LIE SUBALGEBRAS OF VECTOR FIELDS ON AFFINE 2-SPACE AND THE JACOBIAN CONJECTURE

ANDRIY REGETA

Abstract. We study Lie subalgebras L of the vector fields $\mathrm{Vec}^c(\mathbb{A}^2)$ of affine 2-space $\mathbb{A}^2$ of constant divergence, and we classify those L which are isomorphic to the Lie algebra $\mathfrak{aff}_2$ of the group $\mathrm{Aff}_2(K)$ of affine transformations of $\mathbb{A}^2$. We then show that the following statements are equivalent:

(a) The Jacobian Conjecture holds in dimension 2;
(b) All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$;
(c) All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are algebraic.

Finally, we use these results to show that the automorphism groups of the Lie algebras $\mathrm{Vec}(\mathbb{A}^2)$, $\mathrm{Vec}^0(\mathbb{A}^2)$ and $\mathrm{Vec}^c(\mathbb{A}^2)$ are all isomorphic to $\mathrm{Aut}(\mathbb{A}^2)$.

1. Introduction

Let K be an algebraically closed field of characteristic zero. It is a well-known consequence of the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$ that every reductive subgroup $G \subset \mathrm{Aut}(\mathbb{A}^2)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{A}^2)$, i.e. there is a $\psi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\psi G \psi^{-1} \subset \mathrm{GL}_2(\mathbb{C})$ ([Kam79], cf. [Kra96]). The "Linearization Problem" asks whether the same holds for $\mathrm{Aut}(\mathbb{A}^n)$. It was shown by Schwarz in [Sch89] that this is not the case in dimensions $n \geq 4$ (cf. [Kno91]).

In this paper we consider the analogue of the Linearization Problem for Lie algebras. It is known that the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ of the ind-group $\mathrm{Aut}(\mathbb{A}^2)$ is canonically isomorphic to the Lie algebra $\mathrm{Vec}^c(\mathbb{A}^2)$ of vector fields of constant divergence ([Sha66, Sha81], cf. [Kum02]). We will see that the Lie subalgebra

$$L := K(x^2\partial_x - 2xy\partial_y) \oplus K(x\partial_x - y\partial_y) \oplus K\partial_x \subset \mathrm{Vec}^c(\mathbb{A}^2)$$

where $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$, is isomorphic to $\mathfrak{sl}_2$, but not conjugate to the standard $\mathfrak{sl}_2 \subset \mathrm{Vec}^c(\mathbb{A}^2)$ under $\mathrm{Aut}(\mathbb{A}^2)$ (Remark 4.3). However, for some other Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$, the situation is different. Let $\mathrm{Aff}_2(K) \subset \mathrm{Aut}(\mathbb{A}^2)$ be the group of affine transformations and $\mathrm{SAff}_2(K) \subset \mathrm{Aff}_2(K)$ the subgroup of affine transformations with determinant equal to 1, and denote by $\mathfrak{aff}_2$, respectively $\mathfrak{saff}_2$ their Lie algebras which we consider as subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$. The first result we prove is the following (see Proposition 3.9). For $f \in K[x,y]$ we set $D_f := f_x\partial_y - f_y\partial_x \in \mathrm{Vec}^c(\mathbb{A}^2)$.

Theorem A. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$. Then there is an étale map $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $L = \varphi^*(\mathfrak{aff}_2)$. More precisely, if (D_f, D_g) is a basis of the solvable radical of $[L,L]$, then*

$$L = \langle D_f, D_g, D_{f^2}, D_{g^2}, fD_g, gD_f \rangle,$$

Date: February 2014.
The author is supported by a grant from the SNF (Schweizerischer Nationalfonds).

and one can take $\varphi = (f, g)$.

The analogous statements hold for Lie subalgebras isomorphic to $\mathfrak{saff}_2$. As a consequence of this classification we obtain the next result (see Theorem 4.1 and Corollary 4.4). Recall that a Lie subalgebra of $\mathrm{Vec}(\mathbb{A}^2)$ is *algebraic* if it acts locally finitely on $\mathrm{Vec}(\mathbb{A}^2)$.

Theorem B. *The following statements are equivalent:*

(i) *The Jacobian Conjecture holds in dimension 2;*
(ii) *All Lie subalgebras* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{aff}_2$ *are conjugate under* $\mathrm{Aut}(\mathbb{A}^2)$*;*
(iii) *All Lie subalgebras* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{saff}_2$ *are conjugate under* $\mathrm{Aut}(\mathbb{A}^2)$*;*
(iv) *All Lie subalgebras* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{aff}_2$ *are algebraic;*
(v) *All Lie subalgebras* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{saff}_2$ *are algebraic.*

Finally, as a consequence of the theorem above, we can determine the automorphism groups of the Lie algebras of vector fields (Theorem 4.5).

Theorem C. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2)).$$

(Here $\mathrm{Vec}^0(\mathbb{A}^2)$ denotes the vector fields with zero divergence, see section 2).

Acknowledgement: The author would like to thank his thesis advisor Hanspeter Kraft for constant support and help during writing this paper.

2. The Poisson algebra

Definitions. Let K be an algebraically closed field of characteristic zero and let P be the *Poisson algebra*, i.e., the Lie algebra with underlying vector space $K[x, y]$ and with Lie bracket $\{f, g\} := f_x g_y - f_y g_x$ for $f, g \in P$. If $\mathrm{Jac}(f, g)$ denotes the *Jacobian matrix* and $j(f, g)$ the *Jacobian determinant*,

$$\mathrm{Jac}(f,g) := \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}, \quad j(f,g) := \det \mathrm{Jac}(f,g),$$

then $\{f, g\} = j(f, g)$. Denote by $\mathrm{Vec}(\mathbb{A}^2)$ the polynomial vector fields on affine 2-space $\mathbb{A}^2 = K^2$, i.e. the derivations of $K[x, y]$:

$$\mathrm{Vec}(\mathbb{A}^2) := \{p\partial_x + q\partial_y \mid p, q \in K[x, y]\} = \mathrm{Der}(K[x, y]).$$

There is a canonical homomorphism of Lie algebras

$$\mu\colon P \to \mathrm{Vec}(\mathbb{A}^2),\ h \mapsto D_h := h_x\partial_y - h_y\partial_x,$$

with kernel $\ker \mu = K$.

The next lemma lists some properties of the Lie algebra P. These results are essentially known, see e.g. [NN88]. If L is any Lie algebra and $X \subset L$ a subset, we define the *centralizer* of X by

$$\mathfrak{cent}_L(X) := \{z \in L \mid [z, x] = 0 \text{ for all } x \in X\},$$

and we shortly write $\mathfrak{cent}(L)$ for the *center* of L.

Lemma 2.1. (a) *The center of* P *consists exactly of the constants* $K \subset P$.
(b) *If* $f, g \in P$ *are such that* $\{f, g\} = 0$*, then* $f, g \in K[h]$ *for some* $h \in K[x, y]$.

(c) *If $f, g \in P$ such that $\{f, g\} \neq 0$, then f, g are algebraically independent in $K[x, y]$, and $\mathfrak{cent}_P(f) \cap \mathfrak{cent}_P(g) = K$.*
(d) *P is generated, as a Lie algebra, by $\{x, x^3, y^2\}$.*

Proof. (a) is easy and left to the reader.

(b) Consider the morphism $\varphi = (f, g)\colon \mathbb{A}^2 \to \mathbb{A}^2$. Then $C := \overline{\varphi(\mathbb{A}^2)} \subset \mathbb{A}^2$ is an irreducible rational curve, and we have a factorization

$$\varphi\colon \mathbb{A}^2 \xrightarrow{h} \mathbb{A}^1 \xrightarrow{\eta} C \subset \mathbb{A}^2$$

where η is the normalization of C. It follows that $f, g \in K[h]$.

(c) It is clear that f, g are algebraically independent, i.e. $\operatorname{tdeg}_K K(f, g) = 2$. Equivalently, $K(x, y)/K(f, g)$ is a finite algebraic extension. Now assume that $\{h, f\} = \{h, g\} = 0$. Then the derivation D_h vanishes on $K[f, g]$, hence on $K[x, y]$. Thus $D_h = 0$ and so $h \in K$.

(d) Denote by $P_d := K[x, y]_d$ the homogeneous part of degree d. Let $L \subset P$ be the Lie subalgebra generated by $\{x, x^3, y^2\}$. We first use the equations

$$\{x, y\} = 1,\ \{x, y^2\} = 2y,\ \{x^3, y\} = 3x^2,\ \{x^2, y^2\} = 4xy,\ \{x^3, y^2\} = 6x^2y$$

to show that $K \oplus P_1 \oplus P_2 \subset L$ and that $x^2y \in L$. Now the claim follows by induction from the relations

$$\{x^n, x^2y\} = nx^{n+1} \quad \text{and} \quad \{x^r y^s, y^2\} = 2rx^{r-1}y^{s+1}.$$

□

Divergence. The next lemma should also be known. Recall that the *divergence* $\operatorname{Div} D$ of a vector field $D = p\partial_x + q\partial_y \in \operatorname{Vec}(\mathbb{A}^2)$ is defined by $\operatorname{Div} D := p_x + q_y \in K[x, y]$. Define

$$\operatorname{Vec}^0(\mathbb{A}^2) := \{D \in \operatorname{Vec}(\mathbb{A}^2) \mid \operatorname{Div} D = 0\} \subset \operatorname{Vec}^c(\mathbb{A}^2) := \{D \in \operatorname{Vec}(\mathbb{A}^2) \mid \operatorname{Div} D \in K\}.$$

The Lie algebra homomorphism $\mu\colon P \to \operatorname{Vec}(\mathbb{A}^2)$, $f \mapsto D_f$, has its image in $\operatorname{Vec}^0(\mathbb{A}^2)$, because $\operatorname{Div} D_f = 0$.

Lemma 2.2. *Let D be a non-trivial derivation of $K[x, y]$.*

(a) *The kernel $K[x, y]^D$ is either K or $K[f]$ for some $f \in K[x, y]$.*
(b) *If $\operatorname{Div} D = 0$, then $D = D_h$ for some $h \in K[x, y]$. In particular, $\mu(P) = \operatorname{Vec}^0(\mathbb{A}^2)$.*

Now assume that $D = D_f$ for some non-constant $f \in K[x, y]$ and that $D(g) = 1$ for some $g \in K[x, y]$.

(c) *Then $K[x, y]^D = K[f]$.*
(d) *If D is locally nilpotent, then $K[x, y] = K[f, g]$.*

Proof. (a) See [NN88] Theorem 2.8.

(b) Let $D = f\partial_x + g\partial_y$, then $\operatorname{Div} D = f_x + g_y = 0$ implies that there exists $h \in K[x, y]$ such that $f = h_y$, $g = -h_x$.

(c) It is obvious that $\ker(D) \supset K[f]$, hence, by (a), one has $\ker(D) = K[h] \supset K[f]$. Thus $f = F(h)$ for some $F \in K[t]$ and then $D_f(g) = D_{F(h)}(g) = F'(h)D_h(g) = 1$ which implies that F is linear.

(d) Let G be an affine algebraic group, X an affine variety and $\varphi\colon X \to G$ a G-equivariant retraction. Then one has $\mathcal{O}(X) = \varphi^*(\mathcal{O}(G)) \otimes \mathcal{O}(X)^G$. In our case we get $K[x, y] = \mathcal{O}(\mathbb{A}^2) = \mathcal{O}(G) \otimes \mathcal{O}(\mathbb{A}^2)^G = K[g] \otimes K[f]$. □

Automorphisms of the Poisson algebra. Denote by $\mathrm{Aut}_{LA}(P)$ the group of Lie algebra automorphisms of P. There is a canonical homomorphism

$$p\colon \mathrm{Aut}_{LA}(P) \to K^*, \quad \varphi \mapsto \varphi(1),$$

which has a section $s\colon K^* \to \mathrm{Aut}_{LA}(P)$ given by $s(t)|_{K[x,y]_n} := t^{1-n}\,\mathrm{id}_{K[x,y]_n}$ where $K[x,y]_n \subset K[x,y]$ denotes the subspace of homogeneous polynomials of degree n. Thus $\mathrm{Aut}_{LA}(P)$ is a semidirect product $\mathrm{Aut}_{LA}(P) = \mathrm{SAut}_{LA}(P) \rtimes K^*$ where

$$\mathrm{SAut}_{LA}(P) := \ker p = \{\alpha \in \mathrm{Aut}_{LA}(P) \mid \alpha(1) = 1\}.$$

Lemma 2.3. *Every automorphism $\alpha \in \mathrm{Aut}_{LA}(P)$ is determined by $\alpha(1)$, $\alpha(x)$ and $\alpha(y)$, and then $K[x,y] = K[\alpha(x), \alpha(y)]$.*

Proof. Replacing α by the composition $\alpha \circ s(\alpha(1)^{-1})$ we can assume that $\alpha(1) = 1$.

We will show that $\alpha(x^n) = \alpha(x)^n$ and $\alpha(y^n) = \alpha(y)^n$ for all $n \geq 0$. Then the first claim follows from Lemma 2.1(d).

By induction, we can assume that $\alpha(x^j) = \alpha(x)^j$ for $j < n$. We have $\{x^n, y\} = nx^{n-1}$ and so $\{\alpha(x^n), \alpha(y)\} = n\alpha(x^{n-1}) = n\alpha(x)^{n-1}$. On the other hand, we get $\{\alpha(x)^n, \alpha(y)\} = n\alpha(x)^{n-1}\{\alpha(x), \alpha(y)\} = n\alpha(x)^{n-1}$, hence the difference $h := \alpha(x^n) - \alpha(x)^n$ belongs to the kernel of the derivation $D_{\alpha(y)}\colon f \mapsto \{f, \alpha(y)\}$. Since $D_{\alpha(y)}$ is locally nilpotent, we get from Lemma 2.2(c)–(d) that $\ker D_{\alpha(y)} = K[\alpha(y)]$ and that $K[\alpha(x), \alpha(y)] = K[x,y]$. This already proves the second claim and shows that h is a polynomial in $\alpha(y)$.

Since $\{\alpha(x^n), \alpha(x)\} = \alpha(\{x^n, x\}) = 0$ and $\{\alpha(x)^n, \alpha(x)\} = n\alpha(x)^{n-1}\{\alpha(x), \alpha(x)\}$ we get $\{h, \alpha(x)\} = 0$ which implies that $h \in K$.

In the same way, using $\{x, xy\} = x$ and $\{y, xy\} = -y$, we find $\alpha(xy) - \alpha(x)\alpha(y) \in K$. Hence

$$n\alpha(x^n) = \{\alpha(x^n), \alpha(xy)\} = \{\alpha(x)^n, \alpha(x)\alpha(y)\} = n\alpha(x)^n,$$

and so $\alpha(x^n) = \alpha(x)^n$. By symmetry, we also get $\alpha(y^n) = \alpha(y)^n$. □

Automorphisms of affine 2-space. Denote by $\mathrm{Aut}(K[x,y])$ the group of K-algebra automorphisms of $K[x,y]$. We have a canonical identification $\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}(K[x,y])^{op}$ given by $\varphi \mapsto \varphi^*$. For $\rho \in \mathrm{Aut}(K[x,y])$ we will use the notation $\rho = (f, g)$ in case $\rho(x) = f$ and $\rho(y) = g$, which implies that $K[x,y] = K[f,g]$. Note that the Jacobian determinant defines a homomorphism

$$j\colon \mathrm{Aut}(K[x,y]) \to K^*, \quad \rho \mapsto j(\rho) := j(\rho(x), \rho(y))$$

whose kernel is denoted by $\mathrm{SAut}(K[x,y])$.

We can consider $\mathrm{Aut}(K[x,y])$ and $\mathrm{Aut}_{LA}(P)$ as subgroups of the K-linear automorphisms $\mathrm{GL}(K[x,y])$.

Lemma 2.4. *As subgroups of $\mathrm{GL}(K[x,y])$ we have $\mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$.*

Proof. (a) Let μ be an endomorphism of $K[x,y]$ and put $\mathrm{Jac}(\mu) := \mathrm{Jac}(\mu(x), \mu(y))$. For any $f, g \in K[x,y]$ we have $\mathrm{Jac}(\mu(f), \mu(g)) = \mu(\mathrm{Jac}(f,g))\,\mathrm{Jac}(\mu)$, because

$$\begin{aligned}\frac{\partial}{\partial x}\mu(f) &= \frac{\partial f}{\partial x}(\mu(x), \mu(y))\frac{\partial \mu(x)}{\partial x} + \frac{\partial f}{\partial y}(\mu(x), \mu(y))\frac{\partial \mu(y)}{\partial x} \\ &= \mu(\frac{\partial f}{\partial x})\frac{\partial \mu(x)}{\partial x} + \mu(\frac{\partial f}{\partial y})\frac{\partial \mu(y)}{\partial x}.\end{aligned}$$

It follows that $\{\mu(f), \mu(g)\} = \mu(\{f, g\})j(\mu)$. This shows that $\mathrm{SAut}(K[x,y]) \subset \mathrm{SAut}_{LA}(P)$.

(b) Now let $\alpha \in \mathrm{SAut}_{LA}(P)$. Then $j(\alpha(x),\alpha(y)) = \{\alpha(x),\alpha(y)\} = \alpha(1) = 1$ and, by Lemma 2.3, $K[\alpha(x),\alpha(y)] = K[x,y]$. Hence, we can define an automorphism $\rho \in \mathrm{SAut}(K[x,y])$ by $\rho(x) := \alpha(x)$ and $\rho(y) := \alpha(y)$. From (a) we see that $\rho \in \mathrm{SAut}_{LA}(P)$, and from Lemma 2.3 we get $\alpha = \rho$, hence $\alpha \in \mathrm{SAut}(K[x,y])$. □

Remark 2.5. The first part of the proof above shows the following. If $f,g \in P$ are such that $\{f,g\} = 1$, then the K-algebra homomorphism defined by $x \mapsto f$ and $y \mapsto g$ is an injective homomorphism of P as a Lie algebra. (Injectivity follows, because f,g are algebraically independent.)

Lie subalgebras of P. The subspace

$$P_{\leq 2} := K \oplus P_1 \oplus P_2 = K \oplus Kx \oplus Ky \oplus Kx^2 \oplus Kxy \oplus Ky^2 \subset P$$

is a Lie subalgebra. This can be deduced from the following Lie brackets which we note here for later use.

$$\{x^2,xy\} = 2x^2,\ \{x^2,y^2\} = 4xy,\ \{y^2,xy\} = -2y^2; \tag{1}$$

$$\{x^2,x\} = 0,\ \{xy,x\} = -x,\ \{y^2,x\} = -2y, \tag{2}$$

$$\{x^2,y\} = 2x,\ \{xy,y\} = y,\ \{y^2,y\} = 0; \tag{3}$$

$$\{x,y\} = 1. \tag{4}$$

Moreover, $P_2 = Kx^2 \oplus Kxy \oplus Ky^2$ is a Lie subalgebra of $P_{\leq 2}$ isomorphic to $\mathfrak{sl}_2$, and $P_1 = Kx \oplus Ky$ is the two-dimensional simple P_2-module.

From Remark 2.5 we get the following lemma.

Lemma 2.6. *Let $f,g \in K[x,y]$ such that $\{f,g\} = 1$. Then $\langle 1,f,g,f^2,fg,g^2\rangle \subset P$ is a Lie subalgebra isomorphic to $P_{\leq 2}$. An isomorphism is induced from the K-algebra homomorphism $P \to P$ defined by $x \mapsto f, y \mapsto g$.*

Definition 2.7. For $f,g \in K[x,y]$ such that $\{f,g\} \in K^*$ we put

$$P_{f,g} := \langle 1,f,g,f^2,fg,g^2\rangle \subset P.$$

We have just seen that this is a Lie algebra isomorphic to $P_{\leq 2}$. Clearly, $P_{f,g} = P_{f_1,g_1}$ if $\langle 1,f,g\rangle = \langle 1,f_1,g_1\rangle$. Denoting by $\mathfrak{rad}\, L$ the solvable radical of the Lie algebra L we get

$$\mathfrak{rad}\, P_{f,g} = \langle 1,f,g\rangle \quad\text{and}\quad P_{f,g}/\,\mathfrak{rad}\, P_{f,g} \simeq \mathfrak{sl}_2\,.$$

Proposition 2.8. *Let $Q \subset P$ be a Lie subalgebra isomorphic to $P_{\leq 2}$. Then $K \subset Q$, and $Q = P_{f,g}$ for every pair $f,g \in L$ such that $\langle 1,f,g\rangle = \mathfrak{rad}\, Q$. In particular, $\{f,g\} \in K^*$.*

Proof. We first show that $\mathfrak{cent}(Q) = K$. In fact, Q contains elements f,g such that $\{f,g\} \neq 0$. If $h \in \mathfrak{cent}(Q)$, then $h \in \mathfrak{cent}_P(f) \cap \mathfrak{cent}_P(g) = K$, by Lemma 2.1(c).

Now choose an isomorphism $\theta\colon P_{\leq 2} \xrightarrow{\sim} Q$. Then $\theta(K) = K$, and replacing θ by $\theta \circ s(t)$ with a suitable $t \in K^*$ we can assume that $\theta(1) = 1$. Setting $f := \theta(x), g := \theta(y)$ we get $\{f,g\} = 1$, and putting $f_0 := \theta(x^2), f_1 := \theta(xy), f_2 := \theta(y^2)$ we find

$$\{f_1,f\} = \theta\{xy,x\} = \theta(-x) = -f = \{fg,f\}.$$

Similarly, $\{f_1,g\} = \{fg,g\}$, hence $fg = f_1 + c \in Q$, by Lemma 2.1(c). Next we have

$$\{f_0,f\} = 0 \quad\text{and}\quad \{f_0,g\} = \theta(\{x^2,y\}) = \theta(2x) = 2f = \{f^2,g\}.$$

Hence $f^2 = f_0 + d$, and thus $f^2 \in Q$. A similar calculation shows that $g^2 \in Q$, so that we finally get $Q = P_{f,g}$. □

Characterization of $P_{\leq 2}$. The following lemma gives a characterization of the Lie algebras isomorphic to $P_{\leq 2}$.

Lemma 2.9. *Let Q be a Lie algebra containing a subalgebra Q_0 isomorphic to $\mathfrak{sl}_2$. Assume that*

(a) *$Q = Q_0 \oplus V_2 \oplus V_1$ as a Q_0-module where the V_i are simple of dimension i,*
(b) *V_1 is the center of Q, and*
(c) *$[V_2, V_2] = V_1$.*

Then Q is isomorphic to $P_{\leq 2}$.

Proof. Choosing an isomorphism of $P_2 = \langle x^2, xy, y^2 \rangle$ with Q_0 we find a basis (a_0, a_1, a_2) of Q_0 with relations

$$(1') \qquad [a_0, a_1] = 2a_0,\ [a_0, a_2] = 4a_1,\ [a_2, a_1] = -2a_2$$

(see (1) above). Since V_2 is a simple two-dimensional Q_0-module and $Kx \oplus Ky$ a simple two-dimensional P_2-module we can find a basis (b, c) of V_2 such that

$$(2') \qquad [a_0, b] = 0,\ [a_1, b] = -b,\ [a_2, b] = -2c,$$

$$(3') \qquad [a_0, c] = 2b,\ [a_1, c] = c,\ [a_2, c] = 0$$

(see (2) and (3) above). Finally, the last assumption (c) implies that

$$(4') \qquad d := [b, c] \neq 0,\ \text{hence } V_1 = Kd.$$

Comparing the relations (1)–(4) with (1′)–(4′) we see that the linear map $P_{\leq 2} \to Q$ given by $x^2 \mapsto a_0$, $xy \mapsto a_1$, $y^2 \mapsto a_2$, $x \mapsto b$, $y \mapsto c$, $1 \mapsto d$ is a Lie algebra isomorphism. □

3. Vector fields on affine 2-space

The action of $\mathrm{Aut}(\mathbb{A}^2)$ on vector fields. The group $\mathrm{Aut}(\mathbb{A}^2)$ acts on the vector fields $\mathrm{Vec}(\mathbb{A}^2)$. If $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and if the vector fields $\mathrm{Vec}(\mathbb{A}^2)$ are regarded as sections $\xi\colon \mathbb{A}^2 \to T\mathbb{A}^2$ of the tangent bundle, then $\varphi^*(\xi) := (d\varphi)^{-1} \circ \xi \circ \varphi$. Writing $\xi = p\partial_x + q\partial_y$ and $\varphi = (f, g)$, we get

$$(*) \qquad \varphi^*(\xi) = \frac{1}{j(\varphi)} \left((g_y \varphi^*(p) - f_y \varphi^*(q))\, \partial_x + (-g_x \varphi^*(p) + f_x \varphi^*(q))\, \partial_y \right).$$

In particular,

$$\varphi^*(\partial_x) = \frac{1}{j(\varphi)}(g_y \partial_x - g_x \partial_y) \quad \text{and} \quad \varphi^*(\partial_y) = \frac{1}{j(\varphi)}(-f_y \partial_x + f_x \partial_y)$$

In fact, for every $u = (a, b) \in \mathbb{A}^2$ we have $d\varphi_u \circ \varphi^*(\xi)_u = \xi_{\varphi(u)}$. If $\varphi^*(\xi) = \tilde{p}\partial_x + \tilde{q}\partial_y$, this means that

$$\begin{bmatrix} f_x(u) & f_y(u) \\ g_x(u) & g_y(u) \end{bmatrix} \begin{bmatrix} \tilde{p}(u) \\ \tilde{q}(u) \end{bmatrix} = \begin{bmatrix} p(\varphi(u)) \\ q(\varphi(u)) \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \tilde{p}(u) \\ \tilde{q}(u) \end{bmatrix} = \frac{1}{j(\varphi(u))} \begin{bmatrix} g_y(u) & -f_y(u) \\ -g_x(u) & f_x(u) \end{bmatrix} \begin{bmatrix} p(\varphi(u)) \\ q(\varphi(u)) \end{bmatrix}$$

and the claim follows.

Remark 3.1. If $\xi \in \mathrm{Vec}(\mathbb{A}^2)$ is considered as a derivation D of $K[x, y]$, and if $\alpha = \varphi^* \in \mathrm{Aut}(K[x, y])$, then the derivation corresponding to $\varphi^*(\xi)$ is given by $\alpha_* D = \alpha \circ D \circ \alpha^{-1}$.

Remark 3.2. If $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ is étale, i.e. $j(\varphi) \in K^*$, then the pull-back $\varphi^*(\xi)$ is well-defined for every vector field $\xi\colon \mathbb{A}^2 \to T\mathbb{A}^2$. It satisfies the equation $d\varphi \circ \varphi^*(\xi) = \xi \circ \varphi$ and it is given by the formula $(*)$. In terms of derivations, this corresponds to the well-known fact that for an étale extension $\alpha\colon A \hookrightarrow B$ every derivation D of A extends uniquely to a derivation of $\alpha_*(D)$ of B satisfying $\alpha_*(D) \circ \alpha = \alpha \circ D$.

It is not difficult to see that the map

$$\varphi^*\colon \operatorname{Vec}(\mathbb{A}^2) \to \operatorname{Vec}(\mathbb{A}^2),\ \xi \mapsto \varphi^*(\xi),$$

is an injective homomorphism of Lie algebras. In fact, if $\alpha = \varphi^* \in \operatorname{End}(K[x,y])$ and D the derivation of $K[x,y]$ that corresponds to ξ, then we find

$$\begin{aligned}
\alpha_*([D_1, D_2]) \circ \alpha &= \alpha \circ [D_1, D_2] = \alpha \circ D_1 \circ D_2 - \alpha \circ D_2 \circ D_1 \\
&= \alpha_*(D_1) \circ \alpha \circ D_2 - \alpha_*(D_2) \circ \alpha \circ D_1 \\
&= \alpha_*(D_1) \circ \alpha_*(D_2) \circ \alpha - \alpha_*(D_2) \circ \alpha_*(D_1) \circ \alpha \\
&= [\alpha_*(D_1), \alpha_*(D_2)] \circ \alpha,
\end{aligned}$$

hence the claim.

Recall that $\operatorname{Vec}^c(\mathbb{A}^2) \subset \operatorname{Vec}(\mathbb{A}^2)$ are the vector fields D with $\operatorname{Div} D \in K$. Clearly, the divergence $\operatorname{Div}\colon \operatorname{Vec}^c(\mathbb{A}^2) \to K$ is a character with kernel $\operatorname{Vec}^0(\mathbb{A}^2)$, and we have the decomposition

$$\operatorname{Vec}^c(\mathbb{A}^2) = \operatorname{Vec}^0(\mathbb{A}^2) \oplus KE \quad \text{where} \quad E := x\partial_x + y\partial_y \quad \text{is the } \textit{Euler field.}$$

Lemma 3.3. *If $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ is étale, then $\varphi^*(D_h) = j(\varphi)^{-1} D_{\varphi^*(h)}$. Moreover, $\operatorname{Div}(\varphi^*(E)) = 2$, and so $\varphi^*(\operatorname{Vec}^0(\mathbb{A}^2)) \subset \operatorname{Vec}^0(\mathbb{A}^2)$ and $\varphi^*(\operatorname{Vec}^c(\mathbb{A}^2)) \subset \operatorname{Vec}^c(\mathbb{A}^2)$. In particular, the homomorphism $\mu\colon P \to \operatorname{Vec}(\mathbb{A}^2)$ is equivariant with respect to the group $\operatorname{SAut}(K[x,y]) = \operatorname{SAut}_{LA}(P)$.*

Proof. Put $\alpha := \varphi^* \in \operatorname{End}(K[x,y])$. We have $\alpha(D_h) \circ \alpha = \alpha \circ D_h$, hence

$$\alpha(D_h)(\alpha(f)) = \alpha(D_h(f)) = \alpha(j(h,f)) = j(\alpha)^{-1} j(\alpha(h), \alpha(f)) = \\ = j(\alpha)^{-1} D_{\alpha(h)}(\alpha(f)).$$

From formula $(*)$ we get $\alpha(E) = \frac{1}{j(\alpha)}\left((g_y f - f_y g)\partial_x + (-g_x f + f_x g)\partial_y\right)$ which implies that $\operatorname{Div} \alpha(E) = 2$. □

Remark 3.4. Let $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be étale. If $\varphi^*\colon \operatorname{Vec}^0(\mathbb{A}^2) \to \operatorname{Vec}^0(\mathbb{A}^2)$ is an isomorphism, then so is φ. In fact, $\varphi^*(D_{c\cdot h}) = D_{\varphi^*(h)}$ for $c := j(\varphi) \in K^*$, showing that every $f \in K[x,y]$ is of the form $\varphi^*(h)$ up to a constant. It follows that $\varphi^*\colon K[x,y] \to K[x,y]$ is surjective, hence an isomorphism.

Remark 3.5. The lemma above implies that we have canononical homomorphisms

$$\begin{gathered}
\operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}(\mathbb{A}^2)), \\
\operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}^c(\mathbb{A}^2)), \\
\operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}^0(\mathbb{A}^2)).
\end{gathered}$$

We will see in Theorem 4.5 that these are all isomorphisms.

Lie subalgebras of $\mathrm{Vec}(\mathbb{A}^2)$**.** Let $\mathrm{Aff}(\mathbb{A}^2)$ denote the group of *affine transformations* of $\mathbb{A}^2$, $x \mapsto Ax + b$, where $A \in \mathrm{GL}_2(K)$ and $b \in K^2$. The determinant defines a character $\det\colon \mathrm{Aff}(\mathbb{A}^2) \to K^*$ whose kernel will be denoted by $\mathrm{SAff}(\mathbb{A}^2)$. For the corresponding Lie algebras we write $\mathfrak{saff}_2 := \mathrm{Lie}\,\mathrm{SAff}(\mathbb{A}^2) \subset \mathfrak{aff}_2 := \mathrm{Lie}\,\mathrm{Aff}(\mathbb{A}^2)$. There is a canonical embedding $\mathfrak{aff}_2 \subset \mathrm{Vec}(\mathbb{A}^2)$ which identifies $\mathfrak{aff}_2$ with the Lie subalgebra

$$\langle \partial_x, \partial_y, x\partial_x + y\partial_y, x\partial_x - y\partial_y, x\partial_y, y\partial_x \rangle \subset \mathrm{Vec}^c(\mathbb{A}^2),$$

and $\mathfrak{saff}_2$ with

$$\mu(P_{x,y}) = \langle \partial_x, \partial_y, x\partial_x - y\partial_y, x\partial_y, y\partial_x \rangle \subset \mathrm{Vec}^0(\mathbb{A}^2).$$

Note that the Euler field $E = x\partial_x + y\partial_y \in \mathfrak{aff}_2$ is determined by the condition that E acts trivially on $\mathfrak{sl}_2$ and that $[E, D] = -D$ for $D \in \mathfrak{rad}(\mathfrak{saff}_2) = K\partial_x \oplus K\partial_y$. We also remark that the centralizer of $\mathfrak{saff}_2$ in $\mathrm{Vec}(\mathbb{A}^2)$ is trivial:

$$\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^2)}(\mathfrak{saff}_2) = (0).$$

In fact, $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^2)}(\{\partial_x, \partial_y)\} = K\partial_x \oplus K\partial_y$, and $(K\partial_x \oplus K\partial_y)^{\mathfrak{sl}_2} = (0)$.

Let $\varphi = (f, g)\colon \mathbb{A}^2 \to \mathbb{A}^2$ be étale, and assume, for simplicity, that $j(f, g) = 1$. From formula $(*)$ we get

$$\varphi^*(\partial_x) = g_y\partial_x - g_x\partial_y = -D_g, \quad \varphi^*(\partial_y) = -f_y\partial_x + f_x\partial_y = D_f,$$
$$\varphi^*(x\partial_y) = fD_f = \tfrac{1}{2}D_{f^2}, \quad \varphi^*(y\partial_x) = -gD_g = -\tfrac{1}{2}D_{g^2},$$
$$\varphi^*(x\partial_x) = -fD_g, \quad \varphi^*(y\partial_y) = gD_f, \quad \varphi^*(x\partial_x - y\partial_y) = -D_{fg}.$$

This shows that for an étale map $\varphi = (f, g)$ we obtain

$$\varphi^*(\mathfrak{aff}_2) = \langle D_f, D_g, D_{f^2}, D_{g^2}, fD_g, gD_f \rangle,$$
$$\varphi^*(\mathfrak{saff}_2) = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg} \rangle = \mu(P_{f,g})$$

Proposition 3.6. *Let* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *be a Lie subalgebra isomorphic to* $\mathfrak{saff}_2$*. Then there is an étale map* φ *such that* $L = \varphi^*(\mathfrak{saff}_2)$*. More precisely, if* (D_f, D_g) *is a basis of* $\mathfrak{rad}(L)$*, then* $L = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg} \rangle$*, and one can take* $\varphi = (f, g)$*.*

Proof. We first remark that $L \subset \mathrm{Vec}^0(\mathbb{A}^2)$, because $\mathfrak{saff}_2$ has no non-trivial character. By Proposition 2.8 it suffices to show that $Q := \mu^{-1}(L) \subset P$ is isomorphic to $P_{\leq 2}$. We fix a decomposition $L = L_0 \oplus \mathfrak{rad}(L)$ where $L_0 \simeq \mathfrak{sl}_2$. It is clear that the Lie subalgebra $\tilde{Q} := \mu^{-1}(L_0) \subset P$ contains a copy of $\mathfrak{sl}_2$, i.e. $\tilde{Q} = Q_0 \oplus K$ where $Q_0 \simeq \mathfrak{sl}_2$. Hence, as a Q_0-module, we get $Q = Q_0 \oplus V_2 \oplus K$ where V_2 is a two-dimensional irreducible Q_0-module which is isomorphically mapped onto $\mathfrak{rad}(L)$ under μ. Since $\{\mathfrak{rad}(L), \mathfrak{rad}(L)\} = (0)$ we have $\{V_2, V_2\} \subset K$. Now the claim follows from Lemma 2.9 if we show that $\{V_2, V_2\} \neq (0)$.

Assume that $\{V_2, V_2\} = (0)$. Choose a $\mathfrak{sl}_2$-triple (e_0, h_0, f_0) in Q_0 and a basis (f, g) of V_2 such that $\{e_0, f\} = g$ and $\{e_0, g\} = 0$. Since $\{f, g\} = 0$ we get from Lemma 2.1(b) that $f, g \in K[h]$ for some $h \in K[x, y]$, i.e. $f = p(h)$ and $g = q(h)$ for some polynomials $p, q \in K[t]$. But then $0 = \{e_0, g\} = \{e_0, q(h)\} = q'(h)\{e_0, h\}$ and so $\{e_0, h\} = 0$. This implies that $g = \{e_0, f\} = \{e_0, p(h)\} = p'(h)\{e_0, h\} = 0$, a contradiction. □

Remark 3.7. The above description of the Lie subalgebras L isomorphic to $\mathfrak{saff}_2$ also gives a Levi decomposition of L. In fact, (D_f, D_g) is a basis of $\mathfrak{rad}(L)$ and $L_0 := \langle D_{f^2}, D_{g^2}, D_{fg} \rangle$ is a subalgebra isomorphic to $\mathfrak{sl}_2$. The following corollary shows that every Levi decomposition is obtained in this way.

Corollary 3.8. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{saff}_2$, and let $L = \mathfrak{rad}(L) \oplus L_0$ be a Levi decomposition. Then there exist $f, g \in K[x,y]$ such that $\mathfrak{rad}(L) = \langle D_f, D_g\rangle$ and $L_0 = \langle D_{f^2}, D_{fg}, D_{g^2}\rangle$. Moreover, if $L' \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is another Lie subalgebra isomorphic to $\mathfrak{saff}_2$ and if $L' \supset L_0$, then $L' = L$.*

Proof. We can assume that $L = \mathfrak{saff}_2 = \langle D_x, D_y, D_{x^2}, D_{y^2}, D_{xy}\rangle$. Then every Lie subalgebra $L_0 \subset L$ isomorphic to $\mathfrak{sl}_2$ is the image of $\mathfrak{sl}_2 = \langle D_{x^2}, D_{y^2}, D_{xy}\rangle$ under conjugation with an element α of the solvable radical R of SAff_2. As a subgroup of $\mathrm{Aut}(K[x,y])$ the elements of R are the translations $\alpha = (x+a, y+b)$, and we get $\mathfrak{rad}(L) = \langle D_{x+a}, D_{y+b}\rangle$ and $\alpha(\mathfrak{sl}_2) = \langle D_{(x+a)^2}, D_{(y+b)^2}, D_{(x+a)(y+b)}\rangle$ as claimed.

For the last statement, we can assume that $L' = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg}\rangle$ such that $\langle D_{f^2}, D_{g^2}, D_{fg}\rangle = \mathfrak{sl}_2$. This implies that $\langle f^2, g^2, fg, 1\rangle = \langle x^2, y^2, xy, 1\rangle$, and the claim follows. □

Proposition 3.9. *Let $M \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$. Then there is an étale map φ such that $M = \varphi^*(\mathfrak{aff}_2)$. More precisely, if (D_f, D_g) is a basis of $\mathfrak{rad}([M,M])$, then $M = \langle D_f, D_g, fD_f, gD_g, gD_f, fD_g\rangle$, and one can take $\varphi = (f,g)$.*

Proof. The subalgebra $M' := [M,M]$ is isomorphic to $\mathfrak{saff}$, hence, by Proposition 3.6, $M' = \varphi^*(\mathfrak{saff}_2)$ for an étale map $\varphi = (f,g)$ where we can assume that $j(\alpha) = 1$. We want to show that $\varphi^*(\mathfrak{aff}_2) = M$. Consider the decomposition $M = J \oplus M_0 \oplus KD$ where $J = \mathfrak{rad}(M')$, M_0 is isomorphic to $\mathfrak{sl}_2$, and D is the Euler-element acting trivially on M_0. We have $\varphi^*(\mathfrak{aff}_2) = M' \oplus KE$ where E is the image of the Euler element of $\mathfrak{aff}_2$. Since $\mathrm{Vec}^c(\mathbb{A}^2) = \mathrm{Vec}^0(\mathbb{A}^2) \oplus KD'$ for any $D' \in \mathrm{Vec}^c(\mathbb{A}^2)$ with $\mathrm{Div}\, D' \neq 0$ we can write $D = aE + F$ with some $a \in K$ and $F \in \mathrm{Vec}^0(\mathbb{A}^2)$, i.e. $F = D_h$ for some $h \in K[x,y]$.

By construction, $F = D - aE$ commutes with M_0. Since $M_0 = \langle D_{f^2}, D_{g^2}, D_{fg}\rangle$ we get $\{h, f^2\} = c$ where $c \in K$. Thus $c = \{h, f^2\} = 2f\{h,f\}$ which implies that $\{h,f\} = 0$. Similarly, we find $\{h,g\} = 0$, hence h is in the center of $\mu^{-1}(M') = P_{f,g} \subset P$. Thus, by Lemma 2.1(c), $h \in K$ and so $D_h = 0$ which implies $D = aE$. □

4. Vector fields and the Jacobian Conjecture

The Jacobian Conjecture. Recall that the *Jacobian Conjecture* in dimension n says that an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism.

Theorem 4.1. *The following statements are equivalent.*

(i) *The Jacobian Conjecture holds in dimension 2.*
(ii) *All Lie subalgebras of P isomorphic to $P_{\leq 2}$ are equivalent under $\mathrm{Aut}_{LA}(P)$.*
(iii) *All Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$.*
(iv) *All Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$.*

For the proof we need to compare the automorphisms of P with those of the image $\mu(P) = \mathrm{Vec}^0(\mathbb{A}^2) \simeq P/K$. Since K is the center P, we have a canonical homomorphism $F\colon \mathrm{Aut}_{LA}(P) \to \mathrm{Aut}_{LA}(P/K)$, $\varphi \mapsto \bar{\varphi}$.

Lemma 4.2. *The map $F\colon \mathrm{Aut}_{LA}(P) \to \mathrm{Aut}_{LA}(P/K)$ is an isomorphism.*

Proof. If $\varphi \in \ker F$, then $\varphi(x) = x+a, \varphi(y) = y+b$ where $a, b \in K$. By Lemma 2.4, the K-algebra automorphism α of $K[x,y]$ defined by $x \mapsto x+a$, $y \mapsto y+b$ is a Lie algebra automorphism of P, and $\varphi = \alpha$ by Lemma 2.3. But then $\varphi(x^2) = (x+a)^2 = x^2 + 2ax + a^2$, and so $\bar{\varphi}(\overline{x^2}) = \overline{x^2} + 2a\bar{x}$. Therefore, $a = 0$, and similarly we get $b = 0$, hence $\varphi = \mathrm{id}_P$.

Put $\bar{P} := P/K$ and let $\rho\colon \bar{P} \xrightarrow{\sim} \bar{P}$ be a Lie algebra automorphism. Then $\bar{L} := \rho(\bar{P}_{\leq 2}) \subset \bar{P}$ is a Lie subalgebra isomorphic to $\mathfrak{saff}_2$ and thus $L := p^{-1}(\bar{L})$ is a Lie subalgebra of P isomorphic to $P_{\leq 2}$, by Proposition 2.8. Choose $f, g \in L$ such that $\bar{f} = \rho(\bar{x})$ and $\bar{g} = \rho(\bar{y})$. Then $\langle 1, f, g\rangle = \mathfrak{rad}(L)$, and so $L = P_{f,g}$, by Proposition 2.8. It follows that the map $\mu\colon P \to P$ defined by $x \mapsto f, y \mapsto g$ is an injective endomorphism of P (Remark 2.5), and that $\bar{\mu} = \rho$. Since ρ is an isomorphism the same holds for μ. □

Proof of Theorem 4.1. (i)⇒(ii): If $L \subset P$ is isomorphic to $P_{\leq 2}$, then $L = P_{f,g}$ for some $f, g \in K[x,y]$ such that $\{f, g\} = 1$ (Proposition 2.8). By (i) we get $K[x,y] = K[f,g]$, and so the endomorphism $x \mapsto f, y \mapsto g$ of $K[x,y]$ is an isomorphism of P, mapping $P_{\leq 2}$ to L.

(ii)⇒(iii): If $\bar{L} \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is a Lie subalgebra isomorphic to $\mathfrak{saff}_2$, then $\bar{L} = \mu(P_{f,g})$ for some $f, g \in K[x,y]$, by Proposition 3.6. By (ii), $P_{f,g} = \alpha_*(P_{\leq 2})$ for some $\alpha \in \mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$. Hence $\bar{L} = \mu(\alpha_*(\bar{P}_{\leq 2})) = \bar{\alpha}(\mathfrak{saff}_2)$, by Lemma 3.3.

(iii)⇒(iv): Let $M \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$, and set $M' := [M, M] \simeq \mathfrak{saff}_2$. By (iii) there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $M' = \varphi^*(\mathfrak{saff}_2)$. It follows that $\varphi^*(\mathfrak{aff}_2) = M$ since M is determined by $\mathfrak{rad}(M')$ as a Lie subalgebra, by Proposition 3.9.

(iv)⇒(i): Let $\varphi := (f, g)\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an étale morphism. Then $M := \varphi^*(\mathfrak{aff}_2) \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is a Lie subalgebra isomorphic to $\mathfrak{aff}_2$ (see Lemma 3.3). By assumption (iv), there is an automorphism $\psi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\psi^*(\mathfrak{aff}_2) = M$. It follows that $\psi^{-1} \circ \varphi$ is an étale morphism which induces an automorphism of $\mathfrak{aff}_2$, hence of $\mathfrak{saff}_2$, and thus of $\mathfrak{rad}(\mathfrak{saff}_2) = K\partial_x \oplus K\partial_y$. This implies that $\psi^{-1} \circ \varphi$ is an automorphism, and the claim follows. □

Remark 4.3. It is not true that the Lie subalgebras of P or of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{sl}_2$ are equivalent, respectively conjugate. This can be seen from the example $S = Kx^2y \oplus Kxy \oplus Ky \subset P$ which is isomorphic to $\mathfrak{sl}_2$, but not equivalent to $Kx^2 \oplus Kxy \oplus Ky^2$ under $\mathrm{Aut}_{LA}(P)$. In fact, the element x^2y does not act locally finitely on P.

Algebraic Lie algebras. If an algebraic group G acts on an affine variety X we get a canonical anti-homomorphism of Lie algebras $\Phi\colon \mathrm{Lie}\, G \to \mathrm{Vec}(X)$ defined in the usual way:

$$\mathrm{Lie}\, G \ni A \mapsto \xi_A \text{ with } (\xi_A)_x := d\varphi_x(A) \text{ for } x \in X,$$

where $\varphi_x\colon G \to X$ is the orbit map $g \mapsto gx$. A Lie algebra $L \subset \mathrm{Vec}(X)$ is called *algebraic* if L is contained in $\Phi(\mathrm{Lie}\, G)$ for some action of an algebraic group G on X. It is shown in [CD03] that L is algebraic if and only if L acts locally finitely on $\mathrm{Vec}(X)$. With this result we get the following consequence of our Theorem 1.

Corollary 4.4. *The following statements are equivalent.*

(i) *The Jacobian Conjecture holds in dimension 2.*
(ii) *All Lie subalgebras of* $\mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{saff}_2$ *are algebraic.*

(iii) *All Lie subalgebras of* $\mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{aff}_2$ *are algebraic.*

Proof. It is clear that the equivalent statements (i), (ii) or (iii) of Theorem 1 imply (ii) and (iii) from the corollary. It follows from the Propositions 3.6 and 3.9 that every Lie subalgebra L isomorphic to $\mathfrak{saff}_2$ is contained in a Lie subalgebra Q isomorphic to $\mathfrak{aff}_2$, hence (iii) implies (ii). It remains to prove that (ii) implies (i).

We will show that (ii) implies that L is equivalent to $\mathfrak{saff}_2$. Then the claim follows from Theorem 1. By (ii), there is a connected algebraic group G acting faithfully on $\mathbb{A}^2$ such that $\Phi(\mathrm{Lie}\,G)$ contains L. Therefore, $\mathrm{Lie}\,G$ contains a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2$, and so G contains a closed subgroup S such that $\mathrm{Lie}\,S = \mathfrak{s}$. Since every action of SL_2 on $\mathbb{A}^2$ is linearizable (see [KP85]), there is an automorphism φ such that $\varphi^*(\mathfrak{s}) = \mathfrak{sl}_2 = \langle x\partial_y, y\partial_x, x\partial_x - y\partial_y\rangle$. But this implies, by Corollary 3.8, that $\varphi^*(L) = \mathfrak{saff}_2$. □

Automorphisms of vector fields. We have seen in Lemma 2.4 that $\mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$. In this last section we describe the automorphism groups of the Lie algebras $\mathrm{Vec}(\mathbb{A}^2)$, $\mathrm{Vec}^c(\mathbb{A}^2)$ and $\mathrm{Vec}^0(\mathbb{A}^2)$.

Theorem 4.5. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2)).$$

For the proof we need the following two results. The first one is certainly well-known. Recall that $\mathfrak{saff}_2 = [\mathfrak{aff}_2, \mathfrak{aff}_2] \subset \mathfrak{aff}_2$ is invariant under all automorphisms of the Lie algebra $\mathfrak{aff}_2$.

Lemma 4.6. *The canonical homomorphisms*

$$\mathrm{Aff}_2 \xrightarrow[\simeq]{\mathrm{Ad}} \mathrm{Aut}_{LA}(\mathfrak{aff}_2) \xrightarrow[\simeq]{\mathrm{res}} \mathrm{Aut}_{LA}(\mathfrak{saff}_2)$$

are isomorphisms.

Proof. We write the elements of Aff_2 in the form (v, g) with $v \in T = (K^+)^2$ and $g \in \mathrm{GL}_2$ where $(v, g)x = gx + v$ for $x \in \mathbb{A}^2$. It follows that $(v, g)(w, h) = (v + gw, gh)$. Similarly, $(a, A) \in \mathfrak{aff}_2$ means that $a \in \mathfrak{t} = (K)^2$ and $A \in \mathfrak{gl}_2$, and $(a, A)x = Ax + a$. For the adjoint representation of $g \in \mathrm{GL}_2$ and of $v \in T$ on $\mathfrak{aff}_2$ we get

$$\mathrm{Ad}(g)(a, A) = (ga, gAg^{-1}) \quad \text{and} \quad \mathrm{Ad}(v)(a, A) = (a - Av, A),$$

and thus, for $(b, B) \in \mathfrak{aff}_2$,

$$(**) \qquad \mathrm{ad}(B)(a, A) = (Ba, [B, A]) \quad \text{and} \quad \mathrm{ad}(b)(a, A) = (a - Ab, A).$$

Now let θ be an automorphism of the Lie algebra $\mathfrak{saff}_2$. Then $\theta(\mathfrak{t}) = \mathfrak{t}$, because $\mathfrak{t}$ is the solvable radical of $\mathfrak{saff}_2$. Since $g := \theta|_{\mathfrak{t}} \in \mathrm{GL}_2$, composing θ with $\mathrm{Ad}(g^{-1})$, we can assume that θ is the identity on $\mathfrak{t}$. This implies that $\theta(a, A) = (a + \ell(A), \bar{\theta}(A))$ where $\ell\colon \mathfrak{sl}_2 \to \mathfrak{t}$ is a linear map and $\bar{\theta}\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ is a Lie algebra automorphism.

From $(**)$ we get $\mathrm{ad}(b, B)(a, 0) = \mathrm{ad}(B)(a, 0) = (Ba, 0)$ for all $a \in \mathfrak{t}$, hence

$$\begin{aligned}(Ba, 0) = \theta(Ba, 0) = \theta(\mathrm{ad}(B)(a, 0)) = \\ = \mathrm{ad}(\theta(B))(a, 0) = \mathrm{ad}(\bar{\theta}(B))(a, 0) = (\bar{\theta}(B)a, 0).\end{aligned}$$

Thus $\bar{\theta}(B) = B$, i.e. $\theta(a, A) = (a + \ell(A), A)$. For $c := \ell(E)$ we obtain

$$\theta(a, \lambda E) = (a + \lambda c, \lambda E) = \mathrm{Ad}(-c)(a, \lambda E).$$

Thus we can assume that θ is the identity on $KE \subset \mathfrak{aff}_n$. Since M_n is the centralizer of KE in $\mathfrak{aff}_n$ this implies that $\theta(\mathrm{M}_n) = \mathrm{M}_n$, hence $\theta(0, A) = (0, \theta(A)) = (0, \bar{\theta}(A)) = (0, A)$. As a consequence, $\theta = \mathrm{id}$, and the claim follows. □

Lemma 4.7. *If θ is an endomorphism of the Lie algebra $\mathrm{Vec}^0(\mathbb{A}^2)$ which is the identity on $\mathfrak{saff}_2$, then θ is the identity.*

Proof. It follows from Lemma 2.1(d) and Lemma 2.2(b) that $\mathrm{Vec}^0(\mathbb{A}^2)$ is generated by the vector fields ∂_y, $x^2\partial_y$, and $y\partial_x$. So it suffices to show that $\theta(x^2 dy) = x^2 dy$.

Put $D := \theta(x^2 dy)$. Since $[\partial_y, D] = \theta([\partial_y, x^2\partial_y]) = 0$ we see that $D = h(x)\partial_x + f(x)\partial_y$. But $0 = \mathrm{Div}\, D = h_x$, and so $D = a\partial_x + f(x)\partial_y$.

Now $[\partial_x, D] = \theta([\partial_x, a\partial_x + x^2\partial_y]) = \theta(2x\partial_y) = 2x\partial_y = [\partial_x, x^2\partial_y]$. Hence $D = a\partial_x + x^2\partial_y + b\partial_y$. Finally, $[x\partial_y, D] = -a\partial_y = \theta([x\partial_y, x^2\partial_y]) = 0$, hence $a = 0$, and similarly, $[y\partial_x, D] = 2x\partial_y - b\partial_x = \theta([y\partial_x, x^2\partial_y]) = \theta(2x\partial_y) = 2x\partial_y$, hence $b = 0$. □

Proof of Theorem 4.5. (a) The fact that $\mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2))$ is an isomorphism goes back to Kulikov (see proof of theorem 4, [Kul92]). For another proof see [Bav13].

(b) It follows from (a) that we have a canonical homomorphism, by restriction,

$$\mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)),$$

and since $\mathrm{Vec}^0(\mathbb{A}^2) \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is an ideal of finite codimension and is simple as a Lie algebra we also get a homomorphism

$$\mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$$

which is easily seen to be injective. Thus it remains to show that the canonical homomorphism $\omega\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$ is an isomorphism.

(c) It is clear that ω is injective. Let θ be an automorphism of $\mathrm{Vec}^0(\mathbb{A}^2)$. It follows from Proposition 3.6 that there is an étale map φ such that $\varphi^*(\mathfrak{saff}_2) = \theta(\mathfrak{saff}_2)$. Hence the homomorphism $\theta^{-1} \circ \varphi^*$ maps $\mathfrak{saff}_2$ isomorphically onto itself. This implies, by Lemma 4.6, that $(\theta^{-1} \circ \varphi^*)|_{\mathfrak{saff}_2} = \mathrm{Ad}(\psi)$ that for a suitable $\psi \in \mathrm{Aff}_2$. By definition, $\psi^*|_{\mathfrak{saff}_2} = \mathrm{Ad}(\psi)^{-1}$, and so the composition $\theta^{-1} \circ \varphi^* \circ \psi^*$ is the identity on $\mathfrak{saff}_2$, hence the identity on $\mathrm{Vec}^0(\mathbb{A}^2)$, by Lemma 4.7. Therefore, by Remark 3.4, φ is an isomorphism, and so $\theta = \varphi^* \circ \psi^*$ belongs to the image of $\omega\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$. □

Remark 4.8. In [KReg14] our Theorem 4.5 is generalized to any dimension, using a completely different approach.

References

[Bav13] V. V. Bavula, *The group of automorphisms of the Lie algebra of derivations of a polynomial algebra*, preprint (2013), arXiv:1304.3836 [math.RA].

[CD03] Arjeh M. Cohen and Jan Draisma, *From Lie algebras of vector fields to algebraic group actions*, Transform. Groups **8** (2003), no. 1, 51–68.

[Kam79] T. Kambayashi, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra **60** (1979), no. 2, 439–451.

[Kno91] Friedrich Knop, *Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen*, Invent. Math. **105** (1991), no. 1, 217–220.

[KP85] Hanspeter Kraft and Vladimir L. Popov, *Semisimple group actions on the three-dimensional affine space are linear*, Comment. Math. Helv. **60** (1985), no. 3, 466–479.

[Kra96] Hanspeter Kraft, *Challenging problems on affine n-space*, Astérisque (1996), no. 237, Exp. No. 802, 5, 295–317, Séminaire Bourbaki, Vol. 1994/95.

[KReg14] Hanspeter Kraft and Andriy Regeta, *Automorphisms of the Lie algebra of vector fields on affine n-space*, preprint (2014), arxiv.org/abs/1402.5012 [math.AG].

[Kul92] Vik. S. Kulikov, *Generalized and local Jacobian problems*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 5, 1086-1103.

[Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[NN88] Andrzej Nowicki and Masayoshi Nagata, *Rings of constants for k-derivations in* $k[x_1, \cdots, x_n]$, J. Math. Kyoto Univ. **28** (1988), no. 1, 111–118.

[Sch89] Gerald W. Schwarz, *Exotic algebraic group actions*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 2, 89–94.

[Sha66] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) **25** (1966), no. 1-2, 208–212.

[Sha81] ———, *On some infinite-dimensional groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 214–226, 240.

Mathematisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel
E-mail address: Andriy.Regeta@unibas.ch