Multilevel quadrature for elliptic parametric partial differential equations on non-nested meshes

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Abstract Multilevel quadrature methods for parametric operator equations such as the multilevel (quasi-) Monte Carlo method are closely related to the sparse tensor product approximation between the spatial variable and the stochastic variable. In this article, we employ this fact and reverse the multilevel quadrature method via the sparse grid construction by applying differences of quadrature rules to finite element discretizations of different resolution. Besides being more efficient if the underlying quadrature rules are nested, this way of performing the sparse tensor product approximation enables the use of non-nested and even adaptively refined finite element meshes. Especially, the multilevel quadrature is non-intrusive and allows the use of standard finite element solvers. Numerical results are provided to illustrate the approach.

1 Introduction

The present article is concerned with the numerical solution of elliptic parametric second order boundary value problems of the form

$$-\operatorname{div}\big(a(\mathbf{y})\nabla u(\mathbf{y})\big) = f(\mathbf{y}) \text{ in } D, \quad u(\mathbf{y}) = 0 \text{ on } \partial D, \quad \mathbf{y} \in \square, \tag{1}$$

where $D \subset \mathbb{R}^n$ denotes the spatial domain and $\square \subset \mathbb{R}^m$ denotes the parameter domain. Prominent representatives of such problems arise from recasting boundary value problems with random data, like random diffusion coefficients, random right hand sides and even random domains. A high-dimensional parametric boundary

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value problem of the form (1) is then derived by inserting the truncated Karhunen-Loève expansion, see e.g. [2,3,10,22,30]. Thus, the computation of the solution's statistics amounts to a high-dimensional Bochner integration problem which can be dealt with by quadrature methods. Any quadrature method requires the repeated evaluation of the integrand in different sample or quadrature points, corresponding to the solution of (1) with respect to a specific realization of the parameter $\mathbf{y} \in \square$.

An efficient approach to deal with the quadrature problem is the multilevel Monte Carlo method (MLMC) which has been developed in [4,14,16,24,25]. As first observed in [12,20], this approach mimics a certain sparse grid approximation between the physical space and the parameter space. Thus, the extension to the multilevel quasi-Monte Carlo method and even more general multilevel quadrature methods is obvious. In the latter cases, we require extra regularity of the solution in terms of spaces of dominant mixed derivatives, c.f. [20,21,28] for example. This extra regularity is available for important classes of parametric problems, see [9] for the case of affine elliptic diffusion coefficients and [27] for the case of log-normally distributed diffusion coefficients. In this article, for the sake of clarity in presentation, we will focus on affine elliptic diffusion problems as they occur from the discretization of uniformly elliptic random diffusion coefficients. Nevertheless, our ideas can be extended to more general parametric diffusion problems in a straightforward manner.

Based on the observation that a multilevel quadrature scheme resembles a sparse tensor product approximation between the spatial variable and the parametric variable, we can exploit well-known techniques from the sparse tensor product approximation theory. To explain our ideas, we recall the construction of sparse tensor product approximation spaces. Let

$$V_0^{(i)} \subset V_1^{(i)} \subset \cdots \subset V_j^{(i)} \subset \cdots \subset \mathcal{H}_i, \quad i = 1, 2,$$

denote two sequences of finite dimensional sub-spaces with increasing approximation power in some linear spaces $\mathcal{H}_i$. To approximate a given object of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$, one canonically considers the full tensor product spaces $U_j := V_j^{(1)} \otimes V_j^{(2)}$. However, the cost $\dim U_j = \dim V_j^{(1)} \cdot \dim V_j^{(2)}$ is often too expensive. To reduce this cost, one might consider the approximation in so-called *sparse grid spaces*, see e.g. [7,37]. For $\ell \geq 0$, one introduces the complement spaces

$$W_{\ell+1}^{(i)} = V_{\ell+1}^{(i)} \ominus V_\ell^{(i)}, \quad i = 1, 2,$$

which gives rise to the multilevel decompositions

$$V_j^{(i)} = \bigoplus_{\ell=0}^{j} W_\ell^{(i)}, \quad W_0^{(i)} := V_0^{(i)}, \quad i = 1, 2. \tag{2}$$

Then, the sparse grid space is defined by

$$\widehat{U}_j := \bigoplus_{\ell+\ell' \leq j} W_\ell^{(1)} \otimes W_{\ell'}^{(2)}. \tag{3}$$

Under the assumptions that the dimensions of $\{V_\ell^{(1)}\}$ and $\{V_\ell^{(2)}\}$ form geometric series, (3) contains, at most up to a logarithm, only $\mathcal{O}\big(\max\{\dim V_j^{(1)}, \dim V_j^{(2)}\}\big)$ degrees of freedom. Nevertheless, it offers nearly the same approximation power

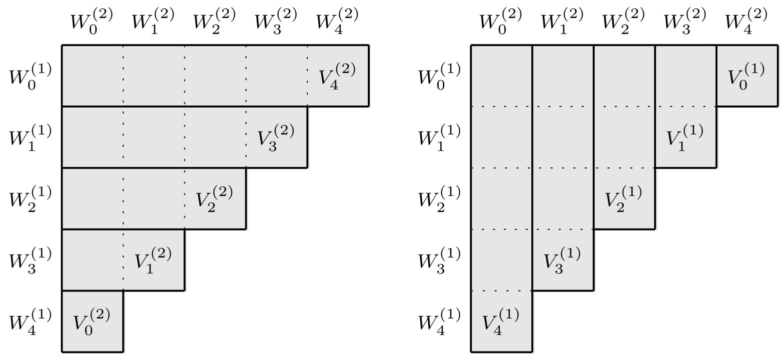

Fig. 1 Different representations of the sparse grid space.

as U_j provided that the object to be approximated has some extra smoothness by means of mixed regularity. For further details, see [17].

In view of (2), factoring out with respect to the first component, one can rewrite (3) according to

$$\widehat{U}_j = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes \left(\bigoplus_{\ell'=0}^{j-\ell} W_{\ell'}^{(2)} \right) = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes V_{j-\ell}^{(2)}. \tag{4}$$

Obviously, in complete analogy there holds

$$\widehat{U}_j = \bigoplus_{\ell'=0}^{j} \left(\bigoplus_{\ell=0}^{j-\ell'} W_\ell^{(1)} \right) \otimes W_{\ell'}^{(2)} = \bigoplus_{\ell=0}^{j} V_{j-\ell}^{(1)} \otimes W_\ell^{(2)}. \tag{5}$$

We refer to Figure 1 for an illustration, where the left plot corresponds to the representation (4) and the right plot corresponds to the representation (5). The advantage of the representation (4) is that we can give up the requirement that the spaces $\{V_\ell^{(2)}\}$ are nested. Likewise, for the representation (5), the spaces $\{V_\ell^{(1)}\}$ need not to be nested any more.

In the context of the parametric diffusion problem (1), one often aims at computing

$$\int_\square \mathcal{F}\big(u(\mathbf{y})\big)\rho(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

where ρ is the density of some measure on $\square$ and $\mathcal{F}$ may denote a (linear) functional or, as in the case of moment computation, it may be defined as $\mathcal{F}\big(u(\mathbf{y})\big) = u^p(\mathbf{y})$ for $p = 1, 2, \ldots$. Here, $\{V_\ell^{(1)}\}$ corresponds to a sequence of finite element spaces and $\{V_\ell^{(2)}\}$ refers to a sequence of quadrature rules. If we denote the finite element solutions of (1) by $\mathbf{u}_\ell(\mathbf{y}) \in V_\ell^{(1)}$ and if we denote the sequence of quadrature rules by $Q_{\ell'}: C(\square) \to \mathbb{R}$, we arrive thus with respect to (4) at the decomposition

$$\int_\square \mathcal{F}\big(u(\mathbf{y})\big)\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \approx \sum_{\ell=0}^{j} Q_{j-\ell} \varDelta\mathcal{F}_\ell\big(u(\mathbf{y})\big), \tag{6}$$

where

$$\Delta\mathcal{F}_\ell\big(u(\mathbf{y})\big) := \mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big) - \mathcal{F}\big(\mathfrak{u}_{\ell-1}(\mathbf{y})\big), \quad \mathcal{F}\big(\mathfrak{u}_{-1}(\mathbf{y})\big) := 0.$$

On the other hand, similarly to (5), we obtain the decomposition

$$\int_\square \mathcal{F}\big(u(\mathbf{y})\big)\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \approx \sum_{\ell=0}^{j} \Delta Q_\ell \mathcal{F}\big(\mathfrak{u}_{j-\ell}(\mathbf{y})\big), \tag{7}$$

where

$$\Delta Q_\ell := Q_\ell - Q_{\ell-1}, \quad Q_{-1} := 0.$$

Both representations are equivalent but have a different impact on its numerical implementation.

Often *multilevel quadrature methods* are interpreted as variance reduction methods, a view which has originally been introduced for the approximation of parametric integrals, cf. [24,25]. Consequently, the representation (4), and thus the decomposition (6), has been used in previous articles, see, for example, [14,15] for stochastic ordinary differential equations and [4,20,34] for partial differential equations with random data. To this end, a nested sequence of approximation spaces is presumed such that the complement spaces $\{W_\ell^{(1)}\}$ are well-defined. In the context of partial differential equations, these complement spaces are given via the difference of Galerkin projections onto subsequent finite element spaces. This circumstance can be avoided in the case of $\mathcal{F}$ being a functional, cf. [34].

The decomposition (6) is well suited if the spatial dimension is small, as it is the case for one-dimensional partial differential equations with random data or for stochastic ordinary differential equations. Nevertheless, in two or three spatial dimensions, the construction of nested approximation spaces might be difficult or even not be possible at all. Sometimes, in view of adaptive refinement strategies, it might be favourable to give up nestedness. In the present article, we will employ the decomposition (7) which is based on the representation (5). It allows for non-nested finite element spaces. Thus, it is conceptually simpler and easy to implement since a black-box finite element solver can be directly employed. Moreover, using nested quadrature formulae, a considerable speed-up is achieved in comparison to the conventional multilevel quadrature which is based on the representation (6), see Theorem 2.

The rest of the article is organized as follows. We introduce the parametric, elliptic model problem of interest in Section 2. It is motivated by considering random diffusion problems in Section 3. Then, the next two sections are dedicated to the discretization, namely the quadrature rule for the parametric variable (Section 4) and the finite element discretization for the physical domain (Section 5). The multilevel quadrature for the model problem is discussed in Section 6. In particular, we show the equivalence of the two representations (6) and (7). Then, in Section 7, we present the error analysis for the latter representation. Finally, in Section 8, we provide numerical results to validate our approach.

Throughout this article, in order to avoid the repeated use of generic but unspecified constants, we mean by $C \lesssim D$ that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \sim D$ as $C \lesssim D$ and $C \gtrsim D$.

2 Problem setting

For some $m \in \mathbb{N}$, let $\square := [-1,1]^m$ denote the parameter domain. Morever, we introduce on $\square$ the measure $\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$ which is induced by the product density function

$$\rho(\mathbf{y}) := \prod_{k=1}^{m} \rho_k(y_k).$$

Next, let $D \subset \mathbb{R}^n$, $n = 2, 3$, be either a convex, polygonal domain or a C^2-domain in order to allow for H^2-regularity of our model problem in the first place. Then, we consider the parametric diffusion problem

$$\begin{aligned} &\text{find } u \in L^p_\rho\big(\square; H^1_0(D)\big) \text{ such that} \\ &\qquad -\operatorname{div}\big(\alpha(\mathbf{y})\nabla u(\mathbf{y})\big) = f \text{ in } D \text{ for almost every } \mathbf{y} \in \square, \end{aligned} \tag{8}$$

where $\alpha\colon D \times \square \to \mathbb{R}$ with

$$\alpha(\mathbf{x}, \mathbf{y}) = \varphi_0(\mathbf{x}) + \sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(\mathbf{x}) y_k \tag{9}$$

and $f \in L^2(D)$. Note that $u \in L^p_\rho\big(\square; H^1_0(D)\big)$ guarantees finite p-th order moments of the solution. Hence, $p = 2$ is sufficient for us in the sequel.

By the Lax-Milgram theorem, unique solvability of the parametric diffusion problem (8) in $L^2_\rho\big(\square; H^1_0(D)\big)$ follows immediately if we impose the condition

$$0 < \alpha_{\min} \leq \alpha(\mathbf{y}) \leq \alpha_{\max} < \infty \text{ in } D \tag{10}$$

for all $\mathbf{y} \in \square$ on the diffusion coefficient. Moreover, we obtain the stability estimate

$$\|u(\mathbf{y})\|_{H^1(D)} \leq \frac{1}{a_{\min}} \|f\|_{H^{-1}(D)} \lesssim \frac{1}{a_{\min}} \|f\|_{L^2(D)} \quad \text{for almost every } \mathbf{y} \in \square.$$

Therefore, the solution to (8) is essentially bounded with respect to $\mathbf{y} \in \square$.

Here and in the sequel, for a given Banach space $\mathcal{X}$, the space $L^p_\rho(\square; \mathcal{X})$, $1 \leq p \leq \infty$, denotes the Bochner space which contains all equivalence classes of strongly measurable functions $v : \square \to \mathcal{X}$ whose norm

$$\|v\|_{L^p_\rho(\square;\mathcal{X})} := \begin{cases} \left(\displaystyle\int_\square \|v(\mathbf{y})\|^p_{\mathcal{X}} \rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\mathbf{y}\in\square} \|v(\mathbf{y})\rho(\mathbf{y})\|_{\mathcal{X}}, & p = \infty \end{cases}$$

is finite. If $p = 2$ and $\mathcal{X}$ is a separable Hilbert space, then the Bochner space is isomorphic to the tensor product space $L^2_\rho(\square) \otimes \mathcal{X}$. Finally, the space $C(\square; \mathcal{X})$ consists of all continuous mappings $v : \square \to \mathcal{X}$.

In [35], it has been proven that the solution u of (8) is analytical as mapping $u\colon \square \to H^1_0(D)$. Moreover, it has been shown there that u is even an analytical mapping $u\colon \square \to \mathcal{W} := H^1_0(D) \cap H^2(D)$ given that the $\{\varphi_k\}$ in (9) belong to $W^{1,\infty}(D)$. This constitutes the necessary mixed regularity for a sparse tensor product discretization, see e.g. [21]. A similar result for diffusion problems with coefficients of the form $\exp\big(\alpha(\mathbf{x}, \mathbf{y})\big)$ has been shown in [27].

Since u is supposed to be in $L^2_\rho\big(\Box; H^1_0(D)\big)$, we can compute its expectation

$$\mathbb{E}[u] = \int_\Box u(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \in H^1_0(D) \tag{11}$$

and its variance

$$\mathbb{V}[u] = \mathbb{E}[u^2] - \mathbb{E}[u]^2 = \int_\Box u^2(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - \mathbb{E}[u]^2 \in W^{1,1}_0(D). \tag{12}$$

We will focus in the sequel on the efficient numerical computation of these possibly high-dimensional integrals.

Remark 1 Note that, in this article, we restrict ourselves to the situation of a fixed dimension m. This means that the constants which appear in our analysis may depend on m. Nevertheless, we emphasize that the presented quadrature methods are also feasible when m tends to infinity if proper modifications are made, see e.g. [9, 28, 36] for details. In the latter case, one has to examine the decay of the sequence $\big\{\|\sqrt{\lambda_k}\varphi_k\|_{W^{1,\infty}(D)}\big\}_k$, cf. (9), in order to derive results that are independent of the dimensionality m.

3 The underlying random model

Let $(\Omega, \Sigma, \mathbb{P})$ be a complete and separable probability space with σ-field $\Sigma \subset 2^\Omega$ and probability measure $\mathbb{P}$. We intend to compute the expectation

$$\mathbb{E}[u] = \int_\Omega u(\omega)\,\mathrm{d}\mathbb{P}(\omega) \in H^1_0(D)$$

and the variance

$$\mathbb{V}[u] = \int_\Omega \big\{u(\omega) - \mathbb{E}[u]\big\}^2\,\mathrm{d}\mathbb{P}(\omega) \in W^{1,1}_0(D)$$

of the random function $u(\omega) \in H^1_0(D)$ which solves the stochastic diffusion problem

$$-\operatorname{div}\big(\alpha(\omega)\nabla u(\omega)\big) = f \text{ in } D \text{ for almost every } \omega \in \Omega. \tag{13}$$

For sake of simplicity, we assume that the stochastic diffusion coefficient is given by a finite Karhunen-Loève expansion

$$\alpha(\mathbf{x},\omega) = \mathbb{E}[\alpha](\mathbf{x}) + \sum_{k=1}^m \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega) \tag{14}$$

with pairwise L^2-orthonormal functions $\varphi_k \in L^\infty(D)$ and stochastically independent random variables $\psi_k(\omega) \in [-1,1]$. Especially, it is assumed that the random variables admit continuous density functions $\rho_k : [-1,1] \to \mathbb{R}$ with respect to the Lebesgue measure.

In practice, one generally has to compute the expansion (14) from the given covariance kernel

$$\mathrm{Cov}[\alpha](\mathbf{x},\mathbf{x}') = \int_\Omega \big\{\alpha(\mathbf{x},\omega) - \mathbb{E}[\alpha](\mathbf{x})\big\}\big\{\alpha(\mathbf{x}',\omega) - \mathbb{E}[\alpha](\mathbf{x}')\big\}\,\mathrm{d}\mathbb{P}(\omega).$$

If the expansion contains infinitely many terms, it has to be appropriately truncated which will induce an additional discretization error. For details, we refer the reader to [13,23,29,33].

The assumption that the random variables $\{\psi_k(\omega)\}$ are stochastically independent implies that the respective joint density function and the joint distribution of the random variables are given by

$$\rho(\mathbf{y}) := \prod_{k=1}^{m} \rho_k(y_k) \quad \text{and} \quad \mathrm{d}\mathbb{P}_\rho(\mathbf{y}) := \rho(\mathbf{y})\mathrm{d}\mathbf{y}.$$

Thus, we are able to reformulate the stochastic problem (13) as a parametric, deterministic problem in $L^2_\rho(\square)$. To this end, the probability space Ω is identified with its image $\square$ with respect to the measurable mapping

$$\boldsymbol{\psi}\colon \Omega \to \square, \quad \omega \mapsto \boldsymbol{\psi}(\omega) := \big(\psi_1(\omega), \ldots, \psi_m(\omega)\big).$$

Hence, the random variables ψ_k are substituted by coordinates $y_k \in [-1,1]$. This leads to an affine diffusion coefficient of the form (9) and finally to the parametric diffusion problem (8).

4 Quadrature in the parameter space

Having the solution $u \in L^2_\rho\big(\square; H^1_0(D)\big)$ of (8) at hand, its expectation and variance are then given by the integrals (11) and (12). To compute these integrals, we shall provide a sequence of quadrature formulae $\{Q_\ell\}$ for the Bochner integral

$$\mathrm{Int}\colon L^1_\rho(\square;\mathcal{X}) \to \mathcal{X}, \quad \mathrm{Int}\, v = \int_\square v(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

where $\mathcal{X} \subset L^2(D)$ denotes a Banach space. The quadrature formula

$$Q_\ell\colon L^1_\rho(\square;\mathcal{X}) \to \mathcal{X}, \quad (Q_\ell v)(\mathbf{x}) = \sum_{i=1}^{N_\ell} \omega_{\ell,i} v(\mathbf{x}, \boldsymbol{\xi}_{\ell,i})\rho(\boldsymbol{\xi}_{\ell,i}) \tag{15}$$

is supposed to provide the error bound

$$\|(\mathrm{Int} - Q_\ell)v\|_{\mathcal{X}} \lesssim \varepsilon_\ell \|v\|_{\mathcal{H}(\square;\mathcal{X})} \tag{16}$$

uniformly in $\ell \in \mathbb{N}$, where $\mathcal{H}(\square;\mathcal{X}) \subset L^2_\rho(\square,\mathcal{X})$ is a suitable Bochner space.

The following particular examples of quadrature rules (15) are considered in our numerical experiments:

- The Monte Carlo method satisfies (16) only with respect to the root mean square error. Namely, it holds

  $$\sqrt{\mathbb{E}\big(\|(\mathrm{Int} - Q_\ell)v\|^2_{\mathcal{X}}\big)} \lesssim \varepsilon_\ell \|v\|_{\mathcal{H}(\square;\mathcal{X})}$$

  with $\varepsilon_\ell = N_\ell^{-1/2}$ and $\mathcal{H}(\square;\mathcal{X}) = L^2_\rho(\square;\mathcal{X})$.

- The standard quasi Monte Carlo method leads typically to $\varepsilon_\ell = N_\ell^{-1}(\log N_\ell)^m$ with the Bochner space $\mathcal{H}(\square;\mathcal{X}) = W^{1,1}_{\mathrm{mix}}(\square;\mathcal{X})$ of all equivalence classes of functions $v\colon \square \to \mathcal{X}$ with finite norm

$$\|v\|_{W^{1,1}_{\mathrm{mix}}(\square;\mathcal{X})} := \sum_{\|\mathbf{q}\|_\infty \leq 1} \int_\square \left\| \frac{\partial^{\|\mathbf{q}\|_1}}{\partial y_1^{q_1} \partial y_2^{q_2} \cdots \partial y_m^{q_m}} v(\mathbf{y}) \right\|_{\mathcal{X}} \mathrm{d}\mathbf{y} < \infty, \tag{17}$$

  see e.g. [31]. Note that this estimate requires that the densities satisfy $\rho_k \in W^{1,\infty}([-1,1])$. For the Halton sequence, cf. [19], it can even be shown that $\varepsilon_\ell = N_\ell^{\delta-1}$ for arbitrary $\delta > 0$ given that the spatial functions in (9) satisfy $\|\sqrt{\lambda_k}\varphi_k\|_{W^{1,\infty}(D)} \lesssim k^{-3-\varepsilon}$ for arbitrary $\varepsilon > 0$, see [28,36].
- Let the densities ρ_k be in $W^{r,\infty}([-1,1])$. If $v\colon \square \to \mathcal{X}$ has mixed regularity of order r with respect to the parameter $\mathbf{y}$, i.e.

$$\|v\|_{W^{r,\infty}_{\mathrm{mix}}(\square;\mathcal{X})} := \max_{\|\boldsymbol{\alpha}\|_\infty \leq r} \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v\right\|_{L^\infty(\square;\mathcal{X})} < \infty, \tag{18}$$

  then one can apply a (sparse) tensor product Clenshaw-Curtis quadrature rule. This yields the convergence rate $\varepsilon_\ell = 2^{-\ell r}\ell^{m-1}$, where $N_\ell \sim 2^\ell \ell^{m-1}$ and $\mathcal{H}(\square;\mathcal{X}) = W^{r,\infty}_{\mathrm{mix}}(\square;\mathcal{X})$, see [32].[1]

For our purposes, we shall assume that the number N_ℓ of points of the quadrature formula Q_ℓ is chosen such that the corresponding accuracy is

$$\varepsilon_\ell = 2^{-\ell}. \tag{19}$$

Then, for the respective difference quadrature $\Delta Q_\ell := Q_\ell - Q_{\ell-1}$, we immediately obtain by combining (16) and (19) the error bound

$$\begin{aligned} \|\Delta Q_\ell v\|_{\mathcal{X}} &= \|(Q_\ell - Q_{\ell-1})v\|_{\mathcal{X}} \\ &\leq \|(\mathrm{Int} - Q_\ell)v\|_{\mathcal{X}} + \|(\mathrm{Int} - Q_{\ell-1})v\|_{\mathcal{X}} \\ &\lesssim 2^{-\ell}\|v\|_{\mathcal{H}(\square;\mathcal{X})}. \end{aligned}$$

5 Finite element approximation in the spatial variable

In order to apply the quadrature formula (15), we shall calculate the solution $u(\mathbf{y}) \in H_0^1(D)$ of the diffusion problem (8) in certain points $\mathbf{y} \in \square$. To this end, consider a not necessarily nested sequence of shape regular and quasi-uniform triangulations or tetrahedralizations $\{\mathcal{T}_j\}$ of the domain D, respectively, each of which with the mesh size $h_j \sim 2^{-j}$. If the domain is not polygonal, then we obtain a polygonal approximation D_j of the domain D by replacing curved edges and faces by planar ones.

In order to deal only with the fixed domain D and not with the different polygonal approximations D_j, we follow [5] and extend functions defined on D_j by zero onto $D \setminus D_j$. Hence, given the triangulation or the tetrahedralization $\{\mathcal{T}_j\}$, we define the spaces

$$\begin{aligned} \mathcal{S}_j^1(D) := \{v \in C(D) : v|_\tau &\text{ is a linear polynomial for all } \tau \in \mathcal{T}_j \\ &\text{and } v(\mathbf{x}) = 0 \text{ for all nodes } \mathbf{x} \in \partial D\} \end{aligned}$$

[1] The Clenshaw-Curtis quadrature converges exponentially if the integrand $v\colon \square \to \mathcal{X}$ and the density ρ are analytic.

of continuous, piecewise linear finite elements. Notice that it does hold $\mathcal{S}_j^1(D) \in H^1(D)$ but not necessarily $\mathcal{S}_j^1(D) \notin H_0^1(D)$. We shall further introduce the finite element solution $\mathfrak{u}_j(\mathbf{y}) \in \mathcal{S}_j^1(D)$ of (8) which satisfies

$$\int_D \alpha(\mathbf{x},\mathbf{y})\nabla \mathfrak{u}_j(\mathbf{x},\mathbf{y})\nabla w_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_D f(\mathbf{x})w_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \text{ for all } w_j \in \mathcal{S}_j^1(D). \tag{20}$$

If $D \neq D_j$, the bilinear form which underlies (20) is also well defined for functions from $S_j^1(D)$ since $S_j^1(D) \subset H^1(D)$. Nevertheless, in order to maintain the ellipticity of the bilinear form, we shall assume that the mesh size h_0 is sufficiently small to ensure that functions in $S_j^1(D)$ are zero on a part of the boundary of D.

It is shown in e.g. [5,6] that the finite element solution $\mathfrak{u}_j(\mathbf{y}) \in \mathcal{S}_j^1(D)$ of (20) admits the following approximation properties.

Lemma 1 *Consider a convex, polygonal domain D or a domain with C^2-smooth boundary and let $f \in L^2(D)$. Then, the finite element solution $\mathfrak{u}_j(\mathbf{y}) \in \mathcal{S}_j^1(D)$ of the diffusion problem* (8) *and respectively its square $\mathfrak{u}_j^2(\mathbf{y})$ satisfy the error estimate*

$$\left\| u^p(\mathbf{y}) - \mathfrak{u}_j^p(\mathbf{y}) \right\|_{\mathcal{X}} \lesssim 2^{-j} \|f\|_{L^2(D)}^p, \tag{21}$$

where $\mathcal{X} = H^1(D)$ for $p = 1$ and $\mathcal{X} = W^{1,1}(D)$ for $p = 2$. The constants hidden in (21) *depend on $\alpha_{\min}$ and $\alpha_{\max}$, but not on $\mathbf{y} \in \square$.*

Proof The parametric diffusion problem (8) is H^2-regular since D is convex or C^2-smooth and $f \in L^2(D)$. Hence, the first error estimate for $p = 1$ immediately follows from the standard finite element theory, see e.g. [5,6]. We further find by the generalized Hölder inequality for $p = 2$ that

$$\begin{aligned} \left\| u^2(\mathbf{y}) - \mathfrak{u}_j^2(\mathbf{y}) \right\|_{W^{1,1}(D)} &= \left\| \big(u(\mathbf{y}) - \mathfrak{u}_j(\mathbf{y})\big)\big(u(\mathbf{y}) + \mathfrak{u}_j(\mathbf{y})\big) \right\|_{W^{1,1}(D)} \\ &\le \|u(\mathbf{y}) - \mathfrak{u}_j(\mathbf{y})\|_{H^1(D)} \big\{ \|u(\mathbf{y})\|_{H^1(D)} + \|\mathfrak{u}_j(\mathbf{y})\|_{H^1(D)} \big\}. \end{aligned}$$

By using

$$\|\mathfrak{u}_j(\mathbf{y})\|_{H^1(D)} \le \|u(\mathbf{y})\|_{H^1(D)} + \|u(\mathbf{y}) - \mathfrak{u}_j(\mathbf{y})\|_{H^1(D)} \lesssim \big(1 + 2^{-j}\big)\|f\|_{L^2(D)},$$

we arrive at the desired estimate (21) for $p = 2$. $\square$

6 The multilevel quadrature method

Taking into account the results from the previous sections, we are now able to introduce the multilevel quadrature in a formal way. To that end, let $u \in \mathcal{H}(\square; H^2(D))$, where the underlying Bochner space is determined by the quadrature under consideration. For the sequence $\{\mathfrak{u}_\ell(\mathbf{y})\}_\ell$ of finite element solutions, there obviously holds

$$\lim_{\ell \to \infty} \mathfrak{u}_\ell(\mathbf{y}) = u(\mathbf{y})$$

uniformly in $\mathbf{y} \in \square$. Thus, if $\mathcal{F}$ is continuous, we obtain

$$\lim_{\ell \to \infty} \mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big) = \mathcal{F}\big(u(\mathbf{y})\big) \tag{22}$$

also uniformly in $\mathbf{y} \in \square$. Moreover, we have for the sequence $\{Q_\ell\}_\ell$ of quadrature rules and for a sufficiently smooth integrand that

$$\lim_{\ell \to \infty} Q_\ell v = \int_\square v(\mathbf{y}) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{23}$$

The combination of the relations (22) and (23) leads to

$$\int_\square \mathcal{F}\big(u(\mathbf{y})\big) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \sum_{\ell=0}^{\infty} \Delta Q_\ell \mathcal{F}\big(u(\mathbf{y})\big) = \sum_{\ell=0}^{\infty} \Delta Q_\ell \sum_{\ell'=0}^{\infty} \Delta \mathcal{F}_{\ell'}\big(u(\mathbf{y})\big).$$

Since ΔQ_ℓ is linear and continuous, we end up with

$$\int_\square \mathcal{F}\big(u(\mathbf{y})\big) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \sum_{\ell, \ell'=0}^{\infty} \Delta Q_\ell \Delta \mathcal{F}_{\ell'}\big(u(\mathbf{y})\big).$$

Truncating this sum in accordance with $\ell + \ell' \leq j$ then yields the multilevel quadrature representation (6) if we recombine the operators ΔQ_ℓ. Analogously, we obtain the representation (7) if we recombine the operators $\Delta \mathcal{F}_\ell$. Note that the sequence of the application of the operators ΔQ_ℓ and $\Delta \mathcal{F}_{\ell'}$ is crucial here. Moreover, we have repeatedly exploited the linearity of ΔQ_ℓ.

In the remainder of this section, for the sake of completeness, we explicitly discuss our multilevel quadrature which is based on the representation (7). We refer to Figure 2 for a graphical illustration of this realization of the multilevel quadrature method. The following theorem shows that the representations (6) and (7) are indeed mathematically equivalent if we set $\mathcal{F}\big(\mathfrak{u}_{-1}(\mathbf{y})\big) := 0$.

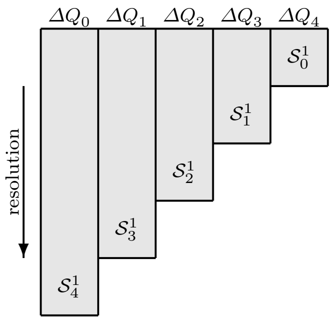

Fig. 2 Combinations of the quadrature operators $\{\Delta Q_\ell\}$ and the finite element spaces $\{\mathcal{S}_\ell^1(D)\}$ in the multilevel quadrature.

Theorem 1 *There holds the identity*

$$\sum_{\ell=0}^{j} Q_{j-\ell} \Delta \mathcal{F}_\ell\big(u(\mathbf{y})\big) = \sum_{\ell=0}^{j} \Delta Q_\ell \mathcal{F}\big(\mathfrak{u}_{j-\ell}(\mathbf{y})\big).$$

Proof Straightforward calculation yields

$$\begin{aligned}\sum_{\ell=0}^{j} Q_{j-\ell}\Delta\mathcal{F}_\ell\big(u(\mathbf{y})\big) &= \sum_{\ell=0}^{j} Q_{j-\ell}\Big(\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big)-\mathcal{F}\big(\mathfrak{u}_{\ell-1}(\mathbf{y})\big)\Big)\\ &= \sum_{\ell=0}^{j} Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big)-\sum_{\ell=0}^{j} Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_{\ell-1}(\mathbf{y})\big)\\ &= \sum_{\ell=0}^{j} Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big)-\sum_{\tilde\ell=-1}^{j-1} Q_{j-(\tilde\ell+1)}\mathcal{F}\big(\mathfrak{u}_{\tilde\ell}(\mathbf{y})\big),\end{aligned}$$

where we substituted $\tilde\ell := \ell-1$. Next, we exploit for $\tilde\ell=-1$ that $Q_j\mathcal{F}\big(\mathfrak{u}_{-1}(\mathbf{y})\big)=0$ and likewise for $\tilde\ell=j$ that $Q_{-1}\mathcal{F}\big(\mathfrak{u}_j(\mathbf{y})\big)=0$, ending up with

$$\begin{aligned}&\sum_{\ell=0}^{j} Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big)-\sum_{\tilde\ell=-1}^{j-1} Q_{j-(\tilde\ell+1)}\mathcal{F}\big(\mathfrak{u}_{\tilde\ell}(\mathbf{y})\big)\\ &\qquad=\sum_{\ell=0}^{j} Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big)-\sum_{\tilde\ell=0}^{j} Q_{j-\tilde\ell-1}\mathcal{F}\big(\mathfrak{u}_{\tilde\ell}(\mathbf{y})\big)=\sum_{\ell=0}^{j}\Delta Q_{j-\ell}\mathcal{F}\big(\mathfrak{u}_\ell(\mathbf{y})\big).\end{aligned}$$

□

Thus, all available results for the representation (6) of the multilevel quadrature, see e.g. [20, 21] and the references therein, carry over to the representation (7). Moreover, (7) now also allows for non-nested meshes and even for adaptively refined meshes.

Besides being more flexible, we emphasize that a further advantage of (7) is an improvement of the cost if nested quadrature formulae are employed. Using (6) implies that each sample of $\Delta\mathcal{F}_\ell\big(u(\mathbf{y})\big)$ involves two solves of the diffusion problem (13) for a specific sample point $\mathbf{y}$. In contrast to this, in (7), we have to solve (13) for each sample point only once. More precisely, there holds the following result.

Theorem 2 *Denote the cost of solving* (8) *for a specific* $\mathbf{y}$ *on level* j*, including the cost of evaluating* $\mathcal{F}$*, by* $\mathcal{C}_j$*, where* $\mathcal{C}_j\sim\sigma^j\mathcal{C}_0$ *for some* $\sigma>1$*. Moreover, assume that the quadrature method satisfies* $N_j\sim\theta^jN_0$ *for some* $\theta>1$*. Then, the computational cost to evaluate the representation* (6) *is of order* $(1+\frac{1}{\sigma})N_0\mathcal{C}_0\sum_{\ell=0}^{j}\theta^{j-\ell}\sigma^\ell$*, whereas the computational cost to evaluate the representation* (7) *is, in case of nested quadrature rules, of order* $N_0\mathcal{C}_0\sum_{\ell=0}^{j}\theta^{j-\ell}\sigma^\ell$.

Proof For evaluating the representation (6), the cost is of the order

$$\sum_{\ell=0}^{j} N_{j-\ell}(\mathcal{C}_\ell+\mathcal{C}_{\ell-1})\sim\left(1+\frac{1}{\sigma}\right)N_0\mathcal{C}_0\sum_{\ell=0}^{j}\theta^{j-\ell}\sigma^\ell.$$

Moreover, the cost for the computation of the difference quadrature $\Delta Q_{j-\ell}$ is of the order $N_{j-\ell}$, since the quadrature points are nested. Thus, we obtain for representation (7) that

$$\sum_{\ell=0}^{j} N_{j-\ell}\mathcal{C}_\ell\sim N_0\mathcal{C}_0\sum_{\ell=0}^{j}\theta^{j-\ell}\sigma^\ell.$$

□

For our setup with shape regular and quasi-uniform meshes, a finite element solver[2] with linear over-all complexity leads to $\sigma = 2, 4, 8$ in one, two, there spatial dimensions, respectively. Thus, in two spatial dimensions, we achieve a speed-up of at least 25%. In three spatial dimensions, we achieve a speed-up of at least 12.5%. This gain stems only from the reordering of the terms in the multilevel quadrature and the application of a nested quadrature method. Nevertheless, we emphasize that non-nested quadrature formulae are feasible in the representation (7) as well. This would result in a combination-technique-like representation of the multilevel quadrature, cf. [18]. In this case, we would end up with the same cost as for evaluating the representation (6).

7 Error analysis

In the sequel, we restrict ourselves for reasons of simplicity to the situations $\mathcal{F}(u) = u$ and $\mathcal{F}(u) = u^2$ which yield the expectation and the second moment of the solution to (8). This means that we consider

$$\operatorname{Int} u^p \approx \sum_{\ell=0}^{j} \Delta Q_\ell \mathfrak{u}_{j-\ell}^p = \sum_{\ell=0}^{j} Q_{j-\ell}\big(\mathfrak{u}_\ell^p - \mathfrak{u}_{\ell-1}^p\big) \quad \text{for } p = 1, 2. \tag{24}$$

We derive a general approximation result for the multilevel quadrature based on the generic estimate

$$\big\|(\operatorname{Int} - Q_\ell)(u^p - \mathfrak{u}_{\ell'}^p)\big\|_{\mathcal{X}} \lesssim 2^{-(\ell+\ell')}\|f\|_{L^2(D)}^p \quad \text{for } p = 1, 2 \tag{25}$$

with f being the right hand side of (8). In particular, any quadrature rule that satisfies this estimate gives rise to a multilevel quadrature method. In the sequel, we provide this estimate for the multilevel (quasi-) Monte Carlo quadrature (MLMC and MLQMC) as well as for the multilevel Clenshaw-Curtis quadrature (MLCC).

Obtaining the generic estimate (25) for the Monte Carlo quadrature is straightforward under the condition that the integrand is square integrable with respect to the parameter $\mathbf{y}$, cf. [4,20]. Nevertheless, since the Monte Carlo quadrature does not provide deterministic error estimates, we have to replace the norm in $\mathcal{X}$ by the $L^2_\rho(\square; \mathcal{X})$-norm. Since the multilevel Monte Carlo quadrature has extensively been studied in numerous articles, see e.g. [4,8,20,34], we skip the error analysis of the method here.

Things become a little more involved for quadrature methods that exploit the smoothness of the integrand with respect to the parameter. The next lemma from [21] provides the smoothness of the Galerkin projection with respect to the parameter $\mathbf{y} \in \square$. Note that straighforward modifications have to be made in the proof if $D \neq D_j$.

Lemma 2 *For the error $\delta_\ell(\mathbf{y}) := (u - \mathfrak{u}_\ell)(\mathbf{y})$ of the Galerkin projection, there holds the estimate*

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_\ell(\mathbf{y})\big\|_{H^1(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|! c^{|\boldsymbol{\alpha}|}\boldsymbol{\gamma}^{\boldsymbol{\alpha}}\|f\|_{L^2(D)} \quad \textit{for all } |\boldsymbol{\alpha}| \geq 0 \tag{26}$$

[2] Here we assume an algorithm with optimal complexity for the solution of the associated discrete systems like e.g. a multiplicative or additive multigrid method etc.

with a constant $c > 0$ dependent on $a_{\min}$ and $a_{\max}$, where $\gamma_k := \|\sqrt{\lambda_k}\varphi_k\|_{W^{1,\infty}(D)}$, cf. (9)*, and $\boldsymbol{\gamma} := \{\gamma_k\}_{k=1}^m$.*

With this lemma, it is straightforward to show the following result related to the second moment, cf. [21].

Lemma 3 *The derivatives of the difference $u^2 - \mathfrak{u}_\ell^2$ satisfy the estimate*

$$\left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\left(u^2 - \mathfrak{u}_\ell^2\right)(\mathbf{y})\right\|_{W^{1,1}(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!c^{|\boldsymbol{\alpha}|}\boldsymbol{\gamma}^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}^2 \quad \textit{for all } |\boldsymbol{\alpha}| \geq 0 \tag{27}$$

with a constant $c > 0$ dependent on $a_{\min}$ and $a_{\max}$.

With the aid of Lemmata 2 and 3 together with the results from [19], we obtain the generic error estimate for the MLQMC with Halton points. Halton points are nested and therefore well suited for our multilevel quadrature method.

Lemma 4 *Let $u \in L_\rho^2\big(\square; H_0^1(D)\big)$ be the solution to* (8) *and $\mathfrak{u}_\ell$ the associated Galerkin projection on level ℓ. Moreover, let $\rho_k \in W^{1,\infty}(-1,1)$ for $k = 1, \ldots, m$. Then, for the quasi-Monte Carlo quadrature based on Halton points, there holds*

$$\left\|(\operatorname{Int} - Q_\ell)(u^p - \mathfrak{u}_{\ell'}^p)\right\|_{\mathcal{X}} \lesssim 2^{-(\ell+\ell')}\|f\|_{L^2(D)}^p \quad \textit{for } p = 1, 2 \tag{28}$$

with $N_\ell \sim 2^{\ell/(1-\delta)}$ for arbitrary $\delta > 0$.

Proof The quadrature error of the quasi-Monte Carlo quadrature based on the point set $P_N = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset [0,1]^m$ satisfies the *Koksma-Hlawka inequality*

$$|(\operatorname{Int} - Q_{P_N})v| \leq \mathcal{D}_\infty^\star(P_N)V_{\mathrm{HK}}(v),$$

where the *star discrepancy* is given by

$$\mathcal{D}_\infty^\star(P_N) := \sup_{\boldsymbol{\xi} \in [0,1]^m}\left|\operatorname{Vol}\big([\mathbf{0}, \boldsymbol{\xi})\big) - \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{[\mathbf{0},\boldsymbol{\xi})}(\boldsymbol{\xi}_i)\right|$$

and the *variation in the sense of Hardy and Krause* is given by

$$V_{\mathrm{HK}}(v) := \sum_{\|\boldsymbol{\alpha}\|_\infty = 1}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\left|\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} v(\boldsymbol{\xi}_{\boldsymbol{\alpha}}, \mathbf{1})\right| \mathrm{d}\boldsymbol{\xi}_{\boldsymbol{\alpha}},$$

cf. [31]. In the above definition, the vector $(\boldsymbol{\xi}_{\boldsymbol{\alpha}}, \mathbf{1})$ is an element of the $|\boldsymbol{\alpha}|$-dimensional face $\{\boldsymbol{\xi} \in [0,1]^m : \xi_k = 1 \text{ if } \alpha_k = 0\}$. For the Halton sequence, it can be shown that $\mathcal{D}_\infty^\star(P_N) = \mathcal{O}\big(N^{-1}\log^m(N)\big)$, where the constant in the big-$\mathcal{O}$-notation also depends on m, cf. [1,31].

We parameterize the parameter domain $\square$ over $[0,1]^m$ by the linear transform $\boldsymbol{\xi} \mapsto \mathbf{y} := 2\boldsymbol{\xi} - \mathbf{1}$ and modify the quasi-Monte Carlo quadrature accordingly. Thus, we obtain with $\hat{u} := u(2\boldsymbol{\xi} - \mathbf{1})$ and $\hat{\mathfrak{u}} := \mathfrak{u}(2\boldsymbol{\xi} - \mathbf{1})$ that

$$\begin{aligned}\left\|(\operatorname{Int} - Q_{2P_N - \mathbf{1}})(u^p - \mathfrak{u}_{\ell'}^p)\right\|_{\mathcal{X}}^2 &= \left\|2^m(\operatorname{Int} - Q_{P_N})(\hat{u}^p - \hat{\mathfrak{u}}_{\ell'}^p)\right\|_{\mathcal{X}}^2 \\ &\leq [\mathcal{D}_\infty^\star(P_N)]^2 2^{2m}\left\|V_{\mathrm{HK}}(\hat{u}^p - \hat{\mathfrak{u}}_{\ell'}^p)\right\|_{\mathcal{X}}^2.\end{aligned} \tag{29}$$

We estimate further by the triangle inequality and Bochner's inequality

$$\begin{aligned}&\big\|V_{\mathrm{HK}}(\hat{u}^p-\hat{\mathrm{u}}^p_{\ell'})\big\|^2_{\mathcal{X}}\\&\quad\le\bigg(\sum_{\|\boldsymbol{\alpha}\|_\infty=1}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\big\|\partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}}\big[(\hat{u}^p-\hat{\mathrm{u}}^p_{\ell'})(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\rho(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\big]\big\|_{\mathcal{X}}\,\mathrm{d}\boldsymbol{\xi}_{\boldsymbol{\alpha}}\bigg)^2.\end{aligned}$$

The Leibniz rule together with Lemmata 2 and 3 yields then

$$\begin{aligned}&\big\|\partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}}\big[(\hat{u}^p-\hat{\mathrm{u}}^p_{\ell'})(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\rho(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\big]\big\|_{\mathcal{X}}\\&\quad\le\sum_{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}\big\|\partial^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'}_{\boldsymbol{\xi}}(\hat{u}^p-\hat{\mathrm{u}}^p_{\ell'})(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\big\|_{\mathcal{X}}\big\|\partial^{\boldsymbol{\alpha}'}_{\boldsymbol{\xi}}\rho(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\big\|_{L^\infty([0,1]^m)}\\&\quad\lesssim 2^{-\ell'}\sum_{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|!(2c)^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}\boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'}\|f\|^p_{L^2(D)}2^{|\boldsymbol{\alpha}'|}\boldsymbol{\rho}^{\boldsymbol{\alpha}'},\end{aligned}$$

where $\boldsymbol{\rho}:=[\|\rho_1\|_{W^{1,\infty}(-1,1)},\ldots,\|\rho_m\|_{W^{1,\infty}(-1,1)}]$. With the identity

$$\sum_{\substack{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}\\|\boldsymbol{\alpha}'|=j}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}=\binom{|\boldsymbol{\alpha}|}{j},$$

we thus arrive at

$$\begin{aligned}&2^{-\ell'}\sum_{\boldsymbol{\alpha}'\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'}|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|!(2c)^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}\boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'}\|f\|^p_{L^2(D)}2^{|\boldsymbol{\alpha}'|}\boldsymbol{\rho}^{\boldsymbol{\alpha}'}\\&\qquad=2^{-\ell'}|\boldsymbol{\alpha}|!2^{|\boldsymbol{\alpha}|}\|f\|^p_{L^2(D)}\sum_{j=0}^{|\boldsymbol{\alpha}|}c^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|}\boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'}\boldsymbol{\rho}^{\boldsymbol{\alpha}'}\\&\qquad\le 2^{-\ell'}(|\boldsymbol{\alpha}|+1)!\|f\|^p_{L^2(D)}\tilde{c}^{|\boldsymbol{\alpha}|}\end{aligned}$$

with $\tilde{c}=2\max_{k=1,\ldots,m}\max\{c\gamma_k,\rho_k\}$.

Next, by reorganizing the summands in the variation, we obtain

$$\begin{aligned}&\sum_{\|\boldsymbol{\alpha}\|_\infty=1}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\big\|\partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}}\big[(\hat{u}^p-\hat{\mathrm{u}}^p_{\ell'})(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\rho(\boldsymbol{\xi}_{\boldsymbol{\alpha}},\mathbf{1})\big]\big\|_{\mathcal{X}}\,\mathrm{d}\boldsymbol{\xi}_{\boldsymbol{\alpha}}\\&\qquad\lesssim 2^{-\ell'}\|f\|^p_{L^2(D)}\sum_{j=1}^m\sum_{\|\boldsymbol{\alpha}\|_1=j}(j+1)!\tilde{c}^j=\sum_{j=1}^m\binom{m}{j}(j+1)!\tilde{c}^j\\&\qquad\le C(m)2^{-\ell'}\|f\|^p_{L^2(D)}\end{aligned}$$

with $C(m):=(m+1)!\frac{1-\tilde{c}^{m+1}}{1-\tilde{c}}$. Inserting this bound into (29) gives us finally

$$\begin{aligned}&\big\|(\mathrm{Int}-Q_{2P_N-\mathbf{1}})(u^p-\mathrm{u}^p_{\ell'})\big\|^2_{\mathcal{X}}\\&\qquad\lesssim[\mathcal{D}^\star_\infty(P_N)]^2 2^{2m}\big(C(m)2^{-\ell'}\|f\|^p_{L^2(D)}\big)^2.\end{aligned}$$

Since it asymptotically holds $\log^m(N) \lesssim N^{-\delta}$ as $N \to \infty$ for every $\delta > 0$, we conclude with $N_\ell \sim 2^{\ell/(1-\delta)}$ that $\mathcal{D}_\infty^\star(P_N) \lesssim 2^{-\ell}$. Inserting this into the last inequality and taking square roots on both sides yields the desired assertion. □

Finally, we show the respective result on the quadrature error for the sparse grid quadrature based on the nested *Clenshaw-Curtis abscissae*, cf. [11,32]. These are given by the extrema of the Chebyshev polynomials

$$\xi_k = \cos\left(\frac{(k-1)\pi}{n-1}\right) \quad \text{for } k = 1, \ldots, n,$$

where $n = 2^{j-1} + 1$ if $j > 1$ and $n = 1$ with $\xi_1 = 0$ if $j = 1$.

Lemma 5 *Let $u \in L^2_\rho\big(\square; H^1_0(D)\big)$ be the solution to (8) and let u_ℓ be the associated Galerkin projection on level ℓ. Moreover, let $\rho_k(y_k) \in C^r([-1,1])$ for $k = 1, \ldots, m$. Then, for the sparse grid quadrature based on Clenshaw-Curtis abscissae, there holds*

$$\big\|(\mathrm{Int} - Q_\ell)(u^p - \mathrm{u}^p_{\ell'})\big\|_{\mathcal{X}} \lesssim 2^{-(\ell r + \ell')}\ell^{m-1}\|f\|^p_{L^2(D)} \quad \text{for } p = 1, 2 \tag{30}$$

provided that $N_\ell \sim 2^\ell \ell^{d-1}$.

Proof It is shown in [32] that the number N_ℓ of quadrature points of the sparse tensor product quadrature Q_ℓ with Clenshaw-Curtis abscissae is of the order $\mathcal{O}(2^\ell \ell^{d-1})$. In addition, we have for functions $v\colon \square \to \mathbb{R}$ with mixed regularity the following error bound:

$$|(\mathrm{Int} - Q_\ell)v| \lesssim 2^{-\ell r}\ell^{(m-1)} \max_{\|\boldsymbol{\alpha}\|_\infty \le r} \big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} v\big\|_{L^\infty(\square)}.$$

Hence, to prove the desired assertion, we have to provide estimates on the derivatives $\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}\big(u^p(\mathbf{y}) - \mathrm{u}^p(\mathbf{y})\big)\rho(\mathbf{y})$. This can be accomplished by the Leibniz formula as in the proof of the previous lemma:

$$\begin{aligned}
&\big\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}\big[(\hat{u}^p - \mathrm{u}^p_{\ell'})(\mathbf{y})\rho(\mathbf{y})\big]\big\|_{\mathcal{X}} \\
&\quad\le \sum_{\boldsymbol{\alpha}' \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \big\|\partial^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'}_{\mathbf{y}}(u^p - \mathrm{u}^p_{\ell'})(\mathbf{y})\big\|_{\mathcal{X}} \big\|\partial^{\boldsymbol{\alpha}'}_{\mathbf{y}}\rho(\mathbf{y})\big\|_{L^\infty(\square)} \\
&\quad\lesssim 2^{-\ell'} \sum_{\boldsymbol{\alpha}' \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} |\boldsymbol{\alpha} - \boldsymbol{\alpha}'|!\, c^{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} \|f\|^p_{L^2(D)} \boldsymbol{\rho}^{\boldsymbol{\alpha}'} \\
&\quad\le 2^{-\ell'}(|\boldsymbol{\alpha}| + 1)!\,\|f\|^p_{L^2(D)} \tilde{c}^{|\boldsymbol{\alpha}|}
\end{aligned}$$

with $\tilde{c} = \max_{k=1,\ldots,m} \max\{c\gamma_k, \rho_k\}$. We set $C(r) := \max_{\|\boldsymbol{\alpha}\|_\infty \le r}(|\boldsymbol{\alpha}| + 1)!\tilde{c}^{|\boldsymbol{\alpha}|}$ and obtain

$$\big\|(\mathrm{Int} - Q_\ell)(u^p - \mathrm{u}^p_{\ell'})\big\|^2_{\mathcal{X}} \lesssim \big(2^{-\ell r}\ell^{(m-1)} 2^{-\ell'} C(r)\|f\|^p_{L^2(D)}\big)^2.$$

Then, exploiting that the integrand is independent of the parameter and taking square roots on both sides completes the proof. □

Remark 2 As for the quasi-Monte Carlo quadrature, by slightly decreasing r in the convergence result for the sparse tensor product quadrature, we may remove the factor ℓ^{m-1} since $\ell^{m-1} \lesssim 2^{\ell\delta}$ for arbitrary $\delta > 0$.

Estimates of the type (25) are crucial to show the following approximation result for the multilevel quadrature. More general, every quadrature that satisfies an estimate of type (25) is feasible for a related multilevel quadrature method.

Theorem 3 *Let $\{Q_\ell\}$ be a sequence of quadrature rules that satisfy an estimate of type (25), where $u \in L^2_\rho\big(\square, H^1_0(D)\big)$ is the solution to (8) that satisfies (21). Then, the error of the multilevel estimator for the mean and the second moment defined in (24) is bounded by*

$$\Big\| \operatorname{Int} u^p - \sum_{\ell=0}^{j} \Delta Q_\ell \mathfrak{u}^p_{j-\ell} \Big\|_{\mathcal{X}} \lesssim 2^{-j} j \|f\|^p_{L^2(D)}, \tag{31}$$

where $\mathcal{X} = H^1(D)$ if $p = 1$ and $\mathcal{X} = W^{1,1}(D)$ if $p = 2$.

Proof We shall apply the following multilevel splitting of the error

$$\begin{aligned}
&\Big\| \operatorname{Int} u^p - \sum_{\ell=0}^{j} \Delta Q_\ell \mathfrak{u}^p_{j-\ell} \Big\|_{\mathcal{X}} \\
&\quad = \Big\| \operatorname{Int} u^p - Q_j u^p + \sum_{\ell=0}^{j} \Delta Q_\ell u^p - \sum_{\ell=0}^{j} \Delta Q_\ell \mathfrak{u}^p_{j-\ell} \Big\|_{\mathcal{X}} \\
&\quad \le \big\| \operatorname{Int} u^p - Q_j u^p \big\|_{\mathcal{X}} + \sum_{\ell=0}^{j} \big\| \Delta Q_\ell \big(u^p - \mathfrak{u}^p_{j-\ell}\big) \big\|_{\mathcal{X}}.
\end{aligned} \tag{32}$$

The first term just reflects the quadrature error and can be bounded with similar arguments as in Lemmata 4 and 5 according to

$$\big\| \operatorname{Int} u^p - Q_j u^p \big\|_{\mathcal{X}} \lesssim 2^{-j} \|f\|^p_{L^2(D)}$$

with a constant that depends on m. The term inside the sum satisfies with (25) that

$$\begin{aligned}
&\big\| \Delta Q_\ell \big(u^p - \mathfrak{u}^p_{j-\ell}\big) \big\|_{\mathcal{X}} \\
&\quad \le \big\| (\operatorname{Int} - Q_\ell) \big(u^p - \mathfrak{u}^p_{j-\ell}\big) \big\|_{\mathcal{X}} + \big\| (\operatorname{Int} - Q_{\ell-1}) \big(u^p - \mathfrak{u}^p_{j-\ell}\big) \big\|_{\mathcal{X}} \\
&\quad \lesssim 2^{-(\ell+j-\ell)} \|f\|^p_{L^2(D)} + 2^{-(\ell-1+j-\ell)} \|f\|^p_{L^2(D)} \lesssim 2^{-j} \|f\|^p_{L^2(D)}.
\end{aligned}$$

Thus, we can estimate (32) as

$$\begin{aligned}
\Big\| \operatorname{Int} u^p - \sum_{\ell=0}^{j} \Delta Q_\ell \mathfrak{u}^p_{j-\ell} \Big\|_{\mathcal{X}} &\lesssim 2^{-j} \|f\|^p_{L^2(D)} + \sum_{\ell=0}^{j} 2^{-j} \|f\|^p_{L^2(D)} \\
&\le 2^{-j} (j+2) \|f\|^p_{L^2(D)}.
\end{aligned}$$

This completes the proof. □

Remark 3 The factor j in the error estimate can obviously be avoided if the quadrature accuracy is chosen in such a way that it provides an additional convergent series, e.g. $\ell^{-1-\delta}$ for arbitrary $\delta > 0$.

Remark 4 Note that we can achieve in our framework also nestedness for the samples in the Monte Carlo method. This is due to the fact that independent samples have to be used only for the estimators Q_ℓ for $\ell = 0, \ldots, j$. But from the proof of the previous theorem, we see that Q_ℓ has not to be sampled independently from $Q_{\ell'}$ for $\ell \neq \ell'$. Thus, we may employ the same underlying set of sample points on each level.

8 Numerical results

The numerical examples in this section are performed in three spatial dimensions. For the finite element discretization, we employ MATLAB and the Partial Differential Equation Toolbox[3]. In both examples, the error is measured by interpolating the obtained solutions on a sufficiently fine grid and comparing it there to a reference solution. In our examples, we consider the MLMC, the MLQMC based on the Halton sequence and the MLCC. Moreover, we set for our problems the density to $\rho(\mathbf{y}) = (1/2)^m$.

8.1 An analytical example

With our first example, we intend to validate the proposed method. To this end, we consider a simple quadrature problem on the unit ball $D = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2 < 1\}$. Figure 3 depicts different tetrahedralizations for this geometry, which are in particular not nested. We aim at computing the expectation of the solution u to the parametric diffusion equation

$$-\operatorname{div}\big(a(\mathbf{y})\nabla u(\mathbf{y})\big) = 1 \text{ in } D, \quad u(\mathbf{y}) = 0 \text{ on } \partial D, \quad \mathbf{y} \in \square,$$

where

$$\alpha(\mathbf{y}) = \left(\prod_{i=1}^{6} \frac{3}{5}\big(2 - y_i^2\big)\right)^{-1}.$$

Since the diffusion coefficient is independent of the spatial variable, we can reformulate the equation according to

$$-\Delta u(\mathbf{y}) = \prod_{i=1}^{6} \frac{3}{5}\big(2 - y_i^2\big) \text{ in } D, \quad u(\mathbf{y}) = 0 \text{ on } \partial D, \quad \mathbf{y} \in \square.$$

Thus, since the Bochner integral interchanges with closed operators, see e.g. [26], we obtain for the expectation of u the equation

$$-\Delta \mathbb{E}[u(\mathbf{y})] = \mathbb{E}\left[\prod_{i=1}^{6} \frac{3}{5}\big(2 - y_i^2\big)\right] = 1 \text{ in } D, \quad u(\mathbf{y}) = 0 \text{ on } \partial D, \quad \mathbf{y} \in \square. \tag{33}$$

Obviously, this equation is solved by

$$\mathbb{E}[u](\mathbf{x}) = \frac{1}{6}\big(1 - \|\mathbf{x}\|_2\big)^2.$$

[3] Release 2015a, The MathWorks, Inc., Natick, Massachusetts, United States.

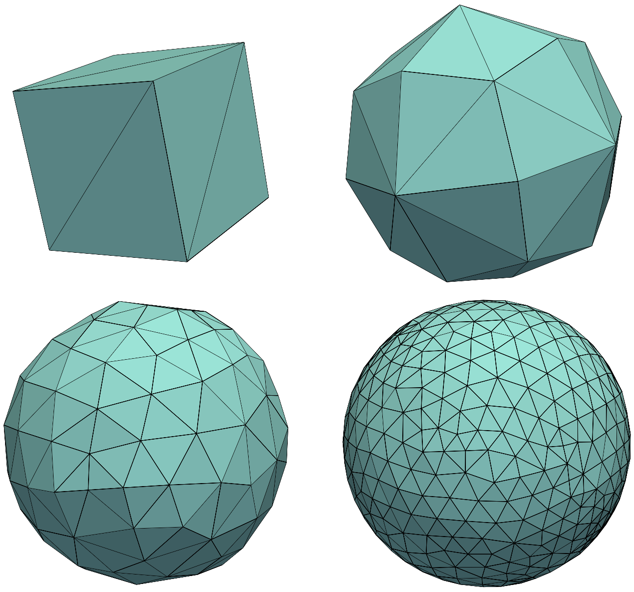

Fig. 3 Tetrahedralizations of four different resolutions for the unit ball.

In order to measure the error to the approximate solution, we interpolate the exact solution to a mesh consisting of 12 047 801 finite elements (this is level $j = 8$). This involves a mesh size of $h_8 = 0.0047$. For the levels $j = 0, \ldots, 7$, the mesh sizes are given in Table 1.

j	0	1	2	3	4	5	6	7
h_j	1.2	0.6	0.3	0.15	0.075	0.0375	0.0188	0.0094

Table 1 Mesh sizes on the different levels for the unit ball.

On the left side of Figure 4, the error for the MLQMC, the MLCC and the MLMC is visualized. It is plotted against the target mesh size for the meshing algorithm in the MATLAB Partial Differential Equation Toolbox. For the MLMC, in order to approximate the root mean square error, we average five realizations of the related approximation error. It turns out that all quadrature methods provide a linear rate of convergence. Especially, the logarithmic factor j from the error estimate (31) is not observed here. Moreover, we chose $N_0 = 10$ for the Monte Carlo quadrature and for the quasi-Monte Carlo quadrature and set $\theta = 4$ and $\theta = 2$, i.e. $N_\ell = 10 \cdot 4^\ell$ and $N_\ell = 10 \cdot 2^\ell$, respectively, cf. Theorem 2. For the Clenshaw-Curtis quadrature, the number of samples are chosen with respect to $r = 1$.[4] The number of samples for the finest level of resolution, i.e. $j = 7$, including

[4] The Clenshaw-Curtis quadrature converges exponentially since the integrand is analytic. The choice $r = 1$ is conservative and reflects the pre-asymptotic regime.

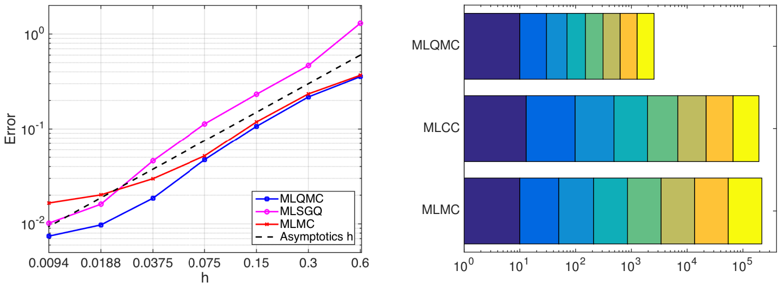

Fig. 4 H^1-errors of the different quadrature methods (left) and number of samples on each level in case of $j = 7$ (right) for the unit ball.

the sparse grid quadrature, is found on the right of Figure 4. It turns out that the quasi-Monte Carlo quadrature requires the least number of quadrature points. In contrast, the number of points for the Monte Carlo quadrature and for the Clenshaw-Curtis quadrature are nearly the same. Nevertheless, for fixed m and $r = 1$, we expect asymptotically $\theta = 2$ for the Clenshaw-Curtis quadrature as well, which is the same rate as for the quasi-Monte Carlo quadrature.

8.2 A more complex example

In our second example, the spatial domain is given by a model of the Zarya module of the International Space Station (ISS), which was the first module to be launched.[5] Figure 5 shows different tetrahedralizations of this geometry with decreasing mesh size. Note that the geometry can be imbedded into a cylinder with radius 0.52 and height 1.58.

j	0	1	2	3	4	5	6
h_j	0.5	0.25	0.125	0.0625	0.0313	0.0156	0.0078

Table 2 Mesh sizes on the different levels for the Zarya geometry.

In this example, the parametric diffusion coefficient is given by

$$\begin{aligned}\alpha(\mathbf{x},\mathbf{y}) = 1 + \frac{\exp(\|\mathbf{x}\|_2^2)}{20}\bigg(&\sin(2\pi x_1)y_1 + \frac{1}{2}\sin(2\pi x_2)y_2 + \frac{1}{4}\sin(2\pi x_3)y_3 \\ &+ \frac{1}{8}\sin(4\pi x_1)\sin(4\pi x_2)y_4 + \frac{1}{16}\sin(4\pi x_1)\sin(4\pi x_3)y_5 \\ &+ \frac{1}{32}\sin(4\pi x_2)\sin(4\pi x_3)y_6\bigg)\end{aligned}$$

and $f = 10$. For $\mathbf{x} \in D$ and $\mathbf{y} \in \square$, the diffusion coefficient varies approximately in the range $[0.19, 1.81]$. Figure 6 shows the mean (left) and the variance (right) of the reference solution. It has been computed on a mesh with 13 069 396 tetrahedrons

[5] We thank Martin Siegel (Rheinbach, Germany) who kindly provided us with this model.

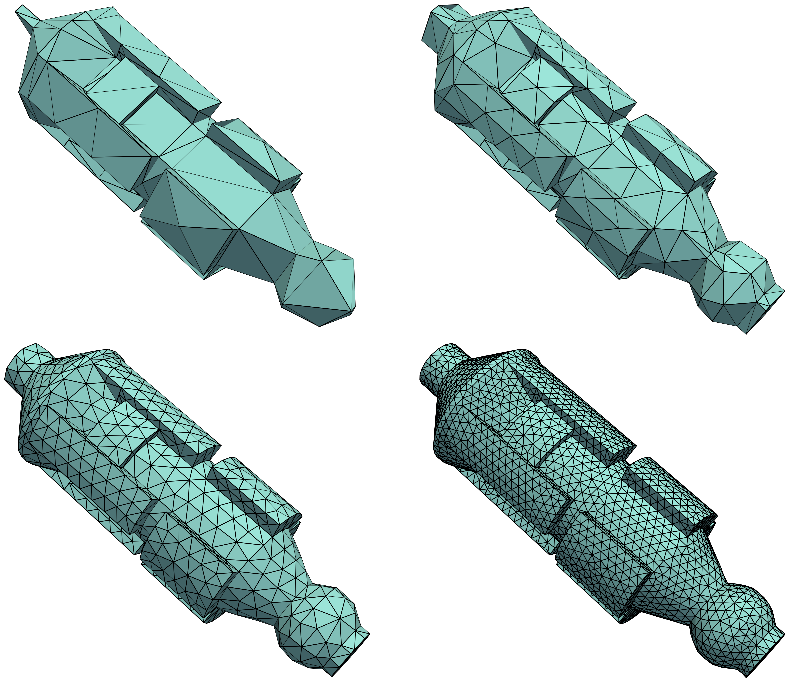

Fig. 5 Tetrahedralizations of four different resolutions for the Zarya geometry.

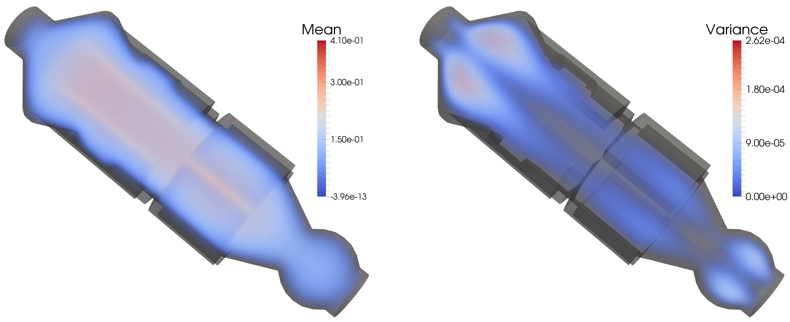

Fig. 6 Mean (left) and variance (right) of the model problem on the Zarya geometry.

resulting in a mesh size of $h = 0.0039$ by 10 000 quasi-Monte Carlo samples based on the Halton sequence. For the levels $j = 0, \ldots, 6$, the mesh sizes are given in Table 2.

Figure 7 visualizes the errors of the approximate expectation and second moment for the different multilevel quadrature methods under consideration. The number of quadrature points for the presented methods are chosen as in the previous example. In the mean, we observe for all methods the theoretical rate of $j2^{-j}$. However, for the second moment, the logarithm in the error seems not to show up.

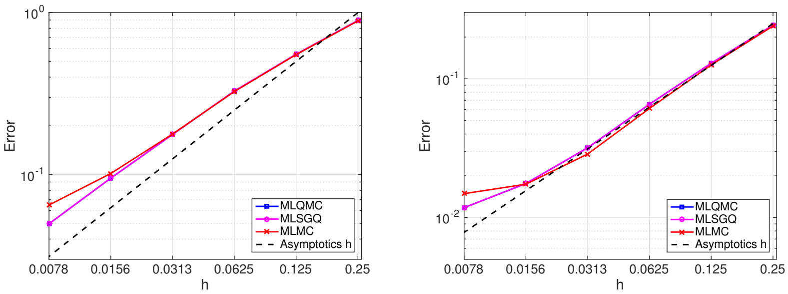

Fig. 7 H^1-errors of the approximate mean (left) and $W^{1,1}$-errors of the approximate second moment (right) on the Zarya geometry for different quadrature methods.

9 Conclusion

In the present article, we have reversed the construction of the conventional multilevel quadrature. This enables us to give up the nestedness of the spatial approximation spaces. Hence, black-box finite element solvers can be directly applied to compute the solution of the underlying boundary value problem. Another aspect of our approach is that the cost is considerably reduced by the application of nested quadrature formulae. Both features have been demonstrated by numerical results for the Clenshaw-Curtis quadrature and the quasi-Monte Carlo quadrature based on Halton points. Of course, other nested quadrature formulae like the Gauss-Patterson quadrature can be used as well. The application of quadrature formulae which are tailored to a possible anisotropy of the integrand is also straightforward. If non-nested quadrature formulae are applied, one arrives at a combination-technique-like representation of the multilevel quadrature. Note finally that adaptively refined finite element meshes could be used here as well.

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