf{y}' \right) \right.$$

$$\left. + \frac{1}{2}\varphi(\mathbf{x}')^2\frac{\partial}{\partial x_n}\left(\int\limits_{\mathbb{R}^{n-1}} \frac{(\mathbf{x}'+x_n\mathbf{e}_n-\mathbf{y}')\cdot\mathbf{e}_n}{|\mathbf{x}'+x_n\mathbf{e}_n-\mathbf{y}'|^n}\,\mathrm{d}\mathbf{y}' \right) \right\}\,\mathrm{d}\mathbf{x}'.$$

We can interchange differentiation and integration in this expression by means of Hadamard's finite part integral. Namely, due to

$$\int\limits_{\mathbb{R}^{n-1}} \frac{(\mathbf{x}'\mp x_n\mathbf{e}_n-\mathbf{y}')\cdot\mathbf{e}_n}{|\mathbf{x}'\mp x_n\mathbf{e}_n-\mathbf{y}'|^n}\,\mathrm{d}\mathbf{y}' = \pm\frac{1}{2}\omega_n$$

with the constant $\omega_n = \frac{1}{c_n}$, it holds

$$\int\limits_{\mathbb{R}^n\setminus\Gamma} |\nabla U_{\boldsymbol{\varphi}}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = (n-2)c_n\int\limits_\Gamma \varphi(\mathbf{x}')\int\limits_\Gamma |\mathbf{x}'-\mathbf{y}'|^{-n}\big(\varphi(\mathbf{y}')-\varphi(\mathbf{x}')\big)\,\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}'$$

$$= (n-2)\int\limits_\Gamma \varphi(\mathbf{x}')(\mathbf{D}\varphi)(\mathbf{x}')\,\mathrm{d}\mathbf{x}',$$

where

$$\mathbf{D}\boldsymbol{\varphi}(\mathbf{x}') = -\,\mathrm{p.f.}\,c_n \int\limits_\Gamma \left(\frac{\partial}{\partial x_n}\,\frac{\partial}{\partial y_n}|\mathbf{x}-\mathbf{y}|^{2-n} \right) \boldsymbol{\varphi}(\mathbf{y}')\,\mathrm{d}\mathbf{y}'$$

is Hadamard's finite part integral with $\mathbf{x} = (\mathbf{x}', 0)$, $\mathbf{y} = (\mathbf{y}', 0)$ and with the hyper-singular kernel function $k_\mathbf{D}(\mathbf{x}, \mathbf{y}) = (n-2)|\mathbf{x}' - \mathbf{y}'|^{-n}c_n$, that is

$$\int\limits_{\mathbb{R}^n\setminus\Gamma} |\nabla U_{\boldsymbol{\varphi}}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = (n-2)c_n \int\limits_\Gamma \mathrm{p.f.} \int\limits_\Gamma |\mathbf{x}'-\mathbf{y}'|^{-n}\boldsymbol{\varphi}(\mathbf{x}')\boldsymbol{\varphi}(\mathbf{y}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{y}'. \quad (2.1)$$

(For the definition of Hadamard's partie finie integral operators, see [7] and [12, Chapter 3.2].)

Hence, the finite part integral on the right of (2.1) which has the Riesz kernel $|\mathbf{x}' - \mathbf{y}'|^{-n}$ for $\mathbf{x}', \mathbf{y}' \in \Gamma \subset \mathbb{R}^{n-1}$ defines the energy of the harmonic double layer potential in $\mathbb{R}^n \setminus \Gamma$ given by the Dirichlet integral on the left of (2.1). Since the Riesz kernel is a homogeneous function of degree $-n$, it defines on $\Gamma \subset \mathbb{R}^{n-1}$ a strongly elliptic pseudodifferential operator $\mathbf{D}$ of order 1 (see [12, Section 7.1.2]).

3 Strongly singular Riesz energy on a manifold

In all that follows, without stated otherwise, we fix $n \geq 2$ and $-1 < \alpha < 1$, and write $\beta := 1 - \alpha$.

In $\mathbb{R}^n$, consider a strongly singular Riesz kernel $|\mathbf{x} - \mathbf{y}|^{\alpha-n}$ and a manifold $\Gamma := \bigcup_{\ell \in I} \Gamma_\ell$, where Γ_ℓ are finitely many compact, connected, mutually disjoint, boundaryless, $(n-1)$-dimensional, oriented C^∞-manifolds, immersed into $\mathbb{R}^n$. Then, the surface measure $\mathrm{d}s$ on Γ is well defined.

In what follows, $(\boldsymbol{\psi}, \boldsymbol{\varphi})_{L^2(\Gamma)}$ will stand for the extension of the L^2-scalar product to dualities as $\boldsymbol{\psi} \in H^{-\beta/2}(\Gamma)$ and $\boldsymbol{\varphi} \in H^{\beta/2}(\Gamma)$ and also to the applications of distributions $\boldsymbol{\psi}$ on Γ operating on $\boldsymbol{\varphi} \in C^\infty(\Gamma)$.

We call the strongly singular partie finie integral of the Riesz kernel

$$(V_\beta\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} = \int\limits_\Gamma \mathrm{p.f.} \int\limits_\Gamma |\mathbf{x}-\mathbf{y}|^{\alpha-n}\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}s_\mathbf{x}\,\mathrm{d}s_\mathbf{y} =: E_\alpha(\boldsymbol{\varphi}) \quad (3.1)$$

with respect to $|\mathbf{x} - \mathbf{y}| \geq \varepsilon_0 \to 0$, $\varepsilon_0 > 0$, operating on $\boldsymbol{\varphi} \in C^\infty(\Gamma)$, the *energy* of the Riesz potential

$$U_{\boldsymbol{\varphi}}(\mathbf{x}) = \mathrm{p.f.} \int\limits_{\mathbf{y}\in\Gamma} |\mathbf{x}-\mathbf{y}|^{\alpha-n}\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}s_\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n,$$

generated by the surface charge $\boldsymbol{\varphi}$ (see e.g. [12]). For $\boldsymbol{\varphi} \in C^\infty(\Gamma)$, the Hadamard partie finie integral operator

$$V_\beta\boldsymbol{\varphi}(\mathbf{x}) = \mathrm{p.f.} \int\limits_{\mathbf{y}\in\Gamma} |\mathbf{x}-\mathbf{y}|^{\alpha-n}\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}s_\mathbf{y}, \quad \mathbf{x} \in \Gamma,$$

which underlies (3.1), is for $0 \leq \alpha < 1$ given by

$$V_\beta \boldsymbol{\varphi}(\mathbf{x}) = \text{ p.v. } \int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| > 0} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x}) \big\} \, \mathrm{d}s_\mathbf{y} + h(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}),$$

where

$$h(\mathbf{x}) = \text{ p.f. } \lim_{\delta \to 0} \int\limits_{\mathbf{y} \in \Gamma \wedge 0 < \delta < |\mathbf{x} - \mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \, \mathrm{d}s_\mathbf{y}, \qquad (3.2)$$

and for $-1 < \alpha < 0$ by

$$V_\beta \boldsymbol{\varphi}(\mathbf{x}) = \text{ p.v. } \int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| > 0} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x}) - (\mathbf{y} - \mathbf{x}) \cdot \nabla \boldsymbol{\varphi}(\mathbf{x}) \big\} \, \mathrm{d}s_\mathbf{y}$$

$$+ h(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) \cdot \nabla \boldsymbol{\varphi}(\mathbf{x}),$$

where

$$\mathbf{h}(\mathbf{x}) = \text{ p.f. } \lim_{\delta \to 0} \int\limits_{\mathbf{y} \in \Gamma \wedge 0 < \delta < |\mathbf{x} - \mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} (\mathbf{y} - \mathbf{x}) \, \mathrm{d}s_\mathbf{y}. \qquad (3.3)$$

The abbrevation p.v. stands for the Calderon–Mikhlin principal value integral (see [16]). (See Appendix A for the explicit computation of the partie finie integrals $h(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$.)

Theorem 3.1 (see [12, Chapter 8]) *The partie finie integral operator V_β is a strongly elliptic pseudodifferential operator of order $\beta = 1 - \alpha \in (0, 2)$ on Γ. The principal symbol of this operator is given by the equivalence class associated with the homogeneous function*

$$a(\boldsymbol{\xi}) = C(n - 1, \beta) |\boldsymbol{\xi}|^\beta, \text{ where } C(n - 1, \beta) = 2^{-\beta} \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{-\beta}{2})}{\Gamma(\frac{n-\alpha}{2})} \text{ and } \xi \in \mathbb{R}^{n-1}. \quad (3.4)$$

V_β defines the linear and continuous mapping $V_\beta : H^s(\Gamma) \to H^{s-\beta}(\Gamma)$ for every $s \in \mathbb{R}$. In particular, for $s = \beta/2$, V_β maps $H^{\beta/2}(\Gamma)$ into $H^{-\beta/2}(\Gamma)$ and there exist $0 < c_0 \leq c_2$ and $c_1 \geq 0$ such that the inequalities

$$c_0 \|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)} - c_1 \|\boldsymbol{\varphi}\|^2_{L^2(\Gamma)} \leq (V_\beta \boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} \leq c_2 \|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)} \qquad (3.5)$$

are satisfied for any $\boldsymbol{\varphi} \in H^{\beta/2}(\Gamma)$.

Proof For justifying the inequalities (3.5), recall that for each of the components of the C^∞-manifolds Γ_ℓ, $\ell \in I$, immersed into $\mathbb{R}^n$, we may associate a family of finite-dimensional atlases $\mathfrak{A}_\ell$ (see [13]). Each atlas $\mathfrak{A}_\ell$ is a family of local charts $(O_{\ell r}, \mathcal{U}_{\ell r}, \mathcal{X}_{\ell r})$, where r ranges through a finite set R_ℓ. The open sets $O_{\ell r} \subset \Gamma_\ell$ define an open covering of Γ_ℓ, while $\mathcal{X}_{\ell r}$ is a C^∞-diffeomorphism of $O_{\ell r}$ onto $\mathcal{U}_{\ell r} \subset \mathbb{R}^{n-1}$. Let $\{\beta_{\ell r}\}_{r \in R_\ell}$ be a C^∞-partition of unity of Γ_ℓ which is subordinate to the atlas $\mathfrak{A}_\ell$. In addition to the partition of unity, let $\{\gamma_{\ell r}\}_{r \in R_\ell}$ be a second system of functions $\gamma_{\ell r} \in C_0^\infty(O_{\ell r})$ with the properties

$$\gamma_{\ell r}(\mathbf{x}) = 1 \text{ for all } \mathbf{x} \in \text{supp } \beta_{\ell r} \text{ and } 0 \leq \gamma_{\ell r}.$$

Thus, it holds that

$$\gamma_{\ell r}(\mathbf{x})\beta_{\ell r}(\mathbf{x}) = \beta_{\ell r}(\mathbf{x}) \text{ and } \beta_{\ell r}(\mathbf{x})\gamma_{\ell r}(\mathbf{x}) = \beta_{\ell r}(\mathbf{x}) \text{ for all } \mathbf{x} \in \Gamma_\ell.$$

With respect to the atlas $\mathfrak{A}_\ell$, let $\mathcal{X}_{\ell r \star}$ denote the corresponding pushforwards and $\mathcal{X}_{\ell r}^\star$ the pullbacks. Then $\mathcal{X}_{\ell r \star}\beta_{\ell r} \in C_0^\infty(\mathcal{U}_{\ell r})$.

Without loss of generality, the local parametric representations can always be chosen in such a way that at one point $\mathbf{x}_{\ell r}^\circ \in O_{\ell r}$ where $\beta_{\ell r}(\mathbf{x}_{\ell r}^\circ) = 1$ we have $\mathcal{X}_{\ell r}(\mathbf{x}_{\ell r}^\circ) = \mathbf{0}$ and, moreover, at this point the tangent bundle

$$\left.\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}\right|_{\mathbf{x}'=\mathbf{0}} = \left.\frac{\partial \mathcal{X}_{\ell r}^{-1}(\mathbf{x}')}{\partial \mathbf{x}'}\right|_{\mathbf{x}'=\mathbf{0}} \text{, where } \mathbf{x}' := \mathcal{X}_{\ell r}(\mathbf{x}),$$

forms a positively oriented system of $n-1$ mutually orthogonal unit vectors. This implies that the Riemannian tensor of Γ_ℓ in the local coordinates at the point $\mathbf{x}_{\ell r}^\circ$ is the unity matrix. Hence, the surface measure satisfies

$$\mathrm{d}s_\ell(\mathbf{x}) = J_{\ell r}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \text{ where } \mathbf{x}' \in \mathcal{U}_{\ell r}, \text{ and } J_{\ell r}(\mathbf{0}) = \mathbf{1}.$$

Given an atlas $\mathfrak{A}_\ell$ on Γ_ℓ, define

$$d_\ell := \min_{r \in R_\ell} \mathrm{diam}\,\mathcal{U}_{\ell r}.$$

Thus, one can choose $\delta_0 > 0$ so that for any given $0 < \delta < \delta_0$ there exists a finite-dimensional atlas $\mathfrak{A}_\ell^\delta$ satisfying all the above formulated properties and $d_\ell = \delta$. Hence, we have a whole family of finite atlases $\mathfrak{A}_\ell^\delta$ with $0 < \delta < \delta_0$ which will be under consideration. (We then shall omit the index δ in the notation.)

Note that the Jacobians $J_{\ell r}$ depend on the geometric properties of Γ_ℓ only, and $J_{\ell r}$ together with their derivatives are uniformly continuous relative to $\delta \in (0, \delta_0)$.

Corresponding to the partition of unity, the pseudodifferential operator V_β on Γ can be decomposed as

$$V_\beta = \sum_{\ell \in I} \sum_{r \in R_\ell} \beta_{\ell r} V_\beta \gamma_{\ell r} + Z_1 = \sum_{\ell \in I} \sum_{r \in R_\ell} \gamma_{\ell r} V_\beta \beta_{\ell r} + Z_2$$

and

$$V_\beta = \sum_{\ell \in I} \sum_{r \in R_\ell} \beta_{\ell r} \mathcal{X}_{\ell r}^\star V_{\ell r} \mathcal{X}_{\ell r \star} \gamma_{\ell r} + Z_1 = \sum_{\ell \in I} \sum_{r \in R_\ell} \gamma_{\ell r} \mathcal{X}_{\ell r}^\star V_{\ell r} \mathcal{X}_{\ell r \star} \beta_{\ell r} + Z_2. \quad (3.6)$$

Herein, Z_1, Z_2 are smoothing operators of order $-\infty$ in view of $|\mathbf{x} - \mathbf{y}|^{\alpha - n} \in C^\infty(\Gamma_p \times \Gamma_k)$ for $p \neq k$ and $\mathrm{supp}\,(1 - \gamma_{\ell r}) \cap \mathrm{supp}\,(\beta_{\ell r}) = \emptyset$ if $\mathbf{x}, \mathbf{y} \in \Gamma_\ell$ (see also [12, Chapter 8]). Moreover,

$$V_{\ell r}\boldsymbol{\varphi}(\mathbf{x}') = \text{ p.f. } \int_{\mathcal{U}_{\ell r}} \left|\mathcal{X}_{\ell r}^{-1}(\mathbf{x}') - \mathcal{X}_{\ell r}^{-1}(\mathbf{y}')\right|^{\alpha - n} \boldsymbol{\varphi}(\mathbf{y}')J_{\ell r}(\mathbf{y}')\,\mathrm{d}\mathbf{y}', \ \boldsymbol{\varphi} \in C_0^\infty(\mathcal{U}_{\ell r}) \quad (3.7)$$

are the localized operators in the parametric domains $\mathcal{U}_{\ell r}$, defined by the operator V_β.

The inequalities (3.5) now follow locally on each chart of the atlas $\mathfrak{A}_\ell$ for the localized operators $V_{\ell r}$ in local coordinates in $\mathcal{U}_{\ell r}$. With Martensen's surface polar

coordinates ((B.5), (B.9) in Appendix B), the kernel of $V_{\ell r}$ admits a pseudohomogeneous asymptotic expansion of the form

$$k(\mathbf{x}', \varrho) = \varrho^{-\beta-(n-1)}\left\{1 + \sum_{j \geq 2} k_{\ell r, j}(\mathbf{x}', \varrho)\right\},$$

where

$$k_{\ell r, j}(\mathbf{x}', t\varrho) = t^j k_{\ell r, j}(\mathbf{x}', \varrho) \text{ for } t > 0, \ \varrho > 0$$

since $|\mathbf{x}' - \mathbf{x}_0'|^2$ satisfies the expansion (B9). Correspondingly, the symbol $a(\mathbf{x}', \boldsymbol{\xi}')$ of $V_{\ell r}$ has the asymptotic expansion

$$a(\mathbf{x}', \boldsymbol{\xi}') = |\boldsymbol{\xi}'|^\beta\left\{1 + \sum_{j \geq 2} a^0_{-j}(\mathbf{x}', \boldsymbol{\xi}')\right\},$$

where $a^0_{-j}(\mathbf{x}', t\boldsymbol{\xi}') = t^{-j} a^0_{-j}(\mathbf{x}', \boldsymbol{\xi}')$ with $t > 0$, $\boldsymbol{\xi}' \neq \mathbf{0}$.

Then, Fourier transform and Parseval's theorem yield

$$(V_{\ell r}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\mathcal{U}_{\ell r})} = \int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}'|^\beta |\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}' + (A^{\beta-2}_{\ell r}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\mathcal{U}_{\ell r})} + (R_{\ell r}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\mathcal{U}_{\ell r})},$$

(3.8)

where $A^{\beta-2}_{\ell r}$ is a pseudodifferential operator of order $\beta - 2$:

$$(A^{\beta-2}_{\ell r}\boldsymbol{\varphi})(\mathbf{x}') = \int_{\mathbb{R}^{n-1}} e^{i\mathbf{x}'\cdot\boldsymbol{\xi}'} \boldsymbol{\Psi}_{\ell r}(\boldsymbol{\xi}')|\boldsymbol{\xi}'|^\beta \sum_{j \geq 2} a^0_{-j}(\mathbf{x}', \boldsymbol{\xi}')\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi}') \, d\boldsymbol{\xi}'.$$

Here, the function $\boldsymbol{\Psi}_{\ell r} \in C^\infty(\mathbb{R}^{n-1})$ is arbitrary but fixed such that $0 \leq \boldsymbol{\Psi}_{\ell r}(\boldsymbol{\xi}') \leq 1$ and

$$\boldsymbol{\Psi}_{\ell r}(\boldsymbol{\xi}') = \mathbf{0} \text{ for } |\boldsymbol{\xi}'| \leq \frac{1}{2} \text{ and } \boldsymbol{\Psi}_{\ell r}(\boldsymbol{\xi}') = 1 \text{ for } |\boldsymbol{\xi}'| \geq 1.$$

The remainder operator $R_{\ell r}\boldsymbol{\varphi}(\mathbf{x}) = \int_\Gamma R_{\ell r}(\mathbf{x}, \mathbf{y})\boldsymbol{\varphi}(\mathbf{y}) \, ds_{\mathbf{y}}$ is a smoothing operator with the smooth kernel function $R_{\ell r} \in C^\infty(\Gamma \times \Gamma)$. Hence, there exists a constant $c_{\ell r} > 0$ such that

$$(V_{\ell r}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\mathcal{U}_{\ell r})} \leq c_{\ell r} \int_{\mathbb{R}^{n-1}} (1 + |\boldsymbol{\xi}'|)^\beta |\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}' = c\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\mathcal{U}_{\ell r})}$$

which implies with some constant $c_2 > 0$ that

$$(V_\beta\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} \leq c_2\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)}.$$

(3.9)

Vice versa, from

$$(V_\beta\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\mathcal{U}_{\ell r})} \geq \int_{\mathbb{R}^{n-1}} (1 + |\boldsymbol{\xi}'|)^\beta |\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}' - c'_{\ell r} \int_{\mathbb{R}^{n-1}} (1 + |\boldsymbol{\xi}'|)^{\beta-2} |\widehat{\boldsymbol{\varphi}}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}',$$

after summation over $\ell \in I$, we obtain the Gårding inequality

$$(V_\beta\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} \geq c_0\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)} - c_1\|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)}.$$

The embedding $H^{\beta/2}(\Gamma) \hookrightarrow L_2(\Gamma)$ is compact since $0 < \beta < 2$. For $\beta = 1 \in \mathbb{N}_0$ the Tricomi condition needs to be satisfied for V_β being a pseudodifferential operator and reads here as

$$\int\limits_{|\Theta|=1} \Theta^{\boldsymbol{\alpha}'} \, \mathrm{d}\omega(\Theta) = 0 \text{ for } |\boldsymbol{\alpha}'| = 1$$

(see [12, Theorem 7.1.7]). In addition, we have $a_{-1}^0(\mathbf{x}', \boldsymbol{\xi}') = 0$. So, V_β is a classical pseudodifferential operator on Γ of order $\beta \in (0,2)$. $\qquad\square$

Now, we introduce a set of so-called *admissible measures* or charges located on Γ. Recall that $\Gamma = \bigcup_{\ell \in I} \Gamma_\ell$, where the finitely many Γ_ℓ are compact, nonintersecting, boundaryless, connected, $(n-1)$-dimensional, orientable C^∞-manifolds, immersed into $\mathbb{R}^n$. With each Γ_ℓ we associate a prescribed sign $\alpha_\ell \in \{-1, 1\}$ where $\alpha_\ell = +1$ for $\ell \in I^+$ and $\alpha_\ell = -1$ for $\ell \in I^-$. Then $I = I^+ \cup I^-$ and $I^+ \cap I^- = \emptyset$, $I^- = \emptyset$ is admitted. Let $\mathfrak{M}(\Gamma)$ denote the σ-algebra of signed Borel measures $\boldsymbol{\nu}$ on Γ equipped with the topology of pointwise convergence on $C(\Gamma)$, the class of all real-valued continuous functions on Γ.

Next, consider the manifold Γ being loaded by charges of the form

$$\boldsymbol{\mu} = \sum_{\ell \in I} \alpha_\ell \mu^\ell \qquad (3.10)$$

where, for every $\ell \in I$, μ^ℓ is a nonnegative Borel measure on Γ_ℓ. The convex cone of all signed measures $\boldsymbol{\mu}$ of the form (3.10) will be denoted by $\mathfrak{M}^+(\Gamma)$.

The following theorem deals with absolutely continuous $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$, i.e. $\mathrm{d}\boldsymbol{\Sigma} = \boldsymbol{\sigma} \, \mathrm{d}s$, with densities $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$,

$$\mathcal{K}^{\beta/2}(\Gamma) := \left\{ \boldsymbol{\sigma} = \sum_{\ell \in I} \alpha_\ell \sigma^\ell, \text{ where } \sigma^\ell \in H^{\beta/2}(\Gamma_\ell) \text{ and } \sigma^\ell \geq 0 \right\}.$$

For brevity, we shall often identify an absolutely continuous Borel measure $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ with $\boldsymbol{\sigma}$, its density. Likewise, the cone of all $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ with $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$ will be denoted by $\mathcal{K}^{\beta/2}(\Gamma)$, provided that this will not cause any misunderstanding. Similar to as it has been done in (3.1), we define *the Riesz energy* of $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$ by

$$E_\alpha(\boldsymbol{\Sigma}) := (V_\beta \boldsymbol{\sigma}, \boldsymbol{\sigma})_{L_2(\Gamma)} = \int\limits_\Gamma \text{ p.f. } \int\limits_\Gamma |\mathbf{x} - \mathbf{y}|^{\alpha-n} \, \mathrm{d}\boldsymbol{\Sigma}(\mathbf{x}) \, \mathrm{d}\boldsymbol{\Sigma}(\mathbf{y}).$$

Theorem 3.2 *For any $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$, the Riesz energy is finite and satisfies the inequalities*

$$c_0' \|\boldsymbol{\sigma}\|_{H^{\beta/2}(\Gamma)}^2 \leq E_\alpha(\boldsymbol{\Sigma}) := (V_\beta \boldsymbol{\sigma}, \boldsymbol{\sigma})_{L_2(\Gamma)} \leq c_1' \|\boldsymbol{\sigma}\|_{H^{\beta/2}(\Gamma)}^2, \qquad (3.11)$$

the constants c_0' and c_1' being strictly positive and independent of $\boldsymbol{\Sigma}$. This means that V_β is continuously invertible on $\mathcal{K}^{\beta/2}(\Gamma)$.

Proof Write

$$\mathcal{K}^\infty(\Gamma) := \left\{ \boldsymbol{\sigma} = \sum_{\ell \in I} \alpha_\ell \sigma^\ell, \text{ where } \sigma^\ell \in C^\infty(\Gamma_\ell) \text{ and } \sigma^\ell \geq 0 \text{ on } \Gamma_\ell \right\}.$$

Let $\psi(\varrho)$ be a $C_0^\infty(\mathbb{R})$-function with $\psi(\varrho) = 1$ for $0 \leq \varrho \leq \delta$, $0 \leq \psi(\varrho) \leq 1$ and $\psi(\varrho) = 0$ for $\varrho \geq 2\delta > 0$. Having the atlas $\mathfrak{A}_\ell^\delta$ at hand, we use the decomposition (3.6) and the representation in local coordinates (3.7) to arrive at

$$(V_{\ell r}\boldsymbol{\sigma})(\mathbf{x}') = \text{ p.f. } \int_{\mathbb{R}^{n-1}} |\mathbf{x}' - \mathbf{y}'|^{-\beta-(n-1)} \psi_\star(|\mathbf{x}' - \mathbf{y}'|) \boldsymbol{\sigma}_\star(\mathbf{y}') J_{\ell r}(\mathbf{y}') \, d\mathbf{y}'$$
$$+ \int_{|\mathbf{x}'-\mathbf{y}'|\geq\delta} |\mathbf{x}' - \mathbf{y}'|^{-\beta-(n-1)} \big(1 - \psi_\star(|\mathbf{x}' - \mathbf{y}'|)\big) \boldsymbol{\sigma}_\star(\mathbf{y}') J_{\ell r}(\mathbf{y}') \, d\mathbf{y}',$$

where $\psi_\star$ and $\boldsymbol{\sigma}_\star$ are the pushforwards of ψ and $\boldsymbol{\sigma}$, respectively. Summation gives

$$(\boldsymbol{\sigma}, V_\beta\boldsymbol{\sigma})_{L_2(\Gamma)} = \sum_{\ell,i \in I} \sum_{r \in R_\ell} \sum_{q \in R_i} \int_{\mathbb{R}^{n-1}} \text{p.f.} \int_{\mathbb{R}^{n-1}} |\mathbf{x}' - \mathbf{y}'|^{-\beta-(n-1)} \psi_\star(|\mathbf{x}' - \mathbf{y}'|)$$
$$\cdot \boldsymbol{\sigma}_\star(\mathbf{y}') J_{\ell r}(\mathbf{y}') \, d\mathbf{y}' \boldsymbol{\sigma}_\star(\mathbf{x}') J_{iq}(\mathbf{x}') \, d\mathbf{x}'$$
$$+ \iint_{|\mathbf{x}-\mathbf{y}|\geq\delta} |\mathbf{x} - \mathbf{y}|^{-\beta-(n-1)} \big(1 - \psi(|\mathbf{x} - \mathbf{y}|)\big) \boldsymbol{\sigma}(\mathbf{y}) \, ds_{\mathbf{y}} \boldsymbol{\sigma}(\mathbf{x}) \, ds_{\mathbf{x}}.$$

Since the localized operators $V_{\ell r}$ are all pseudodifferential operators of order β with positive definite principal symbol (3.4), hence strongly elliptic, one finds with Fourier transform and Parseval's theorem the estimate

$$\sum_{\ell,i \in I} \sum_{r \in R_\ell} \sum_{q \in R_i} \int_{\mathbb{R}^{n-1}} \text{p.f.} \int_{\mathbb{R}^{n-1}} |\mathbf{x}' - \mathbf{y}'|^{-\beta-(n-1)} \psi_\star(|\mathbf{x}' - \mathbf{y}'|)$$
$$\cdot \boldsymbol{\sigma}_\star(\mathbf{y}') J_{\ell r}(\mathbf{y}') \, d\mathbf{y}' \boldsymbol{\sigma}_\star(\mathbf{x}') J_{iq}(\mathbf{x}') \, d\mathbf{x}'$$
$$\geq c_0' \|\boldsymbol{\sigma}\|_{H^{\beta/2}(\Gamma)}^2 - c'' \frac{1}{2-\beta} \delta^{2-\beta} \|\boldsymbol{\sigma}\|_{L_2(\Gamma)}^2 \geq c_0''' \|\boldsymbol{\sigma}\|_{H^{\beta/2}(\Gamma)}^2$$

with $c_0''' > 0$ since $H^{\beta/2}(\Gamma) \hookrightarrow L_2(\Gamma)$ compactly and if $\delta > 0$ is chosen sufficiently small. The remaining quadratic form

$$\iint_{|\mathbf{x}-\mathbf{y}|\geq\delta} |\mathbf{x} - \mathbf{y}|^{-\beta-(n-1)} \big(1 - \psi(|\mathbf{x} - \mathbf{y}|)\big) \boldsymbol{\sigma}(\mathbf{y}) \, ds_{\mathbf{y}} \boldsymbol{\sigma}(\mathbf{x}) \, ds_{\mathbf{x}}$$

has a strictly positive C^∞-kernel. Hence the left inequality in (3.11) is satisfied for $\boldsymbol{\sigma} \in \mathcal{K}^\infty(\Gamma)$, $\boldsymbol{\sigma} \not\equiv \mathbf{0}$. Since $\mathcal{K}^\infty(\Gamma)$ is dense in $\mathcal{K}^{\beta/2}(\Gamma)$, the left inequality in (3.11) also holds for $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$, $\boldsymbol{\sigma} \not\equiv \mathbf{0}$ by completion. The right inequality in (3.11) was already shown in (3.9) for $\boldsymbol{\sigma}$ in place of $\boldsymbol{\varphi}$. This completes the proof. $\square$

We next proceed by defining the notion of the Riesz energy for arbitrary (not necessarily absolutely continuous) measures $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$. Since V_β is a classical pseudodifferential operator on Γ, it maps the distribution given by the Radon measure $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ to $V_\beta \boldsymbol{\Sigma}$ which is a distribution again, and this linear mapping

is continuous in the weak topology of distributions (see Theorem II.1.5 in [19]). Therefore, the action of the measure $V_\beta \boldsymbol{\Sigma}$ on functions $\boldsymbol{\varphi} \in C^\infty(\Gamma)$ is well defined and

$$(V_\beta \boldsymbol{\Sigma}, \boldsymbol{\varphi})_{L_2(\Gamma)} = (\boldsymbol{\Sigma}, V_\beta \boldsymbol{\varphi})_{L_2(\Gamma)},$$

since V_β is symmetric.

Let $\mathcal{E}_\alpha^+(\Gamma)$ consist of all $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ which the property

$$\sup_{\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)} \leq 1} \left| (V_\beta \boldsymbol{\Sigma}, \boldsymbol{\varphi})_{L_2(\Gamma)} \right| < \infty.$$

Hence, for $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^+(\Gamma)$, we can identify $V_\beta \boldsymbol{\Sigma}$ with an associated element $\boldsymbol{\psi} \in H^{-\beta/2}(\Gamma)$ satisfying $\boldsymbol{\psi} \, \mathrm{d}s = \mathrm{d}V_\beta \boldsymbol{\Sigma}$ and

$$\|\boldsymbol{\psi}\|_{H^{-\beta/2}(\Gamma)} = \sup_{\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)} \leq 1} \left| (V_\beta \boldsymbol{\Sigma}, \boldsymbol{\varphi})_{L_2(\Gamma)} \right|.$$

This leads us to the following theorem.

Theorem 3.3 (see also [8, Theorem 3]) *For any $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^+(\Gamma)$ there exists a unique element $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$ such that $\mathrm{d}\boldsymbol{\Sigma} = \boldsymbol{\sigma} \, \mathrm{d}s$ and*

$$\boldsymbol{\Sigma}(\boldsymbol{\varphi}) = \int_\Gamma \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{\Sigma} = (\boldsymbol{\varphi}, \boldsymbol{\sigma})_{L_2(\Gamma)} \text{ for all } \boldsymbol{\varphi} \in C^\infty(\Gamma). \tag{3.12}$$

Moreover, $\mathcal{E}_\alpha^+(\Gamma) = \mathcal{K}^{\beta/2}(\Gamma)$, and the Riesz energy $E_\alpha(\boldsymbol{\Sigma})$ of any $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^+(\Gamma)$ is equivalent to the $H^{\beta/2}(\Gamma)$-norm of the corresponding $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$ in the sense of (3.11).

Proof Choose an arbitrary $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^+(\Gamma)$. As has been observed just above, $V_\beta \boldsymbol{\Sigma} \in H^{-\beta/2}(\Gamma)$ is a linear functional on $C^\infty(\Gamma)$, and it is bounded on $H^{\beta/2}(\Gamma)$ because of

$$|V_\beta \boldsymbol{\Sigma}(\boldsymbol{\varphi})| = \left| (V_\beta \boldsymbol{\Sigma}, \boldsymbol{\varphi})_{L_2(\Gamma)} \right| \leq c \|V_\beta \boldsymbol{\Sigma}\|_{H^{-\beta/2}(\Gamma)} \|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}.$$

Since $C^\infty(\Gamma)$ is dense in $H^{\beta/2}(\Gamma)$, according to the Fischer–Riesz lemma on the representation of bounded linear functionals there exists a unique $\boldsymbol{\sigma}_0 \in H^{\beta/2}(\Gamma)$ with

$$\int_\Gamma V_\beta \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{\Sigma} = \int_\Gamma \boldsymbol{\zeta} \, \mathrm{d}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\zeta}) = \int_\Gamma \boldsymbol{\sigma}_0 \boldsymbol{\zeta} \, \mathrm{d}s = (\boldsymbol{\sigma}_0, \boldsymbol{\zeta})_{L_2(\Gamma)},$$

where $\boldsymbol{\zeta} := V_\beta \boldsymbol{\varphi}$. If $\boldsymbol{\varphi}$ traces $C^\infty(\Gamma)$, so does $\boldsymbol{\zeta}$; hence, (3.12) holds for $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 \in H^{\beta/2}(\Gamma)$ by replacing $\boldsymbol{\zeta}$ by $\boldsymbol{\varphi}$. Since $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$, we actually have $\boldsymbol{\sigma}_0 \in \mathcal{K}^{\beta/2}(\Gamma)$, and the inclusion $\mathcal{E}_\alpha^+(\Gamma) \subseteq \mathcal{K}^{\beta/2}(\Gamma)$ follows.

Now, let $\boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$. Then $\boldsymbol{\psi} = V_\beta \boldsymbol{\sigma} \in H^{-\beta/2}(\Gamma)$ and

$$\|\boldsymbol{\psi}\|_{H^{-\beta/2}(\Gamma)} = \|V_\beta \boldsymbol{\sigma}\|_{H^{-\beta/2}(\Gamma)}$$
$$= \sup_{\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)} \leqslant 1} \left| (V_\beta \boldsymbol{\sigma}, \boldsymbol{\varphi})_{L_2(\Gamma)} \right| \leqslant c \|\boldsymbol{\sigma}\|_{H^{\beta/2}(\Gamma)} < \infty$$

due to the duality $H^{-\beta/2}(\Gamma) \times H^{\beta/2}(\Gamma)$ of $(\cdot, \cdot)_{L_2(\Gamma)}$, V_β being a pseudodifferential operator on Γ. Hence, $\mathcal{K}^{\beta/2}(\Gamma) \subseteq \mathcal{E}_\alpha^+(\Gamma)$, which completes the proof. $\square$

Although we have shown that the distributions in $\mathcal{K}^{\beta/2}(\Gamma)$ all have finite Riesz energy $E_\alpha(\boldsymbol{\mu})$, it is not clear yet whether there are no other measures in $\mathfrak{M}^+(\Gamma)$ whose Riesz energy is finite. To elaborate on this problem, we employ an idea by J. Deny [5]. A measure on Γ can be considered as a distribution on Γ and, hence, can be Fourier transformed. In connection with the localization of V_β on one chart of the atlas on Γ, we have relation (3.8) where the pseudodifferential operator $V_{\ell r}$ is defined via Fourier transform.

If $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ is given, then it becomes via the pushforward $\mathcal{X}_{\ell r\star}$ the localized distribution $\boldsymbol{\Sigma}_{\ell r} := \mathcal{X}_{\ell r\star}\beta_{\ell r}\boldsymbol{\Sigma}$ with compact support in $\mathcal{U}_{\ell r} \subset \mathbb{R}^{n-1}$ (see e.g. [8, Lemma 5]), which can be Fourier transformed to $\widehat{\boldsymbol{\Sigma}}_{\ell r}(\boldsymbol{\xi}')$ on $\mathbb{R}^{n-1}$. The measures in $\mathfrak{M}^+(\Gamma)$ for which

$$\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}'|^\beta |\widehat{\boldsymbol{\Sigma}}_{\ell r}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}' < \infty \tag{3.13}$$

are precisely all those having finite Riesz energy (cf. (3.8); observe that the first summand on the left-hand side of (3.8) is the dominant one). Let $\mathcal{E}_\alpha^\star(\Gamma)$ consist of all $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ satisfying (3.13).

Theorem 3.4 *There holds*

$$\mathcal{E}_\alpha^\star(\Gamma) = \mathcal{K}^{\beta/2}(\Gamma).$$

Proof Let $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$. Then, by Theorem 3.2,

$$(V_\beta\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L_2(\Gamma)} = E_\alpha(\boldsymbol{\sigma}) < \infty.$$

With (3.8) and Parseval's identity, together with $\boldsymbol{\Sigma}_{\ell r} \in H^{\beta/2}(\mathcal{U}_{\ell r})$, we obtain further

$$\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}'|^\beta |\widehat{\boldsymbol{\Sigma}}_{\ell r}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}'$$

$$\leq \left| (V_{\ell r}\boldsymbol{\Sigma}_{\ell r}, \boldsymbol{\Sigma}_{\ell r})_{L^2(\mathcal{U}_{\ell r})} \right| + \left| (A_{\ell r}^{\beta-2}\boldsymbol{\Sigma}_{\ell r}, \boldsymbol{\Sigma}_{\ell r})_{L^2(\mathcal{U}_{\ell r})} \right| + \left| (R_{\ell r}\boldsymbol{\Sigma}_{\ell r}, \boldsymbol{\Sigma}_{\ell r})_{L^2(\mathcal{U}_{\ell r})} \right|$$

$$\leq c\|\boldsymbol{\Sigma}_{\ell r}\|^2_{H^{\beta/2}(\mathcal{U}_{\ell r})} + c'\|\boldsymbol{\Sigma}_{\ell r}\|_{H^{\beta/2+2-\beta}(\mathcal{U}_{\ell r})}\|\boldsymbol{\Sigma}_{\ell r}\|_{H^{\beta/2}(\mathcal{U}_{\ell r})}$$

$$\leq c''\|\boldsymbol{\Sigma}_{\ell r}\|^2_{H^{\beta/2}(\mathcal{U}_{\ell r})} < \infty$$

since $2 - \beta/2 \geq \beta/2$. Consequently, it holds $\boldsymbol{\sigma} = \boldsymbol{\Sigma} \in \mathcal{E}_\alpha^\star(\Gamma)$ and thus $\mathcal{K}^{\beta/2}(\Gamma) \subseteq \mathcal{E}_\alpha^\star(\Gamma)$.

Next, suppose $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^\star(\Gamma)$; then by localization inequality (3.13) holds. Since $|\boldsymbol{\xi}'|^\beta(1 + |\boldsymbol{\xi}'|^2)^{-\beta/2} \leq 1$ we find

$$\|V_{\ell r}\boldsymbol{\Sigma}_{\ell r}\|^2_{H^{-\beta/2}(\mathcal{U}_{\ell r})} = \int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}'|^{2\beta}(1 + |\boldsymbol{\xi}'|^2)^{-\beta/2}|\widehat{\boldsymbol{\Sigma}}_{\ell r}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}'$$

$$\leq \int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}'|^\beta |\widehat{\boldsymbol{\Sigma}}_{\ell r}(\boldsymbol{\xi}')|^2 \, d\boldsymbol{\xi}' < \infty.$$

With pullback to Γ this implies that

$$\|V_\beta\boldsymbol{\Sigma}\|_{H^{-\beta/2}(\Gamma)} < \infty.$$

Application of Theorem 3.3 then gives $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^+(\Gamma) = \mathcal{K}^{\beta/2}(\Gamma)$, which in view of the arbitrary choice of $\boldsymbol{\Sigma} \in \mathcal{E}_\alpha^\star(\Gamma)$ finally yields $\mathcal{K}^{\beta/2}(\Gamma) = \mathcal{E}_\alpha^\star(\Gamma)$. $\square$

Theorem 3.5 (see also [8, Theorem 4]) *Let $\boldsymbol{\Sigma} \in \mathfrak{M}^+(\Gamma)$ with $E_\alpha(\boldsymbol{\Sigma}) < \infty$. Then there exists a sequence of absolutely continuous measures $\boldsymbol{\Sigma}_k \in \mathcal{K}^{\beta/2}(\Gamma)$, where $\mathrm{d}\boldsymbol{\Sigma}_k = \sum_{i \in I} \alpha_i \varphi_k^i \, \mathrm{d}s$ with $\varphi_k^i \in C(\Gamma_i) \cap H^{\beta/2}(\Gamma_i)$ and $\varphi_k^i(\mathbf{x}) \geqslant 0$ for $\mathbf{x} \in \Gamma_i$, such that $\{\boldsymbol{\Sigma}_k\}_{k \in \mathbb{N}}$ converges weakly and strongly in the Hilbert space $H^{\beta/2}(\Gamma)$ to $\boldsymbol{\Sigma}$, i.e.,*

$$\boldsymbol{\Sigma}_k(\boldsymbol{\varphi}) \stackrel{k \to \infty}{\longrightarrow} \boldsymbol{\Sigma}(\boldsymbol{\varphi}) \text{ for all } \boldsymbol{\varphi} \in C(\Gamma) \text{ and } \lim_{k \to \infty} \|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_k\|_{H^{\beta/2}(\Gamma)} = 0.$$

Proof For $\boldsymbol{\Sigma}$ we have $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \in \mathcal{K}^{\beta/2}(\Gamma)$ due to Theorem 3.3. Hence, since $C^\infty(\Gamma) \subset H^{\beta/2}(\Gamma)$ densely, there exists a sequence $\boldsymbol{\sigma}_k = \sum_{i \in I} \alpha_i \sigma_k^i \in C^\infty(\Gamma)$ with $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{H^{\beta/2}(\Gamma)} < \frac{1}{k}$ for all $k \in \mathbb{N}$. We define $\boldsymbol{\Sigma}_k \in \mathfrak{M}^+(\Gamma)$ by

$$\mathrm{d}\boldsymbol{\Sigma}_k = \overline{\boldsymbol{\sigma}}_k \, \mathrm{d}s \quad \text{with} \quad \overline{\boldsymbol{\sigma}}_k = \sum_{i \in I} \alpha_i \overline{\sigma}_k^i,$$

where $\overline{\sigma}_k^i(\mathbf{x}) := \max\{0, \sigma_k^i(\mathbf{x})\}$. Then, $\overline{\sigma}_k^i \in C(\Gamma_i) \cap H^{\beta/2}(\Gamma_i)$ since $\overline{\sigma}_k^i$ is piecewise smooth and

$$\begin{aligned}
\|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_k\|_{H^{\beta/2}(\Gamma)} &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k - \overline{\boldsymbol{\sigma}}_k\|_{H^{\beta/2}(\Gamma)} \\
&\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{H^{\beta/2}(\Gamma)} + \|\boldsymbol{\sigma}_k - \overline{\boldsymbol{\sigma}}_k\|_{H^{\beta/2}(\Gamma)} \\
&= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{H^{\beta/2}(\Gamma)} + \sum_{i \in I} \|\underline{\sigma}_k^i\|_{H^{\beta/2}(\Gamma_i)}.
\end{aligned}$$

Herein, it holds $\underline{\sigma}_k^i(\mathbf{x}) = \min\{0, \sigma_k^i(\mathbf{x})\} \leq 0$ for all $\mathbf{x} \in \Gamma_i$, particularly $\underline{\sigma}_k^i \in C(\Gamma_i) \cap H^{\beta/2}(\Gamma_i)$. From $\sigma^i \geq 0$, it immediately follows that

$$\|\underline{\sigma}_k^i\|_{H^{\beta/2}(\Gamma_i)} \leq \|\sigma^i - \sigma_k^i\|_{H^{\beta/2}(\Gamma_i)}$$

for all $i \in I$ and therefore

$$\|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_k\|_{H^{\beta/2}(\Gamma)} \leq 2\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{H^{\beta/2}(\Gamma)} \leq \frac{2}{k} \stackrel{k \to \infty}{\longrightarrow} 0$$

as desired. $\qquad \square$

4 The Gauss problem

The Gauss variational problem is the problem of minimizing the Riesz energy for particularly signed Borel measures on the given $(n-1)$-dimensional manifold $\Gamma \subset \mathbb{R}^n$, in the presence of an external field. Let $\mathbf{g}$ be a given continuous, positive function on Γ and $\mathbf{a} = (a_i)_{i \in I}$ a given vector with $a_i > 0$, $i \in I$. Then, the set of *admissible charges* for the Gauss problem is defined as

$$\mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g}) := \left\{ \boldsymbol{\mu} \in \mathcal{K}^{\beta/2}(\Gamma) \; : \; \int_{\Gamma_i} g_i \, \mathrm{d}\mu^i = a_i \text{ for all } i \in I \right\}$$

where we set $g_i := \mathbf{g}_{|\Gamma_i}$. Note that the set $\mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$ is convex and weakly and strongly closed in $\mathcal{K}^{\beta/2}(\Gamma)$.

The *Gauss minimal energy problem* reads as follows (see [18] and [14]): To given $\mathbf{a} \in \mathbb{R}_+^{|I|}$, $\mathbf{f} \in C(\Gamma)$ and $\mathbf{g} \in C(\Gamma)$ such that $\mathbf{g} > \mathbf{0}$, find the Borel measure $\boldsymbol{\mu}_0 \in \mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$ which is the minimizer of

$$\inf_{\boldsymbol{\mu} \in \mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})} \mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu}) = \mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu}_0) =: \mathbb{G}_{\mathbf{f}}(\Gamma, \mathbf{a}, \mathbf{g}) \tag{4.1}$$

where the Gauss functional is given by

$$\mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu}) := E_\alpha(\boldsymbol{\mu}) - 2 \int_\Gamma \mathbf{f} \, \mathrm{d}\boldsymbol{\mu}.$$

Since $\mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu})$ is on $\mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$ strictly convex and weakly and strongly continuous, the Gauss problem admits a unique solution $\boldsymbol{\mu}_0 \in \mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$.

Based on Theorem 3.2, the minimization problem (4.1) can also be formulated as a variational problem in $H^{\beta/2}(\Gamma)$. Namely, minimize the functional

$$\mathbb{V}_{\mathbf{f}}(\boldsymbol{\varphi}) := (\boldsymbol{\varphi}, V_\beta \boldsymbol{\varphi})_{L_2(\Gamma)} - 2(\mathbf{f}, \boldsymbol{\varphi})_{L_2(\Gamma)}, \quad \boldsymbol{\varphi} \in H^{\beta/2}(\Gamma), \tag{4.2}$$

over the affine cone

$$\mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g}) := \left\{ \boldsymbol{\varphi} = \sum_{i \in I} \alpha_i \varphi^i \text{ where } \varphi^i \in H^{\beta/2}(\Gamma_i), \right.$$

$$\left. \varphi^i \geq 0 \text{ and } \int_{\Gamma_i} g_i \varphi^i \, \mathrm{d}s = a_i > 0 \text{ for all } i \in I \right\} \subset \mathcal{K}^{\beta/2}(\Gamma) \subset H^{\beta/2}(\Gamma)$$

where $\mathbf{f} \in C(\Gamma)$, $\mathbf{g} > \mathbf{0}$, $\mathbf{g} \in C(\Gamma)$ and $\mathbf{a} \in \mathbb{R}_+^{|I|}$ are given. This minimization problem will be called the *dual Gauss problem*.

Theorem 4.1 *To the unique solution $\boldsymbol{\mu}_0 \in \mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$ of the Gauss problem (4.1), there corresponds a unique element $\boldsymbol{\varphi}_0 \in \mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g}) \subset H^{\beta/2}(\Gamma)$ with the properties*

$$\boldsymbol{\mu}_0(\boldsymbol{\varphi}) = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi})_{L_2(\Gamma)} \text{ for all } \boldsymbol{\varphi} \in C^\infty(\Gamma)$$

and

$$\mathbb{V}_{\mathbf{f}}(\boldsymbol{\varphi}_0) = \mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu}_0) = \mathbb{G}_{\mathbf{f}}(\Gamma, \mathbf{a}, \mathbf{g}).$$

The element $\boldsymbol{\varphi}_0$ is the minimizer of the functional $\mathbb{V}_{\mathbf{f}}$ over $\mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$, i.e.,

$$\mathbb{V}_{\mathbf{f}}(\boldsymbol{\varphi}_0) = \min_{\boldsymbol{\varphi} \in \mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})} \mathbb{V}_{\mathbf{f}}(\boldsymbol{\varphi}) =: \mathbb{V}_{\mathbf{f}}(\Gamma, \mathbf{a}, \mathbf{g}). \tag{4.3}$$

Proof By Theorems 3.2 and 3.3, to any Borel measure $\boldsymbol{\mu} = \sum_{i \in I} \alpha_i \mu^i \in \mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$, there corresponds a unique element $\boldsymbol{\sigma}_\mu = \sum_{i \in I} \alpha_i \sigma_\mu^i \in \mathcal{K}^{\beta/2}(\Gamma)$ satisfying both (3.11) and (3.12). Moreover, since $C^\infty(\Gamma)$ is dense in $C(\Gamma)$, from (3.12) we get

$$(\sigma_\mu^i, g_i)_{L_2(\Gamma_i)} = \mu^i(g_i) = a_i \text{ for all } i \in I.$$

Hence, $\boldsymbol{\sigma}_\mu \in \mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$.

Applying (3.11), for these $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}_\mu$ we also obtain

$$\mathbb{V}_{\mathbf{f}}(\boldsymbol{\sigma}_\mu) = (V_\beta \boldsymbol{\sigma}_\mu, \boldsymbol{\sigma}_\mu)_{L^2(\Gamma)} - 2(\boldsymbol{\sigma}_\mu, \mathbf{f})_{L^2(\Gamma)} = E_\alpha(\boldsymbol{\mu}) - 2\boldsymbol{\mu}(\mathbf{f}) = \mathbb{G}_{\mathbf{f}}(\boldsymbol{\mu}). \tag{4.4}$$

Thus, the correspondence $\boldsymbol{\mu} \mapsto \boldsymbol{\sigma}_\mu$ between $\mathcal{E}_\alpha(\Gamma, \mathbf{a}, \mathbf{g})$ and $\mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$ is one-to-one and satisfies (4.4), which immediately implies

$$\mathbb{G}_{\mathbf{f}}(\Gamma, \mathbf{a}, \mathbf{g}) = \mathbb{V}_{\mathbf{f}}(\Gamma, \mathbf{a}, \mathbf{g}).$$

If now $\boldsymbol{\mu}_0$ is the (unique) solution of the Gauss problem (4.1), then $\boldsymbol{\varphi}_0$, the image of $\boldsymbol{\mu}_0$ under this correspondence, is the unique solution of the minimizing problem (4.3), and vice versa. $\square$

5 The particular case $\alpha = 0$

In the following, we will focus on the particular situation $\alpha = 0$ from the potential theoretic point of view. The double layer energy E_0 of a function $\boldsymbol{\varphi} \in H^{\beta/2}(\Gamma)$, $\beta = 1 - \alpha = 1$, which is harmonic in Ω (see [12, Equation (1.2.17)]) is given by

$$E_0(\boldsymbol{\varphi}) := (\mathbf{D}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)}$$

with the hypersingular integral operator $\mathbf{D}$:

$$\mathbf{D}\boldsymbol{\varphi}(\mathbf{x}) := \text{ p.f.} \int\limits_{\Gamma \setminus \{\mathbf{x}\}} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}) \, ds_{\mathbf{y}},$$

$$k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) = k_{\mathbf{D}}(\mathbf{y}, \mathbf{x}) = c_n \left\{ \frac{\mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} + n \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}(\mathbf{y})(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^{n+2}} \right\}.$$

The Hadamard partie finie integral operator is given by the finite part with respect to $0 < \delta \to 0$ of

$$\int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| > \delta} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}) \, ds_{\mathbf{y}}$$

$$= \int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| > \delta} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\} \, ds_{\mathbf{y}} + \int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| > \delta} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \, ds_{\mathbf{y}} \boldsymbol{\varphi}(\mathbf{x})$$

if $\boldsymbol{\varphi} \in C^\infty(\Gamma)$. The limit

$$\lim_{\delta \to 0} \int\limits_{0 < \delta < |\mathbf{x} - \mathbf{y}|} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\} \, ds_{\mathbf{y}} = \text{ p.v.} \int\limits_{\mathbf{y} \in \Gamma \setminus \{\mathbf{x}\}} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\} \, ds_{\mathbf{y}}$$

exists (as a Cauchy principal value integral), whereas, from $\int_{|\mathbf{x} - \mathbf{y}| > \delta} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \, ds_{\mathbf{y}}$, we have to take the *finite part*

$$\lim_{\delta \to 0} \text{ p.f.} \int\limits_{|\mathbf{x} - \mathbf{y}| > \delta} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \, ds_{\mathbf{y}} =: h(\mathbf{x}).$$

(For the evaluation of $h(\mathbf{x})$, see (A.1).) Hence, we finally arrive at

$$\mathbf{D}\boldsymbol{\varphi}(\mathbf{x}) = \text{ p.v.} \int\limits_{\Gamma \setminus \{\mathbf{x}\}} k_{\mathbf{D}}(\mathbf{x}, \mathbf{y}) \{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\} \, ds_{\mathbf{y}} + h(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}).$$

6 A perturbed Riesz minimal energy problem

Dragnev, Hardin and Saff approximate in e.g. [6] and [10] the Riesz minimal energy problem by a finite number of Dirac measures on Γ. The associated discrete Riesz potential is just defined by removing the self-interactions. In our continuous setting, this corresponds to integrating for a small $\delta > 0$ over $(\Gamma \times \Gamma) \setminus \{|\mathbf{x} - \mathbf{y}| \leq \delta\}$, i.e., by cutting out a set with $|\mathbf{x} - \mathbf{y}| \leq \delta$ near the singularity. In order to explain the computations in the next section, we shall focus first on the following perturbation problem which, for $0 < \varepsilon = \varepsilon(\delta) \to 0$, is essentially the minimization problem we will finally get.

Theorem 6.1 *For $\varepsilon > 0$ sufficiently small, consider the minimization problem:*

$$\left(V_\beta \boldsymbol{\sigma}, \boldsymbol{\sigma}\right)_{L_2(\Gamma)} + \frac{1}{\varepsilon}(\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L_2(\Gamma)} - 2(\mathbf{f}, \boldsymbol{\sigma})_{L_2(\Gamma)} \to \min \tag{6.1}$$

subject to

$$\int_{\Gamma_i} g_i \sigma^i \, \mathrm{d}s = a^i, \ j \in I. \tag{6.2}$$

Let the given data satisfy the additional conditions:

$$\mathbf{f} \in H^{\beta/2}(\Gamma), \ \mathbf{g} \in H^{\frac{3}{2}\beta}(\Gamma) \text{ and } 0 < a^i \in \mathbb{R}, \ i \in I. \tag{6.3}$$

Then, the minimizer $\boldsymbol{\sigma}_\varepsilon^\star \in L_2(\Gamma)$ admits the asymptotic expansion

$$\boldsymbol{\sigma}_\varepsilon^\star = \boldsymbol{\sigma}_0 + \varepsilon \boldsymbol{\sigma}_1 + \varepsilon^2 \boldsymbol{\sigma}_2 \text{ satisfying } \|\boldsymbol{\sigma}_j\|_{\mathcal{H}_\varepsilon} \leq c, \ j = 0, 1, 2 \tag{6.4}$$

with a constant $c > 0$ independent of ε, and where

$$\mathcal{H}_\varepsilon := \left\{ \boldsymbol{\varphi} \in H^{\beta/2}(\Gamma) \text{ with } \|\boldsymbol{\varphi}\|_{\mathcal{H}_\varepsilon}^2 := (\varepsilon V_\beta \boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} + \|\boldsymbol{\varphi}\|_{L_2(\Gamma)}^2 \right\} \subset H^{\beta/2}(\Gamma).$$

In particular, with $f_k := \mathbf{f}_{|\Gamma_k}$, it holds

$$\boldsymbol{\sigma}_0 = (\alpha_k \sigma_0^k)_{k \in I} = \left(\alpha_k g_k a^k (g_k, g_k)_{L_2(\Gamma_k)}^{-1} \right)_{k \in I},$$

$$\boldsymbol{\sigma}_1 = (\alpha_k \sigma_1^k)_{k \in I} = \left(\alpha_k g_k (g_k, g_k)_{L_2(\Gamma_k)}^{-1} (g_k, V_\beta \boldsymbol{\sigma}_0 - f_k)_{L_2(\Gamma_k)} \right)_{k \in I} - V_\beta \boldsymbol{\sigma}_0 + \mathbf{f}.$$

Proof The quadratic form in (6.1) induces for $\varepsilon > 0$ the ε-dependent family of Hilbert spaces $\mathcal{H}_\varepsilon$. Let us denote by $\mathcal{H}_\varepsilon'$ the dual space to $\mathcal{H}_\varepsilon$ whose norm is then defined by

$$\|\mathbf{f}\|_{\mathcal{H}_\varepsilon'} := \sup_{\mathbf{w} \in H^{\beta/2}(\Gamma) \setminus \{0\}} \frac{(\mathbf{f}, \mathbf{w})_{L_2(\Gamma)}}{\|\mathbf{w}\|_{\mathcal{H}_\varepsilon}},$$

satisfying the estimate

$$\|\mathbf{f}\|_{\mathcal{H}_\varepsilon'} \leq \sup_{\mathbf{w} \in H^{\beta/2}(\Gamma) \setminus \{0\}} \frac{(\mathbf{f}, \mathbf{w})_{L_2(\Gamma)}}{\|\mathbf{w}\|_{L_2(\Gamma)}} = \|\mathbf{f}\|_{L_2(\Gamma)} \tag{6.5}$$

since $H^{\beta/2}(\Gamma) \hookrightarrow L_2(\Gamma)$ densely and the unit ball in $\mathcal{H}_\varepsilon$ is contained in the ball $\|\mathbf{w}\|_{L_2(\Gamma)} \leq 1$.

The problem (6.1) can also be written as to minimize

$$\mathbf{J}_\varepsilon(\boldsymbol{\sigma}) := \frac{1}{2}\|\boldsymbol{\sigma}\|_{\mathcal{H}_\varepsilon}^2 - \varepsilon(\mathbf{f},\boldsymbol{\sigma})_{L_2(\Gamma)}$$

subject to (6.2). It possesses the Lagrangian

$$\mathbf{L}_\lambda(\boldsymbol{\sigma}) := \frac{1}{2}\|\boldsymbol{\sigma}\|_{\mathcal{H}_\varepsilon}^2 - \varepsilon(\mathbf{f},\boldsymbol{\sigma})_{L_2(\Gamma)} + \sum_{j\in I}\alpha_j\lambda_j\big(a^j - (g_j,\sigma_j)_{L_2(\Gamma_j)}\big),$$

where $\boldsymbol{\sigma} = \sum\limits_{j\in I}\alpha_j\sigma^j$ and $\alpha_j\sigma^j\,\mathrm{d}s = \boldsymbol{\Sigma}_{|\Gamma_j}$. Thus, the necessary conditions at the minimum read as

$$\partial_{\sigma^k}\mathbf{L}_\lambda(\boldsymbol{\sigma}_\varepsilon) = \varepsilon\alpha_k V_\beta\sigma_\varepsilon^k + \alpha_k\sigma_\varepsilon^k - \varepsilon\alpha_k f_k - \alpha_k\lambda_k g_k = 0,$$

or

$$\varepsilon V_\beta\sigma_\varepsilon^k + \sigma_\varepsilon^k - \varepsilon f_k = \lambda_k g_k \text{ and } (g_k,\sigma_\varepsilon^k)_{L_2(\Gamma_k)} = a^k,\ k \in I. \tag{6.6}$$

Here, $\boldsymbol{\lambda} \neq \mathbf{0}$ since the constraints (6.3) are always active as it follows from (6.5) and will also be seen below.

For $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$, we insert the expansion (6.4) into (6.6) and obtain the system

$$\varepsilon V_\beta\sigma_0^k + \sigma_0^k - \varepsilon f_k + \varepsilon^2 V_\beta\sigma_1^k + \varepsilon\sigma_1^k + \varepsilon^3 V_\beta\sigma_2^k + \varepsilon^2\sigma_2^k = (\lambda_0^k + \varepsilon\lambda_1^k + \varepsilon^2\lambda_2^k)g_k \text{ on } \Gamma_k,$$
$$(g_k,\sigma_0^k)_{L_2(\Gamma_k)} + \varepsilon(g_k,\sigma_1^k)_{L_2(\Gamma_k)} + \varepsilon^2(g_k,\sigma_2^k)_{L_2(\Gamma_k)} = a^k,\ k \in I.$$

Equating equal order terms in ε yields with $\lambda_0^j > 0$:

Order $\underline{\varepsilon^0}$: We find

$$\sigma_0^j = g_j\lambda_0^j \text{ and } (g_j,\sigma_0^j)_{L_2(\Gamma_j)} = (g_j,g_j)_{L_2(\Gamma_j)}\lambda_0^j = a^j > 0.$$

Thus, it follows

$$\lambda_0^j = (g_j,g_j)_{L_2(\Gamma_j)}^{-1}a^j \text{ and } \sigma_0^k = g_k a^k(g_k,g_k)_{L_2(\Gamma_k)}^{-1} \in H^{\frac{3}{2}\beta}(\Gamma_k). \tag{6.7}$$

The assumptions (6.3) imply the properties $\sigma_0^k > 0$. Moreover,

$$\|\sigma_0^k\|_{\mathcal{H}_\varepsilon} \leq c_1\|\sigma_0^k\|_{H^{\frac{3}{2}\beta}(\Gamma_k)} \leq c.$$

Order $\underline{\varepsilon^1}$: It holds

$$-f_k + \sigma_1^k = \lambda_1^k g_k - V_\beta\sigma_0^k$$

and

$$(V_\beta\sigma_0^k,g_k)_{L_2(\Gamma_k)} - (f_k,g_k)_{L_2(\Gamma_k)} + (\sigma_1^k,g_k)_{L_2(\Gamma_k)} = \lambda_1^k(g_k,g_k)_{L_2(\Gamma_k)}.$$

This yields with $(\sigma_1^k,g_k)_{L_2(\Gamma_k)} = 0$:

$$\lambda_1^k = (g_k,g_k)_{L_2(\Gamma_k)}^{-1}(V_\beta\sigma_0^k - f_k,g_k)_{L_2(\Gamma_k)},$$
$$\sigma_1^k = g_k(g_k,g_k)_{L_2(\Gamma_k)}^{-1}(V_\beta\sigma_0^k - f_k,g_k)_{L_2(\Gamma_k)} + f_k - V_\beta\sigma_0^k \in H^{\beta/2}(\Gamma_k). \tag{6.8}$$

Hence,

$$\|\sigma_1^k\|_{\mathcal{H}_\varepsilon} \le c_2 \|\sigma_0^k\|_{H^{\beta/2}(\Gamma_k)} \le c.$$

Order $\underline{\varepsilon^2}$: We derive the identities

$$V_\beta \sigma_1^k + (\varepsilon V_\beta \sigma_2^k + \sigma_2^k) = \lambda_2^k g_k,$$

$$\varepsilon (V_\beta \sigma_2^k, g_k)_{L_2(\Gamma_k)} + (V_\beta \sigma_1^k, g_k)_{L_2(\Gamma_k)} = \lambda_2^k (g_k, g_k)_{L_2(\Gamma_k)} \tag{6.9}$$

since $(\sigma_2^k, g_k)_{L_2(\Gamma_k)} = 0$. Therefore, we conclude

$$(\varepsilon V_\beta + I)_{|\Gamma_k} \sigma_2^k = g_k (g_k, g_k)_{L_2(\Gamma_k)}^{-1}$$
$$\cdot \left\{ \varepsilon (V_\beta \sigma_2^k, g_k)_{L_2(\Gamma_k)} + (V_\beta \sigma_1^k, g_k)_{L_2(\Gamma_k)} \right\} - V_\beta \sigma_1^k \in H^{-\beta/2}(\Gamma_k). \tag{6.10}$$

For every fixed $\varepsilon > 0$ sufficiently small, the mapping $\varepsilon V_\beta + I : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon'$ defines an isomorphism due to (3.5) and the Lax–Milgram lemma. Therefore, (6.10) amounts to the estimate

$$\|\boldsymbol{\sigma}_2\|_{\mathcal{H}_\varepsilon} \le c \left\| \left(g_k (g_k, g_k)_{L_2(\Gamma_k)}^{-1} \left\{ \varepsilon (V_\beta \sigma_2^k, g_k)_{L_2(\Gamma_k)} + (V_\beta \sigma_1^k, g_k)_{L_2(\Gamma_k)} \right\} \right)_{k \in I} - V_\beta \boldsymbol{\sigma}_1 \right\|_{\mathcal{H}_\varepsilon'}$$

$$\le c' \varepsilon \left(\sum_{k \in I} (V_\beta \sigma_2^k, g_k)_{L_2(\Gamma_k)} \right) + c' \|V_\beta \boldsymbol{\sigma}_1\|_{H^{-\beta/2}(\Gamma)}$$

with a constant c' depending on Γ and $\mathbf{g}$ due to (6.5) but *not* on ε. With $\mathbf{g} \in H^\beta(\Gamma)$ and V_β being a pseudodifferential operator of order β on Γ, we further have

$$\left| \sum_{k \in I} (V_\beta \sigma_2^k, g_k)_{L_2(\Gamma_k)} \right| \le c'' \|\boldsymbol{\sigma}_2\|_{L_2(\Gamma)} \|\mathbf{g}\|_{H^\beta(\Gamma)},$$

implying that

$$\|\boldsymbol{\sigma}_2\|_{\mathcal{H}_\varepsilon} \le c''' \varepsilon \|\boldsymbol{\sigma}_2\|_{L_2(\Gamma)} + c'_\mathbf{v} \|\boldsymbol{\sigma}_1\|_{H^{\beta/2}(\Gamma)}$$

$$\le c''' \varepsilon \|\boldsymbol{\sigma}_2\|_{\mathcal{H}_\varepsilon} + c'_\mathbf{v} \|\boldsymbol{\sigma}_1\|_{H^{\beta/2}(\Gamma)}.$$

Consequently, since the constants do not depend on ε, there exists an $\varepsilon_0 > 0$ such that

$$\|\boldsymbol{\sigma}_2\|_{\mathcal{H}_\varepsilon} \le c_\mathbf{v} \|\boldsymbol{\sigma}_1\|_{H^{\beta/2}(\Gamma)} \text{ for all } 0 < \varepsilon < \varepsilon_0$$

and $c_\mathbf{v}$ independent of ε.

With $\mathbf{f} \in H^{\beta/2}(\Gamma)$, $\mathbf{g} \in H^{\frac{3}{2}\beta}(\Gamma)$, we find $\boldsymbol{\sigma}_0 \in H^{\frac{3}{2}\beta}(\Gamma)$ and $\boldsymbol{\sigma}_1 \in H^{\beta/2}(\Gamma)$. Hence, (6.4) is justified which completes the proof of Theorem 6.1. $\qquad\square$

With the help of the previous theorem, we immediately find the following asymptotic behaviour of the minimizer $\boldsymbol{\sigma}_\varepsilon^\star$ if ε tends to zero.

Corollary 6.2 *Under the assumptions in Theorem 6.1, we find that*

$$\|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0\|_{L_2(\Gamma)} \le c\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

with some constant c, independent of $\boldsymbol{\sigma}_0$ and $\varepsilon > 0$, where $\boldsymbol{\sigma}_\varepsilon^\star$ is the minimizer (6.4) *of* (6.1) *for $\varepsilon > 0$.*

Proof Since $\boldsymbol{\sigma}_\varepsilon^\star = \boldsymbol{\sigma}_0 + \varepsilon \boldsymbol{\sigma}_1 + \varepsilon^2 \boldsymbol{\sigma}_2$, with (6.4) we find

$$\|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0\|_{L_2(\Gamma)} \le \|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0\|_{\mathcal{H}_\varepsilon} \le c'(\varepsilon + \varepsilon^2) \le 2c'\varepsilon$$

as proposed. $\qquad\square$

7 Riesz minimal energy without finite part reduction

We consider next the punched hypersingular Riesz potential which is defined by integrating for a small $\delta > 0$ only over $(\Gamma \times \Gamma) \setminus \{|\mathbf{x} - \mathbf{y}| \leq \delta\}$, i.e., by cutting out a set with $|\mathbf{x} - \mathbf{y}| \leq \delta$ near the singularity. Thus, the corresponding Riesz energy is defined as

$$\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\mu}) = \iint\limits_{\Gamma \times \Gamma \wedge 0 < \delta \leq |\mathbf{x}-\mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{\alpha-n} \, \mathrm{d}\boldsymbol{\mu}(\mathbf{x}) \otimes \mathrm{d}\boldsymbol{\mu}(\mathbf{y}) - 2 \int\limits_\Gamma \mathbf{f}(\mathbf{x}) \, \mathrm{d}\boldsymbol{\mu}(\mathbf{x}).$$

In view of Theorems 3.3, 3.5 and 4.1, the associated minimal Riesz energy problem is then equivalent to minimizing the punched functional

$$\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi}) = \iint\limits_{\Gamma \times \Gamma \wedge 0 < \delta \leq |\mathbf{x}-\mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{-m-\beta} \boldsymbol{\varphi}(\mathbf{y}) \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}s_\mathbf{y} \, \mathrm{d}s_\mathbf{x} - 2(\mathbf{f}, \boldsymbol{\varphi})_{L_2(\Gamma)},$$

where $m = n - 1$ and $\beta = 1 - \alpha \in (0, 2)$, over the affine cone $\mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$. Then, the measures satisfy $\mathrm{d}\boldsymbol{\mu}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}s$ with $\mathrm{d}s$ being the surface measure on Γ.

For $\overset{\circ}{\mathbf{J}}_\delta(\varphi)$ one has the following monotonicity property.

Lemma 7.3 *Let $0 < \delta_1 < \delta_2$ and $\boldsymbol{\varphi}_{\delta_1}^\star, \boldsymbol{\varphi}_{\delta_2}^\star \in \mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$ be the minimizers of $\overset{\circ}{\mathbf{J}}_{\delta_1}$ and $\overset{\circ}{\mathbf{J}}_{\delta_2}$, respectively. Then, it holds that*

$$\overset{\circ}{\mathbf{J}}_{\delta_1}(\boldsymbol{\varphi}_{\delta_1}^\star) \geq \overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}_{\delta_1}^\star) \geq \overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}_{\delta_2}^\star). \tag{7.1}$$

Proof Since $\delta_1 < \delta_2$, the minimizer $\boldsymbol{\varphi}_{\delta_1}^\star = \sum_{j \in I} \alpha_j \varphi_{\delta_1}^{\star j}$ with $\varphi_{\delta_1}^{\star j} \geq 0$ is an admissible element for minimizing $\overset{\circ}{\mathbf{J}}_{\delta_2}$. In particular, it holds

$$\overset{\circ}{\mathbf{J}}_{\delta_1}(\boldsymbol{\varphi}_{\delta_1}^\star) = \iint\limits_{|\mathbf{x}-\mathbf{y}|>\delta_1} |\mathbf{x} - \mathbf{y}|^{-m-\beta} \boldsymbol{\varphi}_{\delta_1}^\star(\mathbf{x}) \boldsymbol{\varphi}_{\delta_1}^\star(\mathbf{y}) \, \mathrm{d}s_\mathbf{y} \, \mathrm{d}s_\mathbf{x} - 2(\mathbf{f}, \boldsymbol{\varphi}_{\delta_1}^\star)_{L_2(\Gamma)}$$

$$\geq \iint\limits_{|\mathbf{x}-\mathbf{y}|>\delta_2} |\mathbf{x} - \mathbf{y}|^{-m-\beta} \boldsymbol{\varphi}_{\delta_1}^\star(\mathbf{x}) \boldsymbol{\varphi}_{\delta_1}^\star(\mathbf{y}) \, \mathrm{d}s_\mathbf{y} \, \mathrm{d}s_\mathbf{x} - 2(\mathbf{f}, \boldsymbol{\varphi}_{\delta_1}^\star)_{L_2(\Gamma)} = \overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}_{\delta_1}^\star).$$

We further find

$$\overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}_{\delta_1}^\star) \geq \inf \overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}) = \overset{\circ}{\mathbf{J}}_{\delta_2}(\boldsymbol{\varphi}_{\delta_2}^\star),$$

as proposed in (7.1). $\qquad\square$

In order to see the relation between $\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi})$ and $\mathbb{V}_\mathbf{f}(\boldsymbol{\varphi})$ in (4.2), let us introduce the compensating quadratic functional

$$\mathbf{P}_\delta(\boldsymbol{\varphi}) := \int\limits_\Gamma \left\{ \mathrm{p.f.} \int\limits_{|\mathbf{x}-\mathbf{y}|\leq\delta} |\mathbf{x} - \mathbf{y}|^{-m-\beta} \boldsymbol{\varphi}(\mathbf{y}) \, \mathrm{d}s_\mathbf{y} \right\} \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}s_\mathbf{x}. \tag{7.2}$$

Then, we obtain

$$\mathbf{P}_\delta(\boldsymbol{\varphi}) + \overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi}) = \int_\Gamma \left\{ \text{p.f.} \int_\Gamma |\mathbf{x} - \mathbf{y}|^{-m-\beta} \boldsymbol{\varphi}(\mathbf{y}) \, ds_\mathbf{y} \right\} \boldsymbol{\varphi}(\mathbf{x}) \, ds_\mathbf{x} - 2(\mathbf{f}, \boldsymbol{\varphi})_{L_2(\Gamma)}$$

$$= (V_\beta \boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} - 2(\mathbf{f}, \boldsymbol{\varphi})_{L_2(\Gamma)} = \mathbb{V}_\mathbf{f}(\boldsymbol{\varphi}),$$

and thus

$$\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi}) = \mathbb{V}_\mathbf{f}(\boldsymbol{\varphi}) - \mathbf{P}_\delta(\boldsymbol{\varphi}) = (V_\beta \boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} - 2(\mathbf{f}, \boldsymbol{\varphi})_{L_2(\Gamma)} - \mathbf{P}_\delta(\boldsymbol{\varphi}).$$

For the corresponding functional $\mathbf{P}_\delta$, there holds

Lemma 7.4 *Let $\boldsymbol{\varphi} \in \mathcal{K}(\Gamma, \mathbf{a}, \mathbf{g})$. Then*

$$\mathbf{P}_\delta(\boldsymbol{\varphi}) = -\frac{1}{\beta} \frac{1}{c_m} \delta^{-\beta} \|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)} + \mathbf{P}'_\delta(\boldsymbol{\varphi}),$$

where $\mathbf{P}'_\delta(\boldsymbol{\varphi})$ satisfies

$$|\mathbf{P}'_\delta(\boldsymbol{\varphi})| \leq c \|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)}$$

with a constant c independent of δ. Moreover

$$\lim_{\delta \to 0} \mathbf{P}'_\delta(\boldsymbol{\varphi}) = 0 \text{ for every } \boldsymbol{\varphi} \in H^{\beta/2}(\Gamma). \tag{7.3}$$

Proof Using Martensen's coordinates of Γ in the vicinity of $\mathbf{x} \in \Gamma$ (see Theorem B.2 in Appendix B), for every $\boldsymbol{\varphi} \in C^\infty(\Gamma)$, we have

$$\mathbf{P}_\delta(\boldsymbol{\varphi}) = \int_\Gamma \left\{ \text{p.f.} \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} \boldsymbol{\varphi}(\mathbf{x} + r\boldsymbol{\Theta}) r^{-\beta-1} \, dr \wedge d\boldsymbol{\omega} \right.$$

$$\left. + \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} (\mathbf{x} + r\boldsymbol{\Theta}) a(\mathbf{x}, r) r^{-\beta+1} \, dr \wedge d\boldsymbol{\omega} \right\} \boldsymbol{\varphi}(\mathbf{x}) \, ds_\mathbf{x}$$

$$= \left\{ \text{p.f.} \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} r^{-\beta-1} \, dr \wedge d\boldsymbol{\omega} \right\} \int_\Gamma |\boldsymbol{\varphi}(\mathbf{x})|^2 \, ds_\mathbf{x}$$

$$+ \int_\Gamma \left\{ \text{p.f.} \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} \{\boldsymbol{\varphi}(\mathbf{x} + r\boldsymbol{\Theta}) - \boldsymbol{\varphi}(\mathbf{x})\} r^{-\beta-1} \, dr \wedge d\boldsymbol{\omega} \right\} \boldsymbol{\varphi}(\mathbf{x}) \, ds_\mathbf{x}$$

$$+ \int_\Gamma \left\{ \text{p.f.} \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} \boldsymbol{\varphi}(\mathbf{x} + r\boldsymbol{\Theta}) a(\mathbf{x}, r) r^{-\beta+1} \, dr \wedge d\boldsymbol{\omega} \right\} \boldsymbol{\varphi}(\mathbf{x}) \, ds_\mathbf{x}.$$

Since

$$\text{p.f.} \int_{0 < r \leq \delta \wedge |\boldsymbol{\Theta}| = 1} r^{-\beta-1} \, dr \wedge d\boldsymbol{\omega} = -\frac{1}{\beta} \frac{1}{c_m} \delta^{-\beta}$$

and $\mathbf{P}_\delta(\boldsymbol{\varphi})$ in (7.2) is symmetric, we find

$$\mathbf{P}_\delta(\boldsymbol{\varphi}) = -(\beta c_m)^{-1} \delta^{-\beta} \|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)} + \mathbf{P}'_\delta(\boldsymbol{\varphi}).$$

Herein, $\mathbf{P}'_\delta(\varphi)$ is given by

$$
\begin{aligned}
2\mathbf{P}'_\delta(\varphi) = \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} & \left\{\varphi(\mathbf{x})\big(\varphi(\mathbf{y})-\varphi(\mathbf{x})\big)+\varphi(\mathbf{y})\big(\varphi(\mathbf{x})-\varphi(\mathbf{y})\big)\right\}|\mathbf{x}-\mathbf{y}|^{-\beta-m}\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
& - \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} \big(\varphi(\mathbf{y})-\varphi(\mathbf{x})\big)a(\mathbf{y},r)r^{-\beta+1}\,\mathrm{d}r\wedge\mathrm{d}\boldsymbol{\omega}\varphi(\mathbf{x})\,\mathrm{d}s_{\mathbf{x}} \\
& - \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} \big(\varphi(\mathbf{x})-\varphi(\mathbf{y})\big)a(\mathbf{x},r)r^{-\beta+1}\,\mathrm{d}r\wedge\mathrm{d}\boldsymbol{\omega}\,\varphi(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}} \\
& + \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} \varphi(\mathbf{y})a(\mathbf{x},r)r^{-\beta+1}\,\mathrm{d}r\wedge\mathrm{d}\boldsymbol{\omega}\,\varphi(\mathbf{x})\,\mathrm{d}s_{\mathbf{x}} \\
& + \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} \varphi(\mathbf{x})a(\mathbf{y},r)r^{-\beta+1}\,\mathrm{d}r\wedge\mathrm{d}\boldsymbol{\omega}\varphi(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}. \quad (7.4)
\end{aligned}
$$

We rewrite $\mathbf{P}'_\delta(\varphi)$ according to

$$
\begin{aligned}
\mathbf{P}'_\delta(\varphi) = \frac{1}{2}\iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} & |\varphi(\mathbf{x})-\varphi(\mathbf{y})|^2|\mathbf{x}-\mathbf{y}|^{-\beta-m}\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
& + \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} b(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} + \iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} c(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})^2\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}},
\end{aligned}
$$

where $b(\mathbf{x},\mathbf{y})$ and $c(\mathbf{x},\mathbf{y})$ are kernels which posess pseudohomogeneous expansions of degree $-\beta-m+1$. This means that

$$
\begin{aligned}
\mathbf{P}'_\delta(\varphi) = \frac{1}{2}\iint\limits_{|\mathbf{x}-\mathbf{y}|\le\delta} & |\varphi(\mathbf{x})-\varphi(\mathbf{y})|^2|\mathbf{x}-\mathbf{y}|^{-\beta-m}\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
& + \int_\Gamma (\mathbf{B}_{\beta-1}\varphi)(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}s_{\mathbf{x}} + \int_\Gamma (\mathbf{C}_{\beta-1}1)\varphi(\mathbf{x})^2\,\mathrm{d}s_{\mathbf{x}}
\end{aligned}
$$

with classical pseudodifferential operators $\mathbf{B}_{\beta-1}$ and $\mathbf{C}_{\beta-1}$ of degree $\beta-1$ on Γ. Since the constant charge 1 is a smooth function on Γ and $\mathbf{B}_{\beta-1}:H^{\beta/2}(\Gamma)\to H^{-\beta/2+1}(\Gamma)\hookrightarrow H^{-\beta/2}(\Gamma)$ for $\varphi\in H^{\beta/2}(\Gamma)$, we finally arrive at

$$
\big|\mathbf{P}'_\delta(\varphi)\big| \le c_1\|\varphi\|^2_{H^{\beta/2}(\Gamma)} + c_2\|\varphi\|^2_{L_2(\Gamma)} \le c\|\varphi\|^2_{H^{\beta/2}(\Gamma)},
$$

as proposed.[1]

In order to show (7.3), consider first $\varphi\in C^\infty(\Gamma)$ and use Taylor's expansion about $\mathbf{x}\in\Gamma$ in (7.4). Then, all the integrals on the right hand side are weakly

---

[1] For $\beta=1$, $\mathbf{B}_0$ is a singular Mikhlin–Calderon integral operator with principal part $b(\mathbf{x},\mathbf{x})\boldsymbol{\Theta}(\omega)$ which satisfies the Mikhlin condition $b(\mathbf{x},\mathbf{x})\int_{|\boldsymbol{\Theta}|=1}\boldsymbol{\Theta}(\omega)\,\mathrm{d}\boldsymbol{\omega}=0$.

singular tending to zero with $\delta \to 0$. For $\boldsymbol{\varphi} \in H^{\beta/2}(\Gamma)$ approximate $\boldsymbol{\varphi}$ by $\boldsymbol{\varphi}_\varepsilon \in C^\infty(\Gamma)$, satisfying $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon\|_{H^{\beta/2}(\Gamma)} < \varepsilon$. Then

$$
\begin{aligned}
\left|\mathbf{P}'_\delta(\boldsymbol{\varphi}) - \mathbf{P}'_\delta(\boldsymbol{\varphi}_\varepsilon)\right| \\
\leq \frac{1}{2} \iint\limits_{|\mathbf{x}-\mathbf{y}|\leq\delta} \left\{|\boldsymbol{\varphi}(\mathbf{x}) - \boldsymbol{\varphi}(\mathbf{y})|^2 - |\boldsymbol{\varphi}_\varepsilon(\mathbf{x}) - \boldsymbol{\varphi}_\varepsilon(\mathbf{y})|^2\right\}|\mathbf{x}-\mathbf{y}|^{-\beta-m}\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}s_\mathbf{x} \\
+ \left|(\mathbf{B}_{\beta-1}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\Gamma)} - (\mathbf{B}_{\beta-1}\boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon)_{L_2(\Gamma)}\right| \\
+ \|\mathbf{C}_{\beta-1}\mathbf{1}\|_{L_2(\Gamma)}\big|\|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)} - \|\boldsymbol{\varphi}_\varepsilon\|^2_{L_2(\Gamma)}\big|.
\end{aligned}
$$

With

$$
\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)} - \|\boldsymbol{\varphi}_\varepsilon\|^2_{H^{\beta/2}(\Gamma)} \leq 3\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon\|_{H^{\beta/2}(\Gamma)}
$$

for $\|\boldsymbol{\varphi}_\varepsilon\|_{H^{\beta/2}(\Gamma)} \leq 2\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)}$, one has

$$
\begin{aligned}
\left|\mathbf{P}'_\delta(\boldsymbol{\varphi}) - \mathbf{P}'_\delta(\boldsymbol{\varphi}_\varepsilon)\right| \\
\leq \frac{3}{2}\left\{\iint\limits_{|\mathbf{x}-\mathbf{y}|\leq\delta} |\boldsymbol{\varphi}(\mathbf{x}) - \boldsymbol{\varphi}_\varepsilon(\mathbf{x})||\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}_\varepsilon(\mathbf{y})||\mathbf{x}-\mathbf{y}|^{-\beta-m}\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}s_\mathbf{x}\right\}^{\frac{1}{2}} \|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)} \\
+ \left|\left(\mathbf{B}_{\beta-1}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon), \boldsymbol{\varphi}\right)_{L_2(\Gamma)}\right| + \left|\left(\mathbf{B}_{\beta-1}\boldsymbol{\varphi}_\varepsilon, (\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon)\right)_{L_2(\Gamma)}\right| \\
+ 3\|\mathbf{C}_{\beta-1}\mathbf{1}\|_{L_2(\Gamma)}\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon\|_{H^{\beta/2}(\Gamma)} \\
\leq c\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon\|_{H^{\beta/2}(\Gamma)} \leq c\varepsilon\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}.
\end{aligned}
$$

Then

$$
\overline{\lim_{\delta\to 0}}\left|\mathbf{P}'_\delta(\boldsymbol{\varphi})\right| \leq c\varepsilon\|\boldsymbol{\varphi}\|_{H^{\beta/2}(\Gamma)}
$$

for any $\varepsilon > 0$ which implies (7.3). $\qquad\square$

Since

$$
\begin{aligned}
\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi}) &= \iint\limits_{0<\delta\leq|\mathbf{x}-\mathbf{y}|} |\mathbf{x}-\mathbf{y}|^{-\beta-m}\boldsymbol{\varphi}(\mathbf{y})\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}s_\mathbf{x} - 2(\mathbf{f},\boldsymbol{\varphi})_{L^2(\Gamma)} \\
&= \frac{1}{\varepsilon}\|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)} + (V_\beta\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\Gamma)} - 2(\mathbf{f},\boldsymbol{\varphi})_{L_2(\Gamma)} - \mathbf{P}'_\delta(\boldsymbol{\varphi}),
\end{aligned}
\tag{7.5}
$$

where $\varepsilon = \beta c_m \delta^\beta \to 0$ for $\delta \to 0$, Lemma 7.3 implies

Corollary 7.5 *Let $\beta_0 < 2$ and $\delta_0 > 0$ be given. Then, there exist positive constants $c, c' > 0$ such that the estimate*

$$
\overset{\circ}{\mathbf{J}}_\delta(\boldsymbol{\varphi}) - \frac{1}{\beta}\frac{1}{c_m}\delta^{-\beta}\|\boldsymbol{\varphi}\|^2_{L_2(\Gamma)} \leq c\|\boldsymbol{\varphi}\|^2_{H^{\beta/2}(\Gamma)} + 2\|\mathbf{f}\|_{L_2(\Gamma)}\|\boldsymbol{\varphi}\|_{L_2(\Gamma)} \leq c'
$$

holds uniformly for $\boldsymbol{\varphi} \in \mathcal{K}^{\beta/2}(\Gamma)$, $0 < \delta \leq \delta_0$ and $0 \leq \beta \leq \beta_0$.

The functional (7.5) coincides with the functional from (6.1) except for the perturbation term $\mathbf{P}'_\delta(\boldsymbol{\varphi})$. Hence, we can proceed in complete analogy to the proof of Theorem 6.1.

The Lagrangian to the punched energy functional reads as

$$\overset{\circ}{\mathbf{L}}_\lambda(\boldsymbol{\sigma}) := \frac{1}{2}\big((\varepsilon V_\beta \boldsymbol{\sigma} + \boldsymbol{\sigma}), \boldsymbol{\sigma}\big)_{L_2(\Gamma)} - \varepsilon \mathbf{P}'_\delta(\boldsymbol{\sigma}) - \varepsilon(\mathbf{f}, \boldsymbol{\sigma})_{L_2(\Gamma)}$$
$$+ \sum_{j \in I} \alpha_j \lambda_j \big(a^j - (g_j, \sigma^j)_{L_2(\Gamma_j)}\big),$$

where the first order necessary optimality condition is given by

$$\partial_{\sigma^k} \overset{\circ}{\mathbf{L}}_\lambda(\boldsymbol{\sigma}) = \varepsilon \alpha_k V_\beta \sigma_\varepsilon^k + \alpha_k \sigma_\varepsilon^k - \varepsilon \alpha_k \mathbf{P}''_\delta \sigma_\varepsilon^k - \varepsilon \alpha_k f_k - \alpha_k \lambda_k g_k = 0,$$
$$(g_k, \sigma_\varepsilon^k)_{L_2(\Gamma_k)} = a^k, \ k \in I.$$

Again, $\boldsymbol{\lambda} \neq \mathbf{0}$ since the constraints (6.3) are active. Here,

$$\mathbf{P}''_\delta \sigma_\varepsilon^k = \partial_{\sigma_\varepsilon^k} \mathbf{P}'_\delta(\boldsymbol{\sigma}_\varepsilon) = \int\limits_{|\mathbf{x}-\mathbf{y}| \leq \delta} \big(\sigma_\varepsilon^k(\mathbf{y}) - \sigma_\varepsilon^k(\mathbf{x})\big)|\mathbf{x}-\mathbf{y}|^{-m-\beta} \, ds_\mathbf{y}$$
$$+ \mathbf{B}_{\beta-1}\sigma_\varepsilon^k + \mathbf{B}^\star_{\beta-1}\sigma_\varepsilon^k + 2c_{\beta-1}(\mathbf{x})\sigma_\varepsilon^k(\mathbf{x})$$

with

$$c_{\beta-1}(\mathbf{x}) = \int\limits_{\mathbf{y} \in \Gamma \wedge |\mathbf{x}-\mathbf{y}| \leq \delta} c(\mathbf{x}, \mathbf{y}) \, ds_\mathbf{y}$$

and $\mathbf{B}_{\beta-1}$ and $\mathbf{B}^\star_{\beta-1}$ being bounded linear operators:

$$\mathbf{B}_{\beta-1}\sigma_\varepsilon^k = \int\limits_{\mathbf{y} \in \Gamma \wedge |\cdot-\mathbf{y}| \leq \delta} b(\cdot, \mathbf{y})\sigma_\varepsilon^k(\mathbf{y}) \, ds_\mathbf{y} : H^{\frac{3}{2}\beta}(\Gamma) \to H^{\frac{1}{2}\beta}(\Gamma),$$

$$\mathbf{B}^\star_{\beta-1}\sigma_\varepsilon^k = \int\limits_{\mathbf{y} \in \Gamma \wedge |\cdot-\mathbf{y}| \leq \delta} b(\mathbf{y}, \cdot)\sigma_\varepsilon^k(\mathbf{y}) \, ds_\mathbf{y} : H^{\frac{1}{2}\beta}(\Gamma) \to H^{-\frac{1}{2}\beta}(\Gamma).$$

Under the assumptions of Theorem 6.1 and as in the proof of Theorem 6.1, we finally obtain the asymptotic expansion of the minimizer $\boldsymbol{\sigma}_\varepsilon^\star$ as well as of the Lagrangian multipliers:

$$\boldsymbol{\sigma}_\varepsilon^\star = \boldsymbol{\sigma}_0 + \varepsilon \boldsymbol{\sigma}_1 + \varepsilon^2 \boldsymbol{\sigma}_2 \text{ satisfying } \|\boldsymbol{\sigma}_j\|_{\mathcal{H}_\varepsilon} \leq c, \ j = 0, 1, 2,$$

and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \varepsilon \boldsymbol{\lambda}_1 + \varepsilon^2 \boldsymbol{\lambda}_2$ where

$$(g_k, \sigma_\varepsilon^k)_{L_2(\Gamma_k)} = a^k, \ k \in I.$$

It turns out that σ_0^k and λ_0^k are exactly the same as in (6.7). Moreover, we have to replace V_β by $(V_\beta - \mathbf{P}''_\delta)$ in the equations (6.8) and (6.9), (6.10). Note that $\boldsymbol{\sigma}_0 \in H^{\frac{3}{2}\beta}(\Gamma) \hookrightarrow H^{\frac{1}{2}\beta}(\Gamma)$ and, hence, $\mathbf{P}''_\delta \sigma_0^k$ for $\delta \to 0$ tends to zero in $H^{\beta/2}(\Gamma)$ due to (7.3). (If $\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1 \in C^1(\Gamma)$ for $0 < \beta < 1$ or $\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1 \in C^2(\Gamma)$ for $1 \leq \beta \leq 2$, then $\mathbf{P}''_\delta \sigma_0^k$ and $\mathbf{P}''_\delta \sigma_1^k = \mathcal{O}(\delta^{1-\beta})$, respectively $\mathcal{O}(\delta^{2-\beta})$.)

Collecting these results, we have for the punched energy Riesz minimum problem the following result:

Theorem 7.6 *Under the same assumptions as for Theorem 6.1, the minimization problem*

$$\frac{1}{2} \iint\limits_{\substack{0 < \delta \leq |\mathbf{x} - \mathbf{y}| \\ \mathbf{x}, \mathbf{y} \in \Gamma}} |\mathbf{x} - \mathbf{y}|^{\alpha - n} \boldsymbol{\sigma}(\mathbf{y}) \boldsymbol{\sigma}(\mathbf{x}) \, \mathrm{d}s_{\mathbf{y}} \, \mathrm{d}s_{\mathbf{x}} - \int_{\Gamma} \mathbf{f}(\mathbf{x}) \boldsymbol{\sigma}(\mathbf{x}) \, \mathrm{d}s_{\mathbf{x}} \to \min, \qquad (7.6)$$

subject to (6.2), has for every $\delta > 0$ a unique solution $\boldsymbol{\sigma}_\varepsilon^\star$. It admits for $\varepsilon = \beta c_m \delta^\beta > 0$ the asymptotic expansion

$$\boldsymbol{\sigma}_\varepsilon^\star = \boldsymbol{\sigma}_0 + \varepsilon \boldsymbol{\sigma}_1 + \varepsilon^2 \boldsymbol{\sigma}_2 \text{ satisfying } \|\boldsymbol{\sigma}_j\|_{\mathcal{H}_\varepsilon} \leq c, \; j = 0, 1, 2 \qquad (7.7)$$

with a constant c independent of ε. In particular, with $\boldsymbol{\sigma}_\varepsilon^\star = (\alpha_k \sigma_0^k + \alpha_k \sigma_1^k + \alpha_k \sigma_2^k)_{k \in I}$ there holds

$$\sigma_0^k = g_k a^k (g_k, g_k)_{L_2(\Gamma_k)}^{-1},$$
$$\sigma_1^k = \lambda_1^k g_k - (V_\beta - \mathbf{P}_\delta'') \sigma_0^k + f_k,$$

where

$$\lambda_1^k = (g_k, g_k)^{-1} \big((V_\beta - \mathbf{P}_\delta'') \sigma_0^k - f_k, g_k \big)_{L_2(\Gamma_k)}$$

and

$$\sigma_2^k = \big(I + \varepsilon (V_\beta - P_j'') \big)^{-1} \big\{ \lambda_2^k g_k - (V_\beta - P_j'') \sigma_1^k \big\}$$

with

$$\lambda_2^k = (g_k, g_k)_{L_2(\Gamma_k)}^{-1} \big\{ (V_\beta - P_j'') \sigma_1^k + \varepsilon (V_\beta - P_j'') \sigma_2^k + \sigma_2^k \big\}.$$

Corollary 7.7 *For $\delta \to 0$ and $\varepsilon = \beta c_m \delta^\beta$ one finds*

$$\|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0\|_{L_2(\Gamma)} \leq c\varepsilon \xrightarrow{\delta \to 0} 0$$

with some constant c, independent of $\boldsymbol{\sigma}_0$ and $\varepsilon > 0$, where $\boldsymbol{\sigma}_\varepsilon^\star$ is the minimizer (7.6). Moreover, it holds

$$\mathring{\mathbf{J}}_\delta(\boldsymbol{\sigma}_\varepsilon^\star) = \big(\mathbb{V}_\beta(\boldsymbol{\sigma}_\varepsilon^\star), \boldsymbol{\sigma}_\varepsilon^\star \big)_{L_2(\Gamma)} - P_\delta'(\boldsymbol{\sigma}_\varepsilon^\star) + \frac{1}{\beta} \frac{1}{c_m} \delta^{-\beta} \|\boldsymbol{\sigma}_\varepsilon^\star\|_{L_2(\Gamma)}^2 \xrightarrow{\delta \to 0} \infty. \qquad (7.8)$$

Proof Due to (7.7), we obtain

$$\|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0\|_{L_2(\Gamma)} \leq \|\boldsymbol{\sigma}_\varepsilon^\star - \boldsymbol{\sigma}_0^\star\|_{\mathcal{H}_\varepsilon} \leq \varepsilon \|\boldsymbol{\sigma}_1\|_{\mathcal{H}_\varepsilon} + \varepsilon^2 \|\boldsymbol{\sigma}_2\|_{\mathcal{H}_\varepsilon} \leq c(\varepsilon + \varepsilon^2).$$

The conditions (6.3) imply that $\sigma_0^k > 0$ and, hence, $\|\boldsymbol{\sigma}_\varepsilon^\star\|_{L_2(\Gamma)} \geq \frac{1}{2} \|\boldsymbol{\sigma}_0\|_{L_2(\Gamma)} > 0$ for all ε with $0 < \varepsilon \leq \varepsilon_1$ with some $\varepsilon_1 > 0$. Thus, there holds

$$\mathring{\mathbf{J}}_\delta(\boldsymbol{\sigma}_\varepsilon^\star) = \big(\mathbb{V}_\beta(\boldsymbol{\sigma}_\varepsilon^\star), \boldsymbol{\sigma}_\varepsilon^\star \big)_{L_2(\Gamma)} - P_\delta'(\boldsymbol{\sigma}_\varepsilon^\star) + \frac{1}{\beta} \frac{1}{c_m} \delta^{-\beta} \|\boldsymbol{\sigma}_\varepsilon^\star\|_{L_2(\Gamma)}^2$$

$$\geq \frac{1}{2\beta} \frac{1}{c_m} \delta^{-\beta} \|\boldsymbol{\sigma}_0\|_{L_2(\Gamma)}^2 - c\|\boldsymbol{\sigma}_\varepsilon^\star\|_{\mathcal{H}_\varepsilon}^2$$

with uniformly bounded $\|\boldsymbol{\sigma}_\varepsilon^\star\|_{\mathcal{H}_\varepsilon}^2$. Hence, $\delta \to 0$ implies (7.8). $\qquad \square$

Remark 7.8 *For the torus Γ_1 in $\mathbb{R}^3$, considered in [6] and [10], where $\mathbf{f} \in H^{\beta/2}(\Gamma)$, $a^1 > 0$, $g_1 = 1$, the minimizers $\boldsymbol{\sigma}_\varepsilon^\star$ of the punched minimization problem tend to the constant charge $\boldsymbol{\sigma}_\varepsilon^\star \to \boldsymbol{\sigma}_0 := \frac{1}{|\Gamma|} a^1$.*

A Explicit calculation of particular partie finie integrals

In this appendix, we shall compute the partie finie integrals which define the functions $h(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ from (3.2) and (3.3), respectively.

Lemma A.1 *(i) Let $-1 < \alpha < 1$, $\Gamma \in C^\infty$ and $\boldsymbol{\varphi} \in C^\infty(\Gamma)$. Then, one has*

$$\lim_{\delta \to 0} \text{ p.f.} \int\limits_{|\mathbf{x}-\mathbf{y}|>\delta>0} |\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}\big\{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\big\}\,\mathrm{d}s_\mathbf{y}$$

$$= \text{ p.v.} \int\limits_{\Gamma\backslash\{\mathbf{x}\}} |\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}\big\{\boldsymbol{\varphi}(\mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x})\big\}\,\mathrm{d}s_\mathbf{y},$$

where p.v. *denotes the Mikhlin–Calderon principal value integral.*
    (ii) The function $h(\mathbf{x})$ from (3.2) is given by

$$\begin{aligned}
h(\mathbf{x}) = \int\limits_{\mathbf{y}\in\Gamma\wedge|\mathbf{x}-\mathbf{y}|\leq c} &\big\{|\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)} - r^{-\beta-(n-1)}\big\}\,\mathrm{d}s_\mathbf{y} - (\beta c_n)^{-1}c^{-\beta}\\
&- \text{ p.v.} \int\limits_{\mathbf{y}\in\Gamma\wedge|\mathbf{x}-\mathbf{y}|\leq c} r^{-\beta-(n-1)}\big\{r^{n-2}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega} - \mathrm{d}s_\mathbf{y}\big\}\\
&+ \int\limits_{\mathbf{y}\in\Gamma\wedge|\mathbf{x}-\mathbf{y}|\geq c} |\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}\,\mathrm{d}s_\mathbf{y}
\end{aligned} \tag{A.1}$$

with any $c > 0$ sufficiently small.
    (iii) For the function $\mathbf{h}(\mathbf{x})$ from (3.3), one obtains

$$\begin{aligned}
\mathbf{h}(\mathbf{x}) = \text{ p.v.} \int\limits_{\mathbf{y}\in\Gamma\wedge 0<|\mathbf{x}-\mathbf{y}|\leq c} &\big\{|\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x}) - r^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x})\big\}\,\mathrm{d}s_\mathbf{y}\\
&- \text{ p.v.} \int\limits_{\mathbf{y}\in\Gamma\wedge|\mathbf{x}-\mathbf{y}|\leq c} r^{-\beta-(n-1)}\big\{(\mathbf{y}-\mathbf{x})r^{n-2}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega} - (\mathbf{y}-\mathbf{x})\,\mathrm{d}s_\mathbf{y}\big\}\\
&+ \int\limits_{\mathbf{y}\in\Gamma\wedge 0<|\mathbf{x}-\mathbf{y}|\geq c} |\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x})\,\mathrm{d}s_\mathbf{y}
\end{aligned} \tag{A.2}$$

with any $c > 0$ sufficiently small.

Proof (i) Locally on Γ one has near $\mathbf{x} \in \Gamma$:

$$\boldsymbol{\varphi}(\mathbf{y}) = \boldsymbol{\varphi}(\mathbf{x}) + (\mathbf{y}-\mathbf{x}) \cdot \nabla\boldsymbol{\varphi}(\mathbf{x}) + \mathcal{O}(r^2), \ r = |\mathbf{x}-\mathbf{y}|;$$

$$\boldsymbol{\Theta}(r,\boldsymbol{\omega}) := \frac{1}{r}(\mathbf{y}-\mathbf{x}) \text{ for } \mathbf{x},\mathbf{y} \in \Gamma;$$

$$\boldsymbol{\Theta}(r,\boldsymbol{\omega}) = \boldsymbol{\Theta}(0,\boldsymbol{\omega}) + \mathcal{O}(r), \int\limits_{|\Theta|=1} \boldsymbol{\Theta}(0,\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} = 0,$$

$$\mathrm{d}s_\mathbf{y} = r^{n-2}(1 + \mathcal{O}(r^2))\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega}.$$

$|\boldsymbol{\Theta}(0,\boldsymbol{\omega})| = 1$ describes the $(n-2)$-dimensional unit sphere $\mathbb{S}^{n-2}$ and $\boldsymbol{\omega}$ is its polar coordinate with $\mathrm{d}\boldsymbol{\omega}$ its $(n-2)$-dimensional surface measure. Consequently, with an appropriate constant

$c > 0$ depending on Γ, one has

$$\lim_{\delta \to 0} \text{p.f.} \int_{\mathbf{y} \in \Gamma \wedge 0 < \delta < |\mathbf{x} - \mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \varphi(\mathbf{y}) - \varphi(\mathbf{x}) \big\} \, ds_{\mathbf{y}}$$

$$= \lim_{\delta \to 0+} \text{p.f.} \int_{\mathbf{y} \in \Gamma \wedge 0 < \delta < |\mathbf{x} - \mathbf{y}| \leq c} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)}$$

$$\times \Big\{ (\mathbf{y} - \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) + \big\{ \varphi(\mathbf{y}) - \varphi(\mathbf{x}) - (\mathbf{y} - \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \big\} \Big\} \, ds_{\mathbf{y}}$$

$$+ \int_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| \geq c} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \varphi(\mathbf{y}) - \varphi(\mathbf{x}) \big\} \, ds_{\mathbf{y}}$$

$$= \lim_{\delta \to 0+} \text{p.f.} \int_{0 < \delta \leq r \leq c} r^{-\beta} \, dr \int_{|\boldsymbol{\Theta}| = 1} \boldsymbol{\Theta} \, d\boldsymbol{\omega}(\boldsymbol{\Theta}) \cdot \nabla \varphi(\mathbf{x})$$

$$+ \text{p.v.} \bigg\{ \int_{0 < |\mathbf{x} - \mathbf{y}| \leq c} \big\{ |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} (\mathbf{y} - \mathbf{x}) \, ds_{\mathbf{y}} - r^{-\beta - 1} (\mathbf{y} - \mathbf{x}) \, dr \, d\boldsymbol{\omega} \big\}$$

$$+ \int_{0 < |\mathbf{x} - \mathbf{y}| \leq c} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \varphi(\mathbf{y}) - \varphi(\mathbf{x}) - (\mathbf{y} - \mathbf{x}) \nabla \varphi(\mathbf{x}) \cdot ds_{\mathbf{y}} \big\} \bigg\}$$

$$+ \int_{\mathbf{y} \in \Gamma \wedge |\mathbf{x} - \mathbf{y}| \geq c} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \big\{ \varphi(\mathbf{y}) - \varphi(\mathbf{x}) \big\} \, ds_{\mathbf{y}}.$$

The first integral on the right is zero because of $\int_{|\boldsymbol{\Theta}| = 1} \boldsymbol{\Theta} \, d\boldsymbol{\omega} = \mathbf{0}$. Whereas, the integrand of the remaining integral has at $\mathbf{y} = \mathbf{x}$ a weak singularity $|\mathbf{y} - \mathbf{x}|^{-\beta + 1}$ with $-1 < -\beta + 1 < 1$ whose principal value integral exists. This follows by using e.g. Martensen's surface polar coordinates, cf. Theorem B.1 in Appendix B.

(ii) For the function

$$h(\mathbf{x}) = \lim_{\delta \to 0} \text{p.f.} \int_{0 < \delta < |\mathbf{x} - \mathbf{y}| \wedge \mathbf{y} \in \Gamma} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \, ds_{\mathbf{y}},$$

we find

$$h(\mathbf{x}) = \lim_{\delta \to 0} \text{p.f.} \int_{0 < \delta < |\mathbf{x} - \mathbf{y}| \leq c} \big\{ |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} - r^{-\beta - (n-1)} \big\} \, ds_{\mathbf{y}}$$

$$+ \lim_{\delta \to 0} \text{p.f.} \int_{0 < \delta}^{c} r^{-\beta - (n-1)} r^{n-2} \, dr \int_{|\boldsymbol{\Theta}| = 1} d\boldsymbol{\omega}$$

$$- \lim_{\delta \to 0} \text{p.f.} \int_{0 < \delta}^{c} \int_{|\boldsymbol{\Theta}| = 1} r^{-\beta - (n-1)} \big\{ r^{n-2} \, dr \, d\boldsymbol{\omega} - ds_{\mathbf{y}} \big\}$$

$$+ \int_{c \leq |\mathbf{x} - \mathbf{y}| \wedge \mathbf{y} \in \Gamma} |\mathbf{x} - \mathbf{y}|^{-\beta - (n-1)} \, ds_{\mathbf{y}}.$$

If we choose Martensen's surface polar coordinates on Γ, then the integrand of the first integral on the right is identical zero due to $|\mathbf{x} - \mathbf{y}| = r$. For the second integral on the right, we have with $\int_{|\boldsymbol{\Theta}| = 1} d\boldsymbol{\omega} = c_n^{-1}$ that

$$\lim_{0 < \delta \to 0} \text{p.f.} \int_{\delta}^{c} r^{\beta - 1} \, dr \int_{|\boldsymbol{\Theta}| = 1} d\boldsymbol{\omega} = \lim_{0 < \delta \to 0} \text{p.f.} \frac{1}{c_n} \bigg\{ \frac{\delta^{-\beta}}{\beta} - \frac{1}{\beta} c^{-\beta} \bigg\} = -(c_n \beta)^{-1} c^{-\beta}.$$

For the third integral, Theorem B.2 in Appendix B implies

$$r^{n-2}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega} - \mathrm{d}s_{\mathbf{y}} = \mathcal{O}(r^n)\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega}.$$

Hence, the integrand is of order $\mathcal{O}(r^{-\beta+1})$ and the integral exists as a principal value integral. Collecting these properties proves (A.1).

(iii) We proceed with respect to the vector-valued function

$$\mathbf{h}(\mathbf{x}) = \text{p.f.}\lim_{\delta\to 0}\int\limits_{\mathbf{y}\in\Gamma\wedge 0<\delta<|\mathbf{x}-\mathbf{y}|}|\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x})\,\mathrm{d}s_{\mathbf{y}},$$

in the same manner as for $h(\mathbf{x})$. Inserting $(\mathbf{y}-\mathbf{x}) = r\boldsymbol{\Theta}$ for Martensen's surface polar coordinates, we have:

$$\mathbf{h}(\mathbf{x}) = \lim_{\delta\to 0}\text{p.f.}\int\limits_{0<\delta<|\mathbf{x}-\mathbf{y}|\leq c}\left\{|\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x}) - r^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x})\right\}\mathrm{d}s_{\mathbf{y}}$$

$$+ \lim_{\delta\to 0}\text{p.f.}\int\limits_{0<\delta}^{c}r^{-\beta-(n-1)}r^{n-1}\,\mathrm{d}r\int\limits_{|\boldsymbol{\Theta}|=1}\boldsymbol{\Theta}\,\mathrm{d}\boldsymbol{\omega}$$

$$- \lim_{\delta\to 0}\text{p.f.}\int\limits_{\delta}^{c}\int\limits_{|\boldsymbol{\Theta}|=1}r^{-\beta-n}\boldsymbol{\Theta}\left\{r^{n-2}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega} - \mathrm{d}s_{\mathbf{y}}\right\}$$

$$+ \int\limits_{0\leq|\mathbf{x}-\mathbf{y}|\wedge\mathbf{y}\in\Gamma}|\mathbf{x}-\mathbf{y}|^{-\beta-(n-1)}(\mathbf{y}-\mathbf{x})\,\mathrm{d}s_{\mathbf{y}}.$$

Here, the first integral on the right vanishes if Martensen's surface polar coordinates are used, and the second one vanishes because of $\int_{|\boldsymbol{\Theta}|=1}\boldsymbol{\Theta}\,\mathrm{d}\boldsymbol{\omega} = \mathbf{0}$. The third integral contains an integrand of order $\mathcal{O}(r^{\beta+2})$. Hence, the integrand is bounded and the integral exists. Consequently, also (A.2) holds. $\qquad\square$

B Martensen's surface polar coordinates

For $n = 3$, surface polar coordinates have been introduced by Martensen in [15, Chapter 2.1]. A graphical illustration of these coordinates are found in Figure 2. We generalize this approach to arbitrary spatial dimension $n \geq 2$.
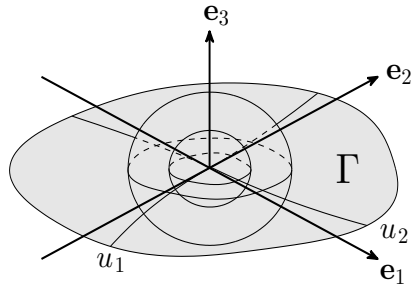

Fig. 2 Illustration of Martensen's surface polar coordinates.

The given surface has locally the parametric representation $\Gamma : \mathbf{x} = \mathbf{x}(\mathbf{u}) \in \mathbb{R}^n$, $\mathbf{u} = (u^1, \ldots, u^{n-1}) \in \mathbb{R}^{n-1}$. For $\mathbf{x} = \overset{\circ}{\mathbf{x}} = \mathbf{x}(\overset{\circ}{\mathbf{u}})$, define the family of $(n-2)$-dimensional manifolds (level sets) as $\{\mathbf{x} \in \mathcal{C}_\varrho \subset \Gamma$ given by $|\mathbf{x} - \overset{\circ}{\mathbf{x}}| = \varrho > 0\}$, where $|\mathbf{x} - \overset{\circ}{\mathbf{x}}|$ denotes the Euclidian distance in $\mathbb{R}^n$ and ϱ is the radial parameter. Let

$$A(\mathbf{u}) := |\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}|,$$

then

$$A^2(\mathbf{u}) = \big(\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}\big) \cdot \big(\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}\big),$$

where $\cdot$ denotes the Euclidian scalar product in $\mathbb{R}^n$. In the local neighborhood of Γ, define

$$\mathbf{x}(\mathbf{u}) = \overset{\circ}{\mathbf{x}} + \mathbf{F}(\mathbf{u}) + G(\mathbf{u})\mathbf{n}(\mathbf{u})$$

with $G(\mathbf{u}) = \big(\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}\big) \cdot \mathbf{n}(\mathbf{u})$, where $\mathbf{n}(\mathbf{u})$ denotes the (exterior) unit normal vector of Γ at $\mathbf{x} = \mathbf{x}(\mathbf{u})$. Then, on Γ, the vector $\mathbf{F}(\mathbf{u})$ is tangential to Γ at $\mathbf{x}(\mathbf{u})$,

$$\mathbf{F} \cdot \mathbf{n} = 0, \ \ \mathbf{F} = \big(\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}\big) - G(\mathbf{u})\mathbf{n}(\mathbf{u}) \ \ \text{and} \ \ A^2 = \mathbf{F} \cdot \mathbf{F} + G^2. \tag{B.1}$$

For $\overset{\circ}{\mathbf{x}} \in \Gamma$ chosen, the surface polar coordinates then locally are given by the family of closed surfaces $\mathcal{C}_\varrho$, i.e., the level sets of the function $A(\mathbf{u})$, and curved radial rays on Γ through $\overset{\circ}{\mathbf{x}}$ which are perpendicular to $\mathcal{C}_\varrho$ for constant ϱ. Let us denote such a radial ray curve by $\mathbf{c}_{\boldsymbol{\Theta}} \in \Gamma$, given by $\mathbf{u} = \mathbf{u}(s)$, $0 \leq s$, where the parametric representation $\mathbf{x}(\mathbf{u}(s))$ at $s = 0$ starts in $\overset{\circ}{\mathbf{x}} = \mathbf{x}(\overset{\circ}{\mathbf{u}})$ in the direction of the unit vector $\mathbf{e}(\boldsymbol{\Theta}) = \mathbf{x}_{|i}(\overset{\circ}{\mathbf{u}})\Theta^i$, $\mathbf{x}_{|i} = \frac{\partial \mathbf{x}}{\partial u^i}$, $i = 1, \ldots, n-1$; $g_{jk} = \mathbf{x}_{|j} \cdot \mathbf{x}_{|k}$, $g_{jk}\Theta^j\Theta^k = 1$. Hence, for the curve $\mathbf{c}_{\boldsymbol{\Theta}} : \mathbf{x} = \mathbf{x}(\mathbf{u}(s))$, we require that

$$\mathbf{x}_{|j}(\overset{\circ}{\mathbf{u}})\frac{\mathrm{d}u^j}{\mathrm{d}s}(\overset{\circ}{\mathbf{u}}) = \mathbf{e}(\boldsymbol{\Theta}).$$

(Note that for $n = 3$ one usually uses $\Theta_1 = \cos\zeta$ and $\Theta_2 = \sin\zeta$.)

Without restriction of generality, we assume in what follows that at $\overset{\circ}{\mathbf{x}}$ we have $g_{jk}(\overset{\circ}{\mathbf{u}}) = \delta_{jk}$ (the Kronecker tensor). Since the radial ray curves $\mathbf{c}_{\boldsymbol{\Theta}} \in \Gamma$ are perpendicular to the level sets, which implicitly are given by $A = \varrho = \text{const}$, they satisfy the ordinary differential equations for fixed $\boldsymbol{\Theta}$:

$$\frac{\partial \mathbf{c}_{\boldsymbol{\Theta}}}{\partial s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\Big|_{\mathbf{c}_{\boldsymbol{\Theta}}} = \mathbf{x}_{|j}\frac{\mathrm{d}u^j}{\mathrm{d}s} = \frac{\mathbf{G}\mathrm{rad}\,A}{|\,\mathbf{G}\mathrm{rad}\,A|^2}(\mathbf{u}) \tag{B.2}$$

where $\mathbf{G}\mathrm{rad}\,A = g^{\ell k}A_{|\ell}\mathbf{x}_{|k}$ is the surface gradient and $g^{\ell k}g_{jk} = \delta_{j\ell}$. Since (B.1) implies on Γ that

$$AA_{|i} = (\mathbf{x} - \overset{\circ}{\mathbf{x}}) \cdot \mathbf{x}_{|i} = F_i = \mathbf{F} \cdot \mathbf{x}_{|i} \ \text{and} \ \mathbf{G}\mathrm{rad}\,A = g^{\ell k}\frac{F_\ell}{A}\mathbf{x}_{|k} = \frac{1}{A}F^k\mathbf{x}_{|k} = \frac{1}{A}\mathbf{F},$$

we find

$$\mathbf{G}\mathrm{rad}\,A \cdot \mathbf{G}\mathrm{rad}\,A = \frac{1}{A^2}\mathbf{F} \cdot \mathbf{F} = \frac{1}{A^2}(A^2 - G^2) = 1 - \frac{G^2}{A^2}$$

and the differential equations (B.2) take the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\Big|_{\mathbf{c}_{\boldsymbol{\Theta}}} = \mathbf{x}_{|j}\frac{\mathrm{d}u^j}{\mathrm{d}s} = \frac{\mathbf{G}\mathrm{rad}\,A}{|\,\mathbf{G}\mathrm{rad}\,A|^2}\Big(1 - \frac{G^2}{A^2}(\mathbf{u})\Big)^{-1}. \tag{B.3}$$

After multiplication with $\mathbf{x}_{|k}g^{\ell k}$, we thus arrive at

$$\frac{\mathrm{d}u^\ell}{\mathrm{d}s} = \frac{1}{A(\mathbf{u})}\Big(1 - \frac{G^2}{A^2}(\mathbf{u})\Big)^{-1}F^\ell(\mathbf{u}), \ \ F^\ell = g^{\ell k}\mathbf{F} \cdot \mathbf{x}_{|k}, \ \ell, k = 1, \ldots, n-1. \tag{B.4}$$

Theorem B.1 *To every $\overset{\circ}{\mathbf{x}} \in \Gamma$ there exists a neighborhood of $\overset{\circ}{\mathbf{x}}$ on Γ where the system (B.4) admits a unique solution*

$$u^\ell(s, \boldsymbol{\Theta}) = s\Theta^\ell + \mathcal{O}(s^2)$$

for $s \geq 0$ which is the solution of the Volterra integral equations

$$u^\ell(s, \boldsymbol{\Theta}) = \overset{\circ}{u}^\ell + s\Theta^\ell + \int_0^s \left\{ \frac{F^\ell\big(\mathbf{u}(\sigma, \boldsymbol{\Theta})\big)}{A\big(\mathbf{u}(\sigma, \boldsymbol{\Theta})\big)} \Big(1 - \frac{G^2\big(\mathbf{u}(\sigma, \boldsymbol{\Theta})\big)}{A^2\big(\mathbf{u}(\sigma, \boldsymbol{\Theta})\big)}\Big)^{-1} - \Theta^\ell \right\} d\sigma. \qquad \text{(B.5)}$$

The transformation $\mathbf{u} \mapsto (\varrho, \boldsymbol{\Theta})$ to surface polar coordinates about $\overset{\circ}{\mathbf{x}}$ is given by $\mathbf{u}(\varrho, \boldsymbol{\Theta})$, i.e., $s = \varrho \geq 0$.

Proof Since

$$\mathbf{F} = (\mathbf{x} - \overset{\circ}{\mathbf{x}}) - \big((\mathbf{x} - \overset{\circ}{\mathbf{x}}) \cdot \mathbf{n}\big)\mathbf{n},$$

the expansion about $\overset{\circ}{\mathbf{x}} = \mathbf{x}(\overset{\circ}{\mathbf{u}})$ gives on the one hand

$$\mathbf{x} - \overset{\circ}{\mathbf{x}} = \overset{\circ}{\mathbf{x}}_{|k} \frac{d\overset{\circ}{u}^k}{ds} s + \frac{1}{2}\left\{ \overset{\circ}{\mathbf{x}}_{|k|\ell} \frac{d\overset{\circ}{u}^k}{ds} \frac{d\overset{\circ}{u}^\ell}{ds} + \overset{\circ}{\mathbf{x}}_{|k} \frac{d^2\overset{\circ}{u}^k}{ds^2} \right\}s^2$$
$$+ \frac{1}{6}\left\{ \overset{\circ}{\mathbf{x}}_{|k|\ell|j} \frac{d\overset{\circ}{u}^k}{ds} \frac{d\overset{\circ}{u}^\ell}{ds} \frac{d\overset{\circ}{u}^j}{ds} + 3\overset{\circ}{\mathbf{x}}_{|k|\ell} \frac{d^2\overset{\circ}{u}^k}{ds^2} \frac{d\overset{\circ}{u}^\ell}{ds} + \mathbf{x}_{|k} \frac{d^3\overset{\circ}{u}^k}{ds^3} \right\}s^3 + \mathcal{O}(s^4). \qquad \text{(B.6)}$$

On the other hand, with the Gaussian equations

$$\mathbf{x}_{|j|k} = \Gamma_{jk}^\ell \mathbf{x}_{|\ell} + L_{jk}\mathbf{n}, \ \ell = 1, \dots, n-1$$

and the Weingarten relations

$$\mathbf{n}_{|j} = -L_j^m \mathbf{x}_{|m},$$

we obtain

$$\mathbf{x}_{|j|k|\ell} = \{\Gamma_{jk|\ell}^m + \Gamma_{jk}^r \Gamma_{r\ell}^m - L_{jk}L_\ell^m\}\mathbf{x}_{|m} + \{L_{jk|\ell} + \Gamma_{jk}^r L_{r\ell}\}\mathbf{n}$$

where Γ_{tj}^ℓ are the Christoffel symbols of the second kind of Γ at $\mathbf{x}$, $\Gamma_{tj|k}^\ell = \frac{\partial}{\partial u^k}\Gamma_{tj}^\ell(\mathbf{u})$, and $L_j^\ell = g^{\ell k}L_{kj}$ with L_{kj} the second fundamental form of Γ at $\mathbf{x}$ (see e.g. [13, p. 90]). Here and in what follows, we abbreviate

$$\overset{\circ}{\mathbf{x}}_{|k} = \mathbf{x}_{|k}(\overset{\circ}{\mathbf{u}}), \ \overset{\circ}{\mathbf{x}}_{|k|\ell} = \mathbf{x}_{|k|\ell}(\overset{\circ}{\mathbf{u}}) \text{ and } \frac{d\overset{\circ}{u}^k}{ds} = \frac{du}{ds}(\overset{\circ}{u}) \text{ etc.}$$

We get thus from (B.6)

$$\mathbf{x} - \overset{\circ}{\mathbf{x}} = \mathbf{e}(\boldsymbol{\Theta})s + \frac{1}{2}\left\{ \big(\overset{\circ}{\Gamma}_{k\ell}^m \overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}_{k\ell}\overset{\circ}{\mathbf{n}}\big)\Theta^k\Theta^\ell + \overset{\circ}{\mathbf{x}}_{|k}\frac{d^2\overset{\circ}{u}^k}{ds^2} \right\}s^2$$
$$+ \frac{1}{6}\left\{ \Big(\big[\overset{\circ}{\Gamma}_{jk|\ell}^t + \overset{\circ}{\Gamma}_{jk}^m \overset{\circ}{\Gamma}_{m\ell}^t - \overset{\circ}{L}_{jk}\overset{\circ}{L}_\ell^t\big]\overset{\circ}{\mathbf{x}}_{|t} + \big[\overset{\circ}{L}_{jk|\ell} + \overset{\circ}{\Gamma}_{jk}^m \overset{\circ}{L}_{m\ell}\big]\overset{\circ}{\mathbf{n}}\Big)\Theta^j\Theta^k\Theta^\ell \qquad \text{(B.7)}$$
$$+ 3\big(\overset{\circ}{\Gamma}_{k\ell}^t \overset{\circ}{\mathbf{x}}_{|t} + \overset{\circ}{L}_{k\ell}\overset{\circ}{\mathbf{n}}\big)\Theta^\ell \frac{d^2\overset{\circ}{u}^k}{ds^2} + \overset{\circ}{\mathbf{x}}_{|t}\frac{d^3\overset{\circ}{u}^t}{ds^3} \right\}s^3 + \mathcal{O}(s^4).$$

By combining this expansion with

$$\mathbf{n}(\mathbf{u}) = \overset{\circ}{\mathbf{n}} + \overset{\circ}{\mathbf{n}}_{|j}\frac{d\overset{\circ}{u}^j}{ds}s + \frac{1}{2}\left\{ \overset{\circ}{\mathbf{n}}_{|j|k}\Theta^k\Theta^j + \overset{\circ}{\mathbf{n}}_{|j}\frac{d^2\overset{\circ}{u}^j}{ds^2} \right\}s^2 + \mathcal{O}(s^3),$$

we arrive at

$$
\begin{aligned}
G &= (\mathbf{x} - \overset{\circ}{\mathbf{x}}) \cdot \mathbf{n}(\mathbf{u})\\[4pt]
&= -\overset{\circ}{\mathbf{x}}_{|k} \cdot \overset{\circ}{\mathbf{x}}_{|m} \Theta^k \overset{\circ}{L}{}^m_j \Theta^j s^2 + \frac{1}{2}\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell s^2 - \frac{1}{2}L_{jk}\Theta^k \frac{\mathrm{d}^2 \overset{\circ}{u}{}^j}{\mathrm{d}s^2} s^3\\[4pt]
&\quad + \left\{\frac{1}{6}\Big(\overset{\circ}{L}_{jk|\ell} + \overset{\circ}{\Gamma}{}^m_{jk}\overset{\circ}{L}_{m\ell}\Big) - \frac{1}{2}\Big(\overset{\circ}{L}_{jm}\overset{\circ}{\Gamma}{}^m_{k\ell} + \overset{\circ}{L}{}^\ell_{j|k} + \overset{\circ}{L}{}^m_j \overset{\circ}{\Gamma}{}^\ell_{mk}\Big)\right\}\Theta^j\Theta^k\Theta^\ell s^3 + \mathcal{O}(s^4)\\[4pt]
&= -\frac{1}{2}\overset{\circ}{L}_{kj}\Theta^k\Theta^j s^2 + \left\{\frac{1}{6}\overset{\circ}{L}_{jk|\ell} - \frac{1}{3}\overset{\circ}{L}_{jm}\overset{\circ}{\Gamma}{}^m_{k\ell} - \frac{1}{2}\overset{\circ}{L}{}^\ell_{j|k} - \frac{1}{2}\overset{\circ}{L}{}^m_j \overset{\circ}{\Gamma}{}^\ell_{mk}\right\}\Theta^j\Theta^k\Theta^\ell s^3\\[4pt]
&\quad - \frac{1}{2}L_{jk}\Theta^k \frac{\mathrm{d}^2 \overset{\circ}{u}{}^j}{\mathrm{d}s^2} s^3 + \mathcal{O}(s^4),\\[6pt]
\mathbf{F} &= \mathbf{e}(\boldsymbol{\Theta})s + \left\{\frac{1}{2}\Big(\big[\overset{\circ}{\Gamma}{}^m_{k\ell}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}_{k\ell}\overset{\circ}{\mathbf{n}}\big]\Theta^k\Theta^\ell + \overset{\circ}{\mathbf{x}}_{|k}\frac{\mathrm{d}^2 \overset{\circ}{u}{}^k}{\mathrm{d}s^2}\Big) + \frac{1}{2}\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell \overset{\circ}{\mathbf{n}}\right\}s^2 + \mathcal{O}(s^3),\\[6pt]
A &= s + \mathcal{O}(s^2).
\end{aligned}
$$

Hence, we conclude

$$
\frac{\mathbf{F}}{A}\Big(1 - \frac{G^2}{A^2}\Big)^{-1} = \mathbf{e}(\boldsymbol{\Theta}) + \mathcal{O}(s)
$$

for any C^2-curve $\mathbf{x}\big(\mathbf{u}(s)\big)$ through $\overset{\circ}{\mathbf{x}}$. Consequently, the kernel function $\{\dots\}$ of the Volterra operator in (B.5) is continuous and there exists a solution $\mathbf{u}(s,\boldsymbol{\Theta})$ for fixed given $\boldsymbol{\Theta}$ in some vicinity of $\overset{\circ}{\mathbf{x}}$ on Γ. This solution is in $C^1([0,S])$ for some $S > 0$ and is as many times continuously differentiable for $s \ge 0$ as is the manifold Γ.

The equation (B.3) implies

$$
1 = \mathbf{G}\mathrm{rad}\, A \cdot \mathbf{x}_{|j}\frac{\mathrm{d}u^j}{\mathrm{d}s} = g^{\ell k}A_{|k}\mathbf{x}_{|\ell}\cdot\mathbf{x}_{|j}\frac{\mathrm{d}u^j}{\mathrm{d}s} = g^{\ell k}g_{\ell j}A_{|k}\frac{\mathrm{d}u^j}{\mathrm{d}s} = A_{|j}\frac{\mathrm{d}u^j}{\mathrm{d}s} = \frac{\mathrm{d}A}{\mathrm{d}s} = \frac{\mathrm{d}\varrho}{\mathrm{d}s}.
$$

Thus, it holds $A = s = \varrho$ and $\mathbf{u}(\varrho,\boldsymbol{\Theta})$, the solution of (B.5), is the desired transformation $(\varrho,\boldsymbol{\Theta}) \mapsto \mathbf{u}$.

By bootstrapping, it follows from (B.4) that $\mathbf{u}(\varrho,\boldsymbol{\Theta})$ is higher order differentiable up to $\varrho = 0$. To see this, consider the Taylor expansions of the left and the right hand sides of the equations (B.4) about $\overset{\circ}{\mathbf{u}}$ up to the order two:

Left hand side of (B.4): (by using (B.3))

$$
\begin{aligned}
\frac{\mathrm{d}u^\ell}{\mathrm{d}\varrho}\mathbf{x}_{|\ell}(\mathbf{u}) &= \frac{\mathrm{d}\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho}\overset{\circ}{\mathbf{x}}_{|\ell} + \left\{\overset{\circ}{\mathbf{x}}_{|\ell}\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2} + \overset{\circ}{\mathbf{x}}_{|\ell|j}\frac{\mathrm{d}\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho}\frac{\mathrm{d}\overset{\circ}{u}{}^j}{\mathrm{d}\varrho}\right\}\varrho\\[4pt]
&\quad + \frac{1}{2}\left\{\overset{\circ}{\mathbf{x}}_{|\ell}\frac{\mathrm{d}^3\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^3} + \overset{\circ}{\mathbf{x}}_{|\ell|j|k}\frac{\mathrm{d}\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho}\frac{\mathrm{d}\overset{\circ}{u}{}^j}{\mathrm{d}\varrho}\frac{\mathrm{d}\overset{\circ}{u}{}^k}{\mathrm{d}\varrho} + 3\overset{\circ}{\mathbf{x}}_{\ell|j}\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2}\frac{\mathrm{d}\overset{\circ}{u}{}^j}{\mathrm{d}\varrho}\right\}\varrho^2 + \mathcal{O}(\varrho^3)\\[4pt]
&= \overset{\circ}{\mathbf{x}}_{|\ell}\Theta^\ell + \left\{\overset{\circ}{\mathbf{x}}_{|\ell}\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2} + \big(\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{\Gamma}_{\ell j}\overset{\circ}{\mathbf{n}}\big)\Theta^\ell\Theta^j\right\}\varrho\\[4pt]
&\quad + \frac{1}{2}\left\{\overset{\circ}{\mathbf{x}}_{|\ell}\frac{\mathrm{d}^3\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^3} + \Big(\big[\overset{\circ}{\Gamma}{}^m_{\ell j|k} + \overset{\circ}{\Gamma}{}^t_{\ell j}\overset{\circ}{\Gamma}{}^m_{tk} - \overset{\circ}{L}_{\ell j}\overset{\circ}{L}{}^m_k\big]\overset{\circ}{\mathbf{x}}_{|m} + \big[\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{L}_{mk} + \overset{\circ}{L}_{\ell j|k}\big]\overset{\circ}{\mathbf{n}}\Big)\Theta^\ell\Theta^j\Theta^k\right.\\[4pt]
&\quad\quad \left. + 3\big(\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}_{\ell j}\overset{\circ}{\mathbf{n}}\big)\Theta^j\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2}\right\}\varrho^2 + \mathcal{O}(\varrho^3).
\end{aligned}
$$

<u>Right hand side of (B.4): (see (B.7))</u>

$$\frac{1}{\varrho}(\mathbf{x} - \overset{\circ}{\mathbf{x}}) = \overset{\circ}{\mathbf{x}}_{|\ell}\Theta^\ell + \frac{1}{2}\Big\{ \big(\overset{\circ}{\Gamma}{}^m_{k\ell}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}_{k\ell}\overset{\circ}{\mathbf{n}}\big)\Theta^k\Theta^\ell + \overset{\circ}{\mathbf{x}}_{|k}\frac{\mathrm{d}^2\overset{\circ}{u}{}^k}{\mathrm{d}\varrho^2} \Big\}\varrho$$

$$+ \frac{1}{6}\Big\{ \overset{\circ}{\mathbf{x}}_{|\ell}\frac{\mathrm{d}^3\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^3} + \big(\big[\overset{\circ}{\Gamma}{}^m_{\ell j|k} + \overset{\circ}{\Gamma}{}^t_{\ell j}\overset{\circ}{\Gamma}{}^m_{tk} - \overset{\circ}{L}_{\ell j}\overset{\circ}{L}{}^m_k\big]\overset{\circ}{\mathbf{x}}_{|m} + \big[\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{L}_{mk} + \overset{\circ}{L}_{\ell j|k}\big]\overset{\circ}{\mathbf{n}}\big)\Theta^\ell\Theta^j\Theta^k$$

$$+ 3\big(\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}_{\ell j}\overset{\circ}{\mathbf{n}}\big)\Theta^j\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2} \Big\}\varrho^2 + \mathcal{O}(\varrho^3),$$

$$\mathbf{n}(\mathbf{u}) = \overset{\circ}{\mathbf{n}} + \overset{\circ}{\mathbf{n}}_{|j}\frac{\mathrm{d}\overset{\circ}{u}{}^j}{\mathrm{d}\varrho}\,\varrho + \frac{1}{2}\Big\{ \overset{\circ}{\mathbf{n}}_{|j|k}\frac{\mathrm{d}\overset{\circ}{u}{}^j}{\mathrm{d}\varrho}\frac{\mathrm{d}\overset{\circ}{u}{}^k}{\mathrm{d}\varrho} + \overset{\circ}{\mathbf{n}}_{|j}\frac{\mathrm{d}^2\overset{\circ}{u}{}^j}{\mathrm{d}\varrho^2} \Big\}\varrho^2 + \mathcal{O}(\varrho^3)$$

$$= \overset{\circ}{\mathbf{n}} - \overset{\circ}{L}{}^m_j\,\overset{\circ}{\mathbf{x}}_{|m}\Theta^j\varrho$$

$$- \frac{1}{2}\Big\{ \big(\overset{\circ}{L}{}^m_{j|k}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}{}^\ell_j\overset{\circ}{\Gamma}{}^m_{\ell k}\overset{\circ}{\mathbf{x}}_{|m} + \overset{\circ}{L}{}^m_j\overset{\circ}{L}_{mk}\overset{\circ}{\mathbf{n}}\big)\Theta^j\Theta^k + \overset{\circ}{L}{}^m_j\frac{\mathrm{d}^2\overset{\circ}{u}{}^j}{\mathrm{d}\varrho^2}\,\overset{\circ}{\mathbf{x}}_{|m} \Big\}\varrho^2 + \mathcal{O}(\varrho^3),$$

$$\frac{1}{\varrho}G = \frac{1}{2}\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell\varrho - \frac{1}{2}\overset{\circ}{L}_{j\ell}\frac{\mathrm{d}^2\overset{\circ}{u}{}^j}{\mathrm{d}\varrho^2}\,\Theta^\ell$$

$$+ \Big\{ \frac{1}{6}\overset{\circ}{L}_{\ell j|k} - \frac{1}{3}\overset{\circ}{\Gamma}{}^m_{\ell j}\overset{\circ}{L}_{mk} - \frac{1}{2}\overset{\circ}{g}_{m\ell}\overset{\circ}{L}{}^m_{j|k} - \frac{1}{2}\overset{\circ}{g}_{m\ell}\overset{\circ}{L}{}^t_j\overset{\circ}{\Gamma}{}^m_{tk} \Big\}\Theta^j\Theta^k\Theta^\ell\varrho^2 + \mathcal{O}(\varrho^3),$$

$$\Big(\frac{1}{\varrho}G\Big)\mathbf{n}(\mathbf{u}) = \Big(\frac{1}{\varrho}G\Big)\Big(\overset{\circ}{\mathbf{n}} - \overset{\circ}{L}{}^m_j\,\Theta^j\overset{\circ}{\mathbf{x}}_{|m}\varrho\Big) + \mathcal{O}(\varrho^3)$$

$$= \overset{\circ}{\mathbf{n}}\frac{1}{2}\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell\varrho + \overset{\circ}{\mathbf{n}}\varrho^2 - \frac{1}{2}\overset{\circ}{L}{}^m_j\,\overset{\circ}{L}_{k\ell}\overset{\circ}{\mathbf{x}}_{|m}\Theta^j\Theta^k\Theta^\ell\varrho^2 - \frac{1}{2}\overset{\circ}{L}_{j\ell}\frac{\mathrm{d}^2\overset{\circ}{u}{}^j}{\mathrm{d}\varrho^2}\,\Theta^\ell\overset{\circ}{\mathbf{n}}\varrho^2 + \mathcal{O}(\varrho^3),$$

$$\frac{G^2}{A^2} = \Big(\frac{1}{\varrho}G\Big)^2 = \frac{1}{4}\big(\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell\big)^2\varrho^2 + \mathcal{O}(\varrho^3),$$

$$\Big(1 - \frac{G^2}{A^2}\Big)^{-1} = 1 + \frac{1}{4}\big(\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell\big)^2\varrho^2 + \mathcal{O}(\varrho^3).$$

Comparing the coefficients of $\mathbf{x}_{|\ell}\varrho$ gives:

$$\underline{\text{lhs:}}\quad \frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2} + \overset{\circ}{\Gamma}{}^\ell_{jk}\Theta^j\Theta^k \quad \text{and} \quad \underline{\text{rhs:}}\quad \frac{1}{2}\overset{\circ}{\Gamma}{}^\ell_{jk}\Theta^j\Theta^k + \frac{1}{2}\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2}\,.$$

Consequently, there holds

$$\frac{\mathrm{d}^2\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^2}\,(\overset{\circ}{\mathbf{u}}) = -\overset{\circ}{\Gamma}{}^\ell_{jk}\Theta^j\Theta^k.$$

Next, compare the coefficients of $\overset{\circ}{\mathbf{x}}_{|\ell}\varrho^2$:

<u>Left hand side of (B.4):</u>

$$\frac{1}{2}\,\frac{\mathrm{d}^3\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^3} + \frac{1}{2}\big(\overset{\circ}{\Gamma}{}^\ell_{mj|k} + \overset{\circ}{\Gamma}{}^t_{mj}\overset{\circ}{\Gamma}{}^\ell_{tk} - \overset{\circ}{L}_{mj}\overset{\circ}{L}{}^\ell_k\big)\Theta^m\Theta^j\Theta^k + \frac{3}{2}\overset{\circ}{\Gamma}{}^\ell_{mj}\Theta^j\frac{\mathrm{d}^2\overset{\circ}{u}{}^m}{\mathrm{d}\varrho^2}$$

$$= \frac{1}{2}\,\frac{\mathrm{d}^3\overset{\circ}{u}{}^\ell}{\mathrm{d}\varrho^3} + \frac{1}{2}\big(\overset{\circ}{\Gamma}{}^\ell_{mj|k} + \overset{\circ}{\Gamma}{}^t_{mj}\overset{\circ}{\Gamma}{}^\ell_{tk} - \overset{\circ}{L}_{mj}\overset{\circ}{L}{}^\ell_k\big)\Theta^m\Theta^j\Theta^k - \frac{3}{2}\overset{\circ}{\Gamma}{}^\ell_{tj}\overset{\circ}{\Gamma}{}^t_{km}\Theta^m\Theta^j\Theta^k.$$

Right hand side of (B.4):

$$\left(\frac{1}{\varrho}(\mathbf{x} - \overset{\circ}{\mathbf{x}}) - \frac{1}{\varrho}G\mathbf{n}\right)\left(1 + \frac{1}{4}\varrho^2(\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell)^2 + \mathcal{O}(\varrho^3)\right)$$

$$= \Theta^\ell\overset{\circ}{\mathbf{x}}_{|\ell} - \varrho\frac{1}{2}\overset{\circ}{\Gamma}{}^{\ell}_{km}\Theta^k\Theta^m\overset{\circ}{\mathbf{x}}_{|\ell}$$

$$+ \left\{\frac{1}{6}\frac{\mathrm{d}^3\overset{\circ}{u}{}^{\ell}}{\mathrm{d}\varrho^3} + \left(\frac{1}{6}\overset{\circ}{\Gamma}{}^{\ell}_{mj|k} - \frac{1}{3}\overset{\circ}{\Gamma}{}^{t}_{mj}\overset{\circ}{\Gamma}{}^{\ell}_{tk} - \frac{1}{3}\overset{\circ}{L}_{j}\overset{\circ}{L}_{km}\right)\Theta^m\Theta^j\Theta^k + \frac{1}{4}(\overset{\circ}{L}_{kj}\Theta^k\Theta^j)^2\Theta^\ell\right\}\overset{\circ}{\mathbf{x}}_{|\ell}.$$

Hence, from equating left hand side and right side, one obtains

$$\frac{\mathrm{d}^3u^\ell}{\mathrm{d}\varrho^3}(\overset{\circ}{\mathbf{u}}) = \left\{-\overset{\circ}{\Gamma}{}^{\ell}_{mj|k} + 2\overset{\circ}{\Gamma}{}^{t}_{mj}\overset{\circ}{\Gamma}{}^{\ell}_{tk} + \frac{1}{2}\overset{\circ}{L}_{mj}\overset{\circ}{L}_{k}{}^{\ell}\right\}\Theta^m\Theta^j\Theta^k + \frac{3}{4}(\overset{\circ}{L}_{kj}\Theta^k\Theta^j)^2\Theta^\ell.$$

With the expressions for $\frac{\mathrm{d}^2\overset{\circ}{u}{}^{\ell}}{\mathrm{d}s^2}$ and $\frac{\mathrm{d}^3\overset{\circ}{u}{}^{\ell}}{\mathrm{d}s^3}$, we recollect the relations for $\mathbf{x} - \overset{\circ}{\mathbf{x}}$ and find instead of (B.6):

$$(\mathbf{x} - \mathbf{x}_0) = \varrho\mathbf{e}(\boldsymbol{\Theta}) + \frac{1}{2}\varrho^2\overset{\circ}{\mathbf{n}}\overset{\circ}{L}_{k\ell}\Theta^k\Theta^\ell$$

$$+ \frac{\varrho^3}{6}\left\{\left(-\frac{1}{2}\overset{\circ}{\mathbf{x}}_{|\ell}\overset{\circ}{L}_{mj}\overset{\circ}{L}_{k}{}^{\ell} + \left[2\overset{\circ}{L}_{mj|k} - 3\overset{\circ}{L}_{tj}\overset{\circ}{\Gamma}{}^{t}_{mk}\right]\overset{\circ}{\mathbf{n}}\right)\Theta^m\Theta^j\Theta^k + \frac{4}{3}(\overset{\circ}{L}_{kj}\Theta^k\Theta^j)^2\boldsymbol{\Theta}\right\} + \mathcal{O}(\varrho^4).$$

Collecting the first three derivatives of u^ℓ at $\overset{\circ}{\mathbf{u}}$ implies that the first terms of the transform $(\varrho, \boldsymbol{\Theta}) \mapsto \mathbf{u}$ read as

$$u^\ell(\varrho, \boldsymbol{\Theta}) = \overset{\circ}{u}{}^{\ell} + \Theta^\ell\varrho - \frac{1}{2}\overset{\circ}{\Gamma}{}^{\ell}_{jk}\Theta^j\Theta^k\varrho^2 \tag{B.8}$$

$$+ \frac{1}{6}\left\{\frac{1}{2}\overset{\circ}{L}_{j}{}^{\ell}\overset{\circ}{L}_{km} + 2\overset{\circ}{\Gamma}{}^{\ell}_{tj}\overset{\circ}{\Gamma}{}^{t}_{km} - \overset{\circ}{\Gamma}{}^{\ell}_{mj|k}\right\}\Theta^j\Theta^k\Theta^m\varrho^3 + \frac{1}{8}(\overset{\circ}{L}_{jk}\Theta^j\Theta^k)^2\Theta^\ell\varrho^3 + \mathcal{O}(\varrho^4),$$

for all $\ell = 1, \ldots, n-1$. Whereas, the inverse mapping $\mathbf{u} \mapsto (\varrho, \boldsymbol{\Theta})$ can be obtained from

$$\varrho = A(\mathbf{u}) = |\mathbf{x}(\mathbf{u}) - \overset{\circ}{\mathbf{x}}|$$

and the nonlinear equations for

$$\Theta^\ell = \frac{1}{\varrho}\left(u^\ell - \overset{\circ}{u}{}^{\ell}\right) + \frac{1}{2}\varrho\overset{\circ}{\Gamma}{}^{\ell}_{jk}\Theta^j\Theta^k$$

$$- \frac{1}{6}\varrho^2\left\{\frac{1}{2}\overset{\circ}{L}_{j}{}^{\ell}\overset{\circ}{L}_{km} + 2\overset{\circ}{\Gamma}{}^{\ell}_{tj}\overset{\circ}{\Gamma}{}^{t}_{km} - \overset{\circ}{\Gamma}{}^{\ell}_{mj|k}\right\}\Theta^j\Theta^k\Theta^m - \frac{1}{8}\varrho^2(\overset{\circ}{L}_{jk}\Theta^j\Theta^k)^2\Theta^\ell + \ldots$$

for all $\ell = 1, \ldots, n-1$ can, for $\varrho > 0$ sufficiently small, be solved via successive iteration. □

Theorem B.2 *Let $\Gamma \in C^4$. Then, the surface measure $\mathrm{d}s_\Gamma$ of Γ in Martensen's surface polar coordinates satisfies*

$$\mathrm{d}s_\Gamma = \varrho^{n-2}\,\mathrm{d}\varrho \wedge \mathrm{d}\boldsymbol{\omega} + \left(\varrho^n a(\boldsymbol{\Theta}) + \mathcal{O}(\varrho^{n+1})\right)\mathrm{d}\varrho \wedge \mathrm{d}\boldsymbol{\omega} \tag{B.9}$$

where

$$a(\boldsymbol{\Theta}) = \sum_{j=1}^{n-1}\Theta^j\left\{\frac{3}{4}\left((\overset{\circ}{L}_{kt}\Theta^k\Theta^t)^2 + \overset{\circ}{L}_{t}{}^{j}\overset{\circ}{L}_{mk}\Theta^t\Theta^m\Theta^k\right) - \Theta^j\overset{\circ}{L}_{m}{}^{\ell}\overset{\circ}{L}_{\ell k}\Theta^m\Theta^k\right\}$$

and

$$\mathrm{d}\boldsymbol{\omega} = \sum_{j=1}^{n-1}(-1)^{j+1}\Theta^j[\mathrm{d}\Theta^1 \wedge \cdots \wedge \widehat{\mathrm{d}\Theta^j} \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}], \quad \sum_{j=1}^{n-1}\Theta^j\Theta^j = 1.$$

Here, $\mathrm{d}\boldsymbol{\omega}$ is the surface measure of the unit sphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^{n-1}$.

Proof For the surface measure of Γ, we use the exterior Pfaffian products, defining the exterior normal vector's components times $\mathrm{d}s_\Gamma$ (see [20, Chapter 11.4]). Then, we multiply scalarly with the exterior unit normal $\mathbf{n}(\mathbf{x}) = \big(n^1(\mathbf{x}), \dots, n^n(\mathbf{x})\big)$ which yields

$$\mathrm{d}s_\Gamma = \sum_{j=1}^{n}(-1)^{j+1}\big[\,\mathrm{d}x^1 \wedge \cdots \wedge \cancel{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n\big]n^j(\mathbf{x}). \tag{B.10}$$

Expansions about $\overset{\circ}{\mathbf{x}}$ up to the order ϱ^2 give

$$\mathbf{n}(\mathbf{x}) = \overset{\circ}{\mathbf{n}} - \varrho \overset{\circ}{L}_j{}^m \overset{\circ}{\mathbf{x}}_{|m}\Theta^j - \frac{1}{2}\varrho^2\Big\{\overset{\circ}{L}_{j|k}^m + \overset{\circ}{L}_j^\ell \overset{\circ}{\varGamma}_{\ell k}^m - \overset{\circ}{\varGamma}_{jk}^t \overset{\circ}{L}_t^m\Big\}\Theta^j\Theta^k\overset{\circ}{\mathbf{x}}_{|m}$$

$$- \frac{1}{2}\varrho^2\overset{\circ}{L}_j^\ell \overset{\circ}{L}_{\ell k}\Theta^j\Theta^k\overset{\circ}{\mathbf{n}} + \mathcal{O}(\varrho^3)$$

and

$$\mathrm{d}x^j = \mathrm{d}\varrho\Theta^j + \varrho\,\mathrm{d}\Theta^j + \frac{1}{2}\varrho^2\,\mathrm{d}\varrho\Big\{\frac{3}{4}\big(\overset{\circ}{L}_{kt}\Theta^k\Theta^t\big)^2 - \frac{1}{2}\overset{\circ}{L}_k^j \overset{\circ}{L}_{mt}\Theta^k\Theta^m\Theta^t\Big\} + \mathcal{O}(\varrho^3)$$

$$= \mathrm{d}\varrho\Theta^j + \varrho\,\mathrm{d}\Theta^j + \varrho^2\,\mathrm{d}\varrho\, c^j(\boldsymbol{\Theta}) + \mathcal{O}(\varrho^3)$$

with

$$c^j(\boldsymbol{\Theta}) = \frac{1}{2}\Big\{\frac{3}{4}\big(\overset{\circ}{L}_{kt}\Theta^k\Theta^t\big)^2 - \frac{1}{2}\overset{\circ}{L}_k^j \overset{\circ}{L}_{mt}\Theta^k\Theta^m\Theta^t\Big\}$$

for $j = 1, \dots, n-1$ and

$$\mathrm{d}x^n = \overset{\circ}{L}_{jk}\Theta^j\Theta^k\varrho\,\mathrm{d}\varrho + \frac{1}{2}\Big(2\overset{\circ}{L}_{mj|k} - 3\overset{\circ}{L}_{\ell j}\overset{\circ}{\varGamma}_{mk}^\ell\Big)\Theta^m\Theta^j\Theta^k\varrho^2\,\mathrm{d}\varrho + \mathcal{O}(\varrho^3).$$

Inserting this into (B.10), yields

$$\mathrm{d}s_\Gamma = (-1)^{n+1}\big[\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}\big]\Big(1 - \frac{1}{2}\varrho^2\overset{\circ}{L}_j^m \overset{\circ}{L}_{mk}\Theta^j\Theta^k + \mathcal{O}(\varrho^3)\Big)$$

$$+ \sum_{j=1}^{n-1}\Big[\big[\,\mathrm{d}x^1 \wedge \cdots \wedge \cancel{\mathrm{d}x^j} \wedge \mathrm{d}x^{n-1}\big] \wedge \mathrm{d}x^n\Big]$$

$$\times \Big\{ - \varrho\overset{\circ}{L}_k^j\Theta^k - \frac{1}{2}\Big(\overset{\circ}{L}_{m|k}^j + \overset{\circ}{L}_m^\ell \overset{\circ}{\varGamma}_{\ell k}^j - \overset{\circ}{\varGamma}_{mk}^t \overset{\circ}{L}_t^j\Big)\Theta^m\Theta^k\varrho^2 + \mathcal{O}(\varrho^3)\Big\}. \tag{B.11}$$

For the first term in (B.11), we obtain (modulo $\mathcal{O}(\varrho^3)$ terms) with the relations for $\mathrm{d}x^j$:

$$\big[\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}\big] = \big[\,\mathrm{d}\varrho\Theta^1 + \varrho\,\mathrm{d}\Theta^1 + \varrho^2\,\mathrm{d}\varrho\,c^1(\boldsymbol{\Theta}) \wedge \mathrm{d}\varrho\Theta^2 + \varrho\,\mathrm{d}\Theta^2 + \varrho^2\,\mathrm{d}\varrho\,c^2(\boldsymbol{\Theta})$$

$$\wedge \cdots \wedge \mathrm{d}\varrho\Theta^{n-1} + \varrho\,\mathrm{d}\Theta^{n-1} + \varrho^2\,\mathrm{d}\varrho\,c^{n-1}(\boldsymbol{\Theta})\big] = \varrho^{n-1}\big[\mathrm{d}\Theta^1 \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}\big]$$

$$+ \varrho^{n-2}\Big[\mathrm{d}\varrho \wedge \sum_{j=1}^{n-1}(-1)^j\big\{\Theta^j + \varrho^2 c^j(\boldsymbol{\Theta})\big\}\big[\mathrm{d}\Theta^1 \wedge \cdots \wedge \cancel{\mathrm{d}\Theta^j} \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}\big]\Big].$$

Since the variables Θ^j vary on the $(n-2)$-dimensional sphere $\mathbb{S}^{n-2}$, where

$$\sum_{j=1}^{n-1}(\Theta^j)^2 = 1,$$

we have

$$\sum_{j=1}^{n-1}\Theta^j\,\mathrm{d}\Theta^j = 0 \tag{B.12}$$

on $\mathbb{S}^{n-2}$. Hence, the differentials $\mathrm{d}\Theta^j$ with $j = 1, \dots, n-1$ are linearly dependent and

$$\big[\mathrm{d}\Theta^1 \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}\big] = 0.$$

Moreover, on $\mathbb{S}^{n-2}$, we have that the exterior unit normal vector $\boldsymbol{\nu}$ to $\mathbb{S}^{n-2}$ satisfies

$$\boldsymbol{\nu}_{\boldsymbol{\omega}} = \sum_{j=1}^{n-1} \Theta^j \mathbf{e}_j.$$

Therefore, it holds

$$\nu_j \, \mathrm{d}\boldsymbol{\omega} = (-1)^{j+1}\big[\mathrm{d}\Theta^1 \wedge \cdots \wedge \cancel{\mathrm{d}\Theta^j} \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}\big] = \Theta^j \, \mathrm{d}\boldsymbol{\omega},$$

and with (B.12) one obtains

$$\mathrm{d}\boldsymbol{\omega} = \sum_{j=1}^{n-1} (-1)^{j+1}\Theta^j \big[\mathrm{d}\Theta^1 \wedge \cdots \wedge \cancel{\mathrm{d}\Theta^j} \wedge \cdots \wedge \mathrm{d}\Theta^{n-1}\big].$$

For the first term in (B.11), we find therefore

$$\mathrm{d}s_\Gamma^1 = (-1)^{n+1}\left\{ \varrho^{n-2} + \varrho^n\Big(\sum_{j=1}^{n-1} c^j(\boldsymbol{\Theta})\Theta^j - \frac{1}{2}\overset{\circ}{L}{}^\ell_m\overset{\circ}{L}_{\ell k}\Theta^m\Theta^k\Big) + \mathcal{O}(\varrho^{n+1})\right\}\mathrm{d}\varrho \wedge \mathrm{d}\boldsymbol{\omega}.$$

For the remaining terms in (B.11), we have (modulo $\mathcal{O}(\varrho^3)$ terms) that

$$
\begin{aligned}
\mathrm{d}s_\Gamma^2 &= \Big[\big[\mathrm{d}x^1 \wedge \cdots \wedge \cancel{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^{n-1}\big] \wedge \mathrm{d}x^n\Big]\Big(-\varrho\overset{\circ}{L}{}^j_t\Theta^t - \frac{1}{2}\varrho^2\{\ldots\}\Theta^m\Theta^k\Big)\\
&= \Big[\big[\mathrm{d}\varrho\Theta^1 + \varrho\,\mathrm{d}\Theta^1 + \varrho^2\,\mathrm{d}\varrho c^1(\boldsymbol{\Theta})\wedge\mathrm{d}\varrho\Theta^2 + \varrho\,\mathrm{d}\Theta^2 + \varrho^2\,\mathrm{d}\varrho c^2(\boldsymbol{\Theta})\\
&\qquad \wedge\cdots\wedge\cancel{\mathrm{d}x^j}\wedge\cdots\wedge\mathrm{d}\varrho^2\Theta^{n-1} + \varrho\,\mathrm{d}\Theta^{n-1} + \varrho^2\,\mathrm{d}\varrho c^{n-1}(\boldsymbol{\Theta})\big]\\
&\qquad \wedge\Big(\varrho\,\mathrm{d}\varrho\overset{\circ}{L}_{jk}\Theta^j\Theta^k + \frac{1}{2}\varrho^2\,\mathrm{d}\varrho\{\ldots\}\Theta^m\Theta^j\Theta^k\Big)\Big]\Big(-\varrho\overset{\circ}{L}{}^k_t\Theta^t - \frac{1}{2}\{\ldots\}\Theta^m\Theta^k\Big)\\
&= \sum_{j=1}^{n-1}(-1)^{n+j}\big(\varrho^n + \mathcal{O}(\varrho^{n+1})\big)\,\mathrm{d}\varrho\wedge\big[\mathrm{d}\Theta^1\wedge\cdots\wedge\cancel{\mathrm{d}\Theta^j}\wedge\cdots\wedge\mathrm{d}\Theta^{n-1}\big]\overset{\circ}{L}_{mk}\Theta^m\Theta^k\overset{\circ}{L}{}^j_t\Theta^t\\
&= (-1)^{n+1}\sum_{j=1}^{n-1}\overset{\circ}{L}_{mk}\Theta^m\Theta^k\overset{\circ}{L}{}^j_t\Theta^t\Theta^j\big(\varrho^n + \mathcal{O}(\varrho^{n+1})\big)\,\mathrm{d}\varrho\wedge\mathrm{d}\boldsymbol{\omega}.
\end{aligned}
$$

Consequently, we finally get in (B.11)

$$\mathrm{d}s_\Gamma = \mathrm{d}s_\Gamma^1 + \mathrm{d}s_\Gamma^2 = \varrho^{n-2}\,\mathrm{d}\varrho \wedge \mathrm{d}\boldsymbol{\omega} + \big(\varrho^n a(\boldsymbol{\Theta}) + \mathcal{O}(\varrho^{n+1})\big)\,\mathrm{d}\varrho \wedge \mathrm{d}\boldsymbol{\omega}$$

with

$$a(\boldsymbol{\Theta}) = \sum_{j=1}^{n-1}\Theta^j\left\{\frac{3}{4}\big(\overset{\circ}{L}_{kt}\Theta^k\Theta^t\big)^2 + \overset{\circ}{L}{}^j_t\overset{\circ}{L}_{mk}\Theta^t\Theta^m\Theta^k - \Theta^j\overset{\circ}{L}{}^\ell_m\overset{\circ}{L}_{\ell k}\Theta^m\Theta^k\right\}.$$

This is the proposed relation (B.9) for the surface measure. $\qquad\square$

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