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**Abstract** Sparse tensor product spaces provide an efficient tool to discretize higher dimensional operator equations. The direct Galerkin method in such ansatz spaces may employ hierarchical bases, interpolets, wavelets or multilevel frames. Besides, an alternative approach is provided by the so-called combination technique. It properly combines the Galerkin solutions of the underlying problem on certain full (but small) tensor product spaces. So far, however, the combination technique has been analyzed only for special model problems. In the present paper, we provide now the analysis of the combination technique for quite general operator equations in sparse tensor product spaces. We prove that the combination technique produces the same order of convergence as the Galerkin approximation with respect to the sparse tensor product space. Furthermore, the order of the cost complexity is the same as for the Galerkin approach in the sparse tensor product space. Our theoretical findings are validated by numerical experiments.

## 1 Introduction

The discretization in sparse tensor product spaces yields efficient numerical methods to solve higher dimensional operator equations. Nevertheless, a Galerkin discretization in these sparse tensor product spaces requires hierarchical bases, interpolets, wavelets, multilevel frames, or other types of multilevel systems [9, 12, 18] which make a direct Galerkin discretization in sparse tensor product spaces quite involved and cumbersome in practical applications. To avoid these issues of the Galerkin dis-

Michael Griebel
Institut für Numerische Simulation, Universität Bonn, Wegelerstr. 6, 53115 Bonn, Germany,
e-mail: griebel@ins.uni-bonn.de

Helmut Harbrecht
Mathematisches Institut, Universität Basel, Rheinsprung 21, 4051 Basel, Switzerland,
e-mail: helmut.harbrecht@unibas.ch

cretization, the *combination technique* has been introduced in [14]. There, only the Galerkin discretizations and solutions in appropriately chosen, full, but small, tensor product spaces need to be computed and combined.

In [16], it has been shown that, in the special case of operator equations which involve a tensor product operator, the approximation produced by the combination technique indeed coincides exactly with the Galerkin solution in the sparse tensor product space. However, for non-tensor product operators, this is no longer the case. Nevertheless, it is observed in practice that still the same order of approximation error is achieved. But theoretical convergence results are only available for specific applications, see for example [3, 14, 20, 21, 22, 24]. Moreover, a general proof of convergence is so far still missing for the combination technique.

In the present paper, we prove optimal convergence rates of the combination technique for arbitrary elliptic operators. To keep the notation and the proofs simple, we restrict ourselves to the case of operator equations which are defined on a two-fold product domain $\Omega_1 \times \Omega_2$. We allow the domains $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ to be of different spatial dimensions and will therefore consider the so-called *generalized sparse tensor product spaces* which have been introduced in [10]. Nevertheless, our proofs can be generalized without further difficulties to arbitrary *L*-fold product domains $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_L$ by employing the techniques from [11] and [24].

The remainder of this paper is organized as follows. We first present the operator equations under consideration in Section 2. Then, in Section 3, we specify the requirements of the multiscale hierarchies on each individual subdomain. In Section 4, we define the generalized sparse tensor product spaces and recall their basic properties. The combination technique is introduced in Section 5 and its convergence is proven in Section 6. Section 7 is dedicated to numerical experiments. They are in good agreement with the presented theory. Finally, in Section 8, we give some concluding remarks.

Throughout this paper, the notion "essential" in the context of complexity estimates means "up to logarithmic terms". Moreover, to avoid the repeated use of generic but unspecified constants, we signify by $C \lesssim D$ that $C$ is bounded by a multiple of $D$ independently of parameters which $C$ and $D$ may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \sim D$ as $C \lesssim D$ and $C \gtrsim D$.

## 2 Operator equations

We consider two sufficiently smooth, bounded domains $\Omega_1 \in \mathbb{R}^{n_1}$ and $\Omega_2 \in \mathbb{R}^{n_2}$, where $n_1, n_2 \in \mathbb{N}$. Moreover, on the product domain $\Omega_1 \times \Omega_2$, let the Hilbert space $\mathscr{H}$ be given such that

$$\mathscr{H} \subset L^2(\Omega_1 \times \Omega_2) \subset \mathscr{H}'$$

forms a Gelfand triple. Now, let $A : \mathscr{H} \to \mathscr{H}'$ denote a differential or pseudo-differential operator. It is assumed that it maps the Hilbert space $\mathscr{H}$ continuously and bijectively onto its dual $\mathscr{H}'$, i.e.,

$$\|Au\|_{\mathcal{H}'} \sim \|u\|_{\mathcal{H}} \text{ for all } u \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}$ is thus the *energy space* of the operator under consideration.

For the sake of simplicity, we further assume that $A$ is $\mathcal{H}$-elliptic. Consequently, the resulting bilinear form

$$a(u,v) := (Au,v)_{L^2(\Omega_1 \times \Omega_2)} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

is continuous

$$a(u,v) \lesssim \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \text{ for all } u,v \in \mathcal{H}$$

and elliptic

$$a(u,u) \gtrsim \|u\|_{\mathcal{H}}^2 \text{ for all } u \in \mathcal{H}.$$

In the following, for given $f \in \mathcal{H}'$, we want to efficiently solve the operator equation $Au = f$ or, equivalently, the variational formulation:

$$\text{find } u \in \mathcal{H} \text{ such that } a(u,v) = (f,v)_{L^2(\Omega_1 \times \Omega_2)} \text{ for all } v \in \mathcal{H}. \tag{1}$$

Of course, since we like to focus on conformal Galerkin discretizations, we should tacitly assume that, for all $j_1, j_2 \geq 0$, the tensor product $V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}$ of the ansatz spaces $V_{j_1}^{(1)}$ and $V_{j_2}^{(2)}$ is contained in the energy space $\mathcal{H}$. Moreover, for the solution $u \in \mathcal{H}$ of (1), we will need a stronger regularity to hold for obtaining decent convergence rates. Therefore, for $s_1, s_2 \geq 0$, we introduce the following Sobolev spaces of dominant mixed derivatives with respect to the underlying space $\mathcal{H}$

$$\mathcal{H}_{mix}^{s_1,s_2} := \left\{ f \in \mathcal{H} : \left\| \frac{\partial^{\alpha+\beta}}{\partial_{\mathbf{x}}^{\alpha} \partial_{\mathbf{y}}^{\beta}} f \right\|_{\mathcal{H}} < \infty \text{ for all } |\alpha| \leq s_1 \text{ and } |\beta| \leq s_2 \right\}.$$

We shall illustrate our setting by the following specific examples.

*Example 1.* A first simple example is the operator $A : L^2(\Omega_1 \times \Omega_2) \to L^2(\Omega_1 \times \Omega_2)$ which underlies the bilinear form

$$a(u,v) = \int_{\Omega_1} \int_{\Omega_2} \alpha(\mathbf{x},\mathbf{y}) u(\mathbf{x},\mathbf{y}) v(\mathbf{x},\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y},$$

where the coefficient function $\alpha$ satisfies

$$0 < \underline{\alpha} \leq \alpha(\mathbf{x},\mathbf{y}) \leq \overline{\alpha} \text{ for all } (\mathbf{x},\mathbf{y}) \in \Omega_1 \times \Omega_2. \tag{2}$$

Here, it holds $\mathcal{H} = L^2(\Omega_1 \times \Omega_2)$. Moreover, our spaces $\mathcal{H}_{mix}^{s_1,s_2}$ coincide with the standard Sobolev spaces of dominant mixed derivatives, i.e.,

$$\mathcal{H}_{mix}^{s_1,s_2} = H_{mix}^{s_1,s_2}(\Omega_1 \times \Omega_2) := H^{s_1}(\Omega_1) \otimes H^{s_2}(\Omega_2).$$

*Example 2.* A second order diffusion equation on the product domain $\Omega_1 \times \Omega_2$ yields the bilinear form

$$a(u,v) = \int_{\Omega_1} \int_{\Omega_2} \alpha(\mathbf{x},\mathbf{y}) \{ \nabla_{\mathbf{x}} u(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{x}} v(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{y}} u(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}} v(\mathbf{x},\mathbf{y}) \} \, d\mathbf{x} \, d\mathbf{y}.$$

If the coefficient $\alpha$ satisfies (2), then the associated operator $A$ is known to be continuous and elliptic with respect to the space $\mathscr{H} = H_0^1(\Omega_1 \times \Omega_2)$. Moreover, our space $\mathscr{H}_{mix}^{s_1,s_2}$ now coincides with $\mathscr{H}_{mix}^{s_1,s_2} = H_0^1(\Omega_1 \times \Omega_2) \cap H_{mix}^{s_1+1,s_2}(\Omega_1 \times \Omega_2) \cap H_{mix}^{s_1,s_2+1}(\Omega_1 \times \Omega_2)$.

*Example 3.* Another example appears in two-scale homogenization. Unfolding ([4]) gives raise to the product of the macroscopic physical domain $\Omega_1$ and the periodic microscopic domain $\Omega_2$ of the cell problem, see [19]. Then, for the first order corrector, one arrives at the bilinear form

$$a(u,v) = \int_{\Omega_1} \int_{\Omega_2} \alpha(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}} u(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}} v(\mathbf{x},\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}.$$

The underlying operator $A$ is continuous and elliptic as a operator in the related energy space $\mathscr{H} = L^2(\Omega_1) \otimes H_0^1(\Omega_2)$ provided that the coefficient $\alpha$ satisfies again (2). Furthermore, our space $\mathscr{H}_{mix}^{s_1,s_2}$ now coincides with $\mathscr{H}_{mix}^{s_1,s_2} = \left( L^2(\Omega_1) \otimes H_0^1(\Omega_2) \right) \cap H_{mix}^{s_1,s_2+1}(\Omega_1 \times \Omega_2)$.

## 3 Approximation on the individual subdomains

On each domain $\Omega_i$, we consider a nested sequence

$$V_0^{(i)} \subset V_1^{(i)} \subset \cdots \subset V_j^{(i)} \subset \cdots \subset L^2(\Omega_i) \tag{3}$$

of finite dimensional spaces

$$V_j^{(i)} = \mathrm{span}\{ \varphi_{j,k}^{(i)} : k \in \Delta_j^{(i)} \}$$

(the set $\Delta_j^{(i)}$ denotes a suitable index set) of piecewise polynomial ansatz functions, such that $\dim V_j^{(i)} \sim 2^{jn_i}$ and

$$L^2(\Omega_i) = \overline{\bigcup_{j \in \mathbb{N}_0} V_j^{(i)}}.$$

We will use the spaces $V_j^{(i)}$ for the approximation of functions. To this end, we assume that the approximation property

$$\inf_{v_j \in V_j^{(i)}} \| u - v_j \|_{H^q(\Omega_i)} \lesssim h_j^{s-q} \| u \|_{H^s(\Omega_i)}, \quad u \in H^s(\Omega_i), \tag{4}$$

holds for $q < \gamma_i$, $q \leq s \leq r_i$ uniformly in $j$. Here we set $h_j := 2^{-j}$, i.e., $h_j$ corresponds to the width of the mesh associated with the subspace $V_j^{(i)}$ on $\Omega_i$. The parameter $\gamma_i > 0$ refers to the *regularity* of the functions which are contained in $V_j^{(i)}$, i.e.,

$$\gamma_i := \sup\{s \in \mathbb{R} : V_j^{(i)} \subset H^s(\Omega_i)\}.$$

The integer $r_i > 0$ refers to the *polynomial exactness*, that is the maximal order of polynomials which are locally contained in the space $V_j^{(i)}$.

Now, let $Q_j^{(i)} : L^2(\Omega_i) \rightarrow V_j^{(i)}$ denote the $L^2(\Omega_i)$-orthogonal projection onto the finite element space $V_j^{(i)}$. By setting $Q_{-1}^{(i)} := 0$, we can define for all $j \geq 0$ the complementary spaces

$$W_j^{(i)} := \left(Q_j^{(i)} - Q_{j-1}^{(i)}\right) L^2(\Omega_i) \subset V_j^{(i)}.$$

They satisfy

$$V_j^{(i)} = V_{j-1}^{(i)} \oplus W_j^{(i)}, \quad V_{j-1}^{(i)} \cap W_j^{(i)} = \{0\},$$

which recursively yields

$$V_J^{(i)} = \bigoplus_{j=0}^{J} W_j^{(i)}. \tag{5}$$

A given function $f \in H^q(\Omega_i)$, where $|q| < \gamma_i$, admits the unique multiscale decomposition

$$f = \sum_{j=0}^{\infty} f_j \text{ with } f_j := \left(Q_j^{(i)} - Q_{j-1}^{(i)}\right) f \in W_j^{(i)}.$$

Especially, it holds the well-known norm equivalence

$$\|f\|_{H^q(\Omega_i)}^2 \sim \sum_{j=0}^{\infty} 2^{2jq} \left\|\left(Q_j^{(i)} - Q_{j-1}^{(i)}\right) f\right\|_{L^2(\Omega_i)}^2, \quad |q| < \gamma_i,$$

see [5]. Finally, for any $f \in H^s(\Omega_i)$ and $|q| < \gamma_i$, the approximation property (4) induces the estimate

$$\left\|\left(Q_j^{(i)} - Q_{j-1}^{(i)}\right) f\right\|_{H^q(\Omega_i)} \lesssim 2^{-j(s-q)} \|f\|_{H^s(\Omega_i)}, \quad q < s \leq r_i.$$

## 4 Generalized sparse tensor product spaces

The canonical approximation method in the Hilbert space $\mathcal{H}$ is the approximation in full tensor product spaces[1]

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[1] Here and in the following, the summation limits are in general no natural numbers and must of course be rounded properly. We leave this to the reader to avoid cumbersome floor/ceil-notations.

$$V_{J/\sigma}^{(1)} \otimes V_{J\sigma}^{(2)} = \bigoplus_{\substack{j_1\sigma \leq J \\ j_2/\sigma \leq J}} W_{j_1}^{(1)} \otimes W_{j_2}^{(2)}.$$

Here, $\sigma > 0$ is a given parameter which can be tuned to optimize the cost complexity. There are $2^{Jn_1/\sigma} \cdot 2^{Jn_2\sigma}$ degrees of freedom in the space $V_{J/\sigma}^{(1)} \otimes V_{J\sigma}^{(2)}$. Moreover, for $f \in \mathscr{H}_{mix}^{s_1,0}(\Omega_1 \times \Omega_2) \cap \mathscr{H}_{mix}^{0,s_2}(\Omega_1 \times \Omega_2)$ and $f_J := (Q_{J/\sigma}^{(1)} \otimes Q_{J\sigma}^{(2)})f \in V_{J/\sigma}^{(1)} \otimes V_{J\sigma}^{(2)}$, an error estimate of the type

$$\|f - f_J\|_{\mathscr{H}} \lesssim 2^{-J\min\{s_1/\sigma, s_2\sigma\}} \|f\|_{\mathscr{H}_{mix}^{s_1,0} \cap \mathscr{H}_{mix}^{0,s_2}} \tag{6}$$

holds for all $0 < s_1 \leq p_1$ and $0 < s_2 \leq p_2$. Note that the upper bounds $p_1$ and $p_2$ are the largest values such that $\mathscr{H}_{mix}^{p_1,0} \subset H_{mix}^{r_1,r_2}(\Omega_1 \times \Omega_2)$ and $\mathscr{H}_{mix}^{0,p_2} \subset H_{mix}^{r_1,r_2}(\Omega_1 \times \Omega_2)$, respectively.

Alternatively, based on the multiscale decompositions (5) on each individual subdomain, one can define the so-called *generalized sparse tensor product space*, see [1] and [10],

$$\widehat{V}_J^\sigma := \bigoplus_{j_1\sigma+j_2/\sigma \leq J} W_{j_1}^{(1)} \otimes W_{j_2}^{(2)} = \sum_{j_1\sigma+j_2/\sigma=J} V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}. \tag{7}$$

Thus, a function $f \in \mathscr{H}$ is represented by the Boolean sum

$$\widehat{f}_J := \sum_{j_1\sigma+j_2/\sigma \leq J} \Delta_{j_1,j_2}^Q f \in \widehat{V}_J^\sigma \tag{8}$$

where, for all $j_1, j_2 \geq 0$, the detail projections $\Delta_{j_1,j_2}^Q$ are given by

$$\Delta_{j_1,j_2}^Q := (Q_{j_1}^{(1)} - Q_{j_1-1}^{(1)}) \otimes (Q_{j_2}^{(2)} - Q_{j_2-2}^{(2)}). \tag{9}$$

Here, we use the convention $Q_{-1}^{(1)} := 0$ and $Q_{-1}^{(2)} := 0$. For further detail on sparse grids we refer the reader to the survey [1] and the references therein.

The dimension of the generalized sparse tensor product space $\widehat{V}_J^\sigma$ is essentially equal to the dimension of the finest univariate finite element spaces which enter its construction, i.e., it is essentially equal to the value of $\max\{\dim V_{J/\sigma}^{(1)}, \dim V_{J\sigma}^{(2)}\}$. Nevertheless, by considering smoothness in terms of mixed Sobolev spaces, its approximation power is essentially the same as in the full tensor product space. To be precise, we have

**Theorem 1 ([10]).** *The generalized sparse tensor product space $\widehat{V}_J^\sigma$ possesses*

$$\dim \widehat{V}_J^\sigma \sim \begin{cases} 2^{J\max\{n_1/\sigma, n_2\sigma\}}, & \text{if } n_1/\sigma \neq n_2\sigma, \\ 2^{Jn_2\sigma}J, & \text{if } n_1/\sigma = n_2\sigma, \end{cases}$$

*degrees of freedom. Moreover, for a given function $f \in \mathcal{H}_{mix}^{s_1,s_2}$ and its $L^2$-orthonormal projection $\widehat{f}_J \in \widehat{V}_J^\sigma$, defined by (8), where $0 < s_1 \leq p_1$ and $0 < s_2 \leq p_2$, there holds the error estimate*

$$\left\| f - \widehat{f}_J \right\|_{\mathcal{H}} \lesssim \begin{cases} 2^{-J\min\{s_1/\sigma,s_2\sigma\}} \|f\|_{\mathcal{H}_{mix}^{s_1,s_2}}, & \text{if } s_1/\sigma \neq s_2\sigma, \\ 2^{-Js_1/\sigma}\sqrt{J}\|f\|_{\mathcal{H}_{mix}^{s_1,s_2}}, & \text{if } s_1/\sigma = s_2\sigma. \end{cases}$$

The optimal choice of the parameter $\sigma$ has been discussed in [10]. It turns out that the best cost complexity rate among all possible values of $s_1, s_2$ is obtained for the choice $\sigma = \sqrt{n_1/n_2}$. This choice induces an equilibration of the degrees of freedom in the extremal spaces $V_{J/\sigma}^{(1)}$ and $V_{J\sigma}^{(2)}$.

We shall consider the Galerkin discretization of (1) in the generalized sparse tensor product space $\widehat{V}_J^\sigma$, that is we want to

$$\text{find } u_J \in \widehat{V}_J^\sigma \text{ such that } a(u_J, v_J) = (f, v_J)_{L^2(\Omega_1 \times \Omega_2)} \text{ for all } v_J \in \widehat{V}_J^\sigma. \qquad (10)$$

In view of Theorem 1, we arrive at the following error estimate due to Céa's lemma.

**Corollary 1.** *The Galerkin solution* (10) *satisfies the error estimate*

$$\|u - u_J\|_{\mathcal{H}} \lesssim \|u - \widehat{u}_J\|_{\mathcal{H}} \lesssim \begin{cases} 2^{-J\min\{s_1/\sigma,s_2\sigma\}} \|u\|_{\mathcal{H}_{mix}^{s_1,s_2}}, & \text{if } s_1/\sigma \neq s_2\sigma, \\ 2^{-Js_1/\sigma}\sqrt{J}\|u\|_{\mathcal{H}_{mix}^{s_1,s_2}}, & \text{if } s_1/\sigma = s_2\sigma, \end{cases}$$

*for all $0 < s_1 \leq p_1$ and $0 < s_2 \leq p_2$ provided that $u \in \mathcal{H}_{mix}^{s_1,s_2}(\Omega_1 \times \Omega_2)$.*

Nevertheless, for the discretization of (10), hierarchical bases, interpolets, wavelets, multilevel frames, or other types of multilevel systems [1, 9, 12, 13, 17, 18, 23, 25] are required which make a direct Galerkin discretization in sparse tensor product spaces quite involved and cumbersome in practical applications.

## 5 Combination technique

The combination technique is a different approach for the discretization in sparse tensor product spaces. It avoids the explicit need of hierarchical bases, interpolets, wavelets or frames for the discretization of (10). In fact, one only has to compute the Galerkin solutions with respect to certain full tensor product spaces $V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}$ and to appropriately combine them afterwords. The related Galerkin solutions $u_{j_1,j_2}$ are given by

$$\text{find } u_{j_1,j_2} \in V_{j_1}^{(1)} \otimes V_{j_2}^{(2)} \text{ such that}$$

$$a(u_{j_1,j_2}, v_{j_1,j_2}) = (f, v_{j_1,j_2})_{L^2(\Omega_1 \times \Omega_2)} \text{ for all } v_{j_1,j_2} \in V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}.$$

This introduces the Galerkin projection

$$P_{j_1,j_2} : \mathscr{H} \to V^{(1)}_{j_1} \otimes V^{(2)}_{j_2}, \quad P_{j_1,j_2} u := u_{j_1,j_2}$$

which especially satisfies the Galerkin orthogonality

$$a(u - P_{j_1,j_2} u, v_{j_1,j_2}) = 0 \text{ for all } v_{j_1,j_2} \in V^{(1)}_{j_1} \otimes V^{(2)}_{j_2}.$$

The Galerkin projection $P_{j_1,j_2}$ is well defined for all $j_1, j_2 \geq 0$ due to the ellipticity of the bilinear form $a(\cdot, \cdot)$. Moreover, as in (6), we conclude the error estimate

$$\|u - P_{j_1,j_2} u\|_{\mathscr{H}} \lesssim \|u - (Q^{(1)}_{j_1} \otimes Q^{(2)}_{j_2}) u\|_{\mathscr{H}} \lesssim 2^{-\min\{j_1 s_1, j_2 s_2\}} \|u\|_{\mathscr{H}^{s_1,0}_{mix} \cap \mathscr{H}^{0,s_2}_{mix}}$$

for all $0 < s_1 \leq p_1$ and $0 < s_2 \leq p_2$ provided that $u \in \mathscr{H}^{s_1,0}_{mix} \cap \mathscr{H}^{0,s_2}_{mix}$. In particular, for fixed $j_1 \geq 0$ and $j_2 \to \infty$, we obtain the Galerkin projection $P_{j_1,\infty}$ onto the space $V_{j_1,\infty} := (Q^{(1)}_{j_1} \otimes I) \mathscr{H} \subset \mathscr{H}$. It satisfies the error estimate

$$\|u - P_{j_1,\infty} u\|_{\mathscr{H}} \lesssim \|u - (Q^{(1)}_{j_1} \otimes I) u\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1} \|u\|_{\mathscr{H}^{s_1,0}_{mix}} \tag{11}$$

for all $0 < s_1 \leq p_1$. Likewise, for fixed $j_2 \geq 0$ and $j_1 \to \infty$, we obtain the Galerkin projection $P_{\infty,j_2}$ onto the space $V_{\infty,j_2} := (I \otimes Q^{(2)}_{j_2}) \mathscr{H} \subset \mathscr{H}$. Analogously to (11), we find

$$\|u - P_{\infty,j_2} u\|_{\mathscr{H}} \lesssim \|u - (I \otimes Q^{(2)}_{j_2}) u\|_{\mathscr{H}} \lesssim 2^{-j_2 s_2} \|u\|_{\mathscr{H}^{0,s_2}_{mix}} \tag{12}$$

for all $0 < s_2 \leq p_2$.

With the help of the Galerkin projections, we can define

$$\Delta^P_{j_1,j_2} u := (P_{j_1,j_2} - P_{j_1-1,j_2} - P_{j_1,j_2-1} + P_{j_1-1,j_2-1}) u \tag{13}$$

where we especially set $P_{j_1,-1} := 0$, $P_{-1,j_2} := 0$, and $P_{-1,-1} := 0$. Then, the combination technique is expressed as the Boolean sum (cf. [6, 7, 8])

$$\widehat{u}_J = \sum_{j_1 \sigma + j_2/\sigma \leq J} \Delta^P_{j_1,j_2} u = u - \sum_{j_1 \sigma + j_2/\sigma > J} \Delta^P_{j_1,j_2} u. \tag{14}$$

Straightforward calculation shows

$$\widehat{u}_J = \sum_{j_1=0}^{\lceil J/\sigma \rceil} (P_{j_1, \lceil J\sigma - j_1 \sigma^2 \rceil} - P_{j_1-1, \lceil J\sigma - j_1 \sigma^2 \rceil}) u \tag{15}$$

if $j_1 \leq j_2 \sigma^2$, and

$$\widehat{u}_J = \sum_{j_2=0}^{\lceil J\sigma \rceil} (P_{\lceil J/\sigma - j_2/\sigma^2 \rceil, j_2} - P_{\lceil J/\sigma - j_2/\sigma^2 \rceil, j_2-1}) u \tag{16}$$

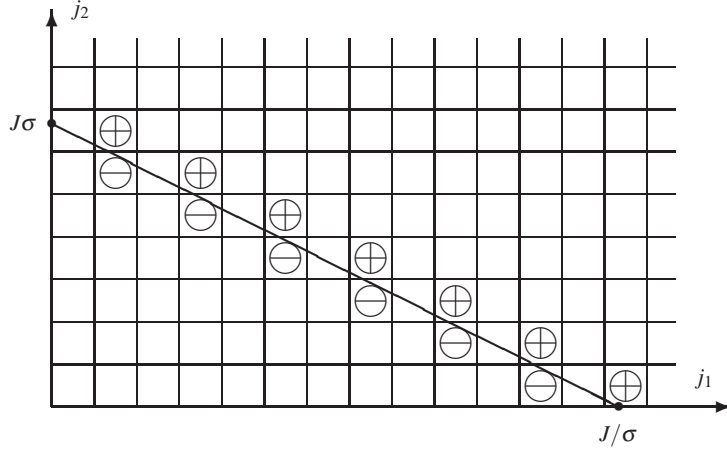if $j_1 > j_2 \sigma^2$. A visualization of the formula (16) is found in Fig. 1.

**Fig. 1** The combination technique in $\widehat{V}_J^\sigma$ combines all the indicated solutions $P_{j_1,j_2}u$ with positive sign ("⊕") and negative sign ("⊖").

Our goal is now to show that the error $\|u - \widehat{u}_J\|_{\mathscr{H}}$ converges as good as the error of the true sparse tensor product Galerkin solution given in Corollary (1).

## 6 Proof of Convergence

To prove the desired error estimate for the combination technique (15) and (16), respectively, we shall prove first the following two helpful lemmata.

**Lemma 1.** *For all* $0 < s_1 \le p_1$ *and* $0 < s_2 \le p_2$, *it holds*

$$\|(P_{j_1,j_2} - P_{j_1-1,j_2})u\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1}\|u\|_{\mathscr{H}_{mix}^{s_1,0}},$$

$$\|(P_{j_1,j_2} - P_{j_1,j_2-1})u\|_{\mathscr{H}} \lesssim 2^{-j_2 s_2}\|u\|_{\mathscr{H}_{mix}^{0,s_2}},$$

*provided that u is sufficiently smooth and provided that*

$$\partial_{\mathbf{x}}^\alpha P_{\infty,j_2}u = P_{\infty,j_2}\partial_{\mathbf{x}}^\alpha u, \quad \partial_{\mathbf{y}}^\beta P_{j_1,\infty}u = P_{j_1,\infty}\partial_{\mathbf{y}}^\beta u \tag{17}$$

*hold for all* $|\alpha| \le s_1$ *and* $|\beta| \le s_2$.

*Proof.* We shall prove only the first estimate, the second one follows in complete analogy. To this end, we split

$$\|(P_{j_1,j_2} - P_{j_1-1,j_2})u\|_{\mathscr{H}} \le \|(P_{j_1,j_2} - P_{\infty,j_2})u\|_{\mathscr{H}} + \|(P_{\infty,j_2} - P_{j_1-1,j_2})u\|_{\mathscr{H}}.$$

Due to $V_{j_1-1,j_2}, V_{j_1,j_2} \subset V_{\infty,j_2}$, the associated Galerkin projections satisfy the identities $P_{j_1,j_2} = P_{j_1,j_2} P_{\infty,j_2}$ and $P_{j_1-1,j_2} = P_{j_1-1,j_2} P_{\infty,j_2}$. Hence, we obtain

$$\|(P_{j_1,j_2} - P_{j_1-1,j_2})u\|_{\mathscr{H}} \leq \|(P_{j_1,j_2} - I)P_{\infty,j_2}u\|_{\mathscr{H}} + \|(I - P_{j_1-1,j_2})P_{\infty,j_2}u\|_{\mathscr{H}}.$$

By employing now the fact that the Galerkin projections $P_{j_1-1,j_2}u$ and $P_{j_1,j_2}u$ are quasi-optimal, i.e., $\|(I - P_{j_1,j_2})u\|_{\mathscr{H}} \lesssim \|(I - Q_{j_1}^{(1)} \otimes Q_{j_2}^{(2)})u\|_{\mathscr{H}}$ and likewise for $P_{j_1-1,j_2}u$, we arrive at

$$\begin{aligned} \|(P_{j_1,j_2} &- P_{j_1-1,j_2})u\|_{\mathscr{H}} \\ &\lesssim \|(Q_{j_1}^{(1)} \otimes Q_{j_2}^{(2)} - I)P_{\infty,j_2}u\|_{\mathscr{H}} + \|(I - Q_{j_1-1}^{(1)} \otimes Q_{j_2}^{(2)})P_{\infty,j_2}u\|_{\mathscr{H}}. \end{aligned}$$

The combination of $Q_{j_1}^{(1)} \otimes Q_{j_2}^{(2)} = (Q_{j_1}^{(1)} \otimes I)(I \otimes Q_{j_2}^{(2)})$ and $(I \otimes Q_{j_2}^{(2)})P_{\infty,j_2} = P_{\infty,j_2}$ yields the operator identity

$$(Q_{j_1}^{(1)} \otimes Q_{j_2}^{(2)})P_{\infty,j_2} = (Q_{j_1}^{(1)} \otimes I)P_{\infty,j_2},$$

and likewise

$$(Q_{j_1-1}^{(1)} \otimes Q_{j_2}^{(2)})P_{\infty,j_2} = (Q_{j_1-1}^{(1)} \otimes I)P_{\infty,j_2}.$$

Hence, we conclude

$$\begin{aligned} \|(P_{j_1,j_2} &- P_{j_1-1,j_2})u\|_{\mathscr{H}} \\ &\lesssim \|((I - Q_{j_1}^{(1)}) \otimes I)P_{\infty,j_2}u\|_{\mathscr{H}} + \|((I - Q_{j_1-1}^{(1)}) \otimes I)P_{\infty,j_2}u\|_{\mathscr{H}} \\ &\lesssim 2^{-j_1 s_1}\|P_{\infty,j_2}u\|_{\mathscr{H}_{mix}^{s_1,0}}. \end{aligned}$$

The condition (17) implies the stability estimate

$$\|P_{\infty,j_2}u\|_{\mathscr{H}_{mix}^{s_1,0}} \lesssim \|u\|_{\mathscr{H}_{mix}^{s_1,0}}$$

which finally yields the desired estimate.                                                                    ∎

*Remark 1.* Condition (17) holds if $A : \mathscr{H} \to \mathscr{H}'$ is also continuous and bijective as a mapping $A : \mathscr{H}_{mix}^{s_1,0} \to (\mathscr{H}')_{mix}^{s_1,0}$ for all $0 < s_1 \leq p_1$ and also as as a mapping $A : \mathscr{H}_{mix}^{0,s_2} \to (\mathscr{H}')_{mix}^{0,s_2}$ for all $0 < s_2 \leq p_2$, respectively. Then, the Galerkin projections $P_{\infty,j_2} : \mathscr{H}_{mix}^{s_2,0} \to V_{\infty,j_2} \subset \mathscr{H}_{mix}^{s_2,0}$ and $P_{j_1,\infty} : \mathscr{H}_{mix}^{0,s_1} \to V_{j_1,\infty} \subset \mathscr{H}_{mix}^{0,s_1}$ are both continuous, which, due to the linearity of the Galerkin projection, indeed implies condition (17).

**Lemma 2.** *If $u \in \mathscr{H}_{mix}^{s_1,s_2}$, then it holds*

$$\|(\Delta_{j_1,j_2}^P - \Delta_{j_1,j_2}^Q)u\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1 - j_2 s_2}\|u\|_{\mathscr{H}_{mix}^{s_1,s_2}}$$

*for all $0 < s_1 \leq p_1$ and $0 < s_2 \leq p_2$ where $\Delta_{j_1,j_2}^Q$ is given by (9) and $\Delta_{j_1,j_2}^P$ is given by (13), respectively.*

*Proof.* Due to $P_{j_1,j_2}(Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2})=Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2}$ for all $j_1,j_2\geq 0$, we obtain

$$\Delta^P_{j_1,j_2}-\Delta^Q_{j_1,j_2}=P_{j_1,j_2}(I-Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2})-P_{j_1-1,j_2}(I-Q^{(1)}_{j_1-1}\otimes Q^{(2)}_{j_2})$$
$$-P_{j_1,j_2-1}(I-Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2-1})+P_{i-1,j-1}(I-Q^{(1)}_{j_1-1}\otimes Q^{(2)}_{j_2-1}).\tag{18}$$

We shall now make use of the identity

$$I-Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2}=I\otimes I-Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2}$$
$$=I\otimes(I-Q^{(2)}_{j_2})+(I-Q^{(1)}_{j_1})\otimes I-(I-Q^{(1)}_{j_1})\otimes(I-Q^{(2)}_{j_2}).$$

Inserting this identity into (18) and reordering the terms yields

$$\Delta^P_{j_1,j_2}-\Delta^Q_{j_1,j_2}=(P_{j_1,j_2}-P_{j_1-1,j_2})\big(I\otimes(I-Q^{(2)}_{j_2})\big)$$
$$-(P_{j_1,j_2-1}-P_{j_1-1,j_2-1})\big(I\otimes(I-Q^{(2)}_{j_2-1})\big)$$
$$+(P_{j_1,j_2}-P_{j_1,j_2-1})\big((I-Q^{(1)}_{j_1})\otimes I\big)$$
$$-(P_{j_1-1,j_2}-P_{j_1-1,j_2-1})\big((I-Q^{(1)}_{j_1-1})\otimes I\big)$$
$$-P_{j_1,j_2}\big((I-Q^{(1)}_{j_1})\otimes(I-Q^{(2)}_{j_2})\big)$$
$$+P_{j_1-1,j_2}\big((I-Q^{(1)}_{j_1-1})\otimes(I-Q^{(2)}_{j_2})\big)$$
$$+P_{j_1,j_2-1}\big((I-Q^{(1)}_{j_1})\otimes(I-Q^{(2)}_{j_2-1})\big)$$
$$-P_{j_1-1,j_2-1}\big((I-Q^{(1)}_{j_1-1})\otimes(I-Q^{(2)}_{j_2-1})\big).$$

The combination of the error estimates

$$\|(P_{j_1,j_2}-P_{j_1-1,j_2})u\|_{\mathcal{H}}\lesssim 2^{-j_1 s_1}\|u\|_{\mathcal{H}^{s_1,0}_{mix}},$$
$$\|(P_{j_1,j_2}-P_{j_1,j_2-1})u\|_{\mathcal{H}}\lesssim 2^{-j_2 s_2}\|u\|_{\mathcal{H}^{0,s_2}_{mix}},$$

cf. Lemma 1, and

$$\big\|(I\otimes(I-Q^{(2)}_{j_2}))u\big\|_{\mathcal{H}^{s_1,0}_{mix}}\lesssim 2^{-j_2 s_2}\|u\|_{\mathcal{H}^{s_1,s_2}_{mix}},$$
$$\big\|((I-Q^{(1)}_{j_1})\otimes I)u\big\|_{\mathcal{H}^{0,s_2}_{mix}}\lesssim 2^{-j_1 s_1}\|u\|_{\mathcal{H}^{s_1,s_2}_{mix}},$$

leads to

$$\big\|(P_{j_1,j_2}-P_{j_1-1,j_2})(I\otimes(I-Q^{(2)}_{j_2}))u\big\|_{\mathcal{H}}\lesssim 2^{-j_1 s_1-j_2 s_2}\|u\|_{\mathcal{H}^{s_1,s_2}_{mix}},$$
$$\big\|(P_{j_1,j_2}-P_{j_1,j_2-1})((I-Q^{(1)}_{j_1})\otimes I)u\big\|_{\mathcal{H}}\lesssim 2^{-j_1 s_1-j_2 s_2}\|u\|_{\mathcal{H}^{s_1,s_2}_{mix}}.\tag{19}$$

Similarly, from the continuity

$$\|P_{j_1,j_2}u\|_{\mathscr{H}} \lesssim \|u\|_{\mathscr{H}}$$

and

$$\left\|\big((I-Q^{(1)}_{j_1})\otimes(I-Q^{(2)}_{j_2})\big)u\right\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1 - j_2 s_2}\|u\|_{\mathscr{H}^{s_1,s_2}_{mix}},$$

we deduce

$$\left\|P_{j_1,j_2}\big((I-Q^{(1)}_{j_1})\otimes(I-Q^{(2)}_{j_2})\big)u\right\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1 - j_2 s_2}\|u\|_{\mathscr{H}^{s_1,s_2}_{mix}}. \tag{20}$$

With (19) and (20) at hand, we can estimate each of the eight different terms which yields the desired error estimate

$$\left\|(\Delta^P_{j_1,j_2} - \Delta^Q_{j_1,j_2})u\right\|_{\mathscr{H}} \lesssim 2^{-j_1 s_1 - j_2 s_2}\|u\|_{\mathscr{H}^{s_1,s_2}_{mix}}.$$

∎

Now, we arrive at our main result which proves optimal convergence rates.

**Theorem 2.** *The solution* (15) *and* (16)*, respectively, of the combination technique satisfies the error estimate*

$$\|u - \widehat{u}_J\|_{\mathscr{H}} \lesssim \begin{cases} 2^{-J\min\{s_1/\sigma,\, s_2\sigma\}}\|u\|_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma \neq s_2\sigma, \\[2mm] 2^{-J s_1/\sigma}\sqrt{J}\|u\|_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma = s_2\sigma, \end{cases}$$

*for all* $0 < s_1 \leq p_1$ *and* $0 < s_2 \leq p_2$ *provided that* $u \in H^{s_1,s_2}_{mix}(\Omega_1 \times \Omega_2)$.

*Proof.* In view of (14), we have

$$\|u - \widehat{u}_J\|^2_{\mathscr{H}} = \left\| \sum_{j_1\sigma + j_2/\sigma > J} \Delta^P_{j_1,j_2}u \right\|^2_{\mathscr{H}}.$$

The Galerkin orthogonality implies the relation

$$\left\| \sum_{j_1\sigma + j_2/\sigma > J} \Delta^P_{j_1,j_2}u \right\|^2_{\mathscr{H}} \sim \sum_{j_1\sigma + j_2/\sigma > J} \left\|\Delta^P_{j_1,j_2}u\right\|^2_{\mathscr{H}}.$$

Thus, we arrive at

$$\|u - \widehat{u}_J\|^2_{\mathscr{H}} \lesssim \sum_{j_1\sigma + j_2/\sigma > J} \left\|\Delta^Q_{j_1,j_2}u\right\|^2_{\mathscr{H}} + \sum_{j_1\sigma + j_2/\sigma > J} \left\|(\Delta^P_{j_1,j_2} - \Delta^Q_{j_1,j_2})u\right\|^2_{\mathscr{H}}.$$

We bound the first sum on the right hand side in complete analogy to [10] from above by

$$\sum_{j_1\sigma+j_2/\sigma>J}\left\|\Delta^{Q}_{j_1,j_2}u\right\|^2_{\mathscr{H}} \lesssim \sum_{j_1\sigma+j_2/\sigma>J}2^{-2j_1s_1-2j_2s_2}\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}$$

$$\lesssim \begin{cases} 2^{-2J\min\{s_1/\sigma,s_2\sigma\}}\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma \neq s_2\sigma, \\ 2^{-2Js_1/\sigma}J\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma = s_2\sigma. \end{cases}$$

Likewise, with the help of Lemma 2, the second sum on the right hand side is bounded from above by

$$\sum_{j_1\sigma+j_2/\sigma>J}\left\|(\Delta^{P}_{j_1,j_2}-\Delta^{Q}_{j_1,j_2})u\right\|^2_{\mathscr{H}} \lesssim \sum_{j_1\sigma+j_2/\sigma>J}2^{-2j_1s_1-2j_2s_2}\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}$$

$$\lesssim \begin{cases} 2^{-2J\min\{s_1/\sigma,s_2\sigma\}}\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma \neq s_2\sigma, \\ 2^{-2Js_1/\sigma}J\|u\|^2_{\mathscr{H}^{s_1,s_2}_{mix}}, & \text{if } s_1/\sigma = s_2\sigma, \end{cases}$$

which, altogether, yields the desired error estimate. ∎

## 7 Numerical results

We now validate our theoretical findings by numerical experiments. Specifically, we will apply the combination technique for the three examples which were mentioned in Section 2. To this end, we consider the most simple case and choose $\Omega_1 = \Omega_2 = (0,1)$, i.e., $n_1 = n_2 = 1$. The ansatz spaces $V_j^{(1)}$ and $V_j^{(2)}$ consist of continuous, piecewise linear ansatz functions on an equidistant subdivision of the interval $(0,1)$ into $2^j$ subintervals. This yields the polynomial exactnesses $r_1 = r_2 = 2$. For the sake of notational convenience, we set $\square = (0,1) \times (0,1)$.

*Example 1.* First, we solve the variational problem

$$\text{find } u \in L^2(\square) \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in L^2(\square)$$

where

$$a(u,v) = \int_{\square} \alpha(x,y)u(x,y)v(x,y)\,\mathrm{d}(x,y)$$

and

$$\ell(v) = \int_{\square} f(x,y)v(x,y)\,\mathrm{d}(x,y). \tag{21}$$

The underlying operator $A$ is the multiplication operator

$$(Au)(x,y) = \alpha(x,y)u(x,y)$$

which is of the order 0. Hence, we have $\mathscr{H} = L^2(\square)$. If the multiplier $\alpha(x,y)$ is a smooth function, then $A$ arbitrarily shifts through the Sobolev scales which implies the condition (17).

Let the solution $u$ be a smooth function, i.e., $u \in \mathscr{H}_{mix}^{s_1,s_2}$, which holds if the right hand side $f$ is sufficiently regular. Then, the best possible approximation rate for the present discretization is obtained for $s_1 = r_1 = 2$ and $s_2 = r_2 = 2$, i.e., for $\mathscr{H}_{mix}^{s_1,s_2} = H_{mix}^{2,2}(\square)$. Thus, the *regular sparse tensor product space*

$$\widehat{V}_J^1 = \bigoplus_{j_1+j_2 \leq J} W_{j_1}^{(1)} \otimes W_{j_2}^{(2)} = \sum_{j_1+j_2=J} V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}. \tag{22}$$

(cf. (7)) is optimal for the discretization, see [10] for a detailed derivation. In particular, with Theorem 2, the combination technique yields the error estimate

$$\|u - \widehat{u}_J\|_{L^2(\square)} \lesssim 4^{-J}\sqrt{J}\|u\|_{H_{mix}^{2,2}(\square)}.$$

For our numerical tests, we choose

$$\alpha(x,y) = 1 + (x+y)^2, \quad f(x,y) = \alpha(x,y)u(x,y), \quad u(x,y) = \sin(\pi x)\sin(\pi y).$$

The resulting convergence history is plotted as the red curve in Fig. 2. As can be seen there, the convergence rate $4^{-J}\sqrt{J}$, indicated by the dashed red line, is indeed obtained in the numerical experiments.
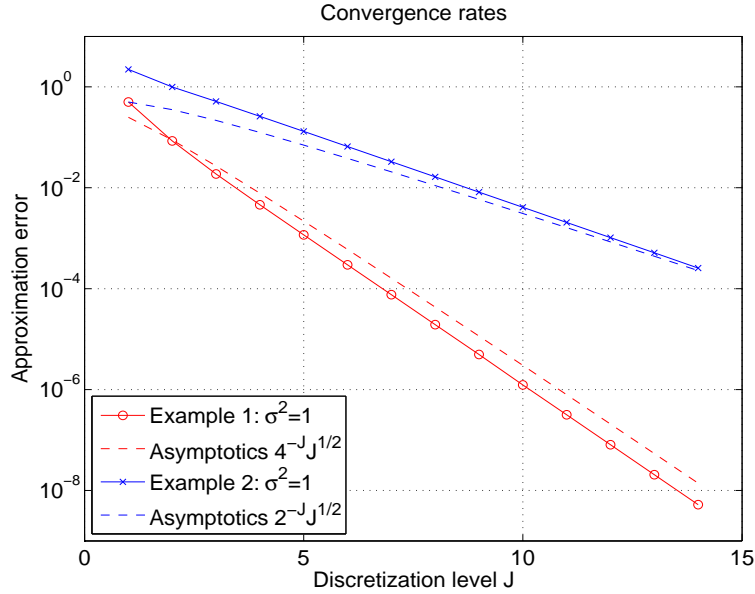


**Fig. 2** Convergence rates in case of the first and second example.

*Example 2.* This example concerns a second order diffusion problem on the domain $\square$. In its weak form, it is given by the variational problem

$$\text{find } u \in H_0^1(\square) \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in H_0^1(\square)$$

where

$$a(u,v) = \int_{\square} \alpha(x,y) \left\{ \frac{\partial u}{\partial x}(x,y) \frac{\partial v}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y) \frac{\partial v}{\partial y}(x,y) \right\} \mathrm{d}(x,y)$$

and $\ell(v)$ as in (21). The diffusion operator $A$ under consideration is of the order 2 and maps $\mathscr{H} = H_0^1(\square)$ bijectively onto $\mathscr{H}' = H^{-1}(\square)$. Since the domain $\square$ is convex, this second order boundary value problem is $H^2$-regular, which implies that $A: L^2(\square) \to H_0^1(\square) \cap H_2(\square)$ is also bijective. By interpolation arguments, we find that $A: \mathscr{H}_{mix}^{1,0} \to (\mathscr{H}')_{mix}^{1,0}$ is continuous and bijective since

$$H^{-1}(\square) \subset (\mathscr{H}')_{mix}^{1,0} \subset L^2(\square) \text{ and } H_0^1(\square) \subset \mathscr{H}_{mix}^{1,0} \subset H_0^1(\square) \cap H_2(\square).$$

Likewise, $A: \mathscr{H}_{mix}^{0,1} \to (\mathscr{H}')_{mix}^{0,1}$ is continuous and bijective. Hence, the condition (17) holds and Lemma 1 applies. Again, the regular sparse tensor product space (22) is optimal for the present discretization. Consequently, Theorem 2 implies as the best possible convergence estimate

$$\|u - \widehat{u}_J\|_{H^1(\square)} \lesssim 2^{-J}\sqrt{J}\|u\|_{H_{mix}^{2,1}(\square) \cap H_{mix}^{1,2}(\square)}$$

provided that $u \in H_{mix}^{2,1}(\square) \cap H_{mix}^{1,2}(\square)$. Here, we exploited that $\mathscr{H}_{mix}^{1,1} = H_0^1(\square) \cap H_{mix}^{2,1}(\square) \cap H_{mix}^{1,2}(\square)$. Nevertheless, in general, we only have $u \in H^2(\square) \not\subset H_{mix}^{2,1}(\square) \cap H_{mix}^{1,2}(\square)$ and can thus only expect a reduced convergence rate.

In our particular numerical computations, we use

$$\alpha(x,y) = 1 + (x+y)^2, \quad u(x,y) = \sin(\pi x)\sin(\pi y),$$

$$f(x,y) = \frac{\partial a}{\partial x}(x,y)\frac{\partial u}{\partial x}(x,y) + \frac{\partial a}{\partial y}(x,y)\frac{\partial u}{\partial y}(x,y) - \alpha(x,y)\Delta u(x,y).$$

Therefore, due to $u \in H_{mix}^{2,1}(\square) \cap H_{mix}^{1,2}(\square)$, we should observe the convergence rate $2^{-J}\sqrt{J}$. The computational approximation errors are plotted as the blue graph in Figure 2. The dashed blue line corresponds to $2^{-J}\sqrt{J}$ and clearly validates the predicted convergence rate. We even observe the slighty better rate $2^{-J}$ which can be explained by the fact that the solution $u$ is even in $H_{mix}^{2,2}(\square)$, see [2] for the details.

*Example 3.* We shall finally consider the variational problem

$$\text{find } u \in L^2(0,1) \otimes H_0^1(0,1) \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in L^2(0,1) \otimes H_0^1(0,1)$$

where

$$a(u,v) = \int_\square \alpha(x,y)\frac{\partial u}{\partial y}(x,y)\frac{\partial v}{\partial y}(x,y)\,\mathrm{d}(x,y)$$

and $\ell(v)$ is again given as in (21). It holds $\mathscr{H} = L^2(0,1) \otimes H_0^1(0,1) \subset H_{mix}^{0,1}(\square)$ and $\mathscr{H}' = L^2(0,1) \otimes H^{-1}(0,1)$. In particular, the operator $A$ shifts as a operator $\mathscr{H}_{mix}^{s_1,s_2+1} \to (\mathscr{H}')_{mix}^{s_1,s_2+1}$ for arbitrary $s_1,s_2 \geq 0$ provided that the coeffcient $\alpha$ is smooth enough. Thus, Theorem 2 holds and predicts the best possible convergence estimate for our underlying discretization if $u$ lies in the space $H_{mix}^{2,2}(\square)$.

According to the theory presented in [10], the optimal cost complexity with respect to the generalized sparse tensor product spaces $\widehat{V}_J^\sigma$ is obtained for the choice

$$\sigma \in \left[\sqrt{\frac{n_1}{n_2}}, \sqrt{\frac{r_1}{r_2-1}}\right] = [1, \sqrt{2}].$$

In order to be able to compare the convergence rates instead of the cost complexities for different choices of $\sigma$, we have to consider the generalized sparse tensor product spaces $\widehat{V}_{\overline{J}}^\sigma$, where $\overline{J} := \sigma J$. Then, for all the above choices of $\sigma$, we essentially expect the convergence rate

$$\|u - \widehat{u}_{\overline{J}}\|_{H_{mix}^{0,1}(\square)} \lesssim 2^{-\overline{J}/\sigma}\|u\|_{H_{mix}^{2,2}(\square)} \sim 2^{-J}\|u\|_{H_{mix}^{2,2}(\square)}$$

while the degrees of freedom of $\widehat{V}_{\overline{J}}^\sigma$ essentially scale like $2^{\overline{J}/\sigma} \sim 2^J$. This setting is employed in our numerical tests, where we further set

$$\alpha(x,y) = 1 + (x+y)^2, \quad u(x,y) = \sin(\pi x)\sin(\pi y),$$
$$f(x,y) = \frac{\partial a}{\partial y}(x,y)\frac{\partial u}{\partial y}(x,y) - \alpha(x,y)\frac{\partial^2 u}{\partial y^2}(x,y).$$

We apply the combination technique for the particular choices

- $\sigma = 1$, which yields an equilibration of the unknowns in all the extremal tensor product spaces $W_{j_1}^{(1)} \otimes W_{\overline{J}-j_1\sigma^2}^{(2)}$,
- $\sigma = \sqrt{2}$, which yields an equilibration of the approximation in all the extremal tensor product spaces $W_{j_1}^{(1)} \otimes W_{\overline{J}-j_1\sigma^2}^{(2)}$, and
- $\sigma = \sqrt{3/2}$, which results in an *equilibrated cost-benefit rate*, see [1, 10] for the details.

The computed approximation errors are found in Fig. 3, where the red curve corresponds to $\sigma = 1$, the black curve corresponds to $\sigma = \sqrt{2}$, and the blue curve corresponds to $\sigma = \sqrt{3/2}$. In the cases $\sigma = 1$ and $\sigma = \sqrt{2}$, we achieve the predicted convergence rate $2^{-J}$ which is indicated by the dashed black line. In the case $\sigma = \sqrt{2}$ the predicted convergence rate is only $2^{-J}\sqrt{J}$ which is also confirmed by Fig. 3.
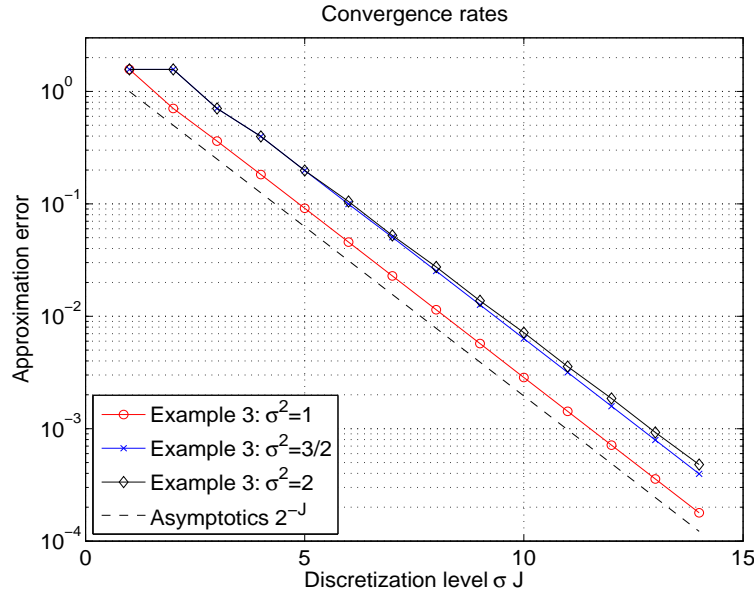
**Fig. 3** Convergence rates in case of the third example.

## 8 Conclusion

In the present paper, we proved the convergence of the combination technique in a rather general set-up. Especially, we considered the combination technique in generalized sparse tensor product spaces. We restricted ourselves here to the case of two-fold tensor product domains. Nevertheless, all our results can straightforwardly be extended to the case of generalized *L*-fold sparse tensor product spaces by applying the techniques from [11] and [24].

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