Tractability of the Quasi-Monte Carlo Quadrature with Halton Points for Elliptic Pdes with Random Diffusion

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TRACTABILITY OF THE QUASI-MONTE CARLO QUADRATURE WITH HALTON POINTS FOR ELLIPTIC PDES WITH RANDOM DIFFUSION

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Abstract. This article is dedicated to the computation of the moments of the solution to stochastic partial differential equations with log-normal distributed diffusion coefficient by the Quasi-Monte Carlo method. Our main result is the polynomial tractability for the Quasi-Monte Carlo method based on the Halton sequence. As a by-product, we obtain also the strong tractability of stochastic partial differential equations with uniformly elliptic diffusion coefficient by the Quasi-Monte Carlo method. Numerical experiments are given to validate the theoretical findings.

1. Introduction

In this article, we analyze the Quasi-Monte Carlo method based on the Halton sequence, cf. [19, 30], to determine the moments of the solution to partial differential equations with stochastic and log-normally distributed diffusion coefficient. Precisely, we consider equations in divergence form, i.e.

$$-\operatorname{div}\big(a(\mathbf{x},\omega)\nabla u(\mathbf{x},\omega)\big) = f(\mathbf{x}), \tag{1}$$

for which we impose for simplicity homogenous boundary conditions.

The efficient treatment of this type of equations has recently been the topic in several works, see e.g. [3, 8, 9, 17, 18, 26, 39]. The method of choice to cope with these equations mainly depends on the quantity of interest. Since the diffusion coefficient is modeled as a stochastic field, it is clear that the solution itself will also be a stochastic field. Therefore, if the solution u itself is of interest, methods like the stochastic Galerkin method, see e.g. [4, 13, 15, 29], or the stochastic collocation method, see e.g. [3, 31], are feasible for its approximation. If one is rather interested in distribution properties of the solution, it might be more convenient to directly approximate the solution's moments, i.e. the expected values of the powers u^p for $p \in \mathbb{N}$ of u. In the latter case, one ends up with a high-dimensional integration problem. The integration can either be performed by stochastic sampling methods, like the well known Monte Carlo method, see e.g. [7], or even randomly shifted deterministic quadrature rules, like randomly shifted lattice rules, see e.g. [41]. The latter quadrature rules belong to the class of Quasi-Monte Carlo methods. Unfortunately, these methods only provide stochastic convergence in the mean square sense. Therefore, one might aim at quadrature methods, which provide deterministic error estimates, like the conventional Quasi-Monte Carlo method based on deterministic point sequences, see e.g. [7, 30], or the sparse grid quadrature, also known as Smolyak's construction, see e.g. [6, 14, 43, 46]. All of these methods depend on the repeated evaluation of the integrand in different sample points or quadrature points. Each such evaluation corresponds to the solution of the equation (1) for a different realization of the parameter ω.

A recently popular approach to keep the computational cost low are multilevel techniques, like the Multilevel Monte Carlo method, cf. [5, 9, 16, 22, 23]. Nevertheless, in [20, 21] it is shown, that arbitrary quadrature rules can be accelerated by multilevel techniques, yielding the related multilevel quadrature methods. Especially faster converging quadrature rules result in a faster converging multilevel quadrature method.

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A common approach for the solution of (1) is based on the separation of the stochastic variable and the spatial variable in the diffusion coefficient a by the so-called Karhunen-Loève expansion, cf. [28]. This is a series expansion of a by L^2-orthogonal functions. Thus the diffusion coefficient depends in principle on infinitely many terms. Depending on the desired accuracy, this series has to be truncated appropriately. Then, the dimensionality of the integration problem is directly coupled to the length of the truncated Karhunen-Loève expansion and increases for higher accuracies. Therefore, it is crucial to construct methods which are as far as possible independent of the length of the Karhunen-Loève expansion and thus do not suffer from the curse of dimensionality. Especially, we want to avoid the exponential dependence of the computational cost on the dimensionality. Methods providing this property are called tractable, see e.g. [32, 33, 34, 42, 45]. One can distinguish between strong tractability, which refers to convergence being completely independent of the dimensionality and polynomial tractability, which describes only a polynomial dependence on the dimensionality. It can be proven that Quasi-Monte Carlo quadrature rules based on the Halton sequence are polynomial tractable. More precisely, they converge up to a linear factor independently of the problem's dimensionality, if the Karhunen-Loève expansion decays sufficiently fast. This is the main result of this article. The idea of the proof is based on the fact that the Halton sequence avoids the boundary of the integration domain, which has been shown in [35].

The rest of this article is organized as follows. Section 2 specifies the diffusion problem under consideration and the corresponding framework. In particular, the parametric reformulation as a high-dimensional deterministic problem is performed here. In Section 3, we derive the crucial regularity estimates of the solution to the stochastic diffusion problem. In Section 4, we elaborate on the Quasi-Monte Carlo quadrature based on the Halton sequence and prove its polynomial tractability. At the end of this section, we refer also to the uniformly elliptic case. Finally, Section 5 validates the theoretical findings by some basic one-dimensional numerical examples. For more sophisticated examples, we refer to the recent work [20].

In the following, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Problem setting

In the following, let $D \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ be a domain with Lipschitz continuous boundary and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with σ-field $\mathcal{F} \subset 2^\Omega$ and a complete probability measure $\mathbb{P}$, i.e. for all $A \subset B$ and $B \in \mathcal{F}$ with $\mathbb{P}[B] = 0$ it follows $A \in \mathcal{F}$. Let the random function $u(\omega) \in H_0^1(D)$ be the solution to the stochastic diffusion problem

$$-\operatorname{div}\big(a(\omega)\nabla u(\omega)\big) = f \text{ in } D \quad \text{ for almost every } \omega \in \Omega \tag{2}$$

with (deterministic) data $f \in L^2(D)$. Instead of directly approximating the probably infinite dimensional solution u itself, we rather intend to compute the solution's moments:

$$\mathcal{M}^p u := \mathbb{E}[u(\cdot, \mathbf{y})^p].$$

Especially, the solution's expectation is given by

$$\mathbb{E}_u(\mathbf{x}) = \int_{\mathbb{R}^m} u(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y} \in H_0^1(D) \tag{3}$$

and its variance by

$$\mathbb{V}_u(\mathbf{x}) = \mathbb{E}_{u^2}(\mathbf{x}) - \mathbb{E}_u^2(\mathbf{x}) = \int_{\mathbb{R}^m} u^2(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \,\mathrm{d}\mathbf{y} - \mathbb{E}_u^2(\mathbf{x}) \in W_0^{1,1}(D). \tag{4}$$

They correspond to the first and the second (centered) moment of the solution u. As we will show later on, it holds more generally for a sufficiently smooth diffusion coefficient a and $f \in L^p(D)$ that $\mathcal{M}^p u \in W_0^{1,1}(D)$. Note, that the knowledge of all moments of u is sufficient to determine the related distribution.

We consider here the log-normal situation, where the logarithm of the diffusion coefficient is a centered Gaussian field which can be represented by a Karhunen-Loève expansion, cf. [28],

$$\log\big(a(\mathbf{x},\omega)\big) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega). \tag{5}$$

The functions $\{\varphi_k\}_k$ are pairwise orthonormal functions which are assumed to be bounded in $L^\infty(D)$ and $\{\psi_k\}_k$ are independent, standard normally distributed random variables, i.e. $\psi_k(\omega) \sim \mathcal{N}(0,1)$. For the convergence of the series in (5), we assume that the sequence

$$\gamma_k := \big\|\sqrt{\lambda_k}\varphi_k\big\|_{L^\infty(D)} \tag{6}$$

satisfies $\{\gamma_k\}_k \in \ell^1(\mathbb{N})$.

For numerical issues it is reasonable to assume that the Karhunen-Loève expansion is either finite of length m or needs to be appropriately truncated after m terms. We will explicitly make use of this assumption in the following. Nevertheless, we allow $m \to \infty$ as the accuracy requirements increase. The error in case of a truncated Karhunen-Loève expansion has been discussed in [8].

The orthogonality of the sequence $\{\psi_k\}_k$ implies the stochastic independence in the Gaussian case. Therefore, the pushforward measure $\mathbb{P}_{\boldsymbol{\psi}} := \mathbb{P} \circ \boldsymbol{\psi}$ with respect to the measurable mapping

$$\boldsymbol{\psi}\colon \Omega \to \mathbb{R}^m, \quad \omega \mapsto \boldsymbol{\psi}(\omega) := \big(\psi_1(\omega), \ldots, \psi_m(\omega)\big)$$

is given by a joint density function with respect to the Lebesgue measure, i.e.

$$\boldsymbol{\rho}(\mathbf{y}) := \prod_{k=1}^{m} \rho(y_k), \quad \text{where} \quad \rho(y) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right). \tag{7}$$

With this representation at hand, we can reformulate the stochastic problem (2) as a parametric deterministic problem. To that end, we substitute the random variables ψ_k by the coordinates $y_k \in \mathbb{R}$. Then, we define the parameterized and truncated diffusion coefficient via

$$a(\mathbf{x},\mathbf{y}) := \exp\left(\sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(\mathbf{x}) y_k\right) \tag{8}$$

for all $\mathbf{x} \in D$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Thus, we arrive at the parametric diffusion problem:

$$\begin{aligned} &\text{find } u \in L^2_{\boldsymbol{\rho}}\big(\mathbb{R}^m; H^1_0(D)\big) \text{ such that} \\ &\qquad -\operatorname{div}\big(a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\big) = f(\mathbf{x}) \text{ in } D \text{ for all } \mathbf{y} \in \mathbb{R}^m. \end{aligned} \tag{9}$$

Here and in the sequel, for a given Banach space X, the Bochner space $L^p_{\boldsymbol{\rho}}(\mathbb{R}^m; X)$, $1 \leqslant p \leqslant \infty$, consists of all functions $v : \mathbb{R}^m \to X$ whose norm

$$\|v\|_{L^p_{\boldsymbol{\rho}}(\mathbb{R}^m;X)} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^m} \|v(\cdot,\mathbf{y})\|_X^p \boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\mathbf{y}\in\mathbb{R}^m} \|v(\cdot,\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\|_X, & p = \infty \end{cases}$$

is finite. If $p = 2$ and X is a Hilbert space, then the Bochner space is isomorphic to the tensor product space $L^2_{\boldsymbol{\rho}}(\mathbb{R}^m) \otimes X$. Note that, for notational convenience, we will always write $v(\mathbf{x},\mathbf{y})$ instead of $\big(v(\mathbf{y})\big)(\mathbf{x})$ if $v \in L^p_{\boldsymbol{\rho}}(\mathbb{R}^m; X)$.

The stochastic diffusion coefficient $a(\mathbf{x},\mathbf{y})$ is neither uniformly bounded away from zero nor uniformly bounded from above for all $\mathbf{y} \in \mathbb{R}^m$. Consequently, it is impossible to show unique solvability in the classical way for elliptic boundary value problems. Especially, the Lax-Milgram theorem does not directly apply to the problem (2). Nevertheless, we have for each fixed $\mathbf{y} \in \mathbb{R}^m$ the bounds

$$0 < a_{\min}(\mathbf{y}) := \operatorname*{ess\,inf}_{\mathbf{x}\in D} a(\mathbf{x},\mathbf{y}) \leqslant \operatorname*{ess\,sup}_{\mathbf{x}\in D} a(\mathbf{x},\mathbf{y}) =: a_{\max}(\mathbf{y}) < \infty. \tag{10}$$

Obviously, it holds

$$a_{\min}(\mathbf{y}) \geqslant \exp\left(-\sum_{k=1}^{m} |\gamma_k y_k|\right) \quad \text{and} \quad a_{\max}(\mathbf{y}) \leqslant \exp\left(\sum_{k=1}^{m} |\gamma_k y_k|\right).$$

Due to (10), for every fixed $\mathbf{y} \in \mathbb{R}^m$, the problem to find $u \in H_0^1(D)$ such that

$$-\operatorname{div}\big(a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\big) = f(\mathbf{x}) \text{ in } D \tag{11}$$

is elliptic and admits a unique solution $u(\cdot,\mathbf{y}) \in H_0^1(D)$ which satisfies

$$\|u(\cdot,\mathbf{y})\|_{H^1(D)} \lesssim \frac{1}{a_{\min}(\mathbf{y})}\|f\|_{L^2(D)}. \tag{12}$$

We refer the reader to e.g. [39] for a proof of this result.

Remark 2.1. *In the framework of* [39]*, the general case of $m \to \infty$ is considered. Therefore, the set*

$$\Gamma := \left\{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} \gamma_k |y_k| < \infty\right\}$$

is introduced which is shown to be of measure $\mathbb{P}_{\boldsymbol{\psi}}(\Gamma) = 1$. Then, the condition $\{\gamma_k\}_k \in \ell^1(\mathbb{N})$ ensures for all $\mathbf{y} \in \Gamma$ that $\big|\log\big(a(\mathbf{x},\mathbf{y})\big)\big| < \infty$ holds uniformly in $\mathbf{x} \in D$. Obviously, we have $\Gamma = \mathbb{R}^m$ for all $m < \infty$. In the following, we make use of the fact that condition (10) *is always satisfied in finite dimensions. Nevertheless, the case of an infinite dimensional stochastics, i.e. $m = \infty$, can be treated by straightforward modifications of the presented arguments.*

3. Regularity of the solution

The topic we address in this article is the computation of the mean and the higher order moments of the solution of (9) by a fully deterministic quadrature rule. Therfore, in order to establish error bounds for the application of Quasi-Monte Carlo quadrature rules, we consider in this section the regularity of the solution u and its powers, i.e., u^p for $p \in \mathbb{N}$. This issue has already been discussed in [3, 8, 26, 39]. We will compile and augment here some of the results which originate from those articles for our framework.

At first, we shall fix some notation. For a multiindex $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$, the related multidimensional derivative is denoted by

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} := \prod_{k=1}^{m}\left(\frac{\partial}{\partial y_k}\right)^{\alpha_k}.$$

Furthermore, we set $|\boldsymbol{\alpha}| := \sum_{k=1}^m \alpha_k$ and, for a vector $\boldsymbol{\beta} \in \mathbb{R}^m$, we define $\boldsymbol{\beta}^{\boldsymbol{\alpha}} := \prod_{k=1}^m \beta_k^{\alpha_k}$.

Remark 3.1. *The Sobolev space $H_0^1(D)$ is considered to be equipped with the norm*

$$\|\cdot\|_{H^1(D)} := \|\nabla\cdot\|_{L^2(D)}.$$

Likewise, we use corresponding norms for the Sobolev spaces $W_0^{1,p}(D)$, i.e.,

$$\|\cdot\|_{W^{1,p}(D)} := \|\nabla\cdot\|_{[L^p(D)]^d}.$$

Since we only consider homogenous Dirichlet problems, by Sobolev's norm equivalence theorem, cf. [2]*, they all induce equivalent norms for these spaces. Of course, all results are straightforwardly extendable to the case of non-homogenous Dirichlet problems.*

The differentiability of u follows straightforwardly from the differentiability of the diffusion coefficient a, cf. [26]. In particular, we shall use the following lemma from [26] which is adjusted for our purposes.

Lemma 3.2. *Set $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \ldots, \gamma_m)$. Then, for the solution u to* (9) *and every $\mathbf{y} \in \mathbb{R}^m$, the following estimate holds*

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot,\mathbf{y})\big\|_{H^1(D)} \leqslant |\boldsymbol{\alpha}|!\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \sqrt{\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}}\|u(\cdot,\mathbf{y})\|_{H^1(D)}.$$

This result shows the regularity of the solution u. For the regularity of u^2, we have then the following proposition.

Proposition 3.3. *The derivatives of u^2, where u is the solution of (9), satisfy*

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u^2(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \lesssim (|\boldsymbol{\alpha}|+1)!\bigg(\frac{\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|_{H^1(D)}^2. \tag{13}$$

Proof. By the Leibniz rule we obtain

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u^2(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \leqslant \sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)}. \tag{14}$$

Each summand can be estimated as follows

$$\begin{aligned}
&\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)}\\
&\quad= \big\|\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y}) + \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{L^1(D)}\\
&\quad\leqslant \big\|\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{L^2(D)} + \big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{L^2(D)}\big\|\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{L^2(D)}\\
&\quad\lesssim \big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{H^1(D)}\big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{H^1(D)} + \big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{H^1(D)}\big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{H^1(D)}.
\end{aligned}$$

Application of Lemma 3.2 yields

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u(\cdot,\mathbf{y})\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \lesssim |\boldsymbol{\beta}|!\,|(\boldsymbol{\alpha}-\boldsymbol{\beta})|!\bigg(\frac{\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|_{H^1(D)}^2.$$

By inserting this inequality into (14), we conclude

$$\begin{aligned}
\big\|\partial_{y_k}^{j} u^2(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} &\lesssim \sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\boldsymbol{\beta}|!\,|(\boldsymbol{\alpha}-\boldsymbol{\beta})|!\bigg(\frac{\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|_{H^1(D)}^2\\
&= \bigg(\frac{\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|_{H^1(D)}^2 \sum_{k=0}^{|\boldsymbol{\alpha}|}(|\boldsymbol{\alpha}|-k)!\,k!\sum_{|\boldsymbol{\beta}|=k}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}.
\end{aligned}$$

In view of

$$\sum_{k=0}^{|\boldsymbol{\alpha}|}(|\boldsymbol{\alpha}|-k)!\,k!\sum_{|\boldsymbol{\beta}|=k}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \sum_{k=0}^{|\boldsymbol{\alpha}|}(|\boldsymbol{\alpha}|-k)!\,k!\binom{|\boldsymbol{\alpha}|}{k} = \sum_{k=0}^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}|! = (|\boldsymbol{\alpha}|+1)!,$$

we finally arrive at the assertion (13). □

For higher order moments, we need some stronger regularity assumptions on the load f.

Proposition 3.4. *Let D be a domain with sufficiently smooth boundary and let $p > 2$. If the load f satisfies $f \in L^p(D)$, then the solution u to (11) is contained in $W_0^{1,p}(D)$ and meets the regularity estimate*

$$\big\|u(\cdot,\mathbf{y})\big\|_{W^{1,p}(D)} \lesssim \frac{1}{a_{\min}(\mathbf{y})}\|f\|_{L^p(D)}. \tag{15}$$

The derivatives of u with respect to the parametric variable $\mathbf{y}$ can be estimated by

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot,\mathbf{y})\big\|_{W^{1,p}(D)} \lesssim |\boldsymbol{\alpha}|!\bigg(\frac{\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\|u(\cdot,\mathbf{y})\|_{W^{1,p}(D)}. \tag{16}$$

Additionally, the derivatives of the powers u^p with respect to the parametric variable $\mathbf{y}$ fulfill

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u^p(\cdot,\mathbf{y})\big\|_{W^{1,1}(D)} \lesssim p|\boldsymbol{\alpha}|!\bigg(\frac{p\boldsymbol{\gamma}}{\log 2}\bigg)^{\boldsymbol{\alpha}} \bigg(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\bigg)^{p}\|u(\cdot,\mathbf{y})\|_{W^{1,p}(D)}^p. \tag{17}$$

Proof. At first, we notice that the bilinear form

$$(u,v)_{H_0^1(D)} := (\nabla v,\nabla u)_{L^2(D)}$$

defines an inner product on the Hilbert space $H_0^1(D)$. Let $1 < p, q < \infty$ be dual exponents, i.e. $1/p + 1/q = 1$. It is proven in [40] that for each function $u \in W_0^{1,p}(D)$ the estimate

$$\|u\|_{W^{1,p}} \leqslant C(p,D) \sup_{0\neq v\in W_0^{1,q}(D)} \frac{(u,v)_{H_0^1(D)}}{\|v\|_{W^{1,q}(D)}}$$

is valid. From this, we derive

$$\|u(\cdot,\mathbf{y})\|_{W^{1,p}(D)} \lesssim \sup_{0\neq v\in W_0^{1,q}(D)} \frac{\big(u(\cdot,\mathbf{y}),v\big)_{H_0^1(D)}}{\|v\|_{W^{1,q}(D)}} \leqslant \frac{1}{a_{\min}(\mathbf{y})} \sup_{0\neq v\in W_0^{1,q}(D)} \frac{\mathcal{B}_{\mathbf{y}}(u,v)}{\|v\|_{W^{1,q}(D)}}.$$

Here, we set

$$\mathcal{B}_{\mathbf{y}}(u,v) := \int_D a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{18}$$

which is the bilinear form related to the variational formulation of (11) for every fixed value of the parameter $\mathbf{y}$. In full, this variational formulation reads

$$\mathcal{B}_{\mathbf{y}}(u,v) = f(v) := \int_D f(x)v(x)\,\mathrm{d}\mathbf{x} \quad \text{for all } v\in H_0^1(D). \tag{19}$$

Regard that $H_0^1(D)\subset W_0^{1,q}(D)$. Since $f\in L^p(D)$, it is easy to verify by a density argument that equation (19) is still valid for $v\in W_0^{1,q}(D)$. Therefore, we have

$$\sup_{0\neq v\in W_0^{1,q}(D)} \frac{\mathcal{B}_{\mathbf{y}}(u,v)}{\|v\|_{W^{1,q}(D)}} = \sup_{0\neq v\in W_0^{1,q}(D)} \frac{f(v)}{\|v\|_{W^{1,q}(D)}} \leqslant \|f\|_{L^p(D)},$$

which follows from the Hölder inequality and the estimate $\|v\|_{L^q(D)} \lesssim \|v\|_{W^{1,q}}$. This establishes the inequality (15).

The second assertion follows in complete analogy to the case $p=2$ that is proven in [26]. We sketch here the essential ideas of the proof which is based on induction. Concretely, we show

$$\big\|a(\cdot,\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u(\cdot,\mathbf{y})\big\|_{[L^p(D)]^d} \lesssim |\boldsymbol{\alpha}|!\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \big\|a(\cdot,\mathbf{y})\nabla u(\cdot,\mathbf{y})\big\|_{[L^p(D)]^d} \tag{20}$$

The case $|\boldsymbol{\alpha}|=0$ is trivial. For $|\boldsymbol{\alpha}|=k>0$ we have

$$\big\|a(\cdot,\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u(\cdot,\mathbf{y})\big\|_{[L^p(D)]^d} \lesssim \sup_{0\neq v\in W_0^{1,q}(D)} \frac{\mathcal{B}_{\mathbf{y}}\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u(\cdot,\mathbf{y}),v\big)}{\|v\|_{W^{1,q}(D)}}.$$

This follows from the fact that $W_0^{1,q}(D)$ is densely embedded into $[L^q(D)]^d$ by the mapping $v\mapsto\nabla v$, cf. [40]. Now, differentiation of the bilinear form (18) with respect to $\mathbf{y}$ yields

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\mathcal{B}_{\mathbf{y}}\big(u(\cdot,\mathbf{y}),v\big) = \mathcal{B}_{\mathbf{y}}\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u(\cdot,\mathbf{y}),v\big) + \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}}a(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}u(\mathbf{x},\mathbf{y})\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Therefore, from the differentiation of the variational formulation (19), we obtain

$$\begin{aligned}\mathcal{B}_{\mathbf{y}}\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u(\cdot,\mathbf{y}),v\big) &\leqslant \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \left\|\frac{\partial_{\mathbf{y}}^{\boldsymbol{\beta}}a(\cdot,\mathbf{y})}{a(\cdot,\mathbf{y})}\right\|_{L^\infty(D)} \int_D \big|a(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}u(\mathbf{x},\mathbf{y})\nabla v(\mathbf{x})\big|\,\mathrm{d}\mathbf{x}\\ &\leqslant \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \boldsymbol{\gamma}^{\boldsymbol{\beta}}\big\|a(\cdot,\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}u(\cdot,\mathbf{y})\big\|_{L^p(D)^d}\|v\|_{W^{1,q}(D)}.\end{aligned}$$

The inequality (20) follows then by inserting the induction hypothesis and some combinatorial estimates as in [26].

Finally, to establish estimate (17), we apply Faà di Bruno's formula, cf. [11]. For $n:=|\boldsymbol{\alpha}|$, this formula provides

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u^p(\mathbf{y}) = \sum_{r=1}^n p(p-1)\cdots(p-r+1)u^{p-r}(\mathbf{y}) \sum_{P(\boldsymbol{\alpha},r)} \boldsymbol{\alpha}! \prod_{j=1}^n \frac{\big(\partial_{\mathbf{y}}^{\boldsymbol{\beta}_j}u(\mathbf{y})\big)^{k_j}}{k_j!\boldsymbol{\beta}_j!}, \tag{21}$$

Here, the set $P(\boldsymbol{\alpha},r)$ contains restricted integer partitions of a multiindex $\boldsymbol{\alpha}$ into r non-vanishing multiindices. For a definition of $P(\boldsymbol{\alpha},r)$ see Appendix A.

Equation (21) together with the generalized Hölder inequality yields

$$\begin{aligned}
&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u^p(\mathbf{y})\|_{W^{1,1}(D)} \\
&\quad\leqslant \sum_{r=1}^{n} p(p-1)\cdots(p-r+1) \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j!\boldsymbol{\beta}_j!} \left\| u^{p-r}(\mathbf{y}) \prod_{j=1}^{n} \left(\partial_{\mathbf{y}}^{\boldsymbol{\beta}^j} u(\mathbf{y})\right)^{k_j} \right\|_{W^{1,1}(D)} \\
&\quad\lesssim p\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \left(\frac{a_{\max}(\mathbf{y})}{a_{\min}(\mathbf{y})}\right)^{p/2} \|u\|_{W^{1,p}(D)}^{p} \sum_{r=1}^{n} \frac{p!}{(p-r)!} \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j!\boldsymbol{\beta}_j!} \prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j}.
\end{aligned}$$

From [11] we know that

$$\sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j!\boldsymbol{\beta}_j!} = S_{n,r},$$

where $S_{n,r}$ are the Stirling numbers of the second kind, cf. [1]. Moreover, since $\prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j} \leqslant |\boldsymbol{\alpha}|!$, we can further estimate

$$\sum_{r=1}^{n} \frac{p!}{(p-r)!} \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j!\boldsymbol{\beta}_j!} \prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j} \leqslant |\boldsymbol{\alpha}|! \sum_{r=1}^{n} \frac{p!}{(p-r)!} S_{n,r} = |\boldsymbol{\alpha}|! p^n.$$

The last inequality follows by the theory of generating functions for the Stirling numbers of the second kind, see e.g. [1]. This completes the proof. □

4. Quasi-Monte Carlo Quadrature for the Stochastic Variable

In this section, we discuss the use of Quasi-Monte Carlo quadrature rules for the integral

$$\mathbf{I}v = \int_{(0,1)^m} v(\mathbf{z})\,\mathrm{d}\mathbf{z}.$$

These quadrature rules are classically of the form

$$\mathbf{Q}v = \frac{1}{N} \sum_{i=1}^{N} v(\cdot, \boldsymbol{\xi}_i),$$

where N denotes the number of samples and $\boldsymbol{\xi}_i \in [0,1]^m$ is one sample point. For the error estimation of the Quasi-Monte Carlo method, it is required that the integrand has integrable, mixed first order derivatives. Then, the error of the standard Quasi-Monte Carlo method over the unit cube $[0,1]^m$ is bounded by means of the L^∞*-star discrepancy*

$$\mathcal{D}_\infty^\star(\Xi) := \sup_{\mathbf{t}\in[0,1]^m} \left| \operatorname{Vol}\left([\mathbf{0},\mathbf{t})\right) - \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{[\mathbf{0},\mathbf{t})}(\boldsymbol{\xi}_i) \right|$$

of the set of sample points $\Xi = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset [0,1]^m$, where $\operatorname{Vol}\left([\mathbf{0},\mathbf{t})\right)$ denotes the Lebesgue measure of the cuboid $[\mathbf{0},\mathbf{t})$, cf. [30]. More precisely, it holds the famous *Koksma-Hlawka-inequality*

$$\|(\mathbf{I}-\mathbf{Q})v\|_X \lesssim \mathcal{D}_\infty^\star(\Xi) \|v\|_{W_{\mathrm{mix}}^{1,1}([0,1]^m;X)},$$

where the Bochner space $W_{\mathrm{mix}}^{1,1}\left([0,1]^m;X\right)$ consists of all functions $v : [0,1]^m \to X$ with finite norm

$$\|v\|_{W_{\mathrm{mix}}^{1,1}([0,1]^m;X)} := \sum_{\|\boldsymbol{\alpha}\|_\infty \leqslant 1} \int_{[0,1]^m} \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\right\|_X \mathrm{d}\mathbf{y} < \infty. \tag{22}$$

In case of certain, so-called *low discrepancy*, point sequences, e.g. the Halton sequence, this discrepancy can typically be estimated to be of the order $\mathcal{O}\left(N^{-1}(\log N)^m\right)$, see e.g. [19, 30].

Remark 4.1. *The estimation of the discrepancy of a set* $\Xi \subset [0,1]^m$*, especially for high dimensions* m*, has been the topic of many publications in the past fifteen years. The aim is to avoid the factor* $(\log N)^m$ *in the estimation of the discrepancy which grows exponential in the dimension* m*. This exponential dependence is called* intractability *in literature, cf.* [32, 42].

In the following, we assume that the sequence of integration points is given by the *Halton sequence*.

Definition 4.2. *Let $b_1, \dots, b_m$ denotes the first m prime numbers. The m-dimensional* Halton sequence *is given by*

$$\boldsymbol{\xi}_i = [h_{b_1}(i), \dots, h_{b_m}(i)]^\intercal, \quad i = 0, 1, 2, \dots,$$

where $h_{b_j}(i)$ denotes the i-th element of the van der Corput sequence *according to b_j. That is, if $i = \cdots c_3 c_2 c_1$ in radix b_j, then $h_{b_j}(i) = 0.c_1 c_2 c_3 \cdots$ in radix b_j.*

We show that the Quasi-Monte Carlo quadrature based on this sequence is polynomial tractable for the determination of the moments of the solution u to (11) under certain decay properties of the sequence $\{\gamma_k\}_k$. The proof is essentially based on the ideas in [35].

To obtain a Quasi-Monte Carlo method for the integration domain $\mathbb{R}^m$, the sample points have to be mapped to $\mathbb{R}^m$ by the inverse distribution function. Therefore, we define the cumulative normal distribution

$$\Phi\colon \mathbb{R} \to (0,1), \quad \text{with} \quad \Phi(y) := \int_{-\infty}^{y} \rho(y')\,\mathrm{d}y'$$

and its inverse

$$\Phi^{-1}\colon (0,1) \to \mathbb{R}.$$

Then, for a function $f \in L^1_\rho(\mathbb{R})$, it is well known that

$$\int_{\mathbb{R}} f(y)\rho(y)\,\mathrm{d}y = \int_0^1 f\big(\Phi^{-1}(z)\big)\,\mathrm{d}z$$

with the substitution $z = \Phi(y)$. Especially, we have $f \circ \Phi^{-1} \in L^1\big((0,1)\big)$. By defining $\boldsymbol{\Phi}(\mathbf{y}) := [\Phi(y_1), \dots, \Phi(y_m)]^\intercal$, we may extend the above integral transform to the multivariate case, i.e. $f \in L^1_\rho(\mathbb{R}^m)$ and

$$\int_{\mathbb{R}^m} f(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{(0,1)^m} f\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\,\mathrm{d}\mathbf{z}.$$

Although we have $f \circ \boldsymbol{\Phi}^{-1} \in L^1\big((0,1)^m\big)$, the integrand might be unbounded in a neighbourhood of the hypercube's boundary in our application, cf. (12). Thus, the corresponding $W^{1,1}_{\text{mix}}$-norm might be unbounded, too. As a consequence, the Koksma-Hlawka inequality is not applicable. The idea of [35] is now to consider subsets K_N such that the first N points $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N$ of the Halton sequence are included in K_N. Due to the definition of the Halton sequence this holds for

$$K_N := \bigtimes_{k=1}^{m} [(b_k N)^{-1}, 1 - (b_k N)^{-1}].$$

Obviously, for the solution to (9), it holds for almost every $\mathbf{x} \in D$

$$\operatorname*{ess\,sup}_{\mathbf{z} \in K_N} u\big(\mathbf{x}, \boldsymbol{\Phi}^{-1}(\mathbf{z})\big) < \infty \quad \text{for all } n \in \mathbb{N}.$$

Let now $\hat{u}(\mathbf{x}, \mathbf{z}) := u\big(\mathbf{x}, \boldsymbol{\Phi}^{-1}(\mathbf{z})\big)$. For $\mathbf{z} \in (0,1)^m \backslash K_N$, we replace $\hat{u}$ by its low-variation extension $\hat{u}_{\text{ext}}$, cf. [35], i.e.

$$\hat{u}_{\text{ext}}(\mathbf{z}) := \hat{u}(\mathbf{z}_0) + \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[(\mathbf{z}_0)_{\boldsymbol{\alpha}}, (\mathbf{z})_{\boldsymbol{\alpha}}]} \mathbb{1}_{\mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{z}_0 \in K_N} \partial^{\boldsymbol{\alpha}} \hat{u}(\mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{z}_0)\,\mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}}. \tag{23}$$

For this formula we introduced some notation. For a vector $\mathbf{z} \in \mathbb{R}^m$, we mean by $(\mathbf{z})_{\boldsymbol{\alpha}} \in \mathbb{R}^{|\boldsymbol{\alpha}|}$ the vector obtained by omitting every entry z_i for which $\alpha_i = 0$ holds. This defines a projection $\Pi_{\boldsymbol{\alpha}} : \mathbb{R}^m \to \mathbb{R}^{|\boldsymbol{\alpha}|}$. For two vectors $\mathbf{y}, \mathbf{z} \in \mathbb{R}^m$, we define $\mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{z} := \big(y_i^{\alpha_i} z_i^{(1-\alpha_i)}\big)_{i=1}^m$. Note that any element $\mathbf{y} \in \Pi_{\boldsymbol{\alpha}}^{-1}\big((\mathbf{y})_{\boldsymbol{\alpha}}\big)$ in the preimage of $(\mathbf{y})_{\boldsymbol{\alpha}}$ is admissible for the occuring $\mathbf{y} \in \mathbb{R}^m$ in (23). Moreover, for a given anchor point $\mathbf{z}_0 \in K_N$, the extension coincides by definition (23) with the function $\hat{u}$ on K_N, i.e. $\hat{u}_{\text{ext}}(\mathbf{z}) = \hat{u}(\mathbf{z})$ for all $\mathbf{z} \in K_N$.

Theorem 4.3. *The Quasi-Monte Carlo quadrature using Halton points for approximating the expectation of the solution u to (11) is polynomial tractable if the sequence $\{\gamma_k\}_k$ satisfies the decay property $\gamma_k \lesssim k^{-4-2\varepsilon}$ for some arbitrary $\varepsilon > 0$. More precisely, there exists for each $\delta > 0$ a sequence $\{\delta_k\}_{k>0} \in \ell^1(\mathbb{N})$ with $\delta_k \sim k^{-1-\varepsilon}$ and a $\tilde{\delta} > 0$ with $\tilde{\delta} + \sum_{k=1}^{\infty} \delta_k < \delta$ such that the error of the Quasi-Monte Carlo quadrature with N Halton points satisfies*

$$\begin{aligned}\big\|(\mathbf{I}-\mathbf{Q})\hat{u}\big\|_{H^1(D)} &\lesssim \|f\|_{L^2(D)}\big(mN^{\max_{k=1,\ldots,m}\delta_k-1} + N^{-1+\tilde{\delta}+\sum_{k=1}^m \delta_k}\big)\\ &\leqslant \|f\|_{L^2(D)}(m+1)N^{-1+\delta}.\end{aligned} \tag{24}$$

The constant hidden in the above inequality depends on the sequence $\{\delta_k\}_{k>0}$, on $\tilde{\delta}$ and on δ.

The proof of this theorem is performed by splitting up the error of integration into three parts. Namely, with respect to the extension $\hat{u}_{\text{ext}}$, we write

$$\big\|(\mathbf{I}-\mathbf{Q})\hat{u}\big\|_{H^1(D)} \leqslant \big\|\mathbf{I}(\hat{u}-\hat{u}_{\text{ext}})\big\|_{H^1(D)} + \big\|\mathbf{Q}(\hat{u}-\hat{u}_{\text{ext}})\big\|_{H^1(D)} + \big\|(\mathbf{I}-\mathbf{Q})\hat{u}_{\text{ext}}\big\|_{H^1(D)}. \tag{25}$$

In this inequality, the second term on the right-hand-side vanishes since $\hat{u}\big|_{K_N} = \hat{u}_{\text{ext}}\big|_{K_N}$. The first term of the right-hand-side of (25) is estimated by Lemma 4.4 and the third term on the right-hand-side of (25) is estimated in Lemma 4.6.

Lemma 4.4. *Let the conditions of Theorem 4.3 hold and let $\hat{u}_{ext}$ be defined according to (23). Then, it holds*

$$\big\|\mathbf{I}(\hat{u}-\hat{u}_{ext})\big\|_{H^1(D)} \lesssim \|f\|_{L^2(D)} N^{\max_j \delta_j - 1} m. \tag{26}$$

Proof. We organize the proof in four steps.
(*i.*) On the one hand, from [12], we know that

$$\Phi^{-1}(z) < \sqrt{-\log\big(2\pi(1-z)^2(1-\log(2\pi(1-z)^2))\big)} \quad \text{for all } z \in [0.9, 1].$$

Furthermore, we have from [36] that

$$\Phi^{-1}(z) \leqslant \sqrt{-2\log(1-z)} - \frac{2.30753 + 0.27061\sqrt{-2\log(1-z)}}{1 + 0.99229\sqrt{-2\log(1-z)} - 0.08962\log(1-z)} + 0.003$$

for all $z \in [0.5, 1]$. These inequalities imply

$$\Phi^{-1}(z) \leqslant \sqrt{-2\log(1-z)} \quad \text{for all } z \in [0.5, 1].$$

Due to the symmetry of the distribution, this shows that

$$|\Phi^{-1}(z)| \leqslant \sqrt{-2\log(\min\{z, 1-z\})} \quad \text{for all } z \in [0, 1].$$

Therefore, we derive

$$\partial_z \Phi^{-1}(z) = \sqrt{2\pi}\exp\left(\frac{\Phi^{-1}(z)^2}{2}\right) \leqslant \sqrt{2\pi}\min\{z, 1-z\}^{-1},$$

which implies the estimate

$$\left|\prod_{k=1}^m \big(\partial_{z_k}\Phi^{-1}(z_k)\big)^{\alpha_k}\right| \leqslant \prod_{k=1}^m \big(\sqrt{2\pi}\min\{z_k, 1-z_k\}^{-1}\big)^{\alpha_k}$$

for all non-negative integers $\alpha_k \geqslant 0$.
(*ii.*) On the other hand, one verifies

$$\exp\big(\gamma_k|\Phi^{-1}(z)|\big) \leqslant C(\delta_k, \gamma_k)\min\{z, 1-z\}^{-\delta_k} \quad \text{for all } \delta_k > 0,$$

with the constant

$$C(\delta_k, \gamma_k) = \begin{cases} \exp\left(\frac{\gamma_k^2}{2\delta_k}\right), & \text{if } \delta_k \leqslant \frac{\gamma_k}{\sqrt{2\log 2}},\\ \frac{\exp(\sqrt{2\log 2}\gamma_k)}{\exp(\delta_k \log 2)}, & \text{else.}\end{cases}$$

Hence, we find

$$\sqrt{\frac{a_{\max}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)}{a_{\min}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)^3}} \leqslant \exp\left(\sum_{k=1}^{m} 2\gamma_k\big|\Phi^{-1}(z_k)\big|\right) \leqslant \prod_{k=1}^{m}\big(C(\delta_k,2\gamma_k)\min\{z_k,1-z_k\}^{-\delta_k}\big).$$

Consequently, with Lemma 3.2 and the stability estimate (12), for any multiindex $\boldsymbol{\alpha}$, we deduce

$$\begin{aligned}
&\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\big\|_{H^1(D)} \\
&\quad\leqslant |\boldsymbol{\alpha}|!\left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}}\sqrt{\frac{a_{\max}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)}{a_{\min}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)}}\big\|u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\big\|_{H^1(D)} \\
&\quad\leqslant |\boldsymbol{\alpha}|!\left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}}\sqrt{\frac{a_{\max}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)}{a_{\min}\big(\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)^3}}\|f\|_{L^2(D)} \\
&\quad\leqslant \|f\|_{L^2(D)}|\boldsymbol{\alpha}|!\left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}}\prod_{k=1}^{m}\Big(C(\delta_k,2\gamma_k)\min\{z_k,1-z_k\}^{-\delta_k}\Big).
\end{aligned}$$

(*iii.*) For an arbitrary multiindex $\boldsymbol{\alpha}$, it holds for all $\mathbf{z}\in(0,1)^m$ that

$$\begin{aligned}
\big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\hat{u}(\cdot,\mathbf{z})\big\|_{H^1(D)} &= \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\big\|_{H^1(D)} \\
&= \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\prod_{k=1}^{m}\big(\partial_{z_k}\Phi^{-1}(z_k)\big)^{\alpha_k}\right\|_{H^1(D)} \\
&= \left|\prod_{k=1}^{m}\big(\partial_{z_k}\Phi^{-1}(z_k)\big)^{\alpha_k}\right|\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z})\big)\big\|_{H^1(D)}.
\end{aligned} \tag{27}$$

From now on, we choose the anchor point $\mathbf{z}_0 = (1/2,1/2,\ldots,1/2)$ and define

$$\tilde{C} := \frac{\sqrt{2\pi}\max_{k\in\mathbb{N}} C(\delta_k,2\gamma_k)}{\log 2}. \tag{28}$$

Note that $\tilde{C}<\infty$ since there is an $n_0\in\mathbb{N}$ such that $C(\delta_k,2\gamma_k)\leqslant 1$ for all $n\geqslant n_0$ under the decay assumptions on the sequences $\{\delta_k\}_k$ and $\{\gamma_k\}_k$. Due to $\Phi^{-1}(1/2)=0$, we easily get from item (*ii.*) that

$$\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u\big(\cdot,\boldsymbol{\Phi}^{-1}(\mathbf{z}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0)\big)\big\|_{H^1(D)} \leqslant \|f\|_{L^2(D)}|\boldsymbol{\alpha}|!\left(\frac{\gamma}{\log 2}\right)^{\boldsymbol{\alpha}}\prod_{k=1}^{m}\Big(C(\delta_k,2\gamma_k)\min\{z_k,1-z_k\}^{-\delta_k}\Big)^{\alpha_k} \tag{29}$$

holds for all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\|_\infty = 1$. Thus, by combining (27) with items (*i.*) and inequality (29), we arrive at the estimate

$$\big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\hat{u}(\cdot,\mathbf{z}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0)\big\|_{H^1(D)} \leqslant |\boldsymbol{\alpha}|!\|f\|_{L^2(D)}\prod_{k=1}^{m}\Big(\gamma_k\tilde{C}\min\{z_k,1-z_k\}^{-1-\delta_k}\Big)^{\alpha_k}. \tag{30}$$

With (23), we obtain the identity

$$\hat{u}(\mathbf{z})-\hat{u}_{\mathrm{ext}}(\mathbf{z}) = \sum_{\|\boldsymbol{\alpha}\|_\infty=1}\int_{[(\mathbf{z}_0)_{\boldsymbol{\alpha}},(\mathbf{z})_{\boldsymbol{\alpha}}]}\mathbb{1}_{\mathbf{y}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0\notin K_N}\,\partial^{\boldsymbol{\alpha}}\hat{u}(\mathbf{y}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0)\,\mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}}.$$

This, together with the estimate (30) on the derivates of $\hat{u}$ yields for $\mathbf{z} \notin K_N$, cf. [35],

$$\begin{aligned}
&\|\hat{u}(\cdot,\mathbf{z}) - \hat{u}_{\text{ext}}(\cdot,\mathbf{z})\|_{H^1(D)} \\
&\quad\leqslant \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \int_{[(\mathbf{z}_0)_{\boldsymbol{\alpha}},(\mathbf{z})_{\boldsymbol{\alpha}}]} \mathbb{1}_{\mathbf{y}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0\notin K_N} \big\|\partial^{\boldsymbol{\alpha}}\hat{u}\big(\mathbf{y}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0\big)\big\|_{H^1(D)} \mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}} \\
&\quad\leqslant \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty=1} |\boldsymbol{\alpha}|! \prod_{k=1}^m (\gamma_k\tilde{C})^{\alpha_k} \int_{[(\mathbf{z}_0)_{\boldsymbol{\alpha}},(\mathbf{z})_{\boldsymbol{\alpha}}]} \mathbb{1}_{\mathbf{y}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0\notin K_N} \prod_{k=1}^m \Big(\min\{y_k,1-y_k\}^{-1-\delta_k}\Big)^{\alpha_k} \mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}} \\
&\quad\leqslant \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty=1} |\boldsymbol{\alpha}|! \prod_{k=1}^m \bigg(\gamma_k\tilde{C}\int_{\min\{z_k,1-z_k\}}^{1/2} y_k^{-1-\delta_k}\,\mathrm{d}y_k\bigg)^{\alpha_k}.
\end{aligned}$$

Herein, the integral can simply be bounded via its lower limit according to

$$\begin{aligned}
\|\hat{u}(\cdot,\mathbf{z}) - \hat{u}_{\text{ext}}(\cdot,\mathbf{z})\|_{H^1(D)} &\leqslant \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty=1} |\boldsymbol{\alpha}|! \prod_{k=1}^m \big(\gamma_k\tilde{C}\min\{z_k,1-z_k\}^{-\delta_k}\big)^{\alpha_k} \\
&\leqslant \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \prod_{k=1}^m \big(k\gamma_k\tilde{C}\min\{z_k,1-z_k\}^{-\delta_k}\big)^{\alpha_k} \\
&= \|f\|_{L^2(D)} \Bigg(\prod_{k=1}^m \bigg(1 + \frac{\min\{z_k,1-z_k\}^{-\delta_k}k\gamma_k\tilde{C}}{\delta_k}\bigg) - 1\Bigg) \\
&\leqslant \|f\|_{L^2(D)} \prod_{k=1}^m \bigg(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\bigg) \min\{z_k,1-z_k\}^{-\delta_k}.
\end{aligned}$$

Now, due to Bochner's inequality, it follows

$$\begin{aligned}
&\big\|\mathbf{I}(\hat{u} - \hat{u}_{\text{ext}})\big\|_{H^1(D)} \\
&\quad\leqslant \int_{(0,1)^m} \big\|\hat{u}(\mathbf{x},\mathbf{z}) - \hat{u}_{\text{ext}}(\mathbf{x},\mathbf{z})\big\|_{H^1(D)}\,\mathrm{d}\mathbf{z} = \int_{(0,1)^m\setminus K_N} \big\|\hat{u}(\mathbf{x},\mathbf{z}) - \hat{u}_{\text{ext}}(\mathbf{x},\mathbf{z})\big\|_{H^1(D)}\,\mathrm{d}\mathbf{z} \\
&\quad\lesssim \|f\|_{L^2(D)} \int_{(0,1)^m\setminus K^N} \prod_{k=1}^m \min\{z_k,1-z_k\}^{-\delta_k}\,\mathrm{d}\mathbf{z} \prod_{k=1}^m \bigg(1+\frac{k\gamma_k\tilde{C}}{\delta_k}\bigg) \\
&\quad\leqslant \|f\|_{L^2(D)} 2^m \sum_{j=1}^m \int_0^{(b_jN)^{-1}} z_j^{-\delta_j}\,\mathrm{d}z_j \prod_{i=1,i\neq j}^m \int_0^{1/2} z_i^{-\delta_i}\,\mathrm{d}z_i \prod_{k=1}^m \bigg(1+\frac{k\gamma_k\tilde{C}}{\delta_k}\bigg) \\
&\quad\leqslant \|f\|_{L^2(D)} 2^m \sum_{j=1}^m (b_jN)^{\delta_j-1} 2^{-m+1} 2^{\sum_{i=1}^m \delta_i} \prod_{k=1}^m \bigg[\bigg(1+\frac{k\gamma_k\tilde{C}}{\delta_k}\bigg)\bigg(\frac{1}{1-\delta_k}\bigg)\bigg] \\
&\quad\lesssim \|f\|_{L^2(D)} N^{\max_j \delta_j - 1} m \prod_{k=1}^m \bigg[\bigg(1+\frac{k\gamma_k\tilde{C}}{\delta_k}\bigg)\bigg(\frac{1}{1-\delta_k}\bigg)2^{\delta_k}\bigg].
\end{aligned}$$

(*iv.*) To prove tractability means to find an error bound which grows at most polynomially in m. Therefore, it is now sufficient to show

$$\prod_{k=1}^\infty \bigg[\bigg(1+\frac{k\gamma_k\tilde{C}}{\delta_k}\bigg)\bigg(\frac{1}{1-\delta_k}\bigg)2^{\delta_k}\bigg] < \infty. \tag{31}$$

Since we may choose $\delta_k > 0$ arbitrarily, we can assume that the sequence $\{\delta_k\}_k$ satisfies the conditions of Theorem 4.3. Then, it holds

$$\prod_{k=1}^\infty 2^{\delta_k} = 2^{\sum_{k=1}^\infty \delta_k} \leqslant 2^\delta \quad \text{and} \quad \prod_{k=1}^\infty \frac{1}{1-\delta_k} = \exp\bigg(-\sum_{k=1}^\infty \log(1-\delta_k)\bigg). \tag{32}$$

We make use of the fact that the Taylor expansion of the logarithm $\log(x)$ at $x = 1$ is given by

$$\log(1-h) = -\sum_{k=1}^{\infty} \frac{h^k}{k} = -h - \mathcal{O}(h^2), \quad h > 0.$$

By inserting this into the equation on the right of (32), we obtain

$$\prod_{k=1}^{\infty} \frac{1}{1-\delta_k} \leqslant \exp\left(\sum_{k=1}^{\infty} \big(\delta_k + \mathcal{O}(\delta_k^2)\big)\right) \lesssim \exp(\delta + c\delta^2)$$

for some $c > 0$. Since the sequence $\{\gamma_k\}_k$ decays asymptotically faster than $k^{-4-2\varepsilon}$, we conclude that

$$\prod_{k=1}^{\infty} \left(1 + \frac{\tilde{C} k \gamma_k}{\delta_k}\right) \lesssim \prod_{k=1}^{\infty} \big(1 + k^{-2-\varepsilon}\big) < \infty$$

is also bounded independently of m. This establishes estimate (31) and, thus, finally the assertion (26). $\square$

Finally, we bound the third term in (25). In [24], the *centered discrepancy* is introduced to establish an estimate for the error of integration. In the sequel, we will also make use of the *extreme discrepancy.*

Definition 4.5. *The* pointwise centered discrepancy *is defined for a given set of N sample points $\Xi \subset [0,1]^m$ and a point $\mathbf{z} \in [0,1]^m$ by, cf.* [25],

$$\mathcal{D}^c(\mathbf{z}, \Xi) := \prod_{k=1}^{m} \big(-z_k + \mathbb{1}_{\{z_k > 1/2\}}\big) - \frac{1}{N} \sum_{\xi \in \Xi} \prod_{k=1}^{m} \big(\mathbb{1}_{\{z_k > 1/2\}} - \mathbb{1}_{\{z_k > \xi_k\}}\big).$$

Then, the centered discrepancy *is given by*

$$\mathcal{D}^c(\Xi) := \sup_{\mathbf{z} \in [0,1]^m} \mathcal{D}^c(\mathbf{z}, \Xi).$$

Furthermore, the extreme discrepancy *is defined by*

$$\mathcal{D}_{\text{extr}}(\Xi) = \sup_{\mathbf{x}, \mathbf{y} \in \mathbb{R}^m} \left| \operatorname{Vol}\big([\mathbf{x}, \mathbf{y})\big) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[\mathbf{x}, \mathbf{y})}(\xi_i) \right|.$$

Obviously, the centered discrepancy can be bounded by the extreme discrepancy. In the following, it is convenient to introduce the projection $(\Xi)_{\boldsymbol{\alpha}}$ of Ξ as $(\Xi)_{\boldsymbol{\alpha}} := \{(\xi)_{\boldsymbol{\alpha}}, \xi \in \Xi\}$.

Lemma 4.6. *Let the conditions of Theorem 4.3 hold and let $\hat{u}_{ext}$ be defined by* (23)*. Then it holds*

$$\big\|(\mathbf{I} - \mathbf{Q})\hat{u}_{\text{ext}}\big\|_{H^1(D)} \lesssim \|f\|_{L^2(D)} N^{-1 + \tilde{\delta} + \sum_{k=1}^{m} \delta_k}. \tag{33}$$

Proof. With the above notation, the following estimate for the quadrature error holds, cf. [24],

$$\big\|(\mathbf{I} - \mathbf{Q})\hat{u}_{\text{ext}}\big\|_{H^1(D)} \leqslant \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{\text{ext}}(\cdot, \mathbf{z} \vee_{\boldsymbol{\alpha}} \mathbf{z}_0)\big\|_{H^1(D)} \, \mathrm{d}(\mathbf{z})_{\boldsymbol{\alpha}} \sup_{(\mathbf{z}_{\boldsymbol{\alpha}}) \in [0,1]^{|\boldsymbol{\alpha}|}} \mathcal{D}^c\big((\mathbf{z})_{\boldsymbol{\alpha}}, (\Xi)_{\boldsymbol{\alpha}}\big).$$

To prove tractability we introduce weights $w_k \in (0, \infty)$ for $k = 1, \ldots, m$ and define the corresponding product weights with respect to a multiindex $\boldsymbol{\alpha}$ by $w_{\boldsymbol{\alpha}} := \prod_{k=1}^{m} w_k^{\alpha_k}$. Later on, we will specify these weights by exploiting the decay properties of the occurring derivatives of the integrand. Now, from the above inequality we deduce

$$\begin{aligned}
&\big\|(\mathbf{I} - \mathbf{Q})\hat{u}_{\text{ext}}\big\|_{H^1(D)} \\
&\quad \leqslant \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} w_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{\text{ext}}(\cdot, \mathbf{z} \vee_{\boldsymbol{\alpha}} \mathbf{z}_0)\big\|_{H^1(D)} \, \mathrm{d}(\mathbf{z})_{\boldsymbol{\alpha}} w_{\boldsymbol{\alpha}}^{1/2} \sup_{(\mathbf{z}_{\boldsymbol{\alpha}}) \in [0,1]^{|\boldsymbol{\alpha}|}} \mathcal{D}^c\big((\mathbf{z})_{\boldsymbol{\alpha}}, (\Xi)_{\boldsymbol{\alpha}}\big) \\
&\quad \leqslant \sup_{\|\boldsymbol{\alpha}\|_\infty = 1} w_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{\text{ext}}(\cdot, \mathbf{z} \vee_{\boldsymbol{\alpha}} \mathbf{z}_0)\big\|_{H^1(D)} \, \mathrm{d}(\mathbf{z})_{\boldsymbol{\alpha}} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} w_{\boldsymbol{\alpha}}^{1/2} \sup_{(\mathbf{z}_{\boldsymbol{\alpha}}) \in [0,1]^{|\boldsymbol{\alpha}|}} \mathcal{D}^c\big((\mathbf{z})_{\boldsymbol{\alpha}}, (\Xi)_{\boldsymbol{\alpha}}\big).
\end{aligned}$$

Due to the definition of $\hat{u}_{\text{ext}}$, cf. (23), the derivative $\partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\hat{u}_{\text{ext}}(\cdot,\mathbf{z}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0)$ vanishes in $[0,1]^{|\boldsymbol{\alpha}|}\backslash\big(K_N\big)_{\boldsymbol{\alpha}}$ and coincides with the derivative of $\hat{u}$ in $\big(K_N\big)_{\boldsymbol{\alpha}}$. Therefore, with $\tilde{C}$ defined as in (28), we can estimate

$$
\begin{aligned}
&\sup_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\hat{u}_{\text{ext}}(\cdot,\mathbf{z}\vee_{\boldsymbol{\alpha}}\mathbf{z}_0)\big\|_{H^1(D)}\,\mathrm{d}(\mathbf{z})_{\boldsymbol{\alpha}}\\
&\leqslant \|f\|_{L^2(D)}\sup_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2}|\boldsymbol{\alpha}|!\int_{(K_N)_{\boldsymbol{\alpha}}}\prod_{k=1}^m\big(\gamma_k\tilde{C}\min\{z_k,1-z_k\}^{-1-\delta_k}\big)^{\alpha_k}\,\mathrm{d}(\mathbf{z})_{\boldsymbol{\alpha}}\\
&\leqslant \|f\|_{L^2(D)}\sup_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2}2^{|\boldsymbol{\alpha}|}\prod_{k=1}^m\left(k\gamma_k\tilde{C}\int_{(b_kN)^{-1}}^{1/2} z_k^{(-1-\delta_k)}\,\mathrm{d}z_k\right)^{\alpha_k}\\
&\leqslant \|f\|_{L^2(D)}\sup_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2}\prod_{k=1}^m\left(\frac{2k\gamma_k\tilde{C}}{\delta_k}\right)^{\alpha_k}(b_kN)^{\alpha_k\delta_k}\\
&= N^{\sum_{k=1}^m\alpha_k\delta_k}\|f\|_{L^2(D)}\prod_{k=1}^m b_k^{\alpha_k\delta_k}\lesssim N^{\sum_{k=1}^m\alpha_k\delta_k}\|f\|_{L^2(D)}.
\end{aligned}\tag{34}
$$

The estimate is valid for the weights

$$
w_{\boldsymbol{\alpha}}^{1/2}:=\prod_{k=1}^m\left(\frac{2k\gamma_k\tilde{C}}{\delta_k}\right)^{\alpha_k}.
$$

Thus, we have

$$
w_k=\frac{8\pi C(\delta_k,2\gamma_k)^2k^2\gamma_k^2}{\delta_k^2\log^2 2}.
$$

The last step in (34) follows since $b_k<2k\log(k+2)$ by the prime number theorem, see e.g. [38], which implies

$$
\prod_{k=1}^\infty b_k^{\delta_k}=\exp\left(\sum_{k=1}^\infty\delta_k\log b_k\right)\lesssim\exp\left(\sum_{k=1}^\infty k^{-1-\varepsilon}\log\big(2k\log(k+2)\big)\right)<\infty.
$$

In order to bound the weighted sum of the centered discrepancies, we use the following result from [30]. It holds

$$
\mathcal{D}_{\text{extr}}(\Xi)\leqslant 2^m\mathcal{D}_\infty^\star(\Xi).
$$

Hence, we have

$$
\sum_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{1/2}\sup_{(\mathbf{z}_{\boldsymbol{\alpha}})\in[0,1]^{|\boldsymbol{\alpha}|}}\mathcal{D}^c\big((\mathbf{z})_{\boldsymbol{\alpha}},(\Xi)_{\boldsymbol{\alpha}}\big)\leqslant\sum_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{1/2}2^{|\boldsymbol{\alpha}|}\mathcal{D}_\infty^\star\big((\Xi)_{\boldsymbol{\alpha}}\big)
$$

Under the decay property

$$
\sum_{k=1}^\infty\tilde{w}_k^{1/2}k\log k<\infty.
$$

of the weights $\tilde{w}_k:=4w_k$, it is shown in [44] that

$$
\sum_{\|\boldsymbol{\alpha}\|_\infty=1}\tilde{w}_{\boldsymbol{\alpha}}^{1/2}\mathcal{D}_\infty^\star\big((\Xi)_{\boldsymbol{\alpha}}\big)\lesssim\big(N^{-1+\tilde{\delta}}\big)
$$

holds for all $\tilde{\delta}>0$ with a constant which is independent of the dimension m. This condition is satisfied, if the weights fulfill $\tilde{w}_k^{1/2}\lesssim k^{-2-\varepsilon}$. Thus, we get the following condition on the decay of γ_k:

$$
\frac{4k\gamma_k\tilde{C}}{\delta_k}\lesssim k^{-2-\varepsilon}\quad\Longrightarrow\quad\gamma_k\lesssim\frac{\delta_k}{4\tilde{C}}k^{-3-\varepsilon}\sim k^{-4-2\varepsilon}.
$$

□

With the preceding two Lemmata at hand, we can establish the estimate (25). This completes the proof of Theorem 4.3.

Remark 4.7. *In this section, we have only shown approximation results of the Quasi-Monte Carlo quadrature based on Halton points for the mean of the function u, i.e. $\mathbb{E}_u$. Note, that if $f \in L^p(D)$ due to the regularity estimates proven in Section 3, the results in this section hold in complete analogy for the p-th moments $\mathcal{M}^p u$ of the solution u to (9). This is due to the similar behaviour of the solutions derivatives and its powers.*

Corollary 4.8. *Let $f \in L^p(D)$ for $p \geqslant 2$. Under the conditions of Theorem 4.3 and the slightly stronger assumption $p\gamma_k \lesssim k^{-4-2\varepsilon}$, the Quasi-Monte Carlo quadrature using Halton points for approximating the p-th moment of the solution u to (11) is polynomial tractable and provides the error estimate*

$$\left\|(\mathbf{I}-\mathbf{Q})\hat{u}^p\right\|_{W^{1,p}(D)} \lesssim p\|f\|^p_{L^p(D)} m N^{-1+\delta}.$$

We want to close this section by a brief note on the uniformly elliptic case. Therefore, let us consider a uniformly elliptic parametric boundary value equation. The problem has the same form as (9), but here the diffusion coefficient itself and not its exponential is given by a Karhunen-Loève expansion i.e.

$$a(\mathbf{x},\mathbf{y}) = \mathbb{E}_a(\mathbf{x}) + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(\mathbf{x}) y_k.$$

The eigenvalues λ_k and the eigenfunctions φ_k are supposed to have the same properties as in the log-normal case and, without loss of generality, we assume that the parameters y_k are uniformly distributed on the interval $[-1/2, 1/2]$. Therefore, the corresponding density function is constant and equal to 1. The uniform ellipticity guarantees the existence of constants $a_{\min}, a_{\max} > 0$ being independent of $\mathbf{y}$, such that for almost every $\mathbf{y} \in [-1/2, 1/2]^\infty$ it holds

$$a_{\min} < a(\mathbf{x},\mathbf{y}) < a_{\max}.$$

For the uniformly elliptic case the proof of tractability of the Quasi-Monte Carlo quadrature with Halton points is much easier since we have to deal neither with an unbounded integration domain nor with unbounded integrands. The sampling points in the cube $[-1/2, 1/2]^\infty$ are straightforwardly obtained from those in $[0, 1]^\infty$ by shifting each coordinate by $1/2$.

Theorem 4.9. *For the uniformly elliptic case there exists for all $\delta > 0$ a constant such that the Quasi-Monte Carlo quadrature based on Halton points for approximating the expectation of the solution u is strongly tractable if the sequence $\{\gamma_k\}_k$ admits at least the decay behaviour $\gamma_k \sim k^{-3-\varepsilon}$ for arbitrary $\varepsilon > 0$.*

Proof. The proof is quite similar to the proof of Lemma 4.6. We can apply the *Zaremba-Hlawka* inequality, cf. [27],

$$(\mathbf{I}-\mathbf{Q})u(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|_\infty=1} (-1)^{|\boldsymbol{\alpha}|} \int_{[-1/2,1/2]^{|\boldsymbol{\alpha}|}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{x}, \mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{1/2}) \operatorname{disc}\big((\Xi)_{\boldsymbol{\alpha}}, (\mathbf{y})_{\boldsymbol{\alpha}}\big) \,\mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}}.$$

Here, the *local discrepancy* is defined by

$$\operatorname{disc}(\Xi, \mathbf{z}) := \operatorname{Vol}\big([-\mathbf{1/2}, \mathbf{y}]\big) - \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{[-\mathbf{1/2},\mathbf{y}]}(\xi_i).$$

We obtain with Hölder's inequality and the introduction of appropriate weights, cf. [27],

$$\begin{aligned}
\|(\mathbf{I}-\mathbf{Q})u\|_{H^1(D)} &\leqslant \sum_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2} \int_{[-1/2,1/2]^{|\boldsymbol{\alpha}|}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot, \mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{1/2})\|_{H^1(D)} \,\mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}} w_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}^\star_\infty\big((\Xi)_{\boldsymbol{\alpha}}\big) \\
&\lesssim \sup_{\|\boldsymbol{\alpha}\|_\infty=1} w_{\boldsymbol{\alpha}}^{-1/2} \int_{[-1/2,1/2]^{|\boldsymbol{\alpha}|}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot, \mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{1/2})\|_{H^1(D)} \,\mathrm{d}(\mathbf{y})_{\boldsymbol{\alpha}} \sum_{|\boldsymbol{\alpha}|=1} w_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}^\star_\infty\big((\Xi)_{\boldsymbol{\alpha}}\big).
\end{aligned}$$

In the uniformly elliptic case, the coefficient is bounded from below and above independently of $\mathbf{y}$. The regularity estimates for the solution are quite similar to those of Section 3. It holds, cf.

[10],

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\cdot, \mathbf{y} \vee_{\boldsymbol{\alpha}} \mathbf{1/2})\|_{H^1(D)} \lesssim |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{a_{\min} \log 2} \right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)} \leqslant \|f\|_{L^2(D)} \prod_{k=1}^{m} \left(\frac{k\gamma_k}{a_{\min} \log 2} \right)^{\alpha_k}.$$

Now, let $w_{\boldsymbol{\alpha}} = \prod_{k=1}^{m} w_k^{\alpha_k}$ with

$$w_k := \frac{k^2 \gamma_k^2}{a_{\min} \log^2 2}.$$

Then, we arrive at the estimate

$$\|(\mathbf{I} - \mathbf{Q})u\|_{H^1(D)} \lesssim \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} w_k^{1/2} \mathcal{D}_\infty^\star\big((\Xi)_{\boldsymbol{\alpha}}\big),$$

which yields the assertion with the same arguments as in the proof of Lemma 4.6. □

Remark 4.10. *In analogy to the log-normal case one can establish estimates on the derivatives of the powers u^p of u. Hence, the approximation of the moments with a Quasi-Monte Carlo quadrature based on Halton points remains strongly tractable in the uniform elliptic case.*

5. Numerical results

In this section, we present numerical examples to validate the theoretical findings. Therefore, we consider the one-dimensional diffusion problem

$$-\partial_x\big(a(x, \mathbf{y})\partial_x u(x, \mathbf{y})\big) = 1 \text{ in } D = (0, 1) \tag{35}$$

with homogenous boundary conditions, i.e. $u(0, \mathbf{y}) = u(1, \mathbf{y}) = 0$. The logarithm of the diffusion coefficient a is given by the Karhunen-Loève expansion,

$$\log\big(a(x, \mathbf{y})\big) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \varphi_k(\mathbf{x}) y_k.$$

Here, the eigenpairs (λ_k, φ_k) are obtained by solving the eigenproblem for the diffusion coefficient's correlation, i.e.

$$\int_0^1 k(x, x')\varphi_k(x') \,\mathrm{d}x' = \lambda_k \varphi_k(x),$$

where we assume that this correlation is given by a positive definite function

$$k(x, x') := \int_\Omega \log\big(a(x, \omega)\big) \log\big(a(x', \omega)\big) \,\mathrm{d}\mathbb{P}(\omega).$$

The knowledge of $k(x, x')$ together with $\mathbb{E}[a(x, \omega)] = 0$ provides the unique description of a since the underlying random process is Gaussian.

Let $r = |x - x'|$. In the sequel, we consider the class of Matérn correlation kernels, i.e.

$$k_\nu(r) := \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{\ell} \right)^\nu K_\nu \left(\frac{\sqrt{2\nu} r}{\ell} \right)$$

with $\ell, \nu \in (0, \infty)$. Here, K_ν denotes the *modified Bessel function of the second kind*, cf. [1]. For half integer values of ν, i.e. $\nu = p + 1/2$ for $p \in \mathbb{N}$, this expression simplifies to

$$k_{p+1/2}(r) = \exp\left(\frac{-\sqrt{2\nu} r}{\ell} \right) \frac{p!}{(2p)!} \sum_{i=0}^{p} \frac{(p+i)!}{i!(p-i)!} \left(\frac{\sqrt{8\nu} r}{\ell} \right)^{p-i}.$$

In the limit case $\nu \to \infty$, we obtain the Gaussian correlation

$$k_\infty(r) = \exp\left(\frac{-r^2}{2\ell^2} \right),$$

cf. [37]. The Sobolev smoothness of the kernel k_ν is controlled by the *smoothness parameter* ν. A visualization of this kernels for different values of ν is given in Figure 1.

The eigenvalues of the Matérn correlation kernels decay like

$$\lambda_k \leqslant C k^{-(1+2\nu/d)}$$

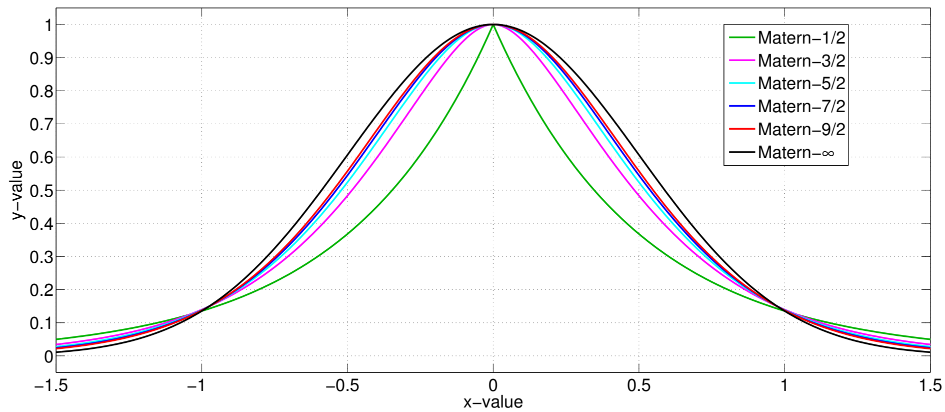

FIGURE 1. Different values for the smoothness parameter ν.

for some $C > 0$, cf. [18]. Hence, we consider $\nu = 5/2, 7/2, 9/2$. For the parameter value $\nu = 5/2$ the eigenvalues of the correlation function decay too slowly and we are thus outside our regime. The parameter value $\nu = 7/2$ is exactly the limit case for the decay of the eigenvalues and the value $\nu = 9/2$ leads to an eigenvalue decay perfectly fitting our assumptions. The correlation length is set to $\ell = 1/2$ for each of the kernels.

We have discretized (35) by piecewise linear finite elements and chose piecewise constant elements for the discretization of the diffusion coefficient. As a reference, we have computed the solution to (35) on level 13, i.e. we have the meshwidth $h = 2^{-13}$, and avaraged 10 runs of a Monte Carlo quadrature each of them with $N = 10^6$ samples. In each example, the Karhunen-Loève expansion is truncated appropriately to sustain the precision of the finite element discretization. Unfortunately, although we observe convergence of our Quasi-Monte Carlo quadrature with respect to this reference solution, the error decay stagnates for increasing number of samples. Thus, we assume that the reference solution is not accurate enough. Therefore, we additionally provide a reference solution computed by the Quasi-Monte Carlo quadrature with Halton points and $N = 7 \cdot 10^6$ samples. The computations for the approximation error were also performed on level 13. This means, we have kept the level fixed and successively increased the number of sample points.

The Matérn kernel for $\nu = 9/2$. For the smoothness parameter $\nu = 9/2$, we have truncated the Karhunen-Loève expansion after $m = 21$ terms. The left plot in Figure 2 shows a visualization of the mean's and the moment's errors with respect to the Monte Carlo reference solution measured in the respective norms for increasing numbers of samples N. The analogous representation for the Quasi-Monte Carlo reference can be found in the right plot of the same figure. The slopes indicated in the plots correspond to a linear least-squares fit for the respective curve. The correlation kernel under consideration, i.e. $k_{9/2}$, perfectly fits our smoothness assumptions. We observe convergence with respect to both references and rates that are higher than that of a Monte Carlo quadrature, at least for the mean and the second moment. The successive decrease of the rate of convergence for the higher moments can be explained by the exponential dependence of the constants in (17) on p in the pre-asymptotic regime. We would like to emphasize, that we observed exactly the same phenomena for a Monte Carlo quadrature. Nevertheless, we saw no additional benefit in showing the related plots here, since we think that this exceeds the scope of this article.

The Matérn kernel for $\nu = 7/2$. For the smoothess parameter $\nu = 7/2$, we have truncated the Karhunen-Loève expansion after $m = 32$ terms. Figure 3 shows the error plots related to the Matérn kernel $k_{7/2}$. As already mentioned, this is the limit case for the required smoothness of the correlation kernel. We observe the same decay of the errors as in the previous example for the smoothness parameter $\nu = 9/2$. Again, the error plot with respect to the Monte Carlo reference

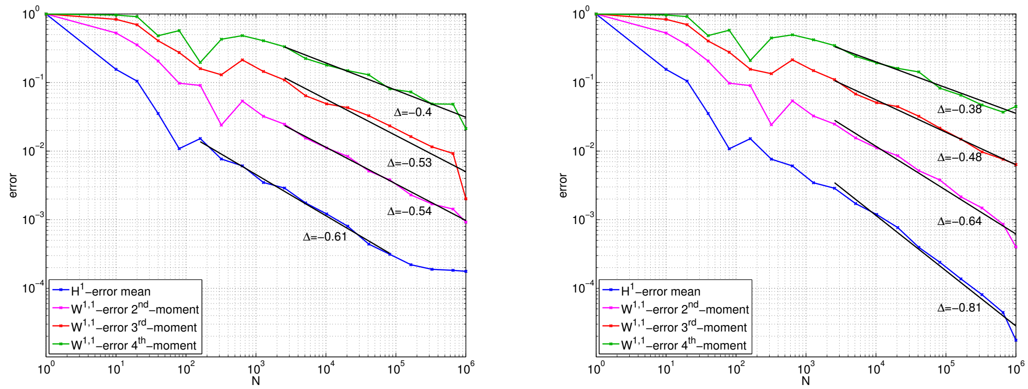

FIGURE 2. Errors for $\nu = 9/2$ with Monte Carlo reference (left) and Quasi-Monte Carlo reference (right).

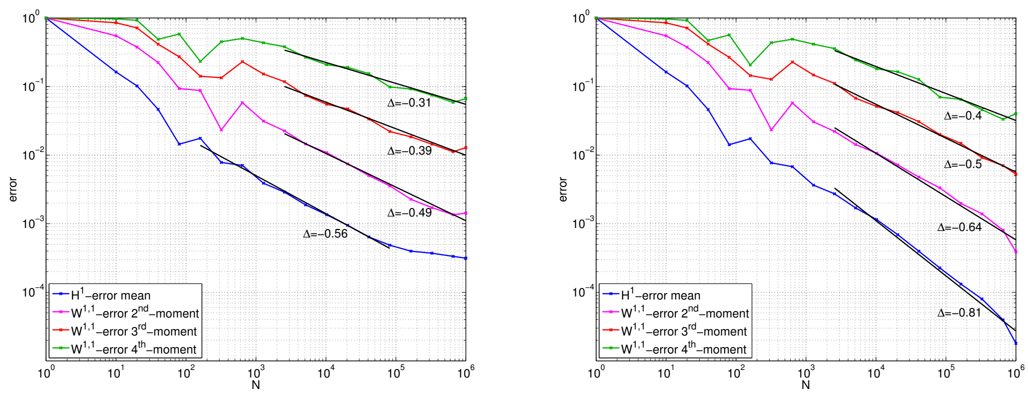

FIGURE 3. Errors for $\nu = 7/2$ with Monte Carlo reference (left) and Quasi-Monte Carlo reference (right).

is found on the left-hand side in Figure 3 and the error with respect to the Quasi-Monte Carlo reference on the right-hand side of this figure.

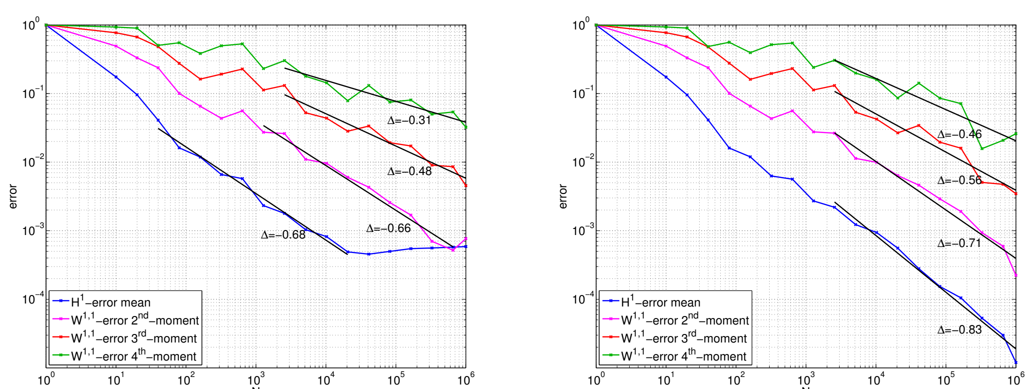

FIGURE 4. Errors for $\nu = 5/2$ with Monte Carlo reference (left) and Quasi-Monte Carlo reference (right).

The Matérn kernel for $\nu = 5/2$. For the smoothness parameter $\nu = 5/2$, we have truncated the Karhunen-Loève expansion after $m = 88$ terms. Although, the correlation kernel $k_{5/2}$ does not meet the required smoothness assumptions anymore, we essentially obtain the same error rates, as in the previous two examples. The same effect has already been observed in [18]. A visualization of the corresponding errors for increasing number of samples is given in Figure 4.

The numerical examples indicate the tractability of the Quasi-Monte Carlo quadrature based on Halton points for log-normal diffusion problems in concordance with our theoretical findings. Nevertheless, the numerical results also imply that the claimed smoothness assumptions can probably be weakened.

Appendix A

The set $P(\boldsymbol{\alpha}, r)$ of restricted integer partitions of a multiindex $\boldsymbol{\alpha}$ into r non vanishing multiindices is defined by

$$P(\boldsymbol{\alpha}, r) := \Big\{ \big((k_1, \boldsymbol{\beta}_1), \ldots, (k_n, \boldsymbol{\beta}_n)\big) \in (\mathbb{N}_0 \times \mathbb{N}_0^m)^n : \sum_{i=1}^n k_i \boldsymbol{\beta}_i = \boldsymbol{\alpha},\ \sum_{i=1}^n k_i = r,\ \text{and } \exists\, 1 \leqslant s \leqslant n \Big| \\ k_i = 0 \text{ and } \boldsymbol{\beta}_i = \mathbf{0} \text{ for all } 1 \leqslant i \leqslant n - s, \\ k_i > 0 \text{ for all } n - s + 1 \leqslant i \leqslant n \text{ and } \mathbf{0} \prec \boldsymbol{\beta}_{n-s+1} \prec \cdots \prec \boldsymbol{\beta}_n \Big\}.$$

Herein, for multiindices $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \mathbb{N}^m$, the relation $\boldsymbol{\beta} \prec \boldsymbol{\beta}'$ means either $|\boldsymbol{\beta}| < |\boldsymbol{\beta}'|$ or, if $|\boldsymbol{\beta}| = |\boldsymbol{\beta}'|$, it denotes the lexicographical order which means that it holds that $\beta_1 = \beta_1', \ldots, \beta_k = \beta_k'$ and $\beta_{k+1} < \beta_{k+1}'$ for some $0 \leqslant k < m$.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Applied mathematics series, Dover Publications, 1964.

[2] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] I. Babuška, F. Nobile, and R. Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal., 45 (2007), pp. 1005–1034.

[4] I. Babuška, R. Tempone, and G. Zouraris, *Galerkin finite element approximations of stochastic elliptic partial differential equations*, SIAM J. Numer. Anal., 42 (2004), pp. 800–825.

[5] A. Barth, C. Schwab, and N. Zollinger, *Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients*, Numer. Math., 119 (2011), pp. 123–161.

[6] H.-J. Bungartz and M. Griebel, *Sparse grids*, Acta Numer., 13 (2004), pp. 147–269.

[7] R. Caflisch, *Monte Carlo and quasi-Monte Carlo methods*, in Acta numerica, 1998, vol. 7 of Acta Numer., Cambridge Univ. Press, Cambridge, 1998, pp. 1–49.

[8] J. Charrier, *Strong and weak error estimates for elliptic partial differential equations with random coefficients*, SIAM J. Numer. Anal., 50 (2012), pp. 216–246.

[9] J. Charrier, R. Scheichl, and A. Teckentrup, *Finite element error analysis of elliptic pdes with random coefficients and its application to multilevel monte carlo methods*, SIAM Journal on Numerical Analysis, 51 (2013), pp. 322–352.

[10] A. Cohen, R. DeVore, and C. Schwab, *Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs*, Found. Comput. Math., 10 (2010), pp. 615–646.

[11] G. M. Constantine and T. H. Savits, *A multivariate Faà di Bruno formula with applications*, Trans. Amer. Math. Soc., 248 (1996), pp. 503–520.

[12] P. Fan, *New inequalities of mill's ratio and its application to the inverse q-function approximation*, CoRR, abs/1212.4899 (2012).

[13] P. Frauenfelder, C. Schwab, and R. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 205–228.

[14] T. Gerstner and M. Griebel, *Numerical integration using sparse grids*, Numer. Algorithms, 18 (1998), pp. 209–232.

[15] R. Ghanem and P. Spanos, *Stochastic finite elements: a spectral approach*, Springer-Verlag, New York, 1991.

[16] M. Giles, *Multilevel Monte Carlo path simulation*, Operations Research, 56 (2008), pp. 607–617.

[17] C. J. Gittelson, *Stochastic Galerkin discretization of the log-normal isotropic diffusion problem.*, Math. Models Methods Appl. Sci., 20 (2010), pp. 237–263.

[18] I. Graham, F. Kuo, J. Nichols, R. Scheichl, C. Schwab, and I. Sloan, *Quasi-Monte Carlo finite element methods for elliptic PDEs with log-normal random coefficient*, Preprint, (2013).

[19] J. H. Halton, *On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals*, Numerische Mathematik, 2 (1960), pp. 84–90.

[20] H. Harbrecht, M. Peters, and M. Siebenmorgen, *Multilevel accelerated quadrature for pdes with log-normal distributed random coefficient*, Preprint, (2013).

[21] ———, *On multilevel quadrature for elliptic stochastic partial differential equations*, Sparse Grids and Applications, 88 (2013), pp. 161–179.

[22] S. Heinrich, *The multilevel method of dependent tests*, in Advances in stochastic simulation methods (St. Petersburg, 1998), Stat. Ind. Technol., Birkhäuser Boston, Boston, MA, 2000, pp. 47–61.

[23] ———, *Multilevel Monte Carlo methods*, in Lect. Notes in Large Scale Scientific Computing, London, 2001, Springer-Verlag, pp. 58–67.

[24] F. J. Hickernell, *A generalized discrepancy and quadrature error bound*, Math. Comp, 67 (1998), pp. 299–322.

[25] F. J. Hickernell and X. Wang, *The error bounds and tractability of quasi-Monte Carlo algorithms in infinite dimension*, Math. Comp, (2002), pp. 1641–1661.

[26] V. A. Hoang and C. Schwab, *N-term Galerkin Wiener chaos approximations of elliptic PDEs with lognormal Gaussian random inputs*, Tech. Report 2011-59, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2011.

[27] F. Kuo, C. Schwab, and I. Sloan, *Quasi-monte carlo methods for high-dimensional integration: the standard (weighted hilbert space) setting and beyond*, ANZIAM Journal, 53 (2012).

[28] M. Loève, *Probability theory. I+II*, vol. 45 of Graduate Texts in Mathematics, Springer-Verlag, New York, 4th ed., 1977.

[29] H. Matthies and A. Keese, *Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 1295–1331.

[30] H. Niederreiter, *Random number generation and quasi-Monte Carlo methods*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1992.

[31] F. Nobile, R. Tempone, and C. G. Webster, *An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal., 46 (2008), pp. 2411–2442.

[32] E. Novak and H. Woźniakowski, *Tractability of Multivariate Problems: Linear Information*, Ems Tracts in Mathematics, European Mathematical Society, 2008.

[33] ———, *Tractability of Multivariate Problems: Standard information for functionals*, vol. 12, European Mathematical Society, 2010.

[34] ———, *Tractability of Multivariate Problems: Standard information for operators*, EMS tracts in mathematics, European Mathematical Society, 2012.

[35] A. B. Owen, *Halton sequences avoid the origin*, SIAM Rev., 48 (2006), pp. 487–503.

[36] J. K. Patel and C. B. Read, *Handbook of the Normal Distribution*, CRC Press, 1996.

[37] C. E. Rasmussen and C. K. I. Williams, *Gaussian Processes for Machine Learning (Adaptive Computation and Machine Learning)*, The MIT Press, 2005.

[38] J. Sándor, D. S. Mitrinovic, and B. Crstici, *Handbook of Number Theory I*, Mathematics and Its Applications, Springer, 1995.

[39] C. Schwab and C. J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer., 20 (2011), pp. 291–467.

[40] C. G. Simader, *On Dirichlet's Boundary Value Problem: LP-Theory based on a Generalization of Garding's Inequality*, Lecture Notes in Mathematics, Springer, 1972.

[41] I. H. Sloan, F. Y. Kuo, and S. Joe, *Constructing randomly shifted lattice rules in weighted sobolev spaces*, SIAM Journal on Numerical Analysis, 40 (2002), pp. 1650–1665.

[42] I. H. Sloan and H. Woźniakowski, *When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?*, J. Complexity, 14 (1997), pp. 1–33.

[43] S. Smolyak, *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Soviet Mathematics, Doklady, 4 (1963), pp. 240–243.

[44] X. Wang, *A constructive approach to strong tractability using quasi-monte carlo algorithms*, J. Complex., 18 (2002), pp. 683–701.

[45] H. Woźniakowski, *Tractability and strong tractability of linear multivariate problems*, Journal of Complexity, 10 (1994), pp. 96–128.

[46] C. Zenger, *Sparse grids*, in Parallel Algorithms for Partial Differential Equations, W. Hackbusch, ed., vol. 31 of Notes on Numerical Fluid Mechanics, Braunschweig/Wiesbaden, 1991, Vieweg-Verlag, pp. 241–251.

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