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NOVEL RESULTS FOR THE ANISOTROPIC SPARSE QUADRATURE AND THEIR IMPACT ON RANDOM DIFFUSION PROBLEMS

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ABSTRACT. This article is dedicated to the anisotropic sparse Gaussian quadrature for functions which are analytically extendable into an anisotropic tensor product domain. Based on a novel estimate for the cardinality of the anisotropic index set, we are able to substantially improve the error versus cost estimates of the anisotropic sparse quadrature. To validate the theoretical findings, we use the anisotropic sparse Gaussian quadrature to compute the moments of elliptic partial differential equations with random diffusion.

1. INTRODUCTION

This article is dedicated to the construction of anisotropic sparse quadrature methods, where we emphasize on Gaussian type quadratures. Anisotropic sparse quadrature methods methods can be seen as a generalization sparse Smolyak type quadratures, cf. [21], since they are explicitly tailored to the anisotropic behaviour of the underlying integrand. Exploiting these anisotropies leads to a remarkable improvement in the complexity of the sparse quadrature.

The main task in estimating the quadrature's complexity is the estimation of the number of multi-indices which are contained in the sparse tensor product index set. For the isotropic variant, the number of indices can easily be determined by combinatorial arguments, see e.g. [7, 18]. Things get more involved if one considers weighted sparse tensor product spaces. In this case, to the best of our knowledge, only very rough estimates on the cardinality of the index set are known, although several estimates can be found in the literature, see e.g. [4]. In fact, this problem is equivalent to the estimation of the number of integer solutions of linear Diophantine inequalities (see [19] and the references therein), which is a problem in number theory, or to the calculation of the integer points in a convex polyhedron. Current estimates are not sharp and do not provide improved complexity results for the anisotropic sparse quadrature in comparison with the anisotropic full tensor product quadrature. In this article, we prove a novel formula to estimate the cardinality of the sparse tensor product index set in the weighted case. This formula is much sharper than the other established formulae.

A very popular application that requires efficient high-dimensional quadrature rules are parametric partial differential equations. They are obtained, for example, from partial differential equations with random data by truncating the series expansions of the underlying random fields and parametrizing with respect to the random fields' distribution. As a representative for such problems, we will consider here elliptic diffusion problems with random coefficients as a specific example to quantify the performance of the anisotropic sparse quadrature. The resulting quadrature approach is very similar to the anisotropic sparse collocation method based on Gaussian collocation points which has been introduced in [16, 17]. This method interpolates the random solution in certain collocation points and represents it in the parameter space with the aid of polynomials. Thus, it belongs to the class of non-intrusive methods, cf. [1]. Instead of representing the random solution, the anisotropic sparse quadrature can be employed to directly compute the solutions statistics, i.e. its moments, and functionals of the solution.

The remainder of this article is organized as follows. Section 2 specifies the quadrature problem under consideration and provides the corresponding framework. The subsequent Section 3 is dedicated to the sparse anisotropic Gaussian quadrature method. Here, we present the construction of the sparse quadrature and provide related error estimates based on a one dimensional, generic

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estimate. Section 5 deals with the cost complexity of the anisotropic sparse quadrature. In particular, we state here a novel estimate on the number of indices in the weighted sparse tensor product and provide a proof of this estimate. In Section 6, we introduce diffusion problems with random coefficients as a relevant application that profits from the improved quadrature methods. A couple of numerical examples that are related to the application under consideration are given in Section 7. Note that, in order to show the asymptotic convergence behaviour of the anisotropic quadrature, we restricted ourselves to one-dimensional examples. This is to avoid dealing with the increased computational complexity of our solver in higher spatial dimensions. Finally, we state concluding remarks in Section 8.

Throughout this article, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of the parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Problem setting

Let $\Gamma \subset \mathbb{R}$ be a bounded or unbounded interval and denote by Γ^∞ the set of all sequences $\psi\colon \mathbb{N} \to \Gamma$, where $\psi = \{\psi_n\}_n$. For a function $f\colon \Gamma^\infty \to \mathbb{R}$ and a suitable product density function $\rho(\psi)$, we are interested in the efficient approximation of the integral

$$\int_{\Gamma^\infty} f(\psi)\rho(\psi)\,\mathrm{d}\psi. \tag{1}$$

At first, we have to state more precisely how this integral has to be understood. To that end, we endow Γ^∞ with the structure of a probability space $\big(\Gamma^\infty, \mathcal{B}^\infty, \rho(\psi)\,\mathrm{d}\psi\big)$ in the usual way: Let $\mathcal{B}$ denote the Borel σ-field on Γ. Then, the Borel σ-field $\mathcal{B}^\infty$ on Γ^∞ is induced by the generating sets

$$\{\psi \in \Gamma^\infty : \psi_1 \in B_1, \ldots, \psi_m \in B_m\} \text{ for } m \geq 1 \text{ and } B_i \in \mathcal{B}.$$

With this construction of $\mathcal{B}^\infty$ at hand, the measure $\rho(\psi)\,\mathrm{d}\psi$ with

$$\rho(\psi) := \prod_{n=1}^{\infty} \rho_n(\psi_n), \quad \text{where} \quad \int_\Gamma \rho_n(\psi_n)\,\mathrm{d}\psi_n = 1 \quad \text{for all } n = 1, 2, \ldots,$$

defines a probability measure on $\mathcal{B}^\infty$.

In order to approximate (1) numerically, we assume that there exists a sequence $f_m\colon \Gamma^m \to \mathbb{R}$ such that

$$\left| \int_{\Gamma^\infty} f(\psi)\rho(\psi)\,\mathrm{d}\psi - \int_{\Gamma^m} f_m(\mathbf{y})\rho_m(\mathbf{y})\,\mathrm{d}\mathbf{y} \right| \lesssim \varepsilon(m), \tag{2}$$

where $\rho_m(\mathbf{y}) = \prod_{n=1}^m \rho_n(y_n)$, with a strictly decreasing null sequence $\varepsilon(m)$. In the sequel, we aim at approximating

$$\mathbf{I} f_m := \left(\bigotimes_{n=1}^{m} I^{(n)} \right) f_m := \int_{\Gamma^m} f_m(\mathbf{y})\rho_m(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{3}$$

by the anisotropic sparse tensor product quadrature. It is evident that the precision of the applied quadrature has to increase when m increases. Moreover, the complexity usually scales exponentially in m, which is referred to as the "curse of dimensionality". Therefore, we have to keep track of the impact of the dimension m on the error estimates. To make this impact as mild as possible, we have to impose a special structure of the function f and the approximations f_m, respectively.

Assumption 2.1. *Let $\Sigma_n = \Sigma(\Gamma, \tau_n) := \{z \in \mathbb{C} : \operatorname{dist}(z, \Gamma) \leq \tau_n\}$ and assume that f is analytically extendable into $\boldsymbol{\Sigma}(\boldsymbol{\tau}) := \bigtimes_{n=1}^\infty \Sigma_n$ for an isotone sequence $\tau_n \to \infty$. In addition, we suppose that f_m is analytically extendable into $\boldsymbol{\Sigma}(\boldsymbol{\tau})$.*

The sequence $\{\tau_n\}_n$ measures the anisotropic dependence of the function f on the different dimensions. Especially, in accordance with e.g. [14, 22] and Section 6, Assumption 2.1 guarantees that an N-point Gaussian quadrature formula constructed with respect to the densities ρ_n satisfies an one-dimensional error estimate of the form

$$\left| \int_\Gamma f_m(y_n, \mathbf{y}^\star)\rho_n(y_n)\,\mathrm{d}y_n - \sum_{k=1}^{N} \omega_k f_m(\eta_k, \mathbf{y}^\star) \right| \le g(\tau_n)\exp\big(-h(\tau_n)(2N-1)\big)\|f_m(\mathbf{y}^\star)\|_{C_{\sigma_n}(\Sigma_n)} \tag{4}$$

for some functions $g, h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathbf{y}^\star := [y_1, \ldots, y_{n-1}, y_{n+1}, \ldots, y_m]$. Here and in the sequel, we set

$$\|f_m(\mathbf{y}^\star)\|_{C_{\sigma_n}(\Sigma_n)} := \max_{z\in\Sigma_n} |\sigma_n(\mathrm{Re}(z)) f_m(z, \mathbf{y}^\star)|$$

for a suitable weight function $\sigma_n\colon \Gamma \to \mathbb{R}_+$. In the following presentation, we restrict ourselves to sparse Gaussian quadrature formulae. Even so, we emphasize that the approach under consideration is not limited to them. Any quadrature is feasible that satisfies a one dimensional error estimate which is similar to (4).

3. Anisotropic sparse Gaussian quadrature

We shall introduce anisotropic sparse Gaussian quadrature formulae which extend the original idea of Smolyak's construction from [21]. To that end, we start by considering an increasing sequence of univariate Gaussian quadrature points

$$\theta_j := \big\{\eta_{i,j}\big\}_{i=1}^{N_j} \subset \mathbb{R}, \quad N_j \in \mathbb{N}, \quad j = 1, 2, \ldots, \tag{5}$$

where $N_1 \le N_2 \le \cdots$. The associated Gaussian quadrature weights are denoted by $\big\{\omega_{i,j}\big\}_{i=1}^{N_j}$ and the associated Gaussian quadrature operators are denoted by Q_j.

Following the notation of [18], we introduce for $j \in \mathbb{N}$ the difference quadrature operator

$$\Delta_j := Q_j - Q_{j-1}, \quad \text{where} \quad Q_{-1} := 0. \tag{6}$$

With the telescoping sum $Q_j = \sum_{\ell=0}^{j} \Delta_\ell$, the isotropic m-fold tensor product quadrature operator, which uses in each direction N_j quadrature points, can be written by

$$Q_j^{(1)} \otimes \cdots \otimes Q_j^{(m)} = \sum_{\|\boldsymbol{\alpha}\|_\infty \le j} \Delta_{\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)}, \tag{7}$$

where the superscript index indicates the particular dimension.

The cost of applying the isotropic full tensor product quadrature operator (7) is obviously given by the number of points N_j^m contained in it. Thus, this isotropic tensor product quadrature extremely suffers from the curse of dimensionality. The classical *sparse Gaussian quadrature*, cf. [5, 7], can overcome this problem up to a certain extent. It is based on linear combinations of tensor product quadrature formulae of relatively small size. To define the sparse Gaussian quadrature, we introduce as in [2, 16] for each *approximation level* q the sets of multi-indices

$$X(q, m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m : \sum_{n=1}^{m} \alpha_n \le q \right\}$$

and

$$Y(q, m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m : q - m + 1 \le \sum_{n=1}^{m} \alpha_n \le q \right\}.$$

The Smolyak quadrature operator, cf. [2, 7, 21], is then given by

$$\mathcal{A}(q, m) := \sum_{\boldsymbol{\alpha}\in X(q,m)} \Delta_{\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)}. \tag{8}$$

An equivalent expression is obtained by the *combination technique* [10]

$$\mathcal{A}(q, m) = \sum_{\boldsymbol{\alpha}\in Y(q,m)} (-1)^{q-|\boldsymbol{\alpha}|} \binom{m-1}{q-|\boldsymbol{\alpha}|} \mathbf{Q}_{\boldsymbol{\alpha}}, \quad \text{where } \mathbf{Q}_{\boldsymbol{\alpha}} := Q_{\alpha_1}^{(1)} \otimes \cdots \otimes Q_{\alpha_m}^{(m)}. \tag{9}$$

A visualization of the set of indices $X(q, m)$ is given in Figure 1.

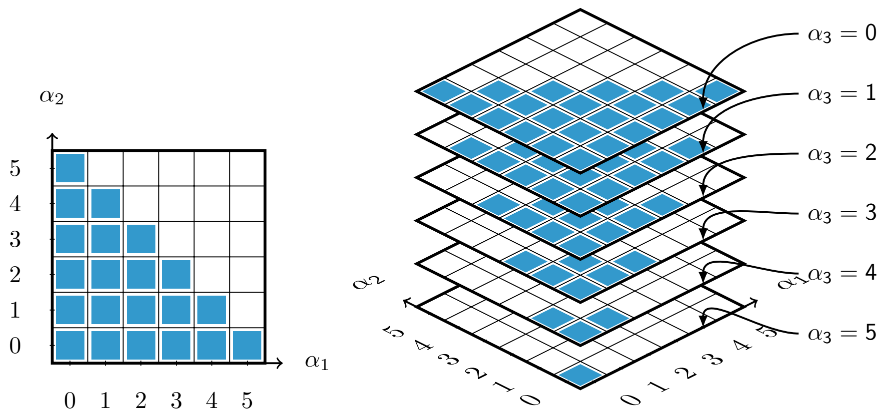

FIGURE 1. The 21 indices contained in the sparse grid $X(5, 2)$ on the left and the 56 indices contained in $X(5, 3)$ on the right.

The number of quadrature points used in (8) or (9) is considerably reduced compared to the full tensor product quadrature. However, the Smolyak quadrature operator does not take into account the fact that the different parameter dimensions are of different importance to the integrand f_m. Indeed, the cardinality of the set $X(q, m)$ is given by

$$\#X(q, m) = \binom{q+m}{m}$$

which still grows exponentially in the dimension m. Thus, we assign a weight to each parameter dimension and use a weighted version of the Smolyak quadrature operator.

Let $\mathbf{w} \in \mathbb{R}_+^m$ denote a weight vector for the different parameter dimensions. We assume in the following that the weight vector is sorted in ascending order, i.e. $w_1 \leq w_2 \leq \ldots \leq w_m$. Otherwise, we would rearrange the parametric dimensions accordingly. We modify the sparse grid sets $X(q, m)$ and $Y(q, m)$ in the following way, see also [17],

$$X_{\mathbf{w}}(q, m) := \left\{ \mathbf{0} \leq \boldsymbol{\alpha} \in \mathbb{N}^m : \sum_{n=1}^{m} \alpha_n w_n \leq q \right\} \tag{10}$$

and

$$Y_{\mathbf{w}}(q, m) := \left\{ \mathbf{0} \leq \boldsymbol{\alpha} \in \mathbb{N}^m : q - \|\mathbf{w}\|_1 < \sum_{n=1}^{m} \alpha_n w_n \leq q \right\}. \tag{11}$$

With this notation at hand, the *anisotropic Smolyak quadrature operator* of level $q \in \mathbb{N}$ is defined by

$$\mathcal{A}_{\mathbf{w}}(q, m) := \sum_{\boldsymbol{\alpha} \in X_{\mathbf{w}}(q,m)} \Delta_{\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)} \tag{12}$$

which can equivalently be expressed as, cf. [17],

$$\mathcal{A}_{\mathbf{w}}(q, m) = \sum_{\boldsymbol{\alpha} \in Y_{\mathbf{w}}(q,m)} c_{\mathbf{w}}(\boldsymbol{\alpha}) \mathbf{Q}_{\boldsymbol{\alpha}}, \quad \text{with} \quad c_{\mathbf{w}}(\boldsymbol{\alpha}) := \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^m \\ \boldsymbol{\alpha}+\boldsymbol{\beta} \in X_{\mathbf{w}}(q,m)}} (-1)^{|\boldsymbol{\beta}|}. \tag{13}$$

The formula (13) can be regarded as the *anisotropic combination technique quadrature*. For the evaluation of this formula, we only need to determine the coefficients $c_{\mathbf{w}}(\boldsymbol{\alpha})$ and to apply tensor product quadrature formulae of relatively small size. Thus, in order to compute the approximation

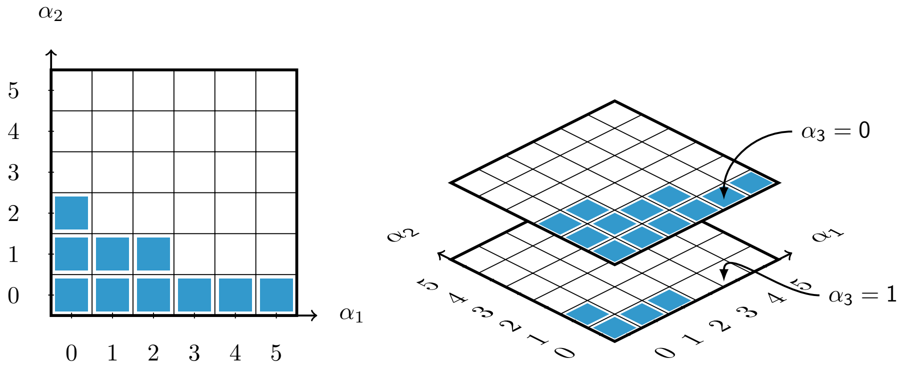

FIGURE 2. The 10 indices contained in the weighted sparse grid $X_{(1,2.5)}(5,2)$ on the left and the 16 indices contained in $X_{(1,2,3)}(5,3)$ on the right.

to (3) with the anisotropic Smolyak quadrature (13), it is sufficient to evaluate the integrand f_m on the *anisotropic sparse grid*

$$\mathcal{J}_{\mathbf{w}}(q,m) := \bigcup_{\boldsymbol{\alpha}\in Y_{\mathbf{w}}(q,m)} \theta_{\alpha_1} \times \cdots \times \theta_{\alpha_m}.$$

Note that the Smolyak quadrature operator (8) coincides with the anisotropic Smolyak quadrature operator (12) for the special weight vector $\mathbf{w} = \mathbf{1} := [1, 1, \ldots, 1]$.

In Figure 2, the indices of the weighted sparse grid $X_{(1,2.5)}(5,2)$ and of the weighted sparse grid $X_{(1,2,3)}(5,3)$ are visualized. We observe that the number of indices is drastically reduced in comparison to the according isotropic sparse grids visualized in Figure 1.

The computation of the anisotropic sparse quadrature formula (12) depends on the choice of the weight vector $\mathbf{w}$ and the sequence $\{N_j\}_j$ in (5). In view of the one-dimensional error estimate (4), the sequence $\{N_j\}_j$ of the number of quadrature points is chosen in accordance with

$$N_j = \left\lceil \frac{1}{2}(j+2) \right\rceil. \tag{14}$$

Then, we can estimate the error of the difference Gaussian quadrature operator $\Delta_j = Q_j - Q_{j-1}$ for all $j \geq 1$ and for all functions $f_1 \colon \Gamma \to \mathbb{R}$ which are analytically extendable in $\Sigma(\Gamma,\tau)$ by

$$\begin{aligned} |\Delta_j f_1| &\leq |f_1 - Q_j f_1| + |f_1 - Q_{j-1} f_1| \\ &\leq g(\tau)\Big(e^{-h(\tau)(j+1)} + e^{-h(\tau)j}\Big)\|f_1\|_{C_\sigma(\Sigma(\Gamma,\tau))} \\ &\leq g(\tau)\Big(1 + e^{-h(\tau)}\Big)e^{-h(\tau)j}\|f_1\|_{C_\sigma(\Sigma(\Gamma,\tau))} \\ &\leq 2g(\tau)e^{-h(\tau)j}\|f_1\|_{C_\sigma(\Sigma(\Gamma,\tau))}. \end{aligned} \tag{15}$$

For $j = 0$, the difference Gaussian quadrature operator coincides with the function evaluation at a particular point z of Γ which implies that

$$|\Delta_0 f_1| = |Q_0 f_1| = |f_1(z)| \leq e^{-h(\tau)\cdot 0}\|f_1\|_{C_\sigma(\Sigma(\Gamma,\tau))}. \tag{16}$$

Note that this estimate is only valid in case that $\sigma(z) \geq 1$ as it is case for the Gauss-Hermite and the Gauss-Legendre quadrature. Analogously, it follows from (14) and (4) that

$$|Iv - Q_j f_1| \leq g(\tau)e^{-h(\tau)(j+1)}\|f_1\|_{C_\sigma(\Sigma(\Gamma;\tau))}. \tag{17}$$

Next, let us consider the multivariate integrand $f_m \colon \Gamma^m \to \mathbb{R}$ which can analytically be extended into the region $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\tau})$. Then, it follows that the error of the tensor product of the operators

Δ_j is bounded by the product of the one-dimensional errors. Indeed, we obtain for a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^m$ that

$$
\begin{aligned}
&\left|\left(\Delta_{\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)}\right) f_m\right| \\
&\le (2g(\tau_1))^{\min(1,\alpha_1)} e^{-h(\tau_1)\alpha_1} \sup_{z\in\Sigma_1} \sigma_1(\operatorname{Re}(z)) \left|\left(\Delta_{\alpha_2}^{(2)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)}\right) f_m(z)\right| \\
&\le \left(\prod_{n=1}^{m} (2g(\tau_n))^{\min(1,\alpha_n)}\right) e^{-\sum_{n=1}^m h(\tau_n)\alpha_n} \|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}
\end{aligned}
\tag{18}
$$

with $\|v\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})} := \sup_{\mathbf{z}\in\boldsymbol{\Sigma}} \boldsymbol{\sigma}(\operatorname{Re}(\mathbf{z}))|v(\mathbf{z})|$ and $\boldsymbol{\sigma}(\operatorname{Re}(\mathbf{z})) := \prod_{n=1}^m \sigma_n(\operatorname{Re}(z_n))$. In addition, we take the minimum in (18) in order to ensure that the constant is 1 if $\alpha_n = 0$ in accordance with (16).

4. Error estimation for the anisotropic sparse Gaussian quadrature

For the estimation of the quadrature error of the anisotropic sparse Gaussian quadrature, we employ the following lemma.

Lemma 4.1. *Let $\{\psi_n\}_n \in \ell^1(\mathbb{N})$ be a summable sequence of positive real numbers. Then, there exists for each $\delta > 0$ a constant $C(\delta)$ independent of $q \ge 1$ such that*

$$
\prod_{n=1}^{\infty} (q\psi_n + 1) \le C(\delta)\exp(\delta q). \tag{19}
$$

Proof. Let $0 < \delta_1, \delta_2 < \delta$ be arbitrary such that $\delta_1 + \delta_2 = \delta$. From the summability of $\{\psi_n\}_n$, it follows that there exists a $j_0 = j_0(\delta_1) \in \mathbb{N}$ such that

$$
\sum_{n=j_0+1}^{\infty} \psi_n \le \delta_1. \tag{20}
$$

We now split the left-hand side in (19) into

$$
\prod_{n=1}^{\infty} (q\psi_n + 1) = \prod_{n=1}^{j_0} (q\psi_n + 1) \prod_{k=j_0+1}^{\infty} (q\psi_n + 1). \tag{21}
$$

Then, the second factor can simply be estimated by

$$
\prod_{n=j_0+1}^{\infty} (q\psi_n + 1) = \exp\left(\sum_{n=j_0+1}^{\infty} \log(q\psi_n + 1)\right) \le \exp(\delta_1 q).
$$

The number of factors j_0 in the first product in (21) is fixed and depends only on the choice of δ_1 and on the decay properties of $\{\psi_k\}_k$. Since j_0 is a fixed natural number, there exists for all $\delta_2 > 0$ a constant $C(\delta_1, \delta_2)$ such that

$$
\prod_{k=1}^{j_0} (q\psi_n + 1) \le C(\delta_1, \delta_2)\exp(\delta_2 q).
$$

Hence, we obtain that

$$
\prod_{n=1}^{\infty} (q\psi_n + 1) \le C(\delta_1, \delta_2)\exp(\delta q).
$$

Since $0 < \delta_1, \delta_2 < \delta$ can be chosen arbitrary with the only limitation that $\delta_1 + \delta_2 = \delta$, the choice $C(\delta) = \inf_{\delta_1+\delta_2=\delta} C(\delta_1, \delta_2)$ yields the desired estimate. □

With the above preliminaries, we are able to establish error estimates for the anisotropic sparse Gaussian quadrature. To that end, we have additionally to exploit some properties of the function g and the sequence $\{\tau_n\}_n$ in (4).

Assumption 4.2. *The sequence $\{\tau_n\}_n$ which describes the regions of analytic extendability of the function f fulfills*

$$\tau_n \gtrsim n^r$$

for some $r > 1$. Hence, the sequence $\{\tau_n^{-1}\}_n$ is summable. Additionally, we suppose that the sequence $\{g(\tau_n)\}_n$ is summable. Moreover, the function h is strictly monotone increasing and satisfies $h(x) \gtrsim log(x+1)$.

For the error estimation, we adapt some parts of the analysis in [17], but then conclude in a different way.

Lemma 4.3. *Let the sequence of quadrature points be chosen as in (14) and let the weight vector $\mathbf{w}$ be given by $w_n = h(\tau_n)$. Then, there exists for each $\delta > 0$ a constant $C(\delta)$ independent of m such that the error of the anisotropic sparse Gaussian quadrature (8) is bounded by*

$$\big|\big(\mathbf{I} - \mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| \lesssim C(\delta)e^{-q(1-\delta)}\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})} \tag{22}$$

The constant hidden in (22) depends on the continuity constant of $\mathbf{I}$ and on $\|\{g(\tau_n)\}_n\|_{\ell^1(\mathbb{N})}$. Note that the constant $C(\delta)$ tends to infinity as δ tends to 0.

Proof. In the same way as in [17], the error of the sparse quadrature is rewritten, with the notation $\mathbf{I} = \bigotimes_{n=1}^m I^{(n)}$, by

$$\mathbf{I} - \mathcal{A}_{\mathbf{w}}(q,m) = \sum_{n=1}^{m} R(q,n) \bigotimes_{k=n+1}^{m} I^{(k)}. \tag{23}$$

The quantity $R(q,n)$ is defined for $n \geq 2$ by

$$R(q,n) := \sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:n-1}}(q,n-1)} \bigotimes_{k=1}^{n-1} \Delta_{\alpha_k}^{(k)} \otimes \left(I^{(n)} - Q_{\left\lfloor\left(q-\sum_{k=1}^{n-1}\alpha_k w_k\right)/w_n\right\rfloor}\right)$$

and for $n = 1$ by

$$R(q,1) := I^{(1)} - Q_{\lfloor q/w_1\rfloor}.$$

For $n > 2$, each summand in (23) can be estimated with (17), (18) and with the continuity of the integration operator by

$$\begin{aligned}
\left|\left(R(q,n)\bigotimes_{k=n+1}^{m} I^{(k)}\right)f_m\right| &\lesssim \sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:n-1}}(q,n-1)} \left(\prod_{k=1}^{n-1}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)e^{-\sum_{k=1}^{n-1}\alpha_k h(\tau_k)} \\
&\qquad \cdot g(\tau_n)e^{-h(\tau_n)\left(\left\lfloor\left(q-\sum_{k=1}^{n-1}\alpha_k w_k\right)/w_n\right\rfloor+1\right)}\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})} \\
&\lesssim g(\tau_n)\sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:n-1}}(q,n-1)} e^{-h(\tau_n)\left(\left\lfloor\left(q-\sum_{k=1}^{n-1}\alpha_k w_k\right)/w_n\right\rfloor+1\right)-\sum_{k=1}^{n-1}\alpha_k h(\tau_k)} \\
&\qquad \cdot\left(\prod_{k=1}^{n-1}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}.
\end{aligned}$$

With the choice $w_k = h(\tau_k)$ for all $k = 1,\dots,m$, it follows that

$$\begin{aligned}
&\left|\left(R(q,n)\bigotimes_{k=n+1}^{m} I^{(k)}\right)v\right| \\
&\lesssim g(\tau_n)\sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:n-1}}(q,n-1)} e^{-q-\sum_{k=1}^{n-1}\alpha_k w_k+\sum_{k=1}^{n-1}\alpha_k w_k}\left(\prod_{k=1}^{n-1}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)\|v\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})} \\
&= g(\tau_n)\sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:n-1}}(q,n-1)} e^{-q}\left(\prod_{k=1}^{n-1}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)\|v\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}.
\end{aligned}$$

For $n=1$, we have that $R(q,1)=I^{(1)}-Q_{\lfloor q/w_1\rfloor}$. We thus deduce that

$$\left|\left(R(q,1)\bigotimes_{k=2}^{m}I^{(k)}\right)v\right|\lesssim g(\tau_1)e^{-h(\tau_1)(\lfloor q/w_1\rfloor+1)}\|v\|_{C_{\boldsymbol\sigma}(\boldsymbol\Sigma)}\lesssim g(\tau_1)e^{-q}\|v\|_{C_{\boldsymbol\sigma}(\boldsymbol\Sigma)}.$$

It remains to estimate

$$\sum_{\boldsymbol\alpha\in X_{\mathbf{w}_{1:n-1}}(q,n-1)}\left(\prod_{k=1}^{n-1}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)\leq\sum_{\boldsymbol\alpha\in X_{\mathbf{w}}(q,m)}\left(\prod_{k=1}^{m}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right).$$

The maximum inside the product is $2g(\tau_k)$ except for the case $\alpha_k=0$. Hence, it follows that

$$\begin{aligned}\sum_{\boldsymbol\alpha\in X_{\mathbf{w}}(q,m)}&\left(\prod_{k=1}^{m}\big(2g(\tau_k)\big)^{\min(1,\alpha_k)}\right)\\&\leq\sum_{\alpha_1=0}^{\lfloor\frac{q}{w_1}\rfloor}(2g(\tau_1))^{\min(\alpha_1,1)}\sum_{\alpha_2=0}^{\lfloor\frac{q}{w_2}\rfloor}(2g(\tau_2))^{\min(\alpha_2,1)}\cdots\sum_{\alpha_m=0}^{\lfloor\frac{q}{w_m}\rfloor}(2g(\tau_m))^{\min(\alpha_m,1)}\\&\leq\prod_{k=1}^{m}\left(\frac{2g(\tau_k)q}{w_k}+1\right)\leq C(\delta)\exp(\delta q).\end{aligned}$$

The last inequality holds since $\{2g(\tau_n)/w_n\}_n$ is summable and, thus, Lemma 4.1 is applicable. Combining our findings yields the estimate (22). □

Lemma 4.3 implies that the anisotropic sparse Gaussian quadrature converges exponentially with respect to the level q. The convergence in Lemma 4.3 is nearly as good as the convergence of the anisotropic tensor product Gaussian quadrature on level q, with $\lceil\frac{q}{2w_n}+\frac{1}{2}\rceil$ quadrature points in the n-th direction.

5. Cost complexity of the anisotropic sparse Gaussian quadrature

5.1. **A preliminary estimate on the cost.**

In order to find an error estimate in terms of the number of quadrature points, we additionally have to estimate the cost of the sparse Gaussian quadrature method on level q. We exploit that the weight vector $\mathbf{w}$ is ordered ascendingly, i.e. $w_1\leq w_2\leq\cdots\leq w_m$. In the following, we establish a bound on the number of quadrature points used in the combination technique formula (13). This number is given by

$$\begin{aligned}\operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q(\epsilon),m)\big)&=\sum_{\boldsymbol\alpha\in Y_{\mathbf{w}}(q,m)}\prod_{n=1}^{m}N_{\alpha_n}=\sum_{\boldsymbol\alpha\in Y_{\mathbf{w}}(q,m)}\prod_{n=1}^{m}\left\lceil\frac{1}{2}\Big((\alpha_n+2)\Big)\right\rceil\\&\leq\sum_{\boldsymbol\alpha\in Y_{\mathbf{w}}(q,m)}\prod_{n=1}^{m}(\alpha_n+1).\end{aligned}\tag{24}$$

Then, we simply use that $Y_{\mathbf{w}}(q,m)\subset X_{\mathbf{w}}(q,m)$, cf. (10) and (11), and estimate the maximum value of the summands in (24). For this, we have to solve the optimization problem

$$\max_{\boldsymbol\alpha\in X_{\mathbf{w}}(q,m)}\prod_{n=1}^{m}\big(\alpha_n+1\big).$$

This is equivalent to the problem

$$\max_{\boldsymbol\alpha\in\mathbb{N}^m}\prod_{n=1}^{m}\big(\alpha_n+1\big)\quad\text{s.t.}\quad\sum_{n=1}^{m}w_n\alpha_n\leq q.$$

We get an upper bound for this optimization problem if we extend the admissible set of multi-indices to arbitrary m-dimensional vectors with positive coefficients

$$\sup_{\boldsymbol\alpha\in\mathbb{R}^m}\prod_{n=1}^{m}(\alpha_n+1)\quad\text{s.t.}\quad\sum_{n=1}^{m}w_n\alpha_n\leq q\quad\text{and}\quad\alpha_n\geq0\quad\text{for}\quad n=1,\dots,m.$$

The problem's solution can be calculated by solving the equivalent optimization problem

$$\sup_{\boldsymbol{\alpha}\in\mathbb{R}_+^m} \sum_{n=1}^m \log(\alpha_n+1) \quad \text{s.t.} \quad \sum_{n=1}^m w_n\alpha_n \le q \quad \text{and} \quad \alpha_n \ge 0 \quad \text{for} \quad n=1,\dots,m.$$

We solve it by means of Lagrangian multipliers and get the optimal solution

$$\alpha_n = \begin{cases} \frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_n} - 1, & \text{if } n \le n_0, \\ 0, & \text{if } n > n_0, \end{cases}$$

where n_0 is determined by

$$n_0 = \operatorname*{argmax}_{n=1,\dots,m} \left\{ q + \sum_{\ell=1}^n w_\ell \ge n w_n \right\}. \tag{25}$$

This implies the following lemma on the upper bound for (24).

Lemma 5.1. *Let the weight vector $\mathbf{w} = [w_1,\dots,w_m]$ be ascendingly ordered. Then, the cost complexity of the anisotropic sparse Gaussian quadrature on level q is, with n_0 from* (25)*, bounded by*

$$\operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m)\big) \le \#X_{\mathbf{w}}(q,m) \prod_{n=1}^{n_0} \left(\frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_n} \right). \tag{26}$$

The product on the right-hand side in (26) can further be estimated.

Lemma 5.2. *Let the weight vector $\mathbf{w} = [w_1,\dots,w_m]$ be ascendingly ordered and $m \le n_0$. Then, it holds that*

$$\prod_{n=1}^{n_0} \left(\frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_n} \right) \le \prod_{n=1}^{m} \left(\frac{q}{n w_n} + 1 \right). \tag{27}$$

Proof. We show for $n = 1,2,\dots,n_0-1$ that

$$\begin{aligned} &\left(\frac{q+\sum_{k=1}^{n_0-n} w_k + n w_{n_0}}{n_0 w_{n_0}} \right) \left(\frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_{n_0-n}} \right) \\ &\qquad \le \left(\frac{q+\sum_{k=1}^{n_0-n-1} w_k + (n+1) w_{n_0}}{n_0 w_{n_0}} \right) \left(\frac{q+\sum_{k=1}^{n_0-1} w_k}{(n_0-1) w_{n_0-n}} \right). \end{aligned} \tag{28}$$

The successive application of this inequality for $n = 1,2,\dots,n_0-1$ leads to

$$\prod_{n=1}^{n_0} \left(\frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_n} \right) \le \left(\frac{q}{n_0 w_{n_0}} + 1 \right) \prod_{n=1}^{n_0-1} \left(\frac{q+\sum_{k=1}^{n_0-1} w_k}{(n_0-1) w_n} \right).$$

Then, it follows by proceeding in the same way for $n_0-1, n_0-2, \dots, 2$ that

$$\begin{aligned} \prod_{n=1}^{n_0} \left(\frac{q+\sum_{k=1}^{n_0} w_k}{n_0 w_n} \right) &\le \left(\frac{q}{n_0 w_{n_0}} + 1 \right) \left(\frac{q}{(n_0-1) w_{n_0-1}} + 1 \right) \prod_{n=1}^{n_0-2} \left(\frac{q+\sum_{k=1}^{n_0-2} w_k}{(n_0-2) w_n} \right) \\ &\le \prod_{n=1}^{n_0} \left(\frac{q}{n w_n} + 1 \right). \end{aligned}$$

Since $n_0 < m$, this would immediately imply the assertion.

To prove (28), we use the abbreviation $\tilde{q} := q + \sum_{k=1}^{n_0-n-1} w_k$ and rewrite this inequality by

$$(n_0-1)\big(\tilde{q} + w_{n_0-n} + n w_{n_0}\big) \left(\tilde{q} + \sum_{k=n_0-n}^{n_0} w_k \right) - n_0\big(\tilde{q} + (n+1) w_{n_0}\big) \left(\tilde{q} + \sum_{k=n_0-n}^{n_0-1} w_k \right) \le 0.$$

After expanding the products, some of the terms vanish and we can simplify this expression to

$$
\begin{aligned}
&n_0\Bigg(\tilde{q}w_{n_0-n}+(n+1)w_{n_0}^2+(w_{n_0-n}-w_{n_0})\sum_{k=n_0-n}^{n_0}w_k\Bigg)\\
&\qquad\qquad-\Bigg(\tilde{q}\bigg(\tilde{q}+\sum_{k=n_0-n}^{n_0}w_k\bigg)+(w_{n_0-n}+nw_{n_0})\bigg(\tilde{q}+\sum_{k=n_0-n}^{n_0}w_k\bigg)\Bigg)\\
&\quad\le n_0\Bigg(\tilde{q}(w_{n_0-n}-w_{n_0})+(w_{n_0-n}-w_{n_0})\sum_{k=n_0-n}^{n_0}w_k+(n+1)w_{n_0}^2-(w_{n_0-n}+nw_{n_0})w_{n_0}\Bigg)\\
&\quad\le n_0\Big(n_0w_{n_0}\big(w_{n_0-n}-w_{n_0}\big)-w_{n_0}\big(w_{n_0-n}-w_{n_0}\big)\Big)\\
&\quad= n_0(n_0-1)\big(w_{n_0-n}-w_{n_0}\big)\le 0.
\end{aligned}
$$

Here, the first and second inequality follow from $\tilde{q}+\sum_{k=n_0-n}^{n_0}w_k=q+\sum_{k=1}^{n_0}w_k\ge n_0w_{n_0}$ and from $w_{n_0-n}\le w_{n_0}$. This completes the proof. □

Next, we can deduce, in view of (26) and (27), that the complexity of the anisotropic sparse Gaussian quadrature is bounded by

$$
\operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m)\big)\le\left(\prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right)\right)\#X_{\mathbf{w}}(q,m). \tag{29}
$$

5.2. **A sharp estimate on the anisotropic sparse index set.** In order to complete the convergence analysis, it remains to estimate the number of indices in the set $X_{\mathbf{w}}(q,m)$. Therefore, we require the following lemma.

Lemma 5.3. *For $L\in\mathbb{N}$, $m\in\mathbb{N}$ and $\delta\in\mathbb{R}_+$, there holds the inequality*

$$
\sum_{j=0}^{L-1}\prod_{n=1}^{m}(n+\delta+j)\le\frac{1}{m+1}\prod_{n=0}^{m}(L+\delta+n)
$$

with equality when $\delta=0$.

Proof. We prove the assertion by induction on L. For $L=1$, we verify

$$
\prod_{n=1}^{m}(n+\delta)=\frac{1}{m+1+\delta}\prod_{n=1}^{m+1}(n+\delta)\le\frac{1}{m+1}\prod_{n=0}^{m}(n+\delta+1).
$$

Let the assertion be fulfilled for L. Then, we conclude for $L+1$ that

$$
\begin{aligned}
\sum_{j=0}^{L}\prod_{n=1}^{m}(n+\delta+j)&\le\frac{L+\delta}{m+1}\prod_{n=1}^{m}(L+\delta+n)+\prod_{n=1}^{m}(L+\delta+n)\\
&=\left(\frac{L+m+1+\delta}{m+1}\right)\prod_{n=1}^{m}(L+n+\delta)\\
&=\left(\frac{1}{m+1}\right)\prod_{n=1}^{m+1}(L+n+\delta)\\
&=\left(\frac{1}{m+1}\right)\prod_{n=0}^{m}(L+1+n+\delta).
\end{aligned}
$$

□

Lemma 5.4. *The cardinality of the set $X_{\mathbf{w}}(q,m)$ in (10), where the weight vector $\mathbf{w}=[w_1,\dots,w_m]$ is ascendingly ordered, i.e. $w_1\le w_2\le\cdots\le w_m$, is bounded by*

$$
\#X_{\mathbf{w}}(q,m)\le\prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right). \tag{30}
$$

Proof. The prove is performed by induction on m. For $m = 1$, the assertion is obviously fulfilled, since

$$\#X_{\mathbf{w}}(q,1) = \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor} 1 = \left\lfloor \frac{q}{w_1} \right\rfloor + 1.$$

Let us assume that (30) is true for $m-1$. For $m \in \mathbb{N}$, the cardinality of $X_{\mathbf{w}}(q,m)$ can be calculated by

$$\#X_{\mathbf{w}}(q,m) = \sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \#X_{\mathbf{w_{1:m-1}}}(q - jw_m, m-1).$$

Inserting the induction hypothesis yields that

$$\begin{aligned}
\#X_{\mathbf{w}}(q,m) &\leq \sum_{j=0}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 + \frac{q - jw_m}{nw_n}\right) \\
&= \left(\prod_{k=1}^{m-1}\left(1 + \frac{q}{kw_k}\right)\right) \sum_{j=0}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \frac{1 + \frac{q-jw_m}{nw_n}}{1 + \frac{q}{nw_n}} \\
&= \left(\prod_{n=1}^{m-1}\left(1 + \frac{q}{nw_n}\right)\right) \sum_{j=0}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 - \frac{jw_m}{nw_n + q}\right) \\
&= \left(\prod_{n=1}^{m-1}\left(1 + \frac{q}{nw_n}\right)\right)\left(1 + \sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 - \frac{jw_m}{nw_n + q}\right)\right).
\end{aligned} \tag{31}$$

Focusing on the last term and since $w_m \geq w_n$ for all $0 \leq n \leq m$, we conclude that

$$\begin{aligned}
\sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 - \frac{jw_m}{nw_n + q}\right) &\leq \sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 - \frac{jw_m}{nw_m + q}\right) \\
&= \prod_{n=1}^{m-1}\left(n + \frac{q}{w_m}\right)^{-1} \sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(n + \frac{q}{w_m} - j\right).
\end{aligned}$$

Applying the previous lemma with $L = \lfloor \frac{q}{w_m} \rfloor$ and $\delta = \frac{q}{w_m} - L$ leads to

$$\sum_{j=1}^{L} \prod_{n=1}^{m-1} (n + L + \delta - j) = \sum_{j=0}^{L-1} \prod_{n=1}^{m-1} (n + \delta + j) \leq \frac{1}{m} \prod_{n=0}^{m-1} (L + \delta + n).$$

Thus, we obtain that

$$\sum_{j=1}^{\lfloor \frac{q}{w_m} \rfloor} \prod_{n=1}^{m-1} \left(1 - \frac{jw_m}{nw_n + q}\right)\Bigg) \leq \frac{L + \delta}{m} = \frac{q}{mw_m}.$$

Inserting this into (31) finishes the proof. □

Remark 5.5. (1) *We would like to point out that estimate* (30) *is sharp in the isotropic case, that is, for the weight* $\mathbf{w} = \mathbf{1}$*. Moreover, the ordering of the weight vector is crucial in this estimate. There are examples where this estimate does not hold if the weights are not in ascending order.*

(2) *At first glance one might claim that even the estimate*

$$\#X_{\mathbf{w}}(q,m) \leq \prod_{n=1}^{m} \frac{\lfloor \frac{q}{w_n} \rfloor + n}{n}$$

is valid. This is true in a lot of cases which we investigated. Nevertheless, there are examples where this estimate fails.

5.3. **Convergence in terms of the number of quadrature points.** The findings of the previous two sections can be summarized to the error estimate of the anisotropic sparse grid quadrature

$$\big|\big(\mathbf{I}-\mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| \lesssim C(\delta)e^{-q(1-\delta)}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}$$

and the complexity estimate

$$\operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m)\big) \leq \left(\prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right)\right)^2. \tag{32}$$

In view of our application to parametric partial differential equations, we have to examine the cost complexity with respect to the properties of the sequence $\{h(\tau_n)\}_n$. In particular, under certain conditions on this sequence, the convergence rate in terms of the number of quadrature points is dimension-independent and algebraic of arbitrary order.

Theorem 5.6. *If the sequence $\{(nh(\tau_n))^{-1}\}_n$ is summable, then there exists a constant $C(\delta,\eta)$, which does not dependent on the dimension m for all $\delta,\eta>0$ but tends to ∞ if $\delta\to 0$ or $\eta\to 0$, such that*

$$\big|\big(\mathbf{I}-\mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| \lesssim C(\delta,\eta)N(q)^{-\frac{1-\delta}{2\eta}}\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})} \tag{33}$$

where $N(q)$ denotes the total number of quadrature points in $\mathcal{A}_{\mathbf{w}}(q,m)$. The constant hidden in the estimate coincides with the constant in Lemma 4.3.

Proof. From the definition of the weights w_n, cf. Lemma 4.3, we know that $w_n = h(\tau_n)$. Since $\{(nh(\tau_n))^{-1}\}$ is summable, it follows with Lemma 4.1 that there exists for each $\eta>0$ a constant $C(\eta)$ independent of m such that

$$\prod_{n=1}^{m-1}\left(\frac{q}{nw_n}+1\right) \leq C(\eta)\exp\big(q\eta\big). \tag{34}$$

Inserting this into (32) implies that

$$N(q) := \operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m) \leq C(\eta)^2\exp(2q\eta) \quad\iff\quad q \geq \frac{1}{2\eta}\log\left(\frac{N(q)}{C(\eta)^2}\right).$$

This yields that the error in terms of $N(q)$ is bounded by

$$\begin{aligned}\big|\big(\mathbf{I}-\mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| &\lesssim C(\delta)\exp\left(-\frac{1-\delta}{2\eta}\log\left(\frac{N(q)}{C(\eta)^2}\right)\right)\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}\\ &= \underbrace{C(\delta)C(\eta)^{\frac{1-\delta}{\eta}}}_{=:C(\delta,\eta)}N(q)^{-\frac{1-\delta}{2\eta}}\|f_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma})}.\end{aligned}$$

□

Remark 5.7. *The condition that $\{(nh(\tau_n))^{-1}\}_n$ is summable implies that $h(\tau_n)$ increases stronger than $\log(n)$. In particular, a rate $\log(n)^{1+\delta}$ for arbitrary $\delta>0$ would be sufficient. Unfortunately, since $h(\tau_n)\eqsim\log(c\tau_n)$ for the Gauss-Legendre and Gauss-Hermite quadrature, cf.* (41) *and* (43)*, any algebraic increase of τ_n is not sufficient for the summability of $\{(nh(\tau_n))^{-1}\}_n$. Nevertheless, if τ_n increases subexponentially, i.e. $\tau_n\eqsim\exp(n^\delta)$ for arbitrary $\delta>0$, summability of $\{(nh(\tau_n))^{-1}\}_n$ is guaranteed, cf.* [9].

In view of this remark, we investigate in the rest of this section how fast the convergence rate deteriorates for an algebraic increase, i.e. $\tau_n \gtrsim n^r$.

Lemma 5.8. *Let the sequence $\{h(\tau_n)\}_n$ increase as $h(\tau_n)\geq\log(cn^r)$ for some $c>1$ and $r\in\mathbb{R}_+$. Then, we obtain that the number of indices in the anisotropic sparse grid is bounded by*

$$\#X_{\mathbf{w}}(q,m) \lesssim \exp\left(\frac{q}{r}\log(\log(m))\right) \tag{35}$$

with a constant which is independent of m.

Proof. From Lemma 5.4, we know that

$$\#X_{\mathbf{w}}(q,m) \le \prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right).$$

Next, we split the product into

$$\prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right) = \left(\frac{q}{w_1}+1\right)\left(\frac{q}{2w_2}+1\right)\left(\frac{q}{3w_3}+1\right)\prod_{n=4}^{m}\left(\frac{q}{nw_n}+1\right). \tag{36}$$

We estimate the last term by

$$\prod_{n=4}^{m}\left(\frac{q}{nw_n}+1\right) \le \exp\left(\sum_{n=4}^{m}\log\left(\frac{q}{nw_n}+1\right)\right) \le \exp\left(\sum_{n=4}^{m}\frac{q}{nw_n}\right).$$

Due to $w_n \ge \log(n^r)$, the sum in this estimate can be bounded by the following integral:

$$\begin{aligned}\sum_{n=4}^{m}\frac{q}{nw_n} \le \int_3^m \frac{q}{x\log(x^r)}\,\mathrm{d}x &= \frac{q}{r}\int_3^m \frac{1}{x\log(x)}\,\mathrm{d}x \\ &= \frac{q}{r}\int_{\log(3)}^{\log(m)}\frac{1}{z}\,\mathrm{d}z = \frac{q}{r}\big(\log(\log(m)) - \log(\log(3))\big).\end{aligned}$$

The first three factors in (36) define a cubic polynomial in q and can thus be estimated by the exponential function according to

$$\left(\frac{q}{w_1}+1\right)\left(\frac{q}{2w_2}+1\right)\left(\frac{q}{3w_3}+1\right) \le C\exp\left(\frac{\log(\log(3))}{r}q\right).$$

Hence, putting all together, we end up with

$$\begin{aligned}\prod_{n=1}^{m}\left(\frac{q}{nw_n}+1\right) &\le C\exp\left(\frac{\log(\log(3))}{r}q\right)\exp\left(\frac{q}{r}\big(\log(\log(m)) - \log(\log(3))\big)\right) \\ &\lesssim \exp\left(\frac{q}{r}\big(\log(\log(m))\big)\right).\end{aligned}$$

□

With Lemma 5.8 at hand, we are able to quantify how the dimensionality m compromises the convergence rate of the anisotropic sparse Gaussian quadrature. In fact, the dimensionality enters only with a factor $\log(\log(m))$ in case of algebraic increasing regions of analyticity.

Theorem 5.9. *Let the conditions of Lemma 4.3 be satisfied and let the assumptions of Lemma 5.8 be fulfilled. Then, the error of the anisotropic sparse Gaussian quadrature $\mathcal{A}_{\mathbf{w}}(q,m)$ is bounded in terms of the total number of quadrature points by*

$$\|v - \mathcal{A}_{\mathbf{w}}(q,m)v\|_X \lesssim N(q)^{-\frac{r}{2\log(\log(m))}}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)}. \tag{37}$$

Proof. Inserting (35) into (32), leads to the complexity estimate

$$N(q) = \mathrm{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m)\big) \lesssim e^{2q\left(\frac{\log(\log(m))}{r}\right)}.$$

Combining this with the error estimate (22) implies the desired bound (37). □

6. Application to diffusion problems with random coefficient

6.1. **Problem setting.** As a practical application of the sparse anisotropic Gaussian quadrature, we consider random diffusion problems with either uniformly or lognormally distributed diffusion coefficients. Since we lay our emphasis on the convergence behavior of the Gaussian quadrature, we will deal here only with one-dimensional problems. Even so, we want to emphasize that all results remains valid also in two and three spatial dimensions.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete and separable probability space. We consider the diffusion equation

$$-\partial_x\big(a(x,\omega)\partial_x u(x,\omega)\big) = 1 \text{ in } D=(0,1) \quad \text{for almost every } \omega\in\Omega \tag{38}$$

with homogenous boundary conditions, i.e. $u(0,\omega) = u(1,\omega) = 0$. The first step towards the solution for this class of problems is the parameterization of the stochastic parameter. To that end, one decomposes the diffusion coefficient with the aid of the Karhunen-Loève expansion. Let the covariance kernel of $a(x,\omega)$ be defined by the positive semi-definite function

$$\mathcal{C}(x,x') := \int_\Omega \big(a(x,\omega) - \mathbb{E}[a](x)\big)\big(a(x',\omega) - \mathbb{E}[a](x')\big) \,\mathrm{d}\mathbb{P}(\omega).$$

Herein, the integral with respect to Ω has to be understood in terms of a Bochner integral, cf. [13]. Now, let (λ_k, φ_k) denote the eigenpairs obtained by solving the eigenproblem for the diffusion coefficient's covariance, i.e.

$$\int_0^1 \mathcal{C}(x,x')\varphi_k(x') \,\mathrm{d}x' = \lambda_k \varphi_k(x).$$

Then, the Karhunen-Loève expansion of $a(x,\omega)$ is given by

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\varphi_n(x) X_n(\omega),$$

where $X_n\colon \Omega \to \Gamma \subset \mathbb{R}$ for $n = 1, 2, \ldots$ are centered, pairwise uncorrelated and L^2-normalized random variables. In the uniformly distributed case, we have $X_n \sim \mathcal{U}([-1,1])$ and in the log-normally distributed case, we have $X_n \sim \mathcal{N}(0,1)$. Note that we compute in the latter case the Karhunen-Loève expansion of $\log\big(a(x,\omega)\big)$ rather than of $a(x,\omega)$ itself and set $\mathbb{E}[\log(a)](x) = 0$. In the lognormal case, the knowledge of $\mathcal{C}(x,x')$ together with $\mathbb{E}[\log(a)](x) = 0$ provides the unique description of $\log\big(a(x,\mathbf{y})\big)$ since the underlying random process is Gaussian. In the uniform case, we have additionally to assume that the random variables are independent and that $\mathbb{E}[a](x) > 0$ such that $a(x,\omega)$ becomes uniformly elliptic.

By substituting the random variables with their image in Γ, we arrive in the uniformly distributed case at the parameterized Karhunen-Loève expansion

$$a(x,\psi) = \mathbb{E}[a](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\varphi_n(x)\sqrt{3}\psi_n,$$

where $\psi_n \in [-1,1]$ and $\rho_n(\psi_n) = 1/2$. Note that the scaling factor $\sqrt{3}$ stems from the normalization of the random varibles' variance. For the lognormally distributed case, we obtain in complete analogy

$$\log\big(a(x,\psi)\big) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\varphi_n(x)\psi_n,$$

where $\psi_n \in \mathbb{R}$ and $\rho_n(\psi_n) = 1/\sqrt{2\pi}\exp(-\psi_n^2/2)$. We define $\gamma_n = \sqrt{\lambda_n}\|\varphi_n\|_{L^\infty(D)}$. The decay of the sequence $\{\gamma_n\}_n$ is important in order to determine the region of analytical extendability of the solution u, cf. Lemma 6.1.

Truncating the respective Karhunen-Loève expansion after $m \in \mathbb{N}$ terms, yields the parametric and truncated diffusion problem

$$-\partial_x\big(a_m(x,\mathbf{y})\partial_x u_m(x,\mathbf{y})\big) = 1 \text{ in } D = (0,1) \quad \text{for almost every } \mathbf{y} \in \Gamma^m. \tag{39}$$

The impact of truncating the Karhunen-Loève expansion on the solution is bounded by

$$\|u - u_m\|_{L^2_\rho(\Gamma^\infty;H_0^1(D))} \lesssim \|a - a_m\|_{L^2_\rho(\Gamma^\infty;L^\infty(D))} = \varepsilon(m),$$

where $\varepsilon(m) \to 0$ montonically as $m \to \infty$, see e.g. [6, 20]. Herein, the Bochner spaces $L^2_\rho(\Gamma^\infty;\mathcal{X})$, where $\mathcal{X}$ is a separable Banach space, consist of all equivalence classes of measurable functions $f\colon \Gamma^\infty \to \mathcal{X}$ with bounded norm

$$\|f\|_{L^2_\rho(\Gamma^\infty;\mathcal{X})} := \left(\int_{\Gamma^\infty} \|f(\psi)\|^2_{\mathcal{X}} \rho(\psi) \,\mathrm{d}\psi\right)^{\frac{1}{2}},$$

see [13] for more details on Bochner spaces and Bochner integrable functions. Since the L^2-norm is stronger than the L^1-norm, this especially yields the approximation estimate (2) for u and u_m, where the modulus has to be replaced by the $H_0^1(D)$-norm.

Given the parametric solution $u_m(x, \mathbf{y})$, we are interested in determining proporties of its distribution. In our numerical examples, we focus on the computation of the solution's moments. These are given by the Bochner integral

$$\mathcal{M}_u^p(x) := \int_{\Gamma^m} u_m^p(x, \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Especially, there holds $\mathcal{M}_u^1(x) = \mathbb{E}_u(x)$.

6.2. **Regularity estimates.** In order to apply the presented quadrature theory to our parametric diffusion problems, we have to provide the related regularity results that allow for an analytic extension of u_m into the complex plane. The extendability is guaranteed by the following lemma from [3], which has slightly been modified to fit our purposes.

Lemma 6.1. *The solution u_m to* (39) *in the uniformly elliptic case admits an analytic extension into the region $\boldsymbol{\Sigma}([-1,1]^m, \boldsymbol{\tau})$ for all $\boldsymbol{\tau}$ with*

$$\tau_k < \frac{a}{C(\delta)k^{1+\delta}\gamma_k}, \quad \textit{where } C(\delta) = \sum_{k=1}^{\infty} k^{-1-\delta} \textit{ for arbitrary } \delta > 0.$$

In addition, it holds that

$$\|u_m\|_{C(\boldsymbol{\Sigma}([-1,1]^m,\boldsymbol{\tau});H_0^1(D))} \lesssim \|f\|_{L^2(D)}.$$

In the lognormal case, the solution u_m to (39) *is analytically extendable into $\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$ provided that*

$$\tau_k < \frac{\log 2}{C(\delta)k^{1+\delta}\gamma_k}.$$

Moreover, the solution is bounded in accordance with

$$\|u_m\|_{C_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});H_0^1(D))} \lesssim \|f\|_{L^2(D)}$$

for the weight function $\boldsymbol{\sigma}(\mathbf{y}) = \exp\big(-2\sum_{n=1}^m \gamma_k |y_k|\big)$.

The constants which are involved in the estimates depend on the choice of $\boldsymbol{\tau}$, but are independent of m.

Lemma 6.1 characterizes the region of analyticity and the according weight function $\boldsymbol{\sigma}(\mathbf{y})$ and, therefore, Assumption 2.1 is satisfied in these cases. It remains to investigate the one-dimensional error estimates of the Gauss-Legendre and the Gauss-Hermite quadrature for functions $v\colon \Gamma \to H_0^1(D)$ which are analytically extendable into a region around the parameter domain Γ. Therefore, we provide the following two lemmata on the best polynomial approximation for analytic extendable and Banach space valued functions, see [1, 14], and the continuity of the Gaussian quadrature operator.

From [1], we have the following result for the Gauss-Legendre quadrature.

Lemma 6.2. *Let X be a Banach space. Suppose that $v \in C([-1,1];X)$ admits an analytic extension in $\Sigma([-1,1],\tau)$ for some $\tau > 0$. Then, the error of the best approximation by polynomials of degree at most n can be bounded by*

$$\inf_{w\in\mathcal{P}_n([-1,1])\otimes X} \|v-w\|_{C([-1,1];X)} \leq \frac{2}{\kappa-1} e^{-n\log\kappa}\|v\|_{C_\sigma(\Sigma([-1,1],\tau);X)} \tag{40}$$

with $\kappa = \tau + \sqrt{1+\tau^2}$.

Thus, in view of our generic error estimate (4), we end up with

$$\Gamma = [-1,1], \quad \sigma(y) \equiv 1, \quad g(\tau) = \frac{4}{\kappa - 1} \quad \text{and} \quad h(\tau) = \log(\kappa) \tag{41}$$

In case of the Gauss-Hermite quadrature, we employ the next lemma from [14].

Lemma 6.3. *Suppose that $v \in C_\sigma^0(\mathbb{R}; X)$ admits an analytic extension in $\Sigma(\mathbb{R}, \tau)$ for some $1/\sqrt{2} < \tau < 1/\gamma$. Then, the error of the best approximation by polynomials of degree at most n can be bounded by*

$$\inf_{w \in \mathcal{P}_n(\mathbb{R}) \otimes X} \|v - w\|_{C_G(\mathbb{R}; X)} \leq \frac{C}{\sqrt{2}\tau - 1} e^{-\log(\sqrt{2}\tau)n} \|v\|_{C_\sigma(\Sigma(\mathbb{R},\tau); X)}, \tag{42}$$

where $C > 0$ is a constant and the weight function $G(y)$ is given by $G(y) := \exp(-y^2/4)$.

Similarly to the Gauss-Legendre quadrature, we obtain the generic error estimate (4) for the Gauss-Hermite quadrature with

$$\Gamma = \mathbb{R}, \quad \sigma(y) = \exp(-2\tau|y|), \quad h(\tau) = \log(\sqrt{2}\tau) \quad \text{and} \quad g(\tau) = C/(\sqrt{2}\tau - 1). \tag{43}$$

Finally, we would like to point out that, with the new estimate (30) on the number of indices $X_{\mathbf{w}}(q, m)$, we are able to get significantly improved results in comparison with the convergence of the anisotropic tensor product Gaussian quadrature. More precisely, we are able to show dimension-independent convergence with an arbitrarily algebraic rate if the regions of analyticity of the integrand grow exponentially like $\tau_k \gtrsim \exp(k^\delta)$ for arbitrary $\delta > 0$. This covers the important case of diffusion coefficients which are derived from Gaussian covariance kernels. In addition, we analyzed the case when τ_k grows algebraically, which covers the case of covariance kernels of the Matèrn class, and obtain that the dimensionality m compromises the convergence rate at most by the term $\log(\log(m))$.

7. Numerical results

7.1. **Setup.** In this section, we present numerical examples to validate the theoretical findings. As a practical application of the sparse anisotropic Gaussian quadrature, we consider random diffusion problems with either uniformly or lognormally distributed diffusion coefficients as defined in the previous section.

In our numerical experiments, we employ two covariance kernels of the Matérn class for $\nu = 5/2$ and $\nu = 7/2$, cf. [15], i.e.

$$\mathcal{C}_{5/2}(r) := \frac{1}{4}\left(1 + \frac{\sqrt{5}r}{\ell} + \frac{5r^2}{3\ell^2}\right) \exp\left(-\frac{\sqrt{5}r}{\ell}\right)$$

and

$$\mathcal{C}_{7/2}(r) := \frac{1}{4}\left(1 + \frac{\sqrt{7}r}{\ell} + \frac{14r^2}{5\ell^2} + \frac{49\sqrt{7}r^3}{15\ell^2}\right) \exp\left(-\frac{\sqrt{7}r}{\ell}\right),$$

where $r := |x - x'|$. The correlation length is in both cases set to $\ell = 1/2$. The spatial discretization is performed with piecewise linear finite elements an a mesh with mesh size $h = 2^{-14}$, which results from 16384 equidistant sub-intervals. A numerical approximation to the Karhunen-Loève expansion is computed by the pivoted Cholesky decomposition of the covariance operator with a trace error of $\varepsilon = 2^{-28}$. This yields an approximation error of the underlying random field of $\varepsilon = 2^{-14}$, see [12] for the details. The related truncation rank is given by $m = 64$ for $\mathcal{C}_{5/2}$ and $m = 30$ for $\mathcal{C}_{7/2}$. In the uniformly distributed case, we set $\mathbb{E}[a](x) = 2.5$. From [8], we know that $\tau_n \gtrsim n^3$ for $\mathcal{C}_{5/2}$ and $\tau_n \gtrsim n^4$ for $\mathcal{C}_{7/2}$.

Since the solution of (38) is not known analytically, we have to provide a reference solution. The error with respect to the reference solution is measured in the $H^1(D)$-norm for the approximation of the mean and in the $W^{1,1}(D)$-norm for the approximations of the higher order moments, respectively. This reference solution is computed by the quasi-Monte Carlo quadrature with Halton points and $N = 10 \cdot 2^{20} \approx 10^7$ samples.

For the anisotropic sparse Gaussian quadrature, we set the weights w_n according to $w_n = h(\tau_n)$ with the same functions $h(\tau)$ and the same quantities τ_n as for a related anisotropic tensor product quadrature for the lognormal and the uniformly elliptic case, respectively. Hence, our anisotropic sparse Gaussian quadrature is essentially a sparsification of the anisotropic tensor product Gaussian quadrature, cf. [11] for more details on the anisotropic tensor product Gaussian quadrature. To choose the same quantity τ_n for the region of analyticity as for the tensor product quadrature seems to be a violation of Lemma 6.1. Indeed, the assertion of this lemma is that the quantities τ_n, which describes the region of analytic extendability in each direction $\Sigma(\Gamma_n, \tau_n)$, should be rescaled to $\tilde{\tau}_n = \tau_n/(C(\delta)n^{1+\delta})$ in order to ensure analytic extendability into the tensor domain $\mathbf{\Sigma}(\tilde{\boldsymbol{\tau}})$. Nevertheless, our experience suggests that the sparsification of the anisotropic Gaussian quadrature yields an error which is nearly as good as the error of the anisotropic Gaussian quadrature itself.

For nearly all numerical examples, it turns out that the convergence rates slightly decrease from the computation of the mean to the computation of the second moment and even successively for the higher order moments. Therefore, we state for all examples the actually obtained convergence rate for the mean and for the fourth moment. The convergence rate of the second and third moment is then between these two convergence rates.

In addition to the convergence studies for the sparse anisotropic Gaussian quadrature, we also provide results on the estimated number of quadratures contained in the sparse grid. We compare the tensor product estimate

$$\text{(TP Formula)} \qquad \#X_{\mathbf{w}}(q,m) \leq \prod_{n=1}^{m} \left(\left\lfloor \frac{q}{w_n} \right\rfloor + 1 \right),$$

the novel estimate proposed in this article

$$\text{(SG Formula)} \qquad \#X_{\mathbf{w}}(q,m) \leq \prod_{n=1}^{m} \frac{\frac{q}{w_n} + n}{n}$$

and finally the well established formula by Beged-Dov, cf. [4],

$$\text{(BD Formula)} \qquad \#X_{\mathbf{w}}(q,m) \leq \prod_{n=1}^{m} \frac{q + \|\mathbf{w}\|_1}{n w_n}.$$

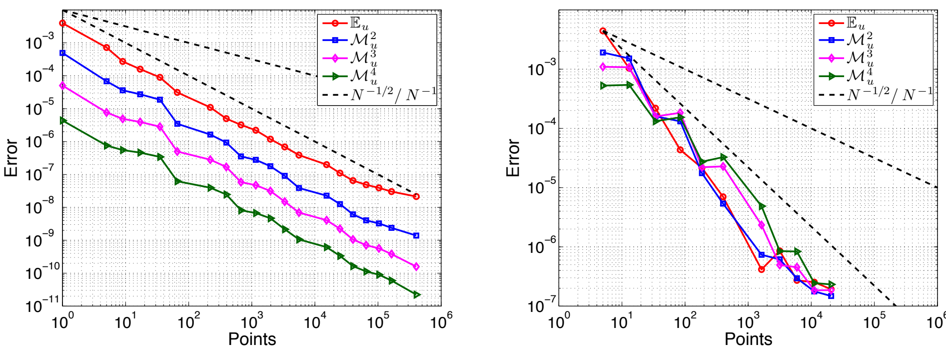

Figure 3. Errors for $\nu = 5/2$ with uniformly distributed coefficient (left) and lognormally distributed coefficient (right)

7.2. **The Matérn kernel for $\boldsymbol{\nu = 5/2}$.** For the smoothness parameter $\nu = 5/2$, we end up with a Karhunen-Loève expansion of length $m = 64$. Figure 3 depicts the convergence rates for both diffusion coefficients. On the left, we see the convergence of the Gauss-Legendre quadrature and on the right the convergence of the Gauss-Hermite quadrature. For the anisotropic sparse Gauss-Legendre quadrature, the convergence rate decreases slightly from N^{-1} to $N^{-0.91}$ for the first to the fourth moment. In case of the Gauss-Hermite quadrature for the lognormally distributed diffusion coefficient, the observed rate is considerably better. For the mean, we observe $N^{-1.25}$ and still N^{-1} for the fourth moment. Note that the stagnation in the convergence might be caused by the accuracy of the reference solution, which is only of order 10^{-7}.

In Figure 4, we see the different estimates for the number of indices in the anisotropic sparse tensor product space. On the left, we have the estimates for the uniformly distributed coefficient

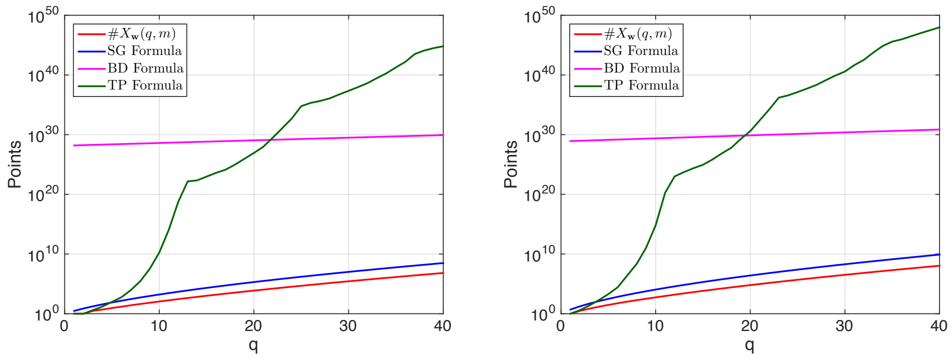

FIGURE 4. Estimates $\nu = 5/2$ with uniformly distributed coefficient (left) and lognormally distributed coefficient (right).

and on the right for the lognormally distributed coefficient. As it turns out, for the uniformly distributed case as well as for the lognormal case, the considered formulae exhibit qualitatively the same behavior. The novel estimate proven in this article only slightly overestimates the number of indices and reflects perfectly the growth of the index set with increasing q. Although the formula of Beged-Dov is asymptotically much better than the crude tensor product estimate, it heavily overestimates the actual number of indices.

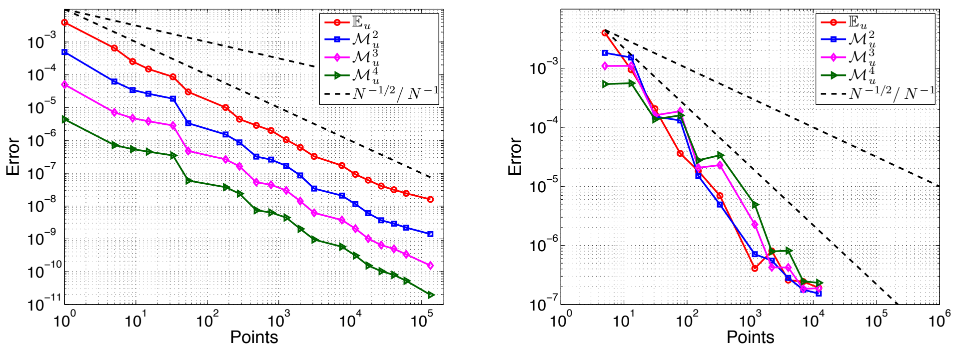

FIGURE 5. Errors for $\nu = 7/2$ with uniformly distributed coefficient (left) and lognormally distributed coefficient (right).

7.3. **The Matérn kernel for $\nu = 7/2$.** In this example, we have to deal with a 30-dimensional integration problem. The convergence rates for the computation of the first four moments of the anisotropic sparse Gaussian quadrature method are depicted in Figure 5. On the left hand side of this figure, we find the convergence rates in case of the uniformly distributed coefficient and on the right hand side for the lognormally distributed coefficient. In the uniformly distributed case, we obtain a convergence rate which is essentially the same for the computation of all considered moments and of order N^{-1}. In the lognormally distributed case, we obtain convergence rates that are considerably higher. For the mean, we observe a rate of $N^{-1.6}$ and still a convergence rate of $N^{-1.2}$ for the fourth moment. Note that the stagnation in the convergence might be caused by the accuracy of the reference solution, which is theoretically only of order 10^{-7}.

In Figure 6, we see the different estimates for the number of indices in the anisotropic sparse tensor product space. On the left, we have the estimates for the uniformly distributed coefficient

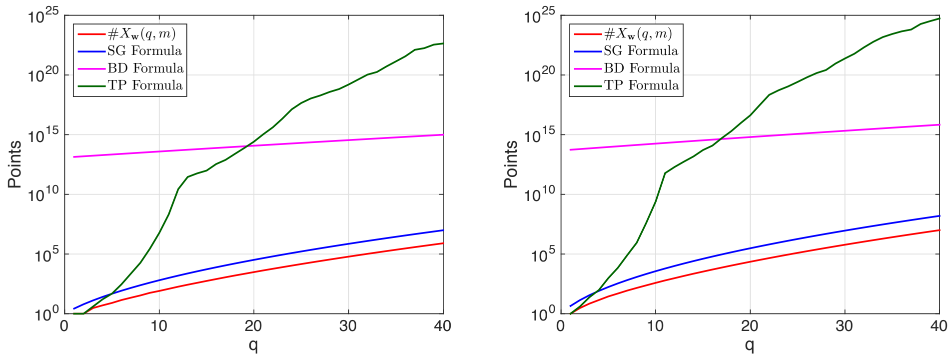

FIGURE 6. Estimates $\nu = 7/2$ with uniformly distributed coefficient (left) and lognormally distributed coefficient (right).

and on the right for the lognormally distributed coefficient. Again, as in the example with $\nu = 5/2$, there is no significant difference between the lognormally distributed and the uniformly distributed case. Again, the novel estimate only slightly overestimates the number of indices and reflects perfectly the growth of the index set with increasing q, whereas the formula by Beged-Dov heavily overestimates the number of indices in $X_{\mathbf{w}}(q, m)$.

8. Conclusion

In the present article, a novel complexity estimate for the anisotropic sparse grid quadrature has been proven. Under the assumption that the dimension weights $\{\tau_n\}$ are increasing at least logarithmically, i.e. $w_n \approx \log(cn^r)$ for some $c > 1$ and $r \in \mathbb{R}_+$, we can prove essentially dimension independent convergence. Our theory has been applied for elliptic diffusion problems with uniformly elliptic random coefficient or lognormally distributed random coefficient. Here, the anisotropic sparse Gauss-Legendre quadrature and the anisotropic sparse Gauss-Hermite quadrature have to be applied, respectively. Nevertheless, the presented results remain also valid for other quadrature rules like e.g. Clenshaw-Curtis or Gauss-Kronrod quadrature formulae.

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