# Nullforms, Polarization and Tensorpowers

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# Overview

**Part I: Singular Spaces of the Nullcone**
Given a complex reductive group $G$ and a complex representation $V$, one of the main goals of invariant theory is to describe - in terms of generators and relations - the ring of invariant polynomial functions, denoted by $\mathcal{O}(V)^G$. However, for most pairs $G$ and $V$, finding explicitly all generators of $\mathcal{O}(V)^G$ is very difficult. An important step in this search is to find homogeneous invariants whose zero set is the *nullcone* $\mathcal{N}_V \subset V$, i.e. the zero set of all homogeneous non-constant invariant functions on $V$. Such invariants are strongly related to $\mathcal{O}(V)^G$ as Hilbert proved the following result: If $f_1, \ldots, f_r$ are homogeneous invariants whose zero set is equal to $\mathcal{N}_V$ then $\mathcal{O}(V)^G$ is a finitely generated module over the subalgebra $\mathbb{C}[f_1, \ldots, f_r]$.

Given some invariants $f_i \in \mathcal{O}(V)^G$ as above one can apply the so called *polarization* process to obtain a set of functions lying in $\mathcal{O}(V^{\oplus k})^G$. Our main interest in this work is to analyze whether the set of functions obtained in this manner defines the nullcone $\mathcal{N}_{V^{\oplus k}}$. Due to an observation of Kraft and Wallach, this is equivalent to the question whether for every linear subspace $H \subset \mathcal{N}_V$ of dimension at most $k$ there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot H = 0$.

For example, for $G = \mathrm{SL}_2$ and $V = V_n$, the binary forms of degree $n$, this amounts to the question whether every subspace $H$ that consists of forms having a root of multiplicity greater than $\frac{n}{2}$ consists of forms having a common root of multiplicity greater than $\frac{n}{2}$. This is indeed the case, as we will see. Furthermore we settle the question for $G = \mathrm{SL}_n$ and $V = S^2(\mathbb{C}^n)^*$ (symmetric bilinear forms), $V = \bigwedge^2(\mathbb{C}^n)^*$ (skew-symmetric bilinear forms) and $G = \mathrm{SL}_3$ and $V = S^3(\mathbb{C}^3)^*$ (ternary cubics).

**Part II: Multiplicities in Tensor Monomials**
There exist a lot of formulas to decompose a tensor product of representations $V \otimes W$ into a direct sum of irreducible representations with respect to an algebraic group $G$. However these formulas usually involve summing over the Weyl-group, which makes explicit calculations often tedious. When considering multiple tensor products, i.e. *tensor monomials* $V_1^{\otimes n_1} \otimes V_2^{\otimes n_2} \cdots V_r^{\otimes n_r}$, then, even with the use of descent computers, an explicit decomposition is mostly impossible because of the complexity that arises. For this reason problems involving tensor monomials remain challenging. The starting point of this work was the following question asked by Finkelberg: For which $(d_1, d_2, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$ does the tensor monomial ${\mathbb{C}^n}^{\otimes d_1} \otimes {\bigwedge^2 \mathbb{C}^n}^{\otimes d_2} \otimes {\bigwedge^3 \mathbb{C}^n}^{\otimes d_3} \otimes \cdots \otimes {\bigwedge^{n-1} \mathbb{C}^n}^{\otimes d_{n-1}}$, considered as $\mathrm{SL}_n$-representation, contain the trivial representation exactly once? We solve this problem and some related generalizations. However, representations occuring with multiplicity one in the decomposition of a tensor monomial ${V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}$ are rather rare as we prove that multiplicities of subrepresentations of tensor monomials grow exponentially with respect to $\sum n_i$. More precisely, we prove, that if $G$ is a simple complex group and $V_1, \ldots, V_r$ and $W$ irreducible non-trivial representations then there is a constant $N$ and a real number $\alpha > 1$ such that if $\sum n_i \geq N$ then $\mathrm{mult}(W, {V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}) \geq \alpha^{\sum n_i}$ unless it is zero.
In its current form, this part is a preprint which evolved from my diploma thesis, where I solved special cases of the two main results Theorem A and Theorem C.

**Part III: The Hilbert Nullcone on Tuples of Matrices and Bilinear Forms**
In this joint work with Jan Draisma we explicitly determine the irreducible components of the nullcone of the representation of $G$ on $M^{\oplus p}$, where either $G = \mathrm{SL}(W) \times \mathrm{SL}(V)$ and $M = \mathrm{Hom}(V, W)$ (linear maps), or $G = \mathrm{SL}(V)$ and $M$ is one of the representations $S^2(V^*)$ (symmetric bilinear forms), $\Lambda^2(V^*)$ (skew bilinear forms), or $V^* \otimes V^*$ (arbitrary bilinear forms). Here $V$ and $W$ are vector spaces over an algebraically closed field $K$ of characteristic zero. We also answer the question of when the nullcone in $M^{\oplus p}$ is defined by the polarisations of the invariants on $M$; typically, this is only the case if either $\dim V$ or $p$ is small. A fundamental tool in our proofs is the Hilbert-Mumford criterion for nilpotency.
This preprint has already been accepted for publication in the Mathematische Zeitschrift. I mainly contributed to the first problem we solved: counting and describing the components of the nullcone of the symmetric bilinear forms. Most other cases evolved from this one, however.

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I would like to thank the following people who all had crucial influence on me in one way or another: Jochen Kuttler, who unfortunately left for the United States when we started having real fun. Simon Straumann, though we could still not solve the $111\ldots1$ problem. Yvonne Gerber for some finest team teaching....Stephan Mohrdieck, who helped me a lot in my early days.

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# Part 1

# Singular Spaces of the Nullcone

# Singular Spaces of the Nullcone

Matthias Bürgin

# Contents

CHAPTER 1

# Introduction and Generalities

## 1. Introduction

Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional complex representation of a complex reductive group $G$ and denote by $\mathcal{N}_V \subset V$ the *nullcone*, i.e. the zero set of the homogeneous non-constant $G$-invariant functions on $V$. The nullcone is in many ways related to the geometry of the representation, in particular it encodes a lot of information about the structure of orbits and their closure. There is also a strong connection to the algebra of invariants $\mathcal{O}(V)^G$ as Hilbert proved the following result ([**Kra85**, Kap. II.4]): If $f_1, \dots f_r$ are homogeneous invariant functions whose zero set is equal to $\mathcal{N}_V$ then $\mathcal{O}(V)^G$ is a finitely generated module over $\mathbb{C}[f_1, \dots, f_r]$. On such $f_1, \dots, f_r$ one can apply the *polarization* process (see below) to obtain a set of invariants in $\mathcal{O}(V^{\oplus k})^G$. The aim of this work is to determine for which $n$ and $k$ this set defines the nullcone $\mathcal{N}_{V^{\oplus k}}$ in the cases $G = \mathrm{SL}_2$ and $V = S^n(\mathbb{C}^2)^*$ (binary forms), $G = \mathrm{SL}_n$ and $V = S^2(\mathbb{C}^n)^*$ (symmetric bilinear forms), $V = \bigwedge^2(\mathbb{C}^n)^*$ (skew-symmetric bilinear forms) and $G = \mathrm{SL}_3$ and $V = S^3(\mathbb{C}^3)^*$ (ternary cubics). Due to an observation of Kraft and Wallach this problem is equivalent to the problem whether for every linear subspace $H \subset \mathcal{N}_V$ of dimension at most $k$ there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot H = 0$.

We have organized our results as follows: For the rest of **Chapter 1** we collect some general facts and explain the two equivalent problems.

In **Chapter 2** we deal with the following situation: If the invariants $f_1, \dots, f_r$ defining the nullcone $\mathcal{N}_{V^{\oplus k}}$ are algebraically independent and multihomogeneous with respect to the multiple copies of $V$, then they are called a *multihomogeneous system of parameters (MHSP)* for $\mathcal{O}(V^{\oplus k})^G$. We prove that MHSP's for $\mathcal{O}(V^{\oplus k})^G$ exist only for small values of $k$ and give an explicit upper bound for $k$. This is of interest as the polarization process yields multihomogeneous functions.

In **Chapter 3** we consider the representation of $\mathrm{SL}_2$ on the binary forms of degree $n$, denoted by $V_n$. We prove that if $f_1, \dots f_r$ define the nullcone $\mathcal{N}_{V_n}$ then the polarizations of the $f_i$'s define the nullcone $\mathcal{N}_{V_n^{\oplus k}}$ for any $k$.

In **Chapter 4** we let $Sym_n$ be the quadratic bilinear forms in $n$ variables under the operation of $\mathrm{SL}_n$. The invariant ring $\mathcal{O}(Sym_n)^{\mathrm{SL}_n}$ is generated by the determinant det. We prove that the polarizations of det define $\mathcal{N}_{Sym_n^{\oplus k}}$ if and only if $n < 5$ or $k = 2$. In addition, we classify linear subspaces of $\mathcal{N}_{Sym_n}$ that fulfill certain rank conditions.

**Chapter 5** deals with the representation of $\mathrm{SL}_n$ on skew-symmetric forms $\mathcal{B}_n$ in $n$ variables. For $n$ even $\mathcal{O}(\mathcal{B}_n)^{\mathrm{SL}_n}$ is generated by the *pfaffian*. We show that the polarizations of the pfaffian define $\mathcal{N}_{\mathcal{B}_n^{\oplus k}}$ if and only if $n = 2$ or 4 or if $k = 2$.

In **Chapter 6** we consider the ternary cubic forms $T$ as representation of $\mathrm{SL}_3$. Its invariant ring $\mathcal{O}(T)^{\mathrm{SL}_3}$ is generated by two invariants $f_4$ and $f_6$. We show that the polarizations of $f_4$ and $f_6$ define $\mathcal{N}_{T^{\oplus k}}$ for any $k$.

In **Appendix A** finally, we examine the operation of $\mathrm{GL}_n$ on the quadratic $n \times n$-matrices $M_n$ by conjugation. For $n > 2$ the polarizations of the invariants $\mathcal{O}(M_n)^{\mathrm{GL}_n}$ do not define any nullcone $\mathcal{N}_{M_n^{\oplus k}}$ but by analyzing the subspaces of $\mathcal{N}_{M_n}$ we found the following theorem on nilpotent rank one matrices: If $A_1, \ldots, A_m$ are nilpotent rank one matrices that span a nilpotent space then all $A_i$ can simultaneously be triangularized.

## 2. Generalities

In this section we collect some basic results to which we will refer throughout our work without further reference.
Our base field is the field of complex numbers $\mathbb{C}$. Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of a connected reductive group $G$. The nullcone $\mathcal{N}_V \subset V$ is defined by

$$\mathcal{N}_V := \{v \in V \mid f(v) = 0 \text{ for all homogeneous non-constant } f \in \mathcal{O}(V)^G\}$$

or equivalently by

$$\mathcal{N}_V := \{v \in V \mid \overline{Gv} \ni 0\}.$$

It is also called *null-fiber* since it is the fiber $\pi_V^{-1}(\pi_V(0))$ of the *quotient morphism* $\pi_V : V \to V/\!\!/G$, where $V/\!\!/G$ is the algebraic quotient (see [**Kra85**, Kap. II.3]). The following theorem is known as Hilbert-Mumford criterion and is the main tool in order to decide whether a $v \in V$ belongs to $\mathcal{N}_V$:

Theorem 1.1 ([**Kra85**, Kap. III.2]). *$v \in V$ belongs to the nullcone $\mathcal{N}_V$ if and only if there exists a one-parameter subgroup (short: 1-PSG) $\lambda : \mathbb{C}^* \to G$ such that* $\lim_{t\to 0} \lambda(t)v = 0$.

This allows to describe the nullcone $\mathcal{N}_V$ without the knowledge of the invariants $\mathcal{O}(V)^G$. From now on for $\lim_{t\to 0} \lambda(t)v = 0$ we will shortly say '$\lambda(t)$ *annihilates* $v$'. Since the torus $\lambda(\mathbb{C}^*) \subset G$ is diagonalizable we can even restrict to diagonal 1-PSG's: A $v \in V$ belongs to the nullcone $\mathcal{N}_V$ if and only if some $v_0$ in the orbit $Gv$ is annihilated by a diagonal 1-PSG.

Our interest in finding functions defining the nullcone comes from the following famous theorem of Hilbert:

Theorem 1.2 ([**Kra85**, Kap. II.4]). *If $f_1, \ldots, f_r$ are homogenous invariants such that the zero set of $f_1, \ldots, f_r$ in $V$ is equal to $\mathcal{N}_V$ then $\mathcal{O}(V)^G$ is a finitely generated module over the subalgebra $\mathbb{C}[f_1, \ldots, f_r]$.*

Note that, as a consequence of the theorem of Hochster-Roberts ([**HoR74**]), this module is even free if $f_1, \ldots, f_r$ are algebraically independent.

Now we consider invariants in $\mathcal{O}(V^{\oplus k})^G$. It is obvious how an $f \in \mathcal{O}(V)^G$ of degree $d$ can be trivially embedded in $k$ different ways into $\mathcal{O}(V^{\oplus k})^G$. However, by the following *polarization* process, $f$ gives rise to a much bigger set of functions in $\mathcal{O}(V^{\oplus k})^G$ : Consider the decomposition

$$f(t_1 v_1 + t_2 v_2 + \cdots + t_k v_k) = \sum_{i_1+\cdots+i_k=d} t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k} P_{i_1,\cdots,i_k} f(v_1, \ldots, v_k).$$

The functions $P_{i_1,\cdots,i_k} f(v_1, \ldots, v_k)$ on the right-hand side are multihomogeneous of different mulitdegree, and since $G$ acts linearly and multidegree-preserving on functions, they are elements of $\mathcal{O}(V^{\oplus k})^G$. We call them *polarizations* of $f$ onto $k$ copies.

Note that functions obtained by polarization are multihomogeneous with respect to the different copies of $V$. Furthermore, restricting a set of functions obtained by polarization onto $k$ copies to $m < k$ copies yields the same functions as those obtained by polarization onto $m$ copies. As a consequence, if $f_1, \ldots, f_s \in \mathcal{O}(V)^G$ are invariants such that the polarizations onto $m$ copies do not define the nullcone $\mathcal{N}_{V^{\oplus m}}$ then, since $\mathcal{N}_{V^{\oplus m}} \times \{0\} \times \ldots \times \{0\} \subset \mathcal{N}_{V^{\oplus k}}$, the polarizations onto any $k > m$ copies do not define the nullcone $\mathcal{N}_{V^{\oplus k}}$ as well.

Now we are ready to state the theorem that gives a relation between *singular spaces*, i.e. linear subspaces of $\mathcal{N}_V$ and polarizations of invariants defining the nullcones $\mathcal{N}_{V^{\oplus k}}$. This theorem is crucial for all following work.

THEOREM 1.3 ([**KrW05**]). *Let $V$ be a representation of a connected reductive group $G$ and let $f_1, f_2, \ldots, f_s$ be homogeneous invariants defining the nullcone $\mathcal{N}_V$. For every integer $m \geq 1$ the following statements are equivalent:*

(1) *Every linear subspace $L \subset \mathcal{N}_V$ of dimension $\leq m$ is annihilated by a 1-PSG of $G$.*
(2) *The polarizations of the $f_i$'s define the nullcones $\mathcal{N}_{V^{\oplus k}}$ for all $k \leq m$.*

PROOF. A point $(a_1, \ldots, a_m) \in V^{\oplus m}$ on which all polarizations vanish gives rise to a subspace $H = \langle a_1, \ldots, a_m \rangle \subset \mathcal{N}_V$ which is annihilated if and only if $(a_1, \ldots, a_m)$ is. $\square$

We conclude this section with the dimension formula for quotients, which we will often use:

THEOREM 1.4 ([**Kra85**, Kap. II.4]). *If $G$ has a finite character group then*

$$\dim V /\!\!/ G = \dim V - \max_{v \in V} \dim Gv.$$

CHAPTER 2

# Polarizing and Homogenous Systems of Parameters

A set of algebraically independent homogeneous functions in $\mathcal{O}(V)^G$ whose zero set equals the nullcone $\mathcal{N}_V$ is called a *homogeneous system of parameters (HSP)* for $\mathcal{O}(V)^G$. Whenever a set of invariants $f_1, \ldots, f_r$ defines the nullcone we can find a HSP by taking $\dim \mathcal{O}(V)^G$ generic linear combinations of $f_1^{d/d_1}, \ldots, f_r^{d/d_r}$ where $d_i := \deg f_i$ and $d$ is the least common multiple of $d_1, \ldots, d_r$. As a drawback, unless all $f_i$ are of the same degree, this procedure increases the degree of the resulting functions, something that one usually wants to avoid in explicit calculations of HSP's.

Polarizing a function $f$ of degree $d$ onto $k$ copies yields $\binom{d+k-1}{d}$ functions (as many as there exist monomials of degree $d$ in $k$ variables) and hence sets of functions that are obtained by the polarization process tend to be rather big. If such a set defines the nullcone $\mathcal{N}_{V^{\oplus k}}$ it is therefore of interest to know, if some subset thereof yields a HSP or if it is necessary to pass over to the generic linear combinations as above.

If indeed such a subset exists then the resulting HSP for $\mathcal{O}(V^{\oplus k})^G$ has the property that its functions are multihomogeneous with respect to the $k$ copies of $V$ (since they are obtained by polarization). Such a HSP is called *multihomogeneous system of parameters (MHSP).*

Remark. The restriction of a multihomogeneous function on $V^{\oplus k}$ to a subdirect sum of $V^{\oplus k}$ either vanishes identically (if it depends also on some other copy) or remains unchanged. Hence MHSP's have the following nice property: if one restricts the functions of a MHSP of $\mathcal{O}(V^{\oplus k})^G$ to a subdirect sum of $V^{\oplus k}$ then the resulting functions are again a MHSP for this subdirect sum.

For $G = \mathrm{SL}_2$ all representations whose invariant rings allow MHSP's are classified in [**Bri82**]. There are only 13 of them. We will see now, that for most invariant rings $\mathcal{O}(V^{\oplus k})^G$ MHSP's exist only for small values of $k$. Generically, they even don't exist for $k > 2$.

Theorem 2.1. *If there exists a MHSP for $\mathcal{O}(V^{\oplus k})^G$ then*

$$k \dim \mathcal{O}(V)^G + \frac{k(k-1)}{2}(\dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G) \leq \dim \mathcal{O}(V^{\oplus k})^G.$$

Proof. Let $H$ be a MHSP for $\mathcal{O}(V^{\oplus k})^G$ and denote by $H_i$ resp. $H_{ij}$ the subset of functions in $H$ that depend exactly on the i-th resp. $i$-th and $j$-th copy of $V$ in $V^{\oplus k}$. Since $\mathcal{V}(H)$ contains the sets $\{\mathcal{N}_V \times \{0\} \times \ldots \times \{0\}\}, \ldots, \{\{0\} \times \ldots \times \{0\} \times \mathcal{N}_V\}$, it follows from our remark above that $|H_i| = \dim \mathcal{O}(V)^G$. To count the bihomogeneous functions $H_{ij}$ consider the points $\{(0, \ldots, 0, a, 0, \ldots, 0, b, 0, \ldots, 0)\} \subset V^{\oplus k}$

with $i$-th coordinate $a$ and $j$-th coordinate $b$ where $a, b \in \mathcal{N}_V$. Indeed all functions depending on one or three or more copies as well as all functions in $H_{p,q}$ with $(p,q) \neq (i,j)$ vanish on these. We conclude $|H_{ij}| = \dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G$ for all $1 \le i < j \le k$. Summing up, we find that

$$\bigcup_i H_i \cup \bigcup_{i<j} H_{ij} \subset H$$

and hence

$$k \dim \mathcal{O}(V)^G + \frac{k(k-1)}{2}(\dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G) \le |H| = \dim \mathcal{O}(V^{\oplus k})^G.$$

□

Corollary 2.2. *If $\mathcal{N}_{V^{\oplus 2}} \neq \mathcal{N}_V \times \mathcal{N}_V$ then there exists $k_0$ such that for all $k \ge k_0$ there is no MHSP for $\mathcal{O}(V^{\oplus k})^G$. In particular, functions obtained by polarizing invariants of $\mathcal{O}(V)^G$ onto $k_0$ or more copies cannot provide homogeneous systems of parameters.*

Proof. For every $k$ satisfying

$$k \dim \mathcal{O}(V)^G + \frac{k(k-1)}{2}(\dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G) > \dim \mathcal{O}(V^{\oplus k})^G$$

no MHSP for $\mathcal{O}(V^k)^G$ can exist, by the above theorem. But since $\mathcal{N}_{V^{\oplus 2}} \neq \mathcal{N}_V \times \mathcal{N}_V$ we have $|H_{ij}| = \dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G > 0$ and so the left-hand side of the inequality grows quadratically in $k$, hence only finitely many $k$ do not satisfy it. □

It is convenient to use the contra positive form of Theorem 2.1 as a numerical criterion:

Corollary 2.3. *If $k$ satisfies*

$$k \dim \mathcal{O}(V)^G + \frac{k(k-1)}{2}(\dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G) > \dim \mathcal{O}(V^{\oplus k})^G$$

*then no MHSP for $\mathcal{O}(V^k)^G$ can exist.*

This has some remarkable consequences. Let us start with an application on binary forms:

Corollary 2.4. *Let $V_d$ be the binary forms of degree $d$. $\mathcal{O}(V_d^{\oplus k})^{\mathrm{SL}_2}$ has a MHSP for a $k > 2$ if and only if the inequality in Corollary 2.3 is dissatisfied.*

Proof. We have $\dim \mathcal{O}(V_1)^{\mathrm{SL}_2} = 0$, $\dim \mathcal{O}(V_1^{\oplus 2})^{\mathrm{SL}_2} = \dim \mathcal{O}(V_2)^{\mathrm{SL}_2} = 1$ and $\dim \mathcal{O}(V_2^{\oplus 2})^{\mathrm{SL}_2} = 3$. One deduces that for $d = 1$ and $d = 2$ the inequality is satisfied for $k \ge 4$ and in both cases the three polarizations of the determinant onto three copies provide MHSP's. For $d = 1$ this is well known and for $d = 2$ we will prove it in the next chapter. As for $d \ge 3$, one has $\dim \mathcal{O}(V_d^k)^{\mathrm{SL}_2} = k(d+1) - 3$ and a simple calculation shows, that for $k > 2$ the inequality is always satisfied. □

Corollary 2.5. *If $\dim \mathcal{O}(V^{\oplus k})^G = k \dim V - \dim G$ for all $k$ then $\mathcal{O}(V^{\oplus m})^G$ has no MHSP for $m > 2$.*

Proof. Under the given condition Corollary 2.3 becomes:

$$k(\dim V - \dim G) + \frac{k(k-1)}{2}((2\dim V - \dim G) - 2(\dim V - \dim G)) > k \dim V - \dim G$$

which simplifies to

$$-(k-1)\dim G + \frac{k(k-1)}{2}\dim G = \frac{1}{2}(k-1)(k-2)\dim G > 0$$

which holds true for $k > 2$. □

This corollary can be applied to a wide variety of representations. For example, due to the dimension formula for quotients, it holds true for every representation $V$ of a semisimple group $G$ whose generic orbit has finite stabilizer. As an application, consider $\mathcal{F}_{n,d}$, the forms in $n$ variables of degree $d$ as representation of $\mathrm{SL}_n$:

COROLLARY 2.6. *For $n \geq 2$ and $d \geq 3$ there exists no MHSP for $\mathcal{O}(\mathcal{F}_{n,d}^{\oplus k})^{\mathrm{SL}_n}$ for $k > 2$.*

PROOF. It is known that for $n \geq 2$ and $d \geq 3$ the stabilizer of a generic element in $\mathcal{F}_{n,d}$ is finite (see for example [**Bri96**]), thus the requirements for Corollary 2.5 are met. □

REMARK. We will see in Chapter 6 that the invariant ring of two copies of ternary cubics $\mathcal{O}(\mathcal{F}_{3,3}^{\oplus 2})^{\mathrm{SL}_3}$ indeed has a MHSP.

CHAPTER 3

# Binary Forms

Let $V_n := \mathbb{C}[x,y]_n$ denote the space of binary forms of degree $n$ with the $\mathrm{SL}_2$ operation $g \cdot f = fg^{-1}$. The invariant rings $\mathcal{O}(V_n^{\oplus k})^{\mathrm{SL}_2}$ for $k=1$, $n \leq 6$ and $k=2$, $n \leq 4$ respectively were already considered by the geometers of the XIX-th century. However only the cases $(k,n) = (1,8)$ by Shioda [**Shi67**] and recently $(2,5)$ by Meulien [**Meu04**] could have been completely solved since then. Even more, several recent results show that the complexity in terms of generators and relations increases dramatically in the unsolved cases and hence no fast progress is expected in this area (see for example [**Meu05, Pop92, DiL85**]).
In this section we prove that for all $n > 1$ the polarizations of the generators of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ onto any $k$ copies generate the nullcone $\mathcal{N}_{V_n^{\oplus k}}$. Recall that from the Hilbert-Mumford criterion follows that a form belongs to $\mathcal{N}_{V_n}$ if and only if it has a linear factor of multiplicity $> \frac{n}{2}$. The main step in the proof is the following lemma.

Lemma 3.1. *Let $f(x)$, $g(x) \in \mathbb{C}[x]$ be two polynomials with the property that for infinitely many $t \in \mathbb{C}$ the polynomial $f(x) + tg(x)$ has a root of order $\ell \geq 2$. Then $f$ and $g$ have a common root of order $\ell$.*

Proof. Consider

$$X = \mathcal{V}(f(x) + yg(x), f'(x) + yg'(x), \ldots, f^{(\ell-1)}(x) + yg^{(\ell-1)}(x)) \subset \mathbb{C}^2.$$

Clearly, $(x_0, y_0) \in X$ if and only if $x_0$ is a root of order $\ell$ of $f(x) + y_0 g(x)$. From the first equation in $X$ follows that if $g(x_0) = 0$ then $x_0$ is a zero of $f(x)$. Otherwise we deduce $y_0 = \frac{-f(x_0)}{g(x_0)}$, combined with the second equation we get

$$f'(x_0) - \frac{f(x_0)}{g(x_0)} g'(x_0) = 0.$$

Thus $x_0$ is a zero of

$$f'(x)g(x) - f(x)g'(x)$$

which vanishes not identically since $f \cdot g^{-1}$ is not a constant. It follows that there are only finitely many values for the $x$-coordinates of points in $X$, therefore we can find two different points $(s, t_1)$ and $(s, t_2)$ in $X$. But then$f(x) + t_1 g(x)$ and $f(x) + t_2 g(x)$ have a common root of order $\ell$ and we are done. □

Theorem 3.2. *Every linear subspace of the nullcone $\mathcal{N}_{V_n}$ is annihilated by an 1-PSG of* $\mathrm{SL}_2$.

Proof. Let $H \subset \mathcal{N}_{V_n}$ be a two dimensional subspace spanned by two forms $f(x,y)$ and $g(x,y)$ and assume that they have no common factor of order $> \frac{n}{2}$. For infinitely many $t \in \mathbb{C}$ the $x$-degree of the factor of multiplicity $> \frac{n}{2}$ in $f(x,y) + tg(x,y)$ is nonzero, hence Lemma 3.1 can be applied to $f(x,1)$ and $g(x,1)$. It follows that $f(x,y)$ and $g(x,y)$ have a common factor of desired multiplicity, a contradiction to our assumption. Since a binary form of degree $n$ can only contain one factor with multiplicity $> \frac{n}{2}$, the case of higher dimensional subspaces follows from an easy induction. □

REMARK 1. After I finished my work on binary forms a preprint [**LMP05**] with a completely different proof of Theorem 3.2 appeared.

REMARK 2. One may have noticed that in the proof of Lemma 3.1 above there is no need for the existence of infinitely many $t$ with the given property. Let us carry out the sharp condition in order that $f$ and $g$ have a common root of order $\ell = 2$. It's harmless to reduce to the case where $f$ and $g$ have no common single roots and $\deg g = m < n = \deg f$. As the proof shows, it would then be sufficient to only require that there exist $\deg(f'(x)g(x) - f(x)g'(x)) + 1 = n + m$ different values of $t$ such that $f(x) + tg(x)$ has a root of order $\ell = 2$. Due to the well-known possibility of expressing the discriminant disc in terms of the resultant res we have $\deg_t \operatorname{disc}(f + tg) = \deg_t \operatorname{res}(f + tg, (f + tg)') \leq 2n - 1 - (n - m) = n + m - 1$, as the resultant in this case is a determinant of a $2n - 1 \times 2n - 1$ matrix such that in $n - m$ columns the variable $t$ does not appear. Hence the requirement of $n + m$ different values of $t$ is optimal. The generalization for roots of order $\ell > 2$ is straightforward.

Let us now turn to the question for which cases polarizing leads to homogenous systems of parameters. As a consequence of Corollary 2.6 we already know that for $n \geq 3$ and $k \geq 3$ no multihomogeneous system of parameters (MHSP) exists for $\mathcal{O}(V_n^{\oplus k})^{\mathrm{SL}_2}$. The precise answer is as follows:

THEOREM 3.3. *Functions obtained by polarizing the generators of* $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ *onto* $k$ *copies yield homogenous systems of parameters if and only if* $(n, k) \in \{(2, 2), (2, 3), (3, 2), (4, 2)\}$.

PROOF. $\mathcal{O}(V_2)^{\mathrm{SL}_2}$ is generated by an invariant $f_2$ (the index indicating the degree), $\mathcal{O}(V_3)^{\mathrm{SL}_2}$ by $g_4$ and $\mathcal{O}(V_4)^{\mathrm{SL}_2}$ by $h_2$ and $h_3$. It is easily verified that in the claimed cases the number of polarizations equals the dimension of the corresponding quotient. To exclude all other pairs $(n, k)$ we remark that all representations of $\mathrm{SL}_2$ that allow MHSP's are classified in [**Bri82**]. The ones of the form $V_n^{\oplus k}$, $n \geq 2$, are exactly the ones given in the theorem. □

CHAPTER 4

# Symmetric Bilinear Forms

Consider the usual action of $\mathrm{SL}_n$ on symmetric bilinear forms in $n$ variables by means of $(gq)(v,w) = q(g^{-1}v, g^{-1}w)$ or, equivalently, on the symmetric $n \times n$ matrices $Sym_n$ by $gA = (g^{-1})^t A g^{-1}$. From classical invariant theory it is known that $\mathcal{O}(Sym_n)^{\mathrm{SL}_n} = \mathbb{C}[\det]$ and that the $n+1$ polarizations of det onto two copies are algebraically independent and generate the invariant rings $\mathcal{O}(Sym_n^{\oplus 2})^{\mathrm{SL}_n}$, see [**AGo77**]. Thus the first natural question to ask in this context is, whether the polarizations of det onto three copies generate $\mathcal{O}(Sym_n^{\oplus 3})^{\mathrm{SL}_n}$, at least for some small $n$. The answer however is negative.

PROPOSITION 4.1. *The polarizations of* det *on* $k > 2$ *copies do not generate the invariant rings* $\mathcal{O}(Sym_n^{\oplus k})^{\mathrm{SL}_n}$.

PROOF. It suffices to prove the claim for $k = 3$ as the restriction of a polarized function onto three copies yields no new function on $\mathcal{O}(Sym_n^{\oplus 3})^{\mathrm{SL}_n}$. Consider the dimension formula for quotients:

$$\dim V/\!\!/G = \dim V - \max_{v \in V} \dim Gv.$$

For $G = \mathrm{SL}_n$ and $V = Sym_n \oplus Sym_n$ this yields

$$n + 1 = (n^2 + n) - (n^2 - 1).$$

Hence the generic orbit is already of maximal dimension and we conclude for $V = Sym_n^{\oplus 3}$:

$$\binom{n+2}{n} = \frac{n^2 + 3n + 2}{2} = 3\frac{n^2 + n}{2} - (n^2 - 1).$$

But the number of functions obtained by polarizing det onto three copies equals $\binom{n+2}{n}$ as well and it follows that if the polarizations would generate all invariants, then $\mathcal{O}(Sym_n^{\oplus 3})^{\mathrm{SL}_n}$ is isomorphic to a polynomial ring. However representations of $\mathrm{SL}_n$ with this property are classified in [**Sch78**] and $Sym_n^{\oplus 3}$ is not one of them. $\square$

MAIN THEOREM. *The nullcone* $\mathcal{N}_{Sym_n^{\oplus k}}$ *is defined by the polarizations of the determinant if and only if* $n \leq 4$ *or* $k = 2$.

As an immediate consequence we get the following corollary:

COROLLARY 4.2. *For* $n \leq 4$ *the functions obtained by polarizing* det *onto three copies are a multihomogeneous system of parameters for* $\mathcal{O}(Sym_n^{\oplus 3})^{\mathrm{SL}_n}$.

PROOF. We have seen in the proof of the above Proposition 4.1 that the number of these functions equals the dimension of the corresponding quotients. $\square$

We will establish the Main Theorem by proving that for $n \leq 4$ every subspace $H \subset \mathcal{N}_{Sym_n}$ is annihilated by a 1-PSG of $\mathrm{SL}_n$ and for $n > 4$ by exhibiting subspaces of $\mathcal{N}_{Sym_n}$ of dimension 3 that are not annihilated. Since the polarizations of det generate the invariant rings $\mathcal{O}(Sym_n^{\oplus 2})^{\mathrm{SL}_n}$ it is clear that every two-dimensional subspace is annihilated. Note that throughout this chapter we often change whether we view an element $A \subset Sym_n$ as matrix or as its corresponding bilinear form. For simplicity's sake however we introduce no additional notation.

The proof of the Main Theorem is divided into several steps. At first we state the two crucial lemmas.

LEMMA 4.3. *Let $H$ be a singular subspace of $Sym_n$ and let $A$ be an element of $H$. Then the restriction of $H$ to* $\operatorname{rad} A$ *is still a singular space.*

PROOF. We can assume that $A$ corresponds to the form $x_1^2 + \ldots + x_m^2$ for some $m < n$ and so $\operatorname{rad} A$ is spanned by $e_{m+1}, \ldots, e_n$. Now take an arbitrary element $B$ of $H$ whose matrix we represent as

$$\begin{bmatrix} B_1 & B_2 \\ B_2^t & B_4 \end{bmatrix}$$

with $B_4$ being an $n - m \times n - m$ block. Now we see that

$$\det(sA + B) = \begin{bmatrix} sE_m + B_1 & B_2 \\ B_2^t & B_4 \end{bmatrix} = s^m \det(B_4) + \{\text{terms of } s\text{-degree} < m\} = 0.$$

Hence $\det(B_4) = 0$ and the claim follows. $\square$

REMARK. Let $A$ and $H$ be as in the proof. The action of $\begin{bmatrix} \mathrm{SL}_m & \\ & \mathrm{SL}_{n-m} \end{bmatrix} \subset \mathrm{SL}_n$ preserves the image as well as the kernel of the restriction map $B \mapsto \bar{B} := B|_{U \times U}$ where $U := \operatorname{rad} A$. Since $\dim \operatorname{rad}(A) < n$, it is clear how the classification of maximal singular spaces of forms on less variables will come into play.

LEMMA 4.4. *Let $H$ be a singular subspace of $Sym_n$ with the property that there exists a $n-1$-dimensional subspace $W$ of $\mathbb{C}^n$ such that the restriction of $H$ to $W$ is still a singular space. Then $H$ lies in a maximal singular subspace containing a rank-one matrix.*

PROOF. We can assume that $W = \{x_1 = 0\}$ and then $\det_{11}(h) = 0$ for all $h \in H$ where $\det_{11}$ means the minor obtained by deleting the first row and column. Let now $A$ be the matrix corresponding to the form $x_1^2$ and then for all $h \in H$ we find $\det(sA + h) = s \det_{11}(h) + \det(h) = 0$. This shows, that $\mathbb{C}A + H$ spans still a singular space. $\square$

## 1. The Case $n = 2$

PROPOSITION 4.5. *Every subspace of $\mathcal{N}_{Sym_2}$ is annihilated by a 1-PSG.*

PROOF. This has already been proved in Theorem 3.2 since symmetric forms in two variables are also binary forms of degree two. It is an easy consequence of Lemma 4.3 as well. $\square$

## 2. The Case $n = 3$

Consider the following subspaces of $\mathcal{N}_{Sym_3}$

$$W_1 = \left\{ \begin{bmatrix} * & * & \\ * & * & \\ & & \end{bmatrix} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} * & * & * \\ * & & \\ * & & \end{bmatrix} \right\}$$

which are clearly annihilated by the one-parameter subgroups

$$\lambda_1(t) = \begin{bmatrix} t & & \\ & t & \\ & & t^{-2} \end{bmatrix} \quad \text{respectively} \quad \lambda_2(t) = \begin{bmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{bmatrix}.$$

We claim that, up to the action of $\mathrm{SL}_3$, these are the two maximal subspaces of $\mathcal{N}_{Sym_3}$ (in the set-theoretic sense).

PROPOSITION 4.6. *Every subspace $H \subset \mathcal{N}_{Sym_3}$ is equivalent to a subspace of $W_1$ or $W_2$ and thus is annihilated by a 1-PSG of* $\mathrm{SL}_3$.

PROOF. We assume first that $H$ contains the rank-one form $x_1^2$. Note that we can annihilate every singular space of $\mathcal{N}_{Sym_2}$ and so by Lemma 4.3 and its remark, we end up with every $h \in H$ being of the form:

$$h = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & 0 \\ h_{13} & 0 & 0 \end{bmatrix}$$

where the right lower $2 \times 2$ block corresponds to the singular space of the restriction. Since $\det(h) = -h_{13}^2 h_{22} = 0$ and $H$ is a linear system we must have either $h_{13} = 0$ for all $h \in H$ and hence $H \subset W_1$ or $h_{22} = 0$ for all $h \in H$ and so $H \subset W_2$. The claim for arbitrary $H$ follows now from the next lemma. □

LEMMA 4.7. *Every maximal subspace $H \subset \mathcal{N}_{Sym_3}$ contains a rank-one element.*

PROOF. Otherwise let $H$ be a maximal singular space of constant rank two. Assume $A \in H$ corresponds to the form $x_1 x_2$. Now for $h \in H$ consider $tA + h$ and due to Lemma 4.3 applied to $A$ we find $tA + h$ being of form

$$tA + h = \begin{bmatrix} h_{11} & t + h_{12} & h_{13} \\ t + h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & 0 \end{bmatrix}.$$

Since for all $h \in H$

$$\det(tA + h) = 2t h_{13} h_{23} + \{\text{terms of } t\text{-degree } 0\} = 0$$

it follows that either $h_{13} = 0$ for all $h \in H$ or $h_{23} = 0$ for all $h \in H$. But then one of the $2 \times 2$-minors $\det_{11}(h) = -h_{23}^2$ or $\det_{22}(h) = -h_{13}^2$ vanishes, a contradiction to Lemma 4.4. □

## 3. The Case $n = 4$

Consider the following subspaces of $\mathcal{N}_{Sym_4}$

$$W_1 = \left\{ \begin{bmatrix} * & * & * & \\ * & * & * & \\ * & * & * & \\ & & & \end{bmatrix} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & & \\ * & & & \\ * & & & \end{bmatrix} \right\}$$

which are clearly annihilated by the one-parameter subgroups

$$\lambda_1(t) = \begin{bmatrix} t & & & \\ & t & & \\ & & t & \\ & & & t^{-3} \end{bmatrix} \quad \text{respectively} \quad \lambda_2(t) = \begin{bmatrix} t^3 & & & \\ & t & & \\ & & t^{-2} & \\ & & & t^{-2} \end{bmatrix}.$$

We claim, that these are the two maximal subspaces of $\mathcal{N}_{Sym_4}$.

Proposition 4.8. *Every subspace $H \subset \mathcal{N}_{Sym_4}$ is equivalent to a subspace of $W_1$ or $W_2$ and thus is annihilated by a 1-PSG of* $\mathrm{SL}_4$.

Proof. As before, assume first $H$ contains the rank one form $x_1^2$. By use of Lemma 4.3 and the fact, that we can annihilate every subspace of $\mathcal{N}_{Sym_3}$, we may assume that $H$ is a subspace of either

$$P = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & \\ * & * & * & \\ * & & & \end{bmatrix} \right\} \quad \text{or} \quad Q = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & & \\ * & * & & \end{bmatrix} \right\}$$

where the right lower $3 \times 3$ blocks correspond to the two types of singular spaces of the restriction. In the first case we find

$$\det(P) = -p_{14}^2 \cdot \det \begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix} = 0$$

and conclude that either $p_{14} = 0$ and hence $H \subset W_1$, or $\det \begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix} = 0$, which means that $H|_U$ is singular where $U := \mathbb{C}e_2 \oplus \mathbb{C}e_3$. Since we can annihilate every subspace of $\mathcal{N}_{Sym_2}$, with a suitable base change in $U$ we get $H|_U \subset \left\{ \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and so $H \subset W_2$.

For the remaining case where $H \subset Q$ we have

$$\det(Q) = -(\det \begin{bmatrix} q_{13} & q_{14} \\ q_{23} & q_{24} \end{bmatrix})^2 = 0$$

and hence $B = \left\{ \begin{bmatrix} q_{13} & q_{14} \\ q_{23} & q_{24} \end{bmatrix} \right\}$ is a rank one space. For suitable $g, h \in \mathrm{SL}_2$ a base change of the form $\begin{bmatrix} g & \\ & h \end{bmatrix}$ replaces $B$ by $gBh^t$ which allows the form $\begin{bmatrix} * & \\ * & \end{bmatrix}$ or $\begin{bmatrix} * & * \\ & \end{bmatrix}$,(see Part III, proof of Theorem 2) and hence $H \subset W_1$ resp. $H \subset W_2$. The claim for arbitrary $H$ follows now from the next lemma. □

Lemma 4.9. *Every maximal subspace $H \subset \mathcal{N}_{Sym_4}$ contains a rank-one element.*

Proof. Let $H$ be a maximal singular subspace not containing rank-one elements. From the normal form for symmetric matrix pencils due to Kronecker and Weierstrass (see [**Gan59**, chap. XII.4 ]) one reads off, that spaces of dimension $> 1$ of constant rank $r$ exist only for even $r$. Hence we find in $H$ a rank-two matrix $A$ which we may assume to correspond to the form $x_1 x_2$. Now for $h \in H$ consider $tA + h$ and due to Lemma 4.3 applied to $A$ we find $tA + h$ being of form

$$tA + h = \begin{bmatrix} h_{11} & t + h_{12} & h_{13} & h_{14} \\ t + h_{12} & h_{22} & h_{23} & h_{24} \\ h_{13} & h_{23} & h_{33} & 0 \\ h_{14} & h_{24} & 0 & 0 \end{bmatrix}.$$

Now one easily finds

$$\det(tA + h) = 2t h_{14} h_{24} h_{33} + \{\text{terms of } t\text{-degree } 0\} = 0$$

imposing $h_{14} h_{24} h_{33} = 0$. But $h_{24} h_{33} = 0$ yields $\det_{11}(h) = 0$ and $h_{14} = 0$ implies $\det_{22}(h) = 0$, both a contradiction to Lemma 4.4. □

## 4. The Case $n = 5$

Consider the following subspaces of $\mathcal{N}_{Sym_5}$

$$W_1 = \begin{bmatrix} * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ & & & & \end{bmatrix} \quad W_2 = \begin{bmatrix} * & * & * & * & * \\ * & * & * & & \\ * & * & * & & \\ * & & & & \\ * & & & & \end{bmatrix} \quad W_3 = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & & & \\ * & * & & & \\ * & * & & & \end{bmatrix}$$

which are annihilated by the one-parameter subgroups

$$\lambda_1(t) = \operatorname{diag}(t, t, t, t, t^{-4}), \lambda_2(t) = \operatorname{diag}(t^4, t, t, t^{-3}, t^{-3})$$

respectively

$$\lambda_2(t) = \operatorname{diag}(t^3, t^3, t^{-2}, t^{-2}, t^{-2}).$$

Proposition 4.10. *Every maximal subspace $H \subset \mathcal{N}_{Sym_5}$ containing a rank-one element is equivalent to a subspace of $W_1$, $W_2$ or $W_3$ and hence annihilated by a 1-PSG of* $\mathrm{SL}_5$.

Proof. Let $H$ contain the rank-one matrix corresponding to the form $x_1^2$. By use of Lemma 4.3 and the fact, that we can annihilate every subspace of $\mathcal{N}_{Sym_4}$, we may assume that $H$ is a subspace of either

$$P = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ * & & & & \end{bmatrix} \quad \text{or} \quad Q = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & & \\ * & * & & & \\ * & * & & & \end{bmatrix}$$

where the right lower $4 \times 4$ blocks correspond to the two types of singular spaces of the restriction. The condition $\det(P) = 0$ implies either $p_{15} = 0$ for all $p \in P$ and thus $H \subset W_1$, or $H|_U$ is singular where $U := \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$. After a suitable base change in $U$ we get $H \subset W_2$ or $H \subset W_3$. The condition $\det(Q) = q_{33}(\det \begin{bmatrix} q_{14} & q_{15} \\ q_{24} & q_{25} \end{bmatrix})^2 = 0$ implies either $q_{33} = 0$ for all $q \in Q$ and so $H \subset W_3$, or else $\left\{ \begin{bmatrix} q_{14} & q_{15} \\ q_{24} & q_{25} \end{bmatrix} \right\}$ is a rank one space. Similarly as in the proof Proposition 4.8, case of $Q$, we conclude that $H \subset W_1$ or $H \subset W_2$. □

In contrast to the cases $n \leq 4$ it is no longer true that every maximal singular space contains a rank one form.

THEOREM 4.11. *There exist maximal spaces in $\mathcal{N}_{Sym_5}$ that contain no elements of rank one and thus cannot be annihilated.*

PROOF. Consider $H = \{A_1, A_2, A_3\}$ with

$$sA_1 + tA_2 + uA_3 = \begin{bmatrix} 0 & 0 & 0 & s & t \\ 0 & 0 & s & t & 0 \\ 0 & s & 0 & 0 & -u \\ s & t & 0 & 2u & 0 \\ t & 0 & -u & 0 & 0 \end{bmatrix}.$$

A direct computation shows that $\det(sA_1 + tA_2 + uA_3) = 0$. Assume now $H$ lies in a maximal singular space containing a rank one element $B$ corresponding to the form $(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5)^2$. The condition that $sA_1 + B$ and $tA_2 + B$ are nullforms implies that $b_5 = b_3 = 0$. Furthermore the nullforms $sA_1 + tA_2 + B$, $tA_2 + uA_3 + B$ and $sA_1 + uA_3 + B$ show that $b_4 = b_1 = b_2 = 0$. □

To finish the proof of the Main Theorem it remains to exhibit singular spaces of dimension 3 for $n > 5$ that cannot be annihilated by a 1-PSG of $\mathrm{SL}_n$. To this end we need the following proposition which will also be of use later. Consider the operation of $\mathrm{SL}_n \times \mathrm{SL}_n$ on the quadratic matrices $M_n$ by means of $(g, h) \cdot A = gAh^{-1}$.

PROPOSITION 4.12. *A subspace $H \subset M_n$ is annihilated by a 1-PSG $(\lambda(t), \mu(t))$ of $\mathrm{SL}_n \times \mathrm{SL}_n$ if and only if there exists a subspace $W \subset \mathbb{C}^n$ such that $\dim HW < \dim W$.*

PROOF. If $\lim_{t\to 0} \lambda(t)H\mu(t)^{-1} = 0$ we may assume $\lambda(t) = \mathrm{diag}(t^{a_1}, \ldots, t^{a_n})$ and $\mu(t)^{-1} = \mathrm{diag}(t^{b_1}, \ldots, t^{b_n})$ with $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$. Since $\lambda(t), \mu(t) \in \mathrm{SL}_n$ we have $\sum_i a_i = \sum_i b_i = 0$. If $a_i + b_{n+1-i} > 0$ for all $i$, then $\sum_i (a_i + b_i) > 0$, a contradiction. Hence we must have $a_s + b_{n+1-s} \leq 0$ for some $s$. But then $a_i + b_j \leq 0$ for $i \geq s$, $j \geq n + 1 - s$, as the sequences of the $a_i$ and $b_i$ are decreasing. Since for an $h \in H$ the $ij$-th entry of $\lambda(t)h\mu(t)^{-1}$ is $h_{ij}t^{a_i+b_j}$ we conclude $h_{ij} = 0$ for $i \geq s$, $j \geq n + 1 - s$ and hence

$$h \in \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & \ldots & 0 \\ * & * & \vdots & \ddots & \\ * & * & 0 & & 0 \end{bmatrix}$$

where the left upper 0 is at position $(s, n+1-s)$. Now set $W = \{e_{n+1-s}, \ldots, e_n\}$ to see $HW \subset \{e_1, \ldots, e_{s-1}\}$ and hence $\dim HW = s - 1 < s = \dim W$.

On the other hand, assume such a $W \subset \mathbb{C}^n$ exists. Given the operation of $\mathrm{SL}_n \times \mathrm{SL}_n$ on $H$ we may assume that $W = \{e_j, \ldots, e_n\}$ and $HW = \{e_1, \ldots, e_i\}$ where $i < n - j + 1$. So we see that

$$H \subset \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & \ldots & 0 \\ * & * & \vdots & \ddots & \\ * & * & 0 & & 0 \end{bmatrix}$$

where the left upper 0 is at position $(i+1, j)$. We construct now a 1-PSG $(\lambda(t), \mu(t))$ of $\mathrm{SL}_n \times \mathrm{SL}_n$ as follows (cf. Part III, first part of proof of Theorem 1): Let $\lambda(t)$ having weights $n - i$ on $e_1, \ldots, e_i$ and $-i$ on $e_{i+1}, \ldots, e_n$ and $\mu(t)^{-1}$ having weights $n - j + 1$ on $e_1, \ldots, e_{j-1}$ and $-j + 1$ on $e_j, \ldots, e_n$. Since $n - i - j + 1 > 0$ it follows that $(\lambda(t), \mu(t))$ annihilates every entry at position $(p, q)$ as soon as $p < i + 1$ or $q < j$ and so $\lim_{t\to 0} \lambda(t) H \mu(t)^{-1} = 0$. $\square$

As for the representation of $\mathrm{SL}_n$ on $Sym_n$, note that the image of $\mathrm{SL}_n$ in $\mathrm{GL}(M_n)$ is contained in the image of $\mathrm{SL}_n \times \mathrm{SL}_n$ under the representation

$$(g, h) \cdot A = gAh^{-1}$$

and hence when viewing the elements of $Sym_n$ as linear maps we can formulate the proposition above for $\mathrm{SL}_n$ on $Sym_n$:

PROPOSITION 4.13. *A subspace $H \subset Sym_n$ is annihilated by a 1-PSG $\lambda(t)$ of $\mathrm{SL}_n$ if and only if there exists a subspace $W \subset \mathbb{C}^n$ such that* $\dim HW < \dim W$.

The space $H$ in Theorem 4.11 cannot be annihilated by a 1-PSG of $\mathrm{SL}_5$ hence there exist no subspace $W \subset \mathbb{C}^5$ such that $\dim HW < \dim W$. It is easy to see that for $n > 5$ for the space

$$sA_1 + tA_2 + uA_3 = \begin{bmatrix} 0 & 0 & 0 & s & t & & & \\ 0 & 0 & s & t & 0 & & & \\ 0 & s & 0 & 0 & -u & & & \\ s & t & 0 & 2u & 0 & & & \\ t & 0 & -u & 0 & 0 & & & \\ & & & & & s & & \\ & & & & & & \ddots & \\ & & & & & & & s \end{bmatrix}$$

there does not exist such a subspace $W \subset \mathbb{C}^n$ as well and thus it cannot be annihilated. This finishes the proof of the Main Theorem.

REMARK 3. Theorem 4.11 is remarkable as C.T.C. WALL in his paper [**Wall78**] claimed that the nullcone $\mathcal{N}_{Sym_n}$ for any $n$ is defined by the polarisations of the determinant. As we have seen this is correct only for $n \leq 4$.

## 5. The Case $n \geq 5$

Under certain rank conditions we can give a somewhat more detailed picture of the structure of singular spaces of $\mathcal{N}_{Sym_n}$ for $n \geq 5$. Note that the orbit structure of $\mathcal{N}_{Sym_n}$ is given by the rank of the elements:

$$\mathcal{N}_{Sym_n} = \bigcup_{m=0}^{n-1} O_m$$

where $O_m = \mathrm{SL}_n \cdot (x_1^2 + \ldots + x_m^2)$ is the orbit of the rank $m$ elements and $\overline{O_m} = O_0 \cup \ldots \cup O_m$. We say a subspace $H$ is *bounded by rank* $m$ if $H \subset \overline{O_m}$. Consider the following variant of Lemma 4.3:

Lemma 4.14. *Let $H$ be a subspace of $Sym_n$ bounded by rank $m$ and let $A \in H$ be an element of rank $m$. Then the restriction of $H$ to* $\mathrm{rad}(A)$ *is zero.*

Proof. As before we may restrict to the case where $A$ corresponds to the form $x_1^2 + \ldots + x_m^2$. Let $B$ be another element of $H$ with entries $b_{ij}$. By assumption, every $m+1 \times m+1$-minor of $tA + B$ vanishes. Construct now minors like this: Delete $n-m-1$ columns but none of the first $m$ ones and delete $n-m-1$ rows but again none of the first $m$ ones. From Lemma 4.3 we see that in every minor the entry in the right bottom corner is zero, hence $b_{ij} = 0$ for $i, j > m$ and we are done. □

Proposition 4.15. *Every singular subspace $H \subset \mathcal{N}_{Sym_n}$ bounded by rank $m := \lfloor \frac{n-1}{2} \rfloor$ is annihilated.*

Proof. By the preceding Lemma 4.14 we can assume $H$ to be of form

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2^t & 0 \end{bmatrix}$$

with $H_1$ being an $m \times m$ block and $H_2$ being an $m \times (n-m)$ block. Now for

$$\lambda(t) = \mathrm{diag}(\overbrace{t^{n-m}, \ldots, t^{n-m}}^{m}, \overbrace{t^{-m}, \ldots, t^{-m}}^{n-m})$$

we see that $\lim_{t \to 0} \lambda(t) \cdot H = 0$. □

In this context one may ask for spaces of low rank that are not annihilated. To construct such spaces the following observation of Jan Draisma is helpful:

Lemma 4.16. *Let $M \subset M_n$ be a subspace that is not annihilated by a 1-PSG of $\mathrm{SL}_n \times \mathrm{SL}_n$. Then $F = \{\begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix} \mid A \in M\} \subset Sym_{2n}$ is not annihilated by a 1-PSG of $\mathrm{SL}_{2n}$ as well.*

Proof. By Proposition 4.13 we have to show, that for all subspaces $W \subset \mathbb{C}^{2n}$ we have $\dim FW \geq \dim W$. Consider $\mathbb{C}^{2n}$ as $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n =: V_1 \oplus V_2$ with projections $p_i : \mathbb{C}^{2n} \to V_i$, and the elements of $M$ as linear maps $V_2 \to V_1$. With this notations we have

$$\dim FW = \dim p_1(FW) + \dim(FW \cap V_2).$$

But $\dim p_1(FW) = \dim(M \cdot p_2(W)) \geq \dim p_2(W)$ since $M$ cannot be annihilated. Furthermore, since $FV_1 \subset V_2$ we see $FW \cap V_2 \supset F(W \cap V_1) = M^t(W \cap V_1)$ and

$\dim M^t(W \cap V_1) \geq \dim W \cap V_1$ as $M^t$ cannot be annihilated as well. Summing up we get

$$\dim FW \geq \dim p_2(W) + \dim(W \cap V_1) = \dim W.$$

□

Consider now the space $M = \{A, B, C\} \subset M_3$ with

$$sA + tB + uC = \begin{bmatrix} 0 & s & t \\ -s & 0 & u \\ -t & -u & 0 \end{bmatrix}.$$

By use of Proposition 4.12 it is easy to see that $H$ cannot be annihilated by an 1-PSG of $\mathrm{SL}_3 \times \mathrm{SL}_3$. Hence by the above Lemma 4.16 the space

$$\begin{bmatrix} & M \\ M^t & \end{bmatrix} \subset Sym_6$$

under the operation of $\mathrm{SL}_6$ cannot be annihilated as well and is bounded by rank 4. Without much effort this can be generalized to spaces

$$H = \{ \begin{bmatrix} & & A \\ & \iddots & \\ A & & \end{bmatrix} \mid A \in M\} \subset M_{3m} \text{ and } \begin{bmatrix} & H \\ H^t & \end{bmatrix} \subset Sym_{6m}$$

and thus for $n = 6m$ we find subspaces bounded by rank $\frac{2n}{3}$ that cannot be annihilated. We conclude that - although there is still some gap - the bound found in Theorem 4.15 is not that bad.

Since every singular space bounded by rank 2 can be annihilated and there exists singular spaces bounded by rank 4 which cannot, the last open case are spaces bounded by rank 3:

THEOREM 4.17. *Every singular subspace $H \subset \mathcal{N}_{Sym_n}$ bounded by rank 3 can be annihilated.*

PROOF. By Theorem 4.15 the remaining cases to consider are $n = 5$ and $n = 6$. For $n = 5$ it is clear that a space $H$ bounded by rank 3 remains singular after the addition of a rank-one element. Thus $H$ is annihilated due to Theorem 4.10.
For $n = 6$ due to Lemma 4.14 we may assume $H$ to be of the form

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2^t & 0 \end{bmatrix}$$

with $H_1, H_2$ being $3 \times 3$ blocks. Now $\det(H) = 0$ imposes $\det(H_2) = 0$ and we conclude that $H_2$ is bounded by rank 1 since $H$ is bounded by rank 3. A suitable base change of form $\begin{bmatrix} g & \\ & h \end{bmatrix}$ with $g, h \in \mathrm{SL}_3$ replaces $H_2$ by $gH_2h^t$ which can be brought in the form

$$\begin{bmatrix} * & & \\ * & & \\ * & & \end{bmatrix} \text{ or } \begin{bmatrix} * & * & * \\ & & \\ & & \end{bmatrix}$$

and so $H$ is annihilated by either $\lambda(t) = \mathrm{diag}(t^4, t^1, t^1, t^{-2}, t^{-2}, t^{-2})$ or $\lambda(t) = \mathrm{diag}(t, t, t, t, t, t^{-5})$.

□

CHAPTER 5

# Skew-Symmetric Forms

Denote by $\mathcal{B}_n$ the skew-symmetric forms in $n$ variables with the usual $\mathrm{SL}_n$ operation. For $n$ even it is well-known that the invariant ring $\mathcal{O}(\mathcal{B}_n)^{\mathrm{SL}_n}$ is generated by the *Pfaffian* which we will denote by Pf and that the polarizations $\mathcal{P}^2(\mathrm{Pf})$ generate the invariants $\mathcal{O}(\mathcal{B}_n \oplus \mathcal{B}_n)^{\mathrm{SL}_n}$, see [**AGo77**]. Since $\mathcal{B}_2$ has no nonzero singular elements we begin our analysis of the singular subspaces with the case $n = 4$:

Proposition 5.1. *Every singular subspace $H \subset \mathcal{N}_{\mathcal{B}_4}$ is annihilated by a 1-PSG of* $\mathrm{SL}_4$.

Proof. Since skew-symmetric forms are of even rank, $H$ is of fixed rank two. By similar arguments as in Lemma 4.14 we can assume $H$ to be of the form

$$H = \begin{bmatrix} H_1 & H_2 \\ -H_2^t & 0 \end{bmatrix}$$

with $H_2$ necessarily being a rank one space. A suitable base change of the form $\begin{bmatrix} g & \\ & h \end{bmatrix}$ with $g, h \in \mathrm{SL}_2$ puts $H$ into form

$$\begin{bmatrix} * & * & * & \\ * & * & * & \\ * & * & & \\ & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & * & * & * \\ * & * & & \\ * & & & \\ * & & & \end{bmatrix}$$

and hence $H$ is annihilated. □

Theorem 5.2. *The polarizations of* Pf *form a multihomogeneous system of parameters (MHSP) for $\mathcal{O}(\mathcal{B}_4^{\oplus k})^{\mathrm{SL}_4}$ if $k \leq 6$. For $k > 6$ no MHSP exists for $\mathcal{O}(\mathcal{B}_4^{\oplus k})^{\mathrm{SL}_4}$.*

Proof. By the dimension formula for quotients we have

$$\dim \mathcal{O}(\mathcal{B}_4^{\oplus 6})^{\mathrm{SL}_4} \geq 6 \cdot 6 - 15 = 21.$$

However the number of functions obtained by polarizing Pf onto 6 copies is $\binom{2+5}{2} = 21$ as well and since these functions define the nullcone they have to be algebraically independent which proves the first claim. For the second, note that $\dim \mathcal{O}(\mathcal{B}_4^{\oplus 6})^{\mathrm{SL}_4} = 6 \cdot 6 - 15 = 21$ implies that $\dim \mathcal{O}(\mathcal{B}_4^{\oplus k})^{\mathrm{SL}_4} = 6k - 15$ for $k \geq 6$ as the generic orbit is of maximal dimension. Thus the inequality in Corollary 2.3:

$$k \dim \mathcal{O}(V)^G + \frac{k(k-1)}{2}(\dim \mathcal{O}(V^2)^G - 2\dim \mathcal{O}(V)^G) > \dim \mathcal{O}(V^k)^G$$

becomes

$$k + \frac{k(k-1)}{2} > 6k - 15$$

respectively

$$(k-5)(k-6) > 0$$

which is satisfied for $k > 6$. □

PROPOSITION 5.3. *For $n \geq 3$ there are singular subspaces $H \subset \mathcal{N}_{\mathcal{B}_{2n}}$ of dimension 3 which are not annihilated by a 1-PSG of* $\mathrm{SL}_{2n}$.

PROOF. Consider the space $H = \{A, B, C\} \subset M_n$ with

$$sA + tB + uC = \begin{bmatrix} 0 & s & t & & & \\ -s & 0 & u & & & \\ -t & -u & 0 & & & \\ & & & s & & \\ & & & & \ddots & \\ & & & & & s \end{bmatrix}.$$

By use of Proposition 4.12 one verifies that $H$ cannot be annihilated by an 1-PSG of $\mathrm{SL}_n \times \mathrm{SL}_n$. Hence by the obvious skew-symmetric version of Lemma 4.16 the space $\begin{bmatrix} & M \\ -M^t & \end{bmatrix} \subset \mathcal{B}_{2n}$ is not annihilated by a 1-PSG of $\mathrm{SL}_{2n}$ as well. □

CHAPTER 6

# Ternary Cubic Forms

Let $T$ be the vector space of ternary cubic forms endowed with the action of $G = \mathrm{SL}_3$. According to [**Kra85**] the nullcone $\mathcal{N}_T$ consists of six orbits:

$$\mathcal{N}_T = \{0\} \cup Gx^3 \cup Gx^2y \cup Gxy(x+y) \cup G(x^2-yz)y \cup G(y^2z-x^3) =: B_0 \cup \ldots \cup B_5$$

and the closure of the orbits is given by $\overline{B_i} = B_0 \cup \ldots \cup B_i$. Inspecting the weight system reveals that, up to the action of the Weyl group, there is only one maximal set of weights that is annihilated by a 1-PSG of $\mathrm{SL}_3$, namely $\{x^3, x^2y, xy^2, y^3, y^2z\}$. We will show that under the action of $\mathrm{SL}_3$ every subspace $H \subset \mathcal{N}_T$ is equivalent to a subspace spanned by these weights.

To this end, we need BERTINI's classical theorem (see [**Har92**]) in the following setting: let $f_0, \ldots, f_s$ be linearly independent projective plane curves of degree three and consider the *linear system* they span, that is the family

$$U_t : t_0f_0 + \ldots + t_sf_s = 0$$

where $t = (t_0, \ldots, t_s) \in \mathbb{P}^s$. The *base points* of $U_t$ are the points in $\mathcal{V}(f_0, \ldots, f_s)$.

BERTINI'S THEOREM. *A generic member of $U_t$ has no singular points outside the base points.*

Now we are ready to prove the following theorem:

THEOREM 6.1. *Every singular subspace $H \subset T$ is annihilated by a 1-PSG of* $\mathrm{SL}_3$.

PROOF. We have to deal with the five different cases where $H$ lies in $\overline{B_i}$ and contains elements of $B_i$, $i = 1, 2, \ldots, 5$.

(1) $H$ lies generically in $\overline{B_5} = \overline{G(y^2z - x^3)}$: Since $B_5 \cap H$ is a dense subset of $H$ we may choose a basis $f_0, \ldots, f_s$ of $H$ such that every $f_i \in B_5$. Note that $B_5$ consists of irreducible elements, hence the corresponding linear system $U_t$ : $t_0f_0 + \ldots + t_sf_s = 0$ has only finitely many base points as long as $s \geq 1$. A generic member of $H$ is a cusp and has therefore exactly one singular point. By BERTINI's Theorem, these singular points are base points and it follows that all the $f_i$ have a common singular point. Without restriction we set $f_0 = y^2z - x^3$ and therefore $[0, 0, 1]$ to be the singular point of the $f_i$. A simple calculation shows, that a ternary cubic having $[0, 0, 1]$ as singular point lies in the span of the monomials $\{x^2z, xyz, x^3, x^2y, xy^2, y^3, y^2z\}$. Thus we have to show, that the $f_i$ don't depend on the first two monomials as we can simultaneously annihilate the others. Indeed the condition that the singularity at $[0, 0, 1]$ is not a double point for $f = ax^2z + bxyz + cy^2z + \ldots$ amounts to the condition $b^2 - 4ac = 0$ (in the via $[x, y, 1]$ dehomogenized coordinates the quadratic terms have to be a square of a linear form). But since $U_t$

is a linear system and the coefficient of $y^2z$ in $f_0$ is nonzero we deduce, that in all $f_i$ the coefficients of the monomials $zx^2$ and $xyz$ indeed are zero and hence $H$ is annihilated.

(2) $H$ lies generically in $\overline{B_4} = \overline{G(x^2 - yz)y}$: We may assume that $f_0 = (x^2 - yz)y \in H$. If the linear system associated to $H$ has only finitely many base points, then the same argument as above shows that $H$ is annihilated. If there are infinitely many base points then there exists a common component. In case the common component is the parabola $x^2 - yz$ then the set of all lines tangent to $p$ is given by $L = \mathbb{C}\{2tx - y - t^2z \mid t \in \mathbb{C}\}$ and it is easy to see that a linear subspace contained in $L$ has dimension one. Otherwise the common component is the line $y = 0$. A parabola $ax^2 + bxz + cz^2 + dyz + ey^2 + lxy$ to which $y = 0$ is a tangent satisfies $b^2 - 4ac = 0$ and since the coefficient of $x^2$ in $f_0$ is nonzero we conclude that $H \subset \{(ax^2 + dyz + ey^2 + lxy)y\}$ and therefore is annihilated.

(3) $H$ lies generically in $\overline{B_3} = \overline{Gxy(x + y)}$: A generic element of the corresponding system $U_t$ consists of the union of three different lines meeting in one point. If $U_t$ contains only finitely many base points then by Bertini's Theorem we may assume that all $f_i$ meet in the same triple point, say $[0, 0, 1]$. This implies that the $f_i$ are independent of the variable $z$, as they consist of lines passing through $[0, 0, 1]$ and thus $H$ is annihilated. If otherwise the base points of $U_t$ consist of two different lines then we may assume $H \subset \{xy(ax + by)\}$ which is annihilated as well. Finally, when only one line $\ell$ lies in the base points we consider the system $U_t\ell^{-1} : t_0f_0\ell^{-1} + \ldots + t_sf_s\ell^{-1} = 0$ which must contain only finitely many base points. But then as above all $f_i\ell^{-1}$ share the same double point and thus all $f_i$ the same triple point and we are in the case as above.

(4) $H$ lies generically in $\overline{B_2} = \overline{Gx^2y}$: We can assume that $f_0 = x^2y \in H$ and then we claim that $H \subset \{x^2(ax + by + cz)\}$. Assume that $f_1 \in B_2$ is an element in $H$ whose quadratic term is linearly independent of $x^2$. After a suitable base change we find, for all $t$,

$$f_0 + tf_1 = x^2p + ty^2q = (a_tx + b_ty + c_tz)^2\ell_t$$

for suitable linear forms $p$, $q$ and $\ell_t$. Note however that $c_t = 0$ for all $t$ since the left-hand side contains no term $z^2$. Furthermore $a_tb_t \neq 0$ for $t \neq 0$ as $f_0$ and $f_1$ are not both divisible by $x^2$ or $y^2$ and hence $\ell_t$ does not depend on $z$ since the left-hand side contains no term $xyz$. So finally $f_0$ and $f_1$ do not depend on $z$ and thus are binary forms of degree three. By Theorem 3.2 they must have a common quadratic factor, a contradiction to our assumption.

(5) $H$ lies generically in $\overline{B_1} = \overline{Gx^3}$: The sum of two linearly independent cubes is not a cube, hence $\dim H = 1$. □

It is classically known that $\mathcal{O}(T)^{\mathrm{SL}_3}$ is generated by two invariants $f_4$ and $f_6$ of degree 4 resp. 6, see [**Aro58**]. From Corollary 2.6 we already know, that for $k > 2$ no homogeneous system of parameters for $\mathcal{O}(T^{\oplus k})^{\mathrm{SL}_3}$ can be found among the polarizations of $f_4$ and $f_6$. However for $k = 2$ the set of polarizations contains $5 + 7 = 12$ functions and indeed we have

$$12 = 2 \cdot 10 - 8 = \dim \mathcal{O}(T^{\oplus 2})^{\mathrm{SL}_3}$$

and thus

THEOREM 6.2. *The polarizations of $f_4$ and $f_6$ form a bihomogeneous system of parameters for $\mathcal{O}(T^{\oplus 2})^{\mathrm{SL}_3}$.*

Using this result we were able to compute the Hilbert series of the invariant ring $\mathcal{O}(T \oplus T)^{\mathrm{SL}_3}$. It is of the form $H(t) = \frac{f(t)}{(1-t^4)^5(1-t^6)^7}$ with

$$\begin{aligned} f(t) = 1 &+ 4t^6 + 9t^8 + 11t^{10} + 30t^{12} + 62t^{14} + 98t^{16} + 125t^{18} + 140t^{20} \\ &+ 140t^{22} + 125t^{24} + 98t^{26} + 62t^{28} + 30t^{30} + 11t^{32} + 9t^{34} + 4t^{36} + t^{38}. \end{aligned}$$

From $H(t)$ one reads off that a set of generators for $\mathcal{O}(T \oplus T)^{\mathrm{SL}_3}$ contains at least 76 elements.

APPENDIX A

# Conjugacy Classes of Matrices

Consider the action of $\mathrm{GL}_n$ on the quadratic $n \times n$-matrices $M_n$ by conjugation. It is well known that the invariant ring $\mathcal{O}(M_n)^{\mathrm{GL}_n}$ is generated by the $n$ traces of the powers : $tr_k : A \mapsto tr(A^k)$, $k = 1 \dots n$ and hence the nullcone $\mathcal{N}_{M_n}$ consist of the nilpotent matrices. A space $H \subset \mathcal{N}_{M_n}$ is annihilated by a 1-PSG of $\mathrm{GL}_n$ if and only if it is triangularizable, that is $gHg^{-1} \subset \mathfrak{N}_n$ for some $g \in \mathrm{GL}_n$, where $\mathfrak{N}_n$ are the strictly upper triangular $n \times n$-matrices.

For $n = 2$ it is easy to see that every subspace $H \subset \mathcal{N}_{M_2}$ can be triangularized ($\dim H \leq 1$ since it consists of nilpotent rank one matrices). But for $n \geq 3$ the two-dimensional space

$$H = sA + tB = \begin{bmatrix} 0 & t & 0 & \dots & 0 \\ s & 0 & -t & & \\ 0 & s & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$$

which consists entirely of nilpotent matrices has some interesting properties. In fact $AB$ is not anymore nilpotent, and for $n = 3$ this space was already used in [**MoT52**] to show that there exist spaces $sA + tB$ of nilpotent matrices such that, as an algebra, $A$ and $B$ generate $M_3$. In our context, this means that for every $n \geq 3$ the space $H$ cannot be triangularized.

In contrast to the symmetric bilinear forms or the skew-symmetric forms, where every singular space - when embedded in some large enough space - could be annihilated (see Prop. 4.15), $H$ has no such property. Moreover, $H$ is of fixed rank 2, thus besides spaces bounded by rank 1 (which are clearly annihilated) not even spaces of small rank can be annihilated in $\mathcal{N}_{M_n}$.

If $H \subset \mathcal{N}_{M_n}$ can be annihilated by a 1-PSG of $\mathrm{GL}_n$ then obviously $H$ can be spanned by rank one elements or is a subspace of a space that is spanned by rank one elements. We prove now that the opposite is also true: If $H \subset \mathcal{N}_{M_n}$ is spanned by rank one elements then $H$ is annihilated. This gives an interesting statement on triangularizability of nilpotent rank one matrices.

Recall that for an element $A \in \mathfrak{N}_n$ of rank one there exist $v, w \in \mathbb{C}^n$ such that $A = v \cdot w^t$. Moreover, if the last non-zero entry of $v$ is $v_i$ then $w_1 = \dots = w_i = 0$.

Lemma. *Let $A_1, \dots, A_k \in \mathfrak{N}_n$ be elements of rank one with the property that*

$$A_1 \cdot A_2 \cdot \ldots \cdot A_k \neq 0.$$

*Then, for all $\sigma \in S_k$, $\sigma \neq id$, we have*

$$A_{\sigma(1)} \cdot A_{\sigma(2)} \cdot \ldots \cdot A_{\sigma(k)} = 0.$$

PROOF. For $i = 1 \dots k$ let $A_i = v_i \cdot w_i^t$ and define $c_i$ to be the index of the last non-zero entry of $v_i$.

Now since

$$A_1 \cdot A_2 \cdot \ldots \cdot A_k = v_1 \cdot (w_1^t \cdot v_2) \cdot \ldots \cdot (w_{k-1}^t \cdot v_k) \cdot w_k^t \neq 0$$

it follows that each $(w_i^t \cdot v_{i+1}) \neq 0$. Since the first $c_1$ entries of $w_1$ are zero, we conclude that $c_2 > c_1$. But then the first $c_2$ entries of $w_2$ are zero, hence $c_3 > c_2$ and so on to finally find $c_1 < c_2 < \ldots < c_k$. Obviously there is no $\sigma \neq id$ satisfying $c_{\sigma(1)} < c_{\sigma(2)} < \ldots < c_{\sigma(k)}$ hence we are done. □

THEOREM. *If $H$ is a nilpotent space spanned by rank one matrices then $H$ can be triangularized.*

PROOF. Let $H$ be spanned by $A_1, \ldots, A_m$. It suffices to prove that every $m+1$-fold monomial in the $A_i$ is already zero: in this case the subalgebra - and hence the Liealgebra - generated by the $A_i$ is nilpotent and so by LIE's theorem, all $A_i$ can simultaneously be triangularized. We proceed by induction on $m$. For $m = 2$, since

$$(t_1 A_1 + t_2 A_2)^3 = t_1^2 t_2 A_1 A_2 A_1 + t_1 t_2^2 A_2 A_1 A_2 = 0$$

we find $A_1 A_2 A_1 = A_2 A_1 A_2 = 0$.

Let's assume the statement has been verified up to $m - 1$, so we have to show that every $m + 1$-fold monomial in $A_1, \ldots, A_m$ vanishes. To this end consider the coefficient of, say, $t_1^2 t_2 \cdots t_m$ in $(t_1 A_1 + \ldots + t_m A_m)^{m+1}$. It consists of the sum of all monomials containing $A_1$ twice and every other $A_i$ exactly once. By induction hypothesis it is clear, that every monomial not being of the form $A_1 A_{i_1} \ldots A_{i_{m-1}} A_1$ is zero (would contain a $l+1$-fold monomial in $l < m$ factors) and also, that we can assume that $A_2, \ldots, A_m \in \mathfrak{N}_n$. If any such monomial is non-zero, then every other of this type is, by the above lemma. Since the coefficient is zero, in fact all of them are. □

REMARK. A famous result in this context is the classical theorem of GERSTENHABER which states that if $H \subset M_n$ is a nilpotent space of dimension $\binom{n}{2}$ then $H$ is triangularizable. This has recently been considerably generalized in [**DKK06**].

# Bibliography


[AGo77] Adamovich, O. M., Golovina, E. O.: *Invariants of a pair of bilinear forms*, Mosc. Univ. Math. Bull. **32(2)** (1977), 11–14.

[Aro58] Aronhold, S.: *Theorie der homogenen Functionen dritten Grades von drei Veränderlichen*, J. reine angew. Math. **55** (1858), 97–191.

[Bri82] Brion, M.: *Invariants de plusieurs formes binaires*, Bull. Soc. math. France **110** (1982), 429–445.

[Bri96] Brion, M..: *Invariants et Covariants des Groupes Algébriques Réductives*, Notes de l' école d'éte, available at http://www-fourier.ujf-grenoble.fr/∼mbrion/monastirrev.pdf

[DiL85] Diximier, J., Lazard, D.: *The minimum number of fundamental invariants for binary forms of degree 7*, Portugal Math. **43** (1985/86), 377–395.

[DKK06] Draisma, J., Kraft, H., Kuttler, J.: *Nilpotent subspaces of maximal dimension in semisimple Lie algebras*, Compositio Math. **142** (2006), 464–476.

[Gan59] Gantmacher, F. R.: *The theory of matrices. Vol. 2.*, Chelsea Publishing, New York, 1959

[Har92] Harris, J.: *Algebraic Geometry*, Springer, New York, 1992

[HoR74] Hochster, M., Roberts, J.: *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. Math. **13** (1974), 115–175.

[Kra85] Kraft, H.: *Geometrische Methoden in der Invariantentheorie*, vol. D1, Vieweg-Verlag, Braunschweig–Wiesbaden, 1985.

[KrW05] Kraft, H., Wallach, N.R.: *Polarizations and nullcone of representations of reductive groups*, preprint, 2005.

[LMP05] Losik, M., Michor, W., Popov, V. L.: *On Polarizations in Invariant Theory*, preprint 2005

[Pop92] Popov, V. L.: *Groups, Generators, Syzygies, and Orbits in Invariant Theory*, Amer. Math. Soc. (1992) .

[Meu05] Meulien, M.: *Sur la complication des algèbres d'invariants combinants*, Journal of Algebra. **284** (2005), 284–295.

[Meu04] Meulien, M.: *Sur les invariants des pinceaux de formes quintiques binaires*, Ann. Inst. Fourier **54**, No. 1 (2004), 21–52.

[MoT52] Motzkin, T. S., Taussky, O.: *Pairs of matrices with property L*, Trans. Amer. Math. Soc. **73** (1952), 108–114.

[Sch78] Schwarz, G. W.: *Representations of Simple Lie Groups with Regular Rings of Invariants* Invent. Math. **49** (1978) 167–191.

[Shi67] Shioda, T.: *On the graded ring of invariants of binary octavics* , Amer. J. Math. **89** (1967).

[Wall78] Wall, C. T. C.: *Nets of quadrics, and theta-characteristics of singular curves*, Philos. Trans. Roy. Soc. London Ser. A **289**(1357), 229–269 (1978)

# Part 2

# Multiplicities in Tensor Monomials

# MULTIPLICITIES IN TENSOR MONOMIALS

MATTHIAS BÜRGIN

ABSTRACT. This paper is about multiplicities occuring in the decomposition of tensor monomials ${V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}$ where the $V_i$ are irreducible representations of a simple complex algebraic group. We show that there is a constant $N$ and a real number $\alpha > 1$ such that if $\sum n_i \geq N$ then $\operatorname{mult}(W, {V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}) \geq \alpha^{\sum n_i}$ unless it is zero.
Also we provide some tools to compute all tensor monomials ${V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}$ that contain a given representation $W$ at most a given time. Especially we find all $(d_1, d_2, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$ such that ${\mathbb{C}^n}^{\otimes d_1} \otimes {\bigwedge^2 \mathbb{C}^n}^{\otimes d_2} \otimes {\bigwedge^3 \mathbb{C}^n}^{\otimes d_3} \otimes \cdots \otimes {\bigwedge^{n-1} \mathbb{C}^n}^{\otimes d_{n-1}}$, considered as $\mathrm{SL}_n$-representation, contains the trivial representation exactly once, a question that was asked by FINKELBERG. Also, using our tools we answer the question of FINKELBERG generalized to the exceptional groups.

## 1. INTRODUCTION

The starting point of this work was the following question asked by FINKELBERG.

**Question.** *For which integers $d_1, d_2, \ldots, d_{n-1}$ does the tensor product*

$$ {\mathbb{C}^n}^{\otimes d_1} \otimes {\bigwedge^2 \mathbb{C}^n}^{\otimes d_2} \otimes {\bigwedge^3 \mathbb{C}^n}^{\otimes d_3} \otimes \cdots \otimes {\bigwedge^{n-1} \mathbb{C}^n}^{\otimes d_{n-1}} $$

*considered as a representation of $\mathrm{SL}_n(\mathbb{C})$ contain the trivial representation $\bigwedge^0 \mathbb{C}^n$ exactly once?*

From classical invariant theory there are some candidates, namely all tensor monomials such that $d_1 + 2d_2 + 3d_3 + \cdots + (n-1)d_{n-1} = n$. They cannot exhaust the list since every solution produces another one, by dualizing. However, both together give the full solution as we will prove in Section 2. A similar result holds for the single occurence of an arbitrary $\bigwedge^i \mathbb{C}^n$.

**Theorem A.** *Considered as a representation of $\mathrm{SL}_n(\mathbb{C})$, the tensor monomial*

$$ {\mathbb{C}^n}^{\otimes d_1} \otimes {\bigwedge^2 \mathbb{C}^n}^{\otimes d_2} \otimes {\bigwedge^3 \mathbb{C}^n}^{\otimes d_3} \otimes \cdots \otimes {\bigwedge^{n-1} \mathbb{C}^n}^{\otimes d_{n-1}} $$

*contains the trivial representation $\bigwedge^0 \mathbb{C}^n$ with multiplicity one if and only if either $\sum_k kd_k = n$ or $\sum_k (n-k)d_k = n$. For $1 \leq i \leq n-1$, it contains $\bigwedge^i \mathbb{C}^n$ with multiplicity one if and only if either $\sum_k kd_k = i$ or $\sum_k (n-k)d_k = n-i$.*

A priory, it is not clear that the number of solutions to the question above is finite. However, in Section 3 we prove the following general result.

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**Theorem B.** *Let $G$ be a simple complex algebraic group and $V_1, V_2, V_3, \cdots, V_r$ non-trivial irreducible representations of $G$. For every integer $k > 0$ and every irreducible representation $W$ of $G$ the following set is finite:*

$$\{(n_1, n_2, \dots, n_r) \in \mathbb{N}^r \mid 1 \leq \operatorname{mult}(W, {V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}) \leq k\}.$$

This statement can be improved considerably. In fact, we will show that the multiplicities "grow exponentially" in the following sense.

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the highest weights of $V_1, V_2, \dots, V_r$ and let $\mu$ be the highest weight of $W$. In order that $W$ appears in the tensor monomial

$${V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}$$

it is necessary that $\mu$ is of the form $\mu = \sum_i n_i \lambda_i - \sum$ positiv roots [Hu72, 21.3 Proposition]. Let us denote by $\Lambda_{\text{root}}$ the *root lattice*, i.e., the sublattice of the weight lattice spanned by the roots.

**Theorem C.** *With the notation above there is a constant $N$ and a real number $\alpha > 1$ such that the following holds:*

$$\operatorname{mult}(W, {V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}) \geq \alpha^{\sum n_i}$$

*if $\mu \in \sum_i n_i \lambda_i + \Lambda_{root}$ and $\sum_i n_i \geq N$.*

Finally, in Section 4 we introduce a tool which can be used to calculate the finite sets

$$\{(n_1, n_2, \dots, n_r) \in \mathbb{N}^r \mid 1 \leq \operatorname{mult}(W, {V_1}^{\otimes n_1} \otimes {V_2}^{\otimes n_2} \otimes \cdots \otimes {V_r}^{\otimes n_r}) \leq k\}$$

that appeared in Theorem B above. As an application, we solve the question of Finkelberg for the exceptional groups $\mathtt{G}_2$, $\mathtt{F}_4$, $\mathtt{E}_6$, $\mathtt{E}_7$ and $\mathtt{E}_8$ by use of the computer program LiE [Li96].

## 2. Tensor monomials of exterior powers

The irreducible representations of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ are parametrized by the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of *height* $\leq n$ where $\lambda$ corresponds to the $\mathrm{GL}_n$-module $V_\lambda$ of highest weight $\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n$ in the usual way (see[Kr85, III.1.4 Satz 1] , cf. [GW98, 5.2.1] ). In particular, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, and the *height* of $\lambda$ is the maximal index $i$ such that $\lambda_i > 0$.

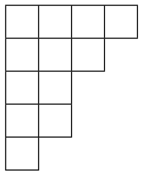
Restricting to $\mathrm{SL}_n$ the module $V_\lambda$ remains irreducible, and $V_\lambda|_{\mathrm{SL}_n}$ is trivial if and only if $\lambda = (m, m, \dots, m)$ for some $m \geq 0$. Finally, define the *degree* of $\lambda$ by $|\lambda| := \sum_i \lambda_i$.

The *fundamental weights* are given by

$$\omega_k := (\underbrace{1, 1, \dots, 1}_{k \text{ times}}, 0, 0, \dots, 0) \text{ for } k = 1, 2, \dots, n$$

and the corresponding fundamental representations are $V_{\omega_k} = \bigwedge^k \mathbb{C}^n$. For simplicity's sake we use in this section the notation $V_k := \bigwedge^k \mathbb{C}^n$. Furthermore we identify $V_0$ with the trivial representation.

We will represent a partition $\lambda$ by a *Young diagram* consisting of square boxes whose $i$-th row has length $\lambda_i$, e.g.



represents the partition $(4, 3, 2, 2, 1)$ of height 5 and degree 12.

*Remark* 1. By what was said above, we see that $V_\lambda$ and $V_{\lambda'}$ are isomorphic as $\mathrm{SL}_n$ modules if and only if $\lambda'$ is obtained from $\lambda$ by adding or removing columns of lenght $n$.

Most of the combinatorics involved has a nice description in terms of Young diagrams. As an example let us recall Pieri's formula (see [FH91, page 79, formula (6.9)].

**Proposition 1.** *For any partition $\lambda$ of height $\leq n$ we have the decomposition as* $\mathrm{GL}_n$*-module*

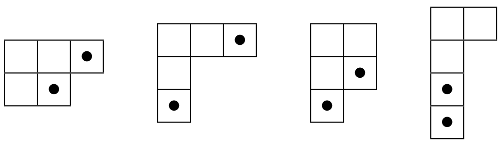
$$V_\lambda \otimes \bigwedge^k \mathbb{C}^n \;\simeq\; \bigoplus_\nu V_\nu$$

*where $\nu$ runs through all partitions of degree $|\nu| = |\lambda| + k$ and height $\leq n$ whose Young diagrams are obtained form the Young diagram of $\lambda$ by adding $k$ boxes, at most one to each row.*

**Example 1.** We have the following decomposition

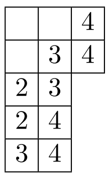
$$V_{(2,1)} \otimes \bigwedge^2 \mathbb{C}^4 \simeq \; V_{(3,2)} \oplus V_{(3,1,1)} \oplus V_{(2,2,1)} \oplus V_{(2,1,1,1)}$$

according to the Young diagrams



When decomposing a multiple tensor product $V_\lambda \otimes V_{k_1} \otimes \cdots \otimes V_{k_s}$ via Pieri's formula above, then one Young diagram can always be constructed in a canonical way which is best described as 'move as many boxes as possible in the leftmost columns'. An example will make everything clear:

**Example 2.** As a $\mathrm{GL}_5$-module the canonical Young diagram obtained from the four-fold tensor product $V_{(2,1)} \otimes V_2 \otimes V_3 \otimes V_4$ is

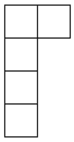



where the boxes coming from the module $V_i$ are labeled with $i$.

Using this construction of the canonical YOUNG diagram it is obvious that for $0 \leq i \leq n-1$ a tensor monomial ${V_1}^{\otimes m_1} \otimes {V_2}^{\otimes m_2} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$ contains the representation $V_i$ with respect to $\mathrm{SL}_n$ if and only if $\sum_k m_k k \equiv i \mod n$. Let us write now $(2, 1^\ell)$ for the partition

$$(2, \underbrace{1, 1, \ldots, 1}_{\ell \text{ times}}, 0, 0, \ldots, 0)$$

whose Young diagram has one row of length 2 and $\ell$ rows of length 1, e.g.

$(2, 1^3) =$ 

Using again the construction of the canonical YOUNG diagram the following lemma is clear:

**Lemma 1.** *Consider the partition $\mu = (2, 1^\ell)$ of height $\ell + 1 \leq n$ and let $k_1, \ldots, k_s < n$ be positive integers such that $\sum_i k_i + |\mu| \equiv 0 \mod n$ and $\sum_i k_i + |\mu| \geq 2n$. Then as an $\mathrm{SL}_n$-module $V_\mu \otimes V_{k_1} \otimes V_{k_2} \otimes \cdots \otimes V_{k_s}$ contains the trivial representation.*

Now we can prove our first main theorem.

**Theorem A.** *Considered as a representation of $\mathrm{SL}_n(\mathbb{C})$, the tensor monomial*

$${V_1}^{\otimes m_1} \otimes {V_2}^{\otimes m_2} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$$

*contains the trivial representation with multiplicity one if and only if either $\sum_k k m_k = n$ or $\sum_k (n-k) m_k = n$. For $1 \leq i \leq n-1$, it contains $V_i$ with multiplicity one if and only if either $\sum_k k m_k = i$ or $\sum_k (n-k) m_k = n - i$.*

*Proof.* (a) We already know that the trivial representation occurs in a tensor monomial ${V_1}^{\otimes m_1} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$ if and only if $\sum_k k m_k \equiv 0 \mod n$. If $\sum_k k m_k = n$ then the tensor monomial ${V_1}^{\otimes m_1} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$ is a quotient of $V^{\otimes n}$. It is well known from the SCHUR-WEYL duality that $(V^{\otimes n})^{\mathrm{SL}_n} = \bigwedge^n V$ which shows that the trivial representation occurs exactly once in ${V_1}^{\otimes m_1} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$. By duality, i.e. $V_d^* \simeq V_{n-d}$, the same holds if $\sum_k (n-k) k = n$.

(b) Assume now that $\sum_k k m_k \equiv 0 \mod n$ and that $\sum_k k m_k$ and $\sum_k (n-k) m_k$ are both $\geq 2n$. Write the tensor monomial ${V_1}^{\otimes m_1} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$ in the form $V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_N}$ where $0 < r_1 \leq r_2 \leq \cdots \leq r_N < n$. If $r_1 + r_2 \leq n$ then $V_{r_1} \otimes V_{r_2}$ contains the irreducible summands $V_{r_1+r_2}$ and $V_{(2, 1^{r_1+r_2-2})}$. Lemma 1 above implies that $V_{(2,1^{r_1+r_2-2})} \otimes V_{r_3} \otimes \cdots \otimes V_{r_N}$ contains a trivial summand, and the same holds for $V_{r_1+r_2} \otimes V_{r_3} \otimes \cdots \otimes V_{r_N}$. Thus the multiplicity of the trivial representation in $V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_N}$ is at least two. By duality the same argument applies to the case when $(n - r_{N-1}) + (n - r_N) \leq n$.

(c) It remains to show that if $\sum_j r_j$ and $\sum_j (n - r_j)$ are both $\geq 2n$ then either $r_1 + r_2 \leq n$ or $(n - r_{N-1}) + (n - r_N) \leq n$. In fact, $r_1 + r_2 > n$ implies that $r_j > \frac{n}{2}$ for $j \geq 2$. Moreover, $N \geq 3$ because $\sum_j r_j \geq 2n$. Now the claim for the trivial representation follows.

(d) For $1 \leq i \leq n-1$ let $T := {V_1}^{\otimes m_1} \otimes \cdots \otimes {V_{n-1}}^{\otimes m_{n-1}}$ be a tensor monomial with $\sum_k k m_k = i$. We have

$$\begin{aligned}\operatorname{mult}(V_i, T) = 1 &\Leftrightarrow \operatorname{mult}(V_0, T \otimes V_{n-i}) = 1 \\ &\Leftrightarrow \sum_k k m_k + n - i = n \quad \text{or} \quad \sum_k (n-k) m_k + i = n \\ &\Leftrightarrow \sum_k k m_k = i \quad \text{or} \quad \sum_k (n-k) m_k = n - i\end{aligned}$$

□

We like to prove a similar result for tensor products of *symmetric* powers $S^k\mathbb{C}^n = V_{n\omega_1}$. First of all there is also a Pieri formula for this situation (see [FH91, page 79, formula (6.8)] ).

**Proposition 2.** *For any partition $\lambda$ of height $\leq n$ we have the decomposition as* $\mathrm{GL}_n$*-module*

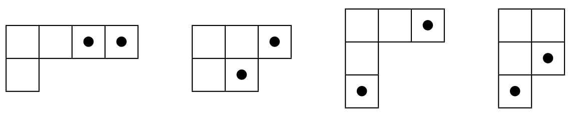
$$V_\lambda \otimes S^k\mathbb{C}^n \simeq \bigoplus_\nu V_\nu$$

*where $\nu$ runs through all partitions of degree $|\nu| = |\lambda| + k$ and height $\leq n$ whose Young diagrams are obtained form the Young diagram of $\lambda$ by adding $k$ boxes, at most one to each column.*

**Example 3.** We have the following decomposition

$$V_{(2,1)} \otimes S^2\mathbb{C}^4 \simeq V_{(4,1)} \oplus V_{(3,2)} \oplus V_{(3,1,1)} \oplus V_{(2,2,1)}$$

according to the Young diagrams



Recall that the *dual partition* $\lambda^\vee$ of $\lambda$ is obtained by interchanging rows and columns of the corresponding Young diagrams, i.e.

$$\lambda_j^\vee := \#\{i \mid \lambda_i \geq j\}.$$

In particular, $|\lambda| = |\lambda^\vee|$ and $\lambda^\vee$ has heigt $\lambda_1$. By definition, $\bigwedge^k\mathbb{C}^n$ and $S^k\mathbb{C}^n$ correspond to dual partitions, and the two Pieri formulas (Proposition 1 and 2) are dual with respect to this duality of irreducible representations. As a consequence we get the following result.

**Proposition 3.** *Let $m, n$ be two positive integers and $\lambda$ a partition of height $\leq n$ such that $\lambda_1 \leq m$. For any finite set of integers $0 < k_1, k_2, \ldots, k_r \leq n$ we have*

$$\begin{aligned}&\operatorname{mult}_{\mathrm{GL}_n}(V_\lambda, \textstyle\bigwedge^{k_1}\mathbb{C}^n \otimes \bigwedge^{k_2}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_r}\mathbb{C}^n) = \\ &\operatorname{mult}_{\mathrm{GL}_m}(V_\lambda^\vee, S^{k_1}\mathbb{C}^m \otimes S^{k_2}\mathbb{C}^m \otimes \cdots \otimes S^{k_r}\mathbb{C}^m)\end{aligned}$$

This enables us to carry over the results from Theorem A. We do it explicitely for the case of the trivial representation. For this we consider, under the assumption above, the partition $\lambda = (n^m) := (\underbrace{n, n, \ldots, n}_{m \text{ times}})$.

**Corollary.** (a) *The tensor product* $S^{k_1}\mathbb{C}^m \otimes S^{k_2}\mathbb{C}^m \otimes \cdots \otimes S^{k_r}\mathbb{C}^m$ *contains the trivial representation with respect to* $\mathrm{SL}_m$ *if and only if there is an integer* $n \geq 0$ *such that* $\sum_j k_j = m \cdot n$ *and* $k_j \leq n$ *for all* $j$.

(b) *If* $\sum_j k_j = m \cdot n$ *and* $0 < k_1, \dots k_r \leq n$ *then* $S^{k_1}\mathbb{C}^m \otimes S^{k_2}\mathbb{C}^m \otimes \cdots \otimes S^{k_r}\mathbb{C}^m$ *contains the trivial representation (with respect to* $\mathrm{SL}_m$*) with multiplicity one if and only if* $\bigwedge^{k_1}\mathbb{C}^n \otimes \bigwedge^{k_2}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_r}\mathbb{C}^n)$ *does (with respect to* $\mathrm{SL}_n$*).*

Summing up we obtain the following result.

**Theorem.** *Let* $k_1 \leq k_2 \leq \cdots \leq k_r$ *be positive integers. Then the tensor product* $S^{k_1}\mathbb{C}^m \otimes S^{k_2}\mathbb{C}^m \otimes \cdots \otimes S^{k_r}\mathbb{C}^m$ *contains the trivial representation of* $\mathrm{SL}_m$ *with multiplicity one if and only if there is an integer* $n \geq 0$ *such that the following holds:*

(i) $\sum_j k_j = n \cdot m$ *and* $k_j \leq n$ *for all* $j$.

(ii) $k_i = n$ *for all* $i$ *and hence* $r = m$ *or* $\sum_{k_j<n} k_j = n$ *or* $\sum_{k_j<n}(n - k_j) = n$.

## 3. Growth of Multiplicities

Let $G$ be a simple complex algebraic group, with maximal torus $T \subset G$ and Lie algebra $\mathfrak{g}$. We choose a set $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ of simple roots and denote by $\{\omega_1, \omega_2, \dots, \omega_\ell\}$ the corresponding fundamental weights. They span the *weight lattice* $\Lambda := \sum_i \mathbb{Z}\omega_i$ which contains the *root lattice* $\Lambda_{\text{root}} := \sum_i \mathbb{Z}\alpha_i$. For every simple root $\alpha_i$ we choose a root vector $X_i \in \mathfrak{g}_{\alpha_i}$. If $W$ is a representation of $G$ and $\lambda \in \Lambda$ then $W(\lambda)$ denotes the weight space of weight $\lambda$.

A basic tool in the study of multiplicities of tensor monomials is the following result due to Zelobenko [Ze73].

**Proposition 4.** *Let* $V_\mu, V_\delta$ *be irreducible representations of* $G$ *of highest weights* $\mu, \delta$ *where* $\delta = \sum_i r_i\omega_i$. *For any representation* $W$ *of* $G$ *we have*

$$\mathrm{mult}(V_\mu, V_\delta \otimes W) = \dim\{w \in W(\mu - \delta) \mid X_i^{r_i+1}w = 0 \text{ for } i = 1, 2, \dots, \ell\}.$$

The following corollary was pointed out by Evgueni Tevelev:

**Corollary 1.** *Let* $V_\delta$ *be an irreducible representation of highest weight* $\delta = \sum_i r_i\omega_i$ *and let* $W_1, W_2, \dots, W_s$ *be arbitrary representations of* $G$ *such that* $r_i \geq \dim W_j$ *for all* $i, j$. *Then*

$$\mathrm{mult}(V_\delta, V_\delta \otimes W_1^{\otimes n_1} \otimes W_2^{\otimes n_2} \otimes \cdots \otimes W_s^{\otimes n_s}) \geq \prod_j (\dim W_j(0))^{n_j}.$$

*Proof.* The proposition above shows that $\mathrm{mult}(V_\delta, V_\delta \otimes W_j) = \dim W_j(0)$ since $X_i^{r_i+1}|_{W_j} = 0$ by assumption. Now the claim follows by induction. □

**Proposition 5.** *Let* $V_\delta$ *be an irreducible representation of highest weight* $\delta = \sum_i r_i\omega_i$ *and let* $W_1, W_2, \dots, W_s$ *be arbitrary representations of* $G$. *Assume:*

(1) $\delta \in \Lambda_{root}$;
(2) $r_i \geq \dim W_j$ *for all* $i$ *and* $j$;
(3) $W_j(0) \neq 0$ *for all* $j$.

*Then there exists an $N_0 > 0$ such that $V_\delta$ occurs in the tensor monomial*

$$W_1^{\otimes m_1} \otimes W_2^{\otimes m_2} \otimes \cdots \otimes W_s^{\otimes m_s}$$

*as soon as $\sum_j m_j \geq N_0$.*

*Proof.* It follows from Lemma 2 below that, for every $j$, $V_\delta$ occurs in some tensor power $W_j^{\otimes N_j}$, and Corollary 1 above implies that $V_\delta$ occurs in $V_\delta \otimes W_1^{\otimes n_1} \otimes W_2^{\otimes n_2} \otimes \cdots \otimes W_s^{\otimes n_s}$ for all $n_1, \ldots, n_s \geq 0$. Define $N_0 := \sum_j N_j$. If $\sum_j m_j \geq N_0$ then $m_j \geq N_j$ for at least one $j$ and so

$$V_\delta \subset V_\delta \otimes W_1^{\otimes m_1} \otimes \cdots \otimes W_j^{\otimes m_j - N_j} \otimes \cdots \otimes W_s^{\otimes m_s} \subset W_1^{\otimes m_1} \otimes W_2^{\otimes m_2} \otimes \cdots \otimes W_s^{\otimes m_s}$$

which proves the claim. □

The following well-known result was used in the proof above.

**Lemma 2.** *Let $V$ be an irreducible representation and $W$ a faithful representation of a semisimple group $G$. Then $V$ occurs in $W^{\otimes N}$ for some $N$.*

Now we are ready to formulate the main result of this section. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the highest weights of $V_1, V_2, \ldots, V_r$ and let $\mu$ be the highest weight of $W$. In order that $W$ appears in the tensor monomial

$$V_1{}^{\otimes n_1} \otimes V_2{}^{\otimes n_2} \otimes \cdots \otimes V_r{}^{\otimes n_r}$$

it is necessary that $\mu$ is a weight of the tensor product, hence of the form $\mu = \sum_i n_i \lambda_i - \sum$ positiv roots [Hu72, 21.3 Proposition].

**Theorem C.** *Let $V_1, V_2, \ldots, V_r$ and $W$ be irreducible representations of highest weights $\lambda_1, \lambda_2, \ldots, \lambda_r \neq 0$ and $\mu$. Then there is a constant $N$ and a real number $\alpha > 1$ such that the following holds:*

$$\operatorname{mult}(W, V_1{}^{\otimes n_1} \otimes V_2{}^{\otimes n_2} \otimes \cdots \otimes V_r{}^{\otimes n_r}) \geq \alpha^{\sum n_i}$$

*if $\mu \in \sum_i n_i \lambda_i + \Lambda_{root}$ and $\sum_i n_i \geq N$.*

*Proof.* Since

$$\operatorname{mult}(W, V_1{}^{\otimes n_1} \otimes \cdots \otimes V_r{}^{\otimes n_r}) = \operatorname{mult}(\mathbb{C}, V_1{}^{\otimes n_1} \otimes \cdots \otimes V_r{}^{\otimes n_r} \otimes W^*)$$

it suffices to consider the case $W = \mathbb{C}$, the 1-dimensional trivial representation. Define

$$M := \{a = (a_1, \ldots, a_r) \in \mathbb{N}^r \mid \sum a_i \lambda_i \in \Lambda_{\text{root}}\}.$$

By Gordan's Lemma this is a finitely generated monoid: $M = \sum_{j=1}^s \mathbb{N} a^{(j)}$. Put

$$W_j := V_1^{\otimes a_1^{(j)}} \otimes V_2^{\otimes a_2^{(j)}} \otimes \cdots \otimes V_r^{\otimes a_r^{(j)}}$$

for $j = 1, 2, \ldots, s$. By construction, the weights of every $W_j$ are in the root lattice. In particular, $W_j(0) \neq 0$. Moreover, $\dim(W_j \otimes W_j)(0) \geq 2$, because $W_j \otimes W_j$ is not irreducible.

Now choose an irreducible representation $V_\delta$ of highest weight $\delta = \sum r_i \omega_i \in \Lambda_{\text{root}}$ such that $r_i \geq (\dim W_j)^2 \geq \dim W_j$ for all $i, j$. It follows from Proposition 5 above that there is an $N_0 > 0$ such that $V_\delta$ and $V_\delta^*$ occur in every tensor product $W_1^{\otimes m_1} \otimes W_2^{\otimes m_2} \otimes \cdots \otimes W_s^{\otimes m_s}$ as soon $\sum m_j \geq N_0$.

If $\sum m_i \geq 2N_0 + 2$ then $m := (m_1, \ldots, m_s) \in \mathbb{N}^s$ can be written in the form

$$m = p + q + 2r, \quad p = (p_1, \ldots, p_s), q = (q_1, \ldots, q_s), r = (r_1, \ldots, r_s) \in \mathbb{N}^s$$

where $\sum p_j = N_0, \sum q_j = N_0$ or $N_0 + 1$ and $\sum r_j > 0$. Then we see that $W^{\otimes m} = W^{\otimes p} \otimes W^{\otimes q} \otimes W^{\otimes 2r}$ contains $V_\delta^* \otimes V_\delta \otimes \bigotimes_j (W_j \otimes W_j)^{\otimes r_j}$. Since

$$\mathrm{mult}(V_\delta, V_\delta \otimes \bigotimes_j (W_j \otimes W_j)^{\otimes r_j}) = \prod_j (\dim(W_j \otimes W_j)(0))^{r_j} \geq \prod_j 2^{r_j}$$

we get $\mathrm{mult}(\mathbb{C}, W_1^{\otimes m_1} \otimes \cdots \otimes W_s^{\otimes m_s}) \geq 2^{|r|}$ where $|r| := \sum_j r_j = \lfloor \frac{\sum_j m_j - 2N_0}{2} \rfloor$.

Now start with a tensor monomial $V^{\otimes n} = V_1^{\otimes n_1} \otimes \cdots \otimes V_r^{\otimes n_r}$ where $n = (n_1, \ldots, n_r) \in M$, and write $n = \sum_j m_j a^{(j)}$. Then $V^{\otimes n} = W^{\otimes m}$ where $m = (m_1, \ldots, m_s)$, and we have the estimate $|n| \leq |m| \cdot A$ where $A := \max_j |a^{(j)}|$. This implies that

$$\mathrm{mult}(\mathbb{C}, V_1{}^{\otimes n_1} \otimes V_2{}^{\otimes n_2} \otimes \cdots \otimes V_r{}^{\otimes n_r}) \geq 2^t$$

for $(n_1, \ldots, n_r) \in M$, $|n| := \sum n_i \geq (2N_0 + 2)A$ and $t = \lfloor \frac{|n|}{2A} - N_0 \rfloor$. From this the claim follows easily. □

## 4. Computing Tensormonomials (jointly with H. Kraft)

In this section we provide tools to compute the finite sets

$$M_k^W = \{(n_1, n_2, \ldots, n_r) \in \mathbb{N}^r \mid 1 \leq \mathrm{mult}(W, V_1{}^{\otimes n_1} \otimes V_2{}^{\otimes n_2} \otimes \cdots \otimes V_r{}^{\otimes n_r}) \leq k\}$$

from Theorem B in Section 1. Using these tools we can answer the question of Finkelberg for the exceptional groups $\mathtt{G}_2$, $\mathtt{F}_4$, $\mathtt{E}_6$, $\mathtt{E}_7$ and $\mathtt{E}_8$ by use of the computer program LiE [Li96].

*to be written*

## References


[FH91] Fulton, W., Harris, J.: *Representation Theory*, Graduate Texts in Math., vol. 129, Springer-Verlag, New York–Heidelberg–Berlin, 1991.

[GW98] Goodman, R., Wallach, N. R.: *Representations and Invariants of the Classical Groups*, Cambridge University Press, 1998.

[Hu72] Humphreys, J.E.: *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Math., vol. 9, Springer-Verlag, New York–Heidelberg–Berlin, 1972.

[Kr85] Kraft, H.: *Geometrische Methoden in der Invariantentheorie*, vol. D1, Vieweg-Verlag, Braunschweig–Wiesbaden, 1985.

[Li96] *LiE, A computer package for Lie group computations*, available at http://www-math.univ-poitiers.fr/∼maavl/LiE/

[Ze73] Zelobenko, D.P.: *Compact Lie Groups and their Representations*, Trans. AMS, vol. 40, 1973.



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# Part 3

# The Hilbert Nullcone on Tuples of Matrices and Bilinear Forms

# THE HILBERT NULL-CONE ON TUPLES OF MATRICES AND BILINEAR FORMS

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Abstract. We describe the null-cone of the representation of $G$ on $M^p$, where either $G = \mathrm{SL}(W) \times \mathrm{SL}(V)$ and $M = \mathrm{Hom}(V, W)$ (linear maps), or $G = \mathrm{SL(V)}$ and $M$ is one of the representations $S^2(V^*)$ (symmetric bilinear forms), $\Lambda^2(V^*)$ (skew bilinear forms), or $V^* \otimes V^*$ (arbitrary bilinear forms). Here $V$ and $W$ are vector spaces over an algebraically closed field $K$ of characteristic zero and $M^p$ is the direct sum of $p$ of copies of $M$.

More specifically, we explicitly determine the irreducible components of the null-cone on $M^p$. Results of Kraft and Wallach predict that their number stabilises at a certain value of $p$, and we determine this value. We also answer the question of when the null-cone in $M^p$ is defined by the polarisations of the invariants on $M$; typically, this is only the case if either $\dim V$ or $p$ is small. A fundamental tool in our proofs is the Hilbert-Mumford criterion for nilpotency (also known as unstability).

## 1. Introduction

For a group $G$ and a finite-dimensional $G$-module $M$ over an algebraically closed field $K$, we denote by $K[M]^G$ the algebra of $G$-invariant polynomials on $M$. An element $m \in M$ is called *nilpotent* (or *unstable*) if it cannot be distinguished from 0 by $K[M]^G$, or, in other words, if all $G$-invariant polynomials on $M$ without constant term vanish on $m$. The nilpotent elements in $M$ form a (Zariski-)closed cone in $M$, called the *null-cone* in $M$ ($G$ being understood) and denoted $\mathcal{N}(M) = \mathcal{N}_G(M)$; it is a central object of study in representation theory. In this paper we will describe the *irreducible components* of the null-cone in some concrete representations.

We will, in fact, be studying the null-cone in a direct sum $M^p$ of $p$ copies of $M$, regarded as a $G$-module with the diagonal action. We recall some relations between the invariants and the null-cone of $M^q$ and those of $M^p$, where $p$ and $q$ are natural numbers. It is convenient, for this purpose, to identify $M^p$ with $K^p \otimes M$ where $G$ acts trivially on the first factor, and also, given a linear map $\pi : K^p \to K^q$, to use the same letter $\pi$ for the $G$-homomorphism $M^p \to M^q$ determined by $\pi(x \otimes m) = \pi(x) \otimes m,\ x \in K^p, m \in M$.

First, from an invariant $f \in K[M^q]^G$ we can construct $G$-invariants on $M^p$ as follows: for any linear map $\pi : K^p \to K^q$ the function $f \circ \pi$ is an invariant on $M^p$. The functions obtained in this way as $\pi$ varies are usually called *polarisations* of $f$ if $q \leq p$ and *restitutions* of $f$ if $q \geq p$. Using this construction, due to Weyl [18], it is easy to see that any linear map $\pi : K^p \to K^q$ maps $\mathcal{N}(M^p)$ into $\mathcal{N}(M^q)$:

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indeed, an element $v$ of the former null-cone cannot be distinguished from 0 by any $G$-invariants on $M^p$, let alone by those of the form $f \circ \pi$ with $f \in K[M^q]^G$; hence $\pi(v) \in \mathcal{N}(M^q)$. Using this observation, we can prove that the number $c(M^p)$ of irreducible components of the $\mathcal{N}(M^p)$ behaves as follows.

**Proposition 1.** *If $p \geq q$, then $c(M^p) \geq c(M^q)$. If in addition $q \geq \dim M$, then $c(M^p) = c(M^q)$ and the polarisations to $M^p$ of the invariants on $M^q$ without constant term define the null-cone set-theoretically.*

*Proof.* Fix any surjective linear map $\pi : K^p \to K^q$; we claim that it maps $\mathcal{N}(M^p)$ surjectively onto $\mathcal{N}(M^q)$. Indeed, if $\sigma : K^q \to K^p$ is a right inverse of $\pi$, then any $v \in \mathcal{N}(M^q)$ is the image under $\pi$ of $\sigma v \in \mathcal{N}(M^p)$. This shows the first statement. For the second statement it suffices to prove that the map

$$\phi : \mathrm{Hom}(K^q, K^p) \times \mathcal{N}(M^q) \to \mathcal{N}(M^p),\ (\sigma, v) \mapsto \sigma v$$

is surjective for $q \geq \dim M$, because the right-hand side has precisely $c(M^q)$ irreducible components. To prove surjectivity of $\phi$, let $v = (m_1, \dots, m_p) \in \mathcal{N}(M^p)$. As $q \geq \dim M$, we can find a $w \in M^q$ whose components span the $K$-subspace $\langle m_1, \dots, m_p \rangle_K$ in $M$. It follows that there exist linear maps $\pi : K^p \to K^q$ and $\sigma : K^q \to K^p$ such that $\pi v = w$ and $\sigma w = v$. We conclude that $w = \pi v$ lies in $\mathcal{N}(M^q)$ and $v = \phi(\sigma, w)$. The last statement is proved by a similar argument: suppose that all polarisations $f \circ \pi$ with $\pi \in \mathrm{Hom}(K^p, K^q)$ and $f \in K[M^q]^G$ without constant term vanish on $v \in M^p$, and let $h \in K[M^p]^G$ be without constant term. We can choose $\pi$ and $\sigma$ with $\sigma\pi v = v$ as before, and we find that $h(v) = ((h \circ \sigma) \circ \pi)v = 0$, because $(h \circ \sigma) \circ \pi$ is a polarisation of the $G$-invariant $h \circ \sigma$ on $M^q$. □

*Remark* 1. In characteristic zero the last statement of Proposition 1 also follows from from Weyl's stronger result that the invariant ring on $M^p$ is generated by the polarisations of invariants on $M^q$ for $q \geq \dim V$ [18]. Weyl's theorem no longer holds in positive characteristic, though a weaker statement is still true [12]. However, an analogue of Weyl's theorem, for *separating* invariants, is true in arbitrary characteristic [5]—and, again, implies the last statement of Proposition 1.

Proposition 1 shows that $c(M^p)$ is an ascending function of $p$ that stabilises at some finite $p \leq \dim M$. This phenomenon was first observed by Kraft and Wallach in the case of reductive group representations [14], to which we turn our attention now. Suppose that $G$ is a connected, reductive affine algebraic group over $K$ and $M$ is a rational finite-dimensional $G$-module. One of the most important results on the null-cone in this setting is the *Hilbert-Mumford criterion* [15, 16] for nilpotency: $v \in M$ lies in $\mathcal{N}(M)$ if and only if there exists a one-parameter subgroup $\lambda : K^* \to G$ such that $\lim_{t \to 0} \lambda(t)v = 0$; we then say that $\lambda$ *annihilates* $v$. In this setting much more can be said about the irreducible components of the null-cone in $M^p$: one verifies that for every one-parameter subgroup $\lambda$, the set

$$G \cdot \{v \in M^p \mid \lim_{t \to 0} \lambda(t)v = 0\} \tag{1}$$

is a closed $G$-stable irreducible subset of $\mathcal{N}(M^p)$, and that a finite number of them cover $\mathcal{N}(M^p)$. Moreover, for $p$ sufficiently large, there are only the "obvious" inclusions among these sets [14] and this observations gives rise to a combinatorial algorithm for counting the irreducible components of $\mathcal{N}(M^p), p >> 0$ [3]. However, for smaller values of $p$, there are usually many more inclusions, and our goal in this paper is to determine the exact "stabilising" value of $c(M^p)$ for the pairs $(G, M)$ in the abstract.

We note that the notion of "optimal" one-parameter subgroups for elements of the null-cone gives yet a finer description of the geometry of $\mathcal{N}(M)$ [10, 16]—but this notion is not needed here.

Summarising, we will settle the following two fundamental problems for the pairs $(G, M)$ of the abstract: first, we describe the irreducible components of $\mathcal{N}(M^p)$ and determine at which value of $p$ their number stabilises; and second, we determine when $\mathcal{N}(M^p)$ is defined by the polarisations of the invariants on $M$. Note that in this case, by a result of Hilbert, the invariant ring of $M^p$ is finite over the subring generated by these polarisations [13, Section II.4.3]. The remainder of this paper has the following transparent organisation: Sections 2, 3, 4, and 5 deal with tuples of linear maps, symmetric bilinear forms, skew bilinear forms, and arbitrary bilinear forms, respectively. In the rest of the text we assume that $K$ has characteristic 0; this allows for the use of some "differential" arguments in the case of linear maps, while avoiding problems in small characteristics in the case of bilinear forms. However, most of what is proved here remains valid in arbitrary characteristic.

## 2. Nilpotent tuples of linear maps

For an $m$-dimensional vector space $V$ and an $n$-dimensional vector space $W$, both over our fixed algebraically closed field $K$ of characteristic 0, the group $G = \mathrm{SL}(W) \times \mathrm{SL}(V)$ acts on the space $M = \mathrm{Hom}(V, W)$ of linear maps by $(g, h)A := gAh^{-1}$. By duality we may assume that $0 < m \leq n$, and we let $q := \lceil \frac{n}{m} \rceil$ be the smallest integer $\geq n/m$. Then $\mathcal{N}(M^p)$ is as follows.

**Theorem 1.** *The null-cone of* $\mathrm{SL}(W) \times \mathrm{SL}(V)$ *in* $M^p = \mathrm{Hom}(V, W)^p$ *consists of all* $p$*-tuples* $(A_1, \ldots, A_p)$ *of linear maps for which there exist subspaces* $V'$ *of* $V$ *and* $W'$ *of* $W$ *such that* $n \cdot \dim V' > m \cdot \dim W'$ *and* $A_i V' \subseteq W'$ *for all* $i$.

*The* $p$*-tuples for which* $V'$ *can be chosen of a fixed dimension* $k \in \{1, \ldots, m\}$ *form a closed irreducible subset of* $\mathcal{N}(M^p)$, *denoted* $C_k^{(p)}$. *For* $p < q$ *the sets* $C_k^{(p)}$ *are all equal to* $M^p$, *and for* $p > q$ *they are precisely the distinct irreducible components of* $\mathcal{N}(M^p)$. *For* $p = q$ *there are still inclusions among the* $C_k^{(q)}$, *unless* $m = 1$, *in which case* $C_1^{(q)} = C_1^{(n)} = \mathcal{N}(M^n)$ *is the irreducible null-cone consisting of singular* $n \times n$*-matrices; or* $n = (q-1)m + 1$ *with* $q \geq 3$, *in which case the* $C_k^{(q)}$ *are already the distinct components of the null-cone.*

This theorem does not completely answer the question of how many irreducible components the null-cone on $q$ copies has. Some remarks on this matter can be found after the proof of the theorem, just before Example 2.

Somewhat prematurely, we will from now on call a pair $V', W'$ as in the theorem a *witness* for the nilpotency of $(A_1, \ldots, A_p)$. In the proof that follows we use a theorem from elementary optimisation theory, the *max-flow-min-cut theorem*, which states that the maximal size of a flow from a source $s$ to a sink $t$ in a network equals the minimal capacity of a cut disconnecting $s$ from $t$. Here a *network* is a directed graph with two distinguished vertices $s$ and $t$ and a prescribed real-valued capacity function $c$ on the arrows; a *flow* is a real-valued function $f$ on the arrows that is bounded by $c$ and for which at every vertex other than $s$ and $t$ the sum of the $f$-values on the incoming arrows equals the sum of the $f$-values on the outgoing arrows; a *cut* is a set of arrows whose removal disconnects $s$ from $t$; and

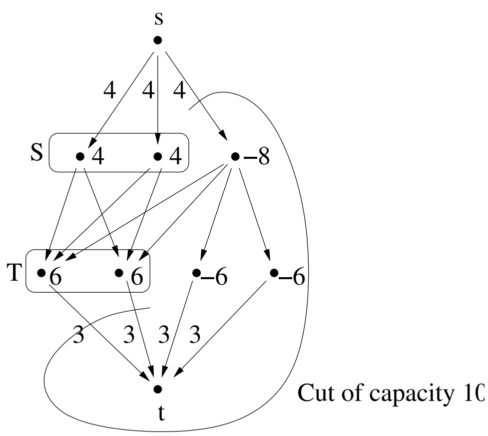


FIGURE 1. The graph $\Gamma$ with a cut.

the *capacities* of a flow and of a cut are defined in the obvious manner. See [2, Chapter 3] for details.

*Proof of Theorem 1, part one.* Suppose that $A = (A_1, \ldots, A_p)$ lies in the null-cone and let $(\mu, \lambda) : K^* \to \mathrm{SL}(V) \times \mathrm{SL}(W)$ be a one-parameter subgroup annihilating $A$. Let $v_1, \ldots, v_m$ be a basis of $V$ with $\lambda(t)v_j = t^{a_j} v_j$, where $a_j \in \mathbb{Z}$, let $w_1, \ldots, w_n$ be a basis of $W$ with $\mu(t)w_i = t^{b_i} w_i$, where $b_i \in \mathbb{Z}$, and note that $\det \lambda(t) = \det \mu(t) = 1$ implies $\sum_j a_j = \sum_i b_i = 0$.

Now construct a directed graph $\Gamma$ with arrows of capacity $n$ from a source $s$ to $m$ vertices $1, \ldots, m$, arrows of capacity $m$ from $n$ vertices $\hat{1}, \ldots, \hat{n}$ to a sink $t$, and an arrow—for convenience, of infinite capacity—from $j$ to $\hat{i}$ if and only if $b_i - a_j > 0$. See Figure 1 for an example with $m = 4$ and $n = 6$. From

$$\lim_{t \to 0} \mu(t) A_k \lambda(t)^{-1} v_j = \lim_{t \to 0} \mu(t) A_k t^{-a_j} v_j = 0$$

it is clear that each $A_k$ maps $v_j$ into the space spanned by the $w_i$ with $j \to \hat{i}$ in $\Gamma$. We claim that the maximal flow from $s$ to $t$ in $\Gamma$ is strictly smaller than the obvious upper bound $mn$. Indeed, suppose that this upper bound were attained by a flow in which $c_{j,i}$ is the flow from $j$ to $\hat{i}$. Then $\sum_i c_{j,i} = n$ for all $j$ and $\sum_j c_{j,i} = m$ for all $i$, so that

$$0 = m \sum_i b_i - n \sum_j a_j = \sum_{j,i} c_{j,i}(b_i - a_j);$$

but $c_{j,i} = 0$ whenever $b_i - a_j \leq 0$, so that the right-hand side is strictly positive, a contradiction. Now the max-flow-min-cut theorem assures the existence of a cut of capacity strictly smaller than $mn$ and in particular not containing edges of infinite capacity. Let $T \subseteq \{\hat{1}, \ldots, \hat{n}\}$ be the set of vertices cut off from $t$, and let $S \subseteq \{1, \ldots, m\}$ be the set of vertices *not* cut off from $s$. By definition of a cut, no vertex $j$ of $S$ is connected to any vertex $\hat{i}$ outside of $T$, so that $V' := \langle v_j \mid j \in S \rangle_K$ is mapped by every $A_k$ into $W' := \langle w_i \mid \hat{i} \in T \rangle_K$. Finally, the capacity of the cut is equal to

$$m|T| + n(m - |S|) \text{ and by assumption } < mn,$$

so that $m \dim W' < n \dim V'$ as required.

Conversely, suppose that $V', W'$ is a witness for the nilpotency of $A$, set $(k,l) := (\dim V', \dim W')$, and choose complements $V''$ and $W''$ of $V'$ and $W'$, respectively. Let $\lambda$ be the one-parameter subgroup of $\mathrm{SL}(V)$ having weights $a_1 := n(m-k)$ on $V'$ and $a_2 := -nk$ on $V''$; note that $ka_1 + (n-k)a_2 = 0$. Similarly, let $\mu$ be the one-parameter subgroup of $\mathrm{SL}(W)$ having weights $b_1 := m(n-l)$ on $W'$ and $b_2 := -ml$ on $W''$. From the inequalities

$$b_1 - a_1 > 0,\ b_1 - b_2 > 0,\ b_2 - a_1 \leq 0, \text{ and } b_2 - a_2 > 0$$

we infer that $(\mu, \lambda)$ annihilates any linear map sending $V'$ into $W'$, so that $A \in \mathcal{N}(M^p)$. This proves the first statement of the theorem. □

The sets $C_k^{(p)}$ from Theorem 1 are closed and irreducible by a general argument: they are of the form (1). Hence to prove the theorem we need only determine for what values of $p$ there are inclusions among the $C_k^{(p)}$. For this we need some auxiliary notation and results, which are of independent interest and which also give a formula for the dimensions of the irreducible components of $\mathcal{N}(M^p)$. We write $M_{a,b}$ for the space of $a \times b$-matrices with entries in $K$.

**Definition 1.** Let $a, b, c, d$, and $p$ be non-negative integers and let

$$X_i \in M_{c,a} \text{ and } Y_i \in M_{b,d} \text{ for } i = 1, \ldots, p.$$

Define the *cut-and-paste map* $\mathrm{CP} = \mathrm{CP}_{(X_i,Y_i)_i} : M_{a,b} \to M_{c,d}$ by

$$\mathrm{CP}\, A = \sum_{i=1}^{p} X_i A Y_i.$$

Now the rank of the linear map CP is clearly a lower semi-continuous function of the $p$-tuple $(X_i, Y_i)_i$, and we let $\mathrm{cp}^{(p)}(a,b,c,d)$, the *cut-and-paste rank*, be the maximal possible rank of CP, i.e., the rank for a generic $p$-tuple $(X_i, Y_i)_i$.

*Remark* 2. The following properties of the cut-and-paste rank are easy to check:

$$\mathrm{cp}^{(p)}(c,d,a,b) = \mathrm{cp}^{(p)}(a,b,c,d) = \mathrm{cp}^{(p)}(b,a,d,c).$$

Indeed, the second equality comes from the fact that, upon composition with transposition on both sides, the cut-and-paste map $\mathrm{CP}_{(X_i,Y_i)_i} : M_{a,b} \to M_{c,d}$ yields $\mathrm{CP}_{(Y_i^t, X_i^t)_i} : M_{b,a} \to M_{d,c}$; and the first equality reflects the fact that the transpose of $\mathrm{CP}_{(X_i,Y_i)_i}$ can be identified, via the trace form, with $\mathrm{CP}_{(X_i^t, Y_i^t)_i} : M_{c,d} \to M_{a,b}$. Moreover, if $a \leq c$ and $b \leq d$ then $\mathrm{cp}^{(p)}(a,b,c,d) = ab$ for all $p \geq 1$. Thus we reduce the computation of the cut-and-paste-rank to the case where $ab \leq cd, a \geq c$, and $b \leq d$. Then each of the maps $A \mapsto X_i A Y_i$ generically has rank $bc$, so that

$$\mathrm{cp}^{(p)}(a,b,c,d) \leq \min\{ab, pbc\}$$

Moreover, for $p \leq a/c$ it is easy to see that $\mathrm{cp}^{(p)}(a,b,c,d)$ is in fact equal to $pbc$: by using suitable $X_i$ and $Y_i$, one can "cut" p non-overlapping $c \times b$-blocks from an $a \times b$-matrix, and "paste" them in a non-overlapping way into a $c \times d$-matrix. The same argument shows that for $p$ sufficiently large $\mathrm{cp}^{(p)}(a,b,c,d)$ equals $ab$; this is the case, for example, as soon as one can cut an $a \times b$-matrix into $p$ non-overlapping rectangular blocks that fit without overlap into a $c \times d$-matrix. One might think that the inequality for the cut-and-paste-rank given above is always an equality, but this is not true: for $(a,b,c,d) = (5,4,3,7)$, for instance, we find cut-and-paste-ranks $12, 19, 20$ for $p = 1, 2, 3$, respectively. In short, we have no closed formula for cp and it would be interesting—but too much of a digression at this point in the paper—to find such a formula. In small concrete cases, however, the cut-and-paste rank can be computed easily; see below for some examples.

**Proposition 2.** *Let $k, l, m, n, p$ be integers satisfying $0 < k \leq m$, $0 \leq l < n$, and $p \geq 0$. Then*

$$Q := \{(A_1, \ldots, A_p) \in M_{n,m}^p \mid \exists U \subseteq K^m : \dim U = k \text{ and } \dim(\sum_{i=1}^{p} A_i U) \leq l\}$$

*is an irreducible closed subvariety of $M_{n,m}^p$ , and a sufficient condition for $Q$ to be strictly smaller than $M_{n,m}^p$ is*

$$p > \frac{l}{k} + \frac{m-k}{n-l}.$$

*Moreover, $\dim Q$ equals $pmn$ if $pk \leq l$ and*

$$pmn - (pk - l)(n - l) + \mathrm{cp}^{(p)}(m - k, k, \min\{p(m - k), n - l\}, pk - l)$$

*otherwise.*

*Proof.* The set $Q$ is an irreducible closed variety because it is of the form (1), that is, the result of a vector space stable under a Borel subgroup of $G = \mathrm{SL}_n \times \mathrm{SL}_m$ being "smeared" around by $G$. For $pk \leq l$ the proposition is evident: any $p$-tuple maps any $k$-space into an $l$-space. Suppose therefore that $pk \geq l$. In the diagram

$$\begin{array}{ccc} M_{n,m}^p \times (M_{m,k})_{\mathrm{reg}} & \xrightarrow{\mu} & M_{n,pk} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \\ M_{n,m}^p & & \end{array}$$

$\mu$ maps $(A_1, \ldots, A_p, B)$ to $(A_1 B| \ldots |A_p B)$, $\tilde{\pi}$ is the projection, and $(M_{n,k})_{\mathrm{reg}}$ is the set of rank $k$ matrices. Hence $Q = \tilde{\pi}(\mu^{-1}(X_l))$, where $X_l$ is the variety of matrices in $M_{n,pk}^p$ having rank at most $l$. We will first compute the dimension of $Z := \mu^{-1}(X_l)$ and then the dimension of a generic fibre of $\pi := \tilde{\pi}|_Z : Z \to Q$; the difference between these numbers is the dimension of $Q$.

First, $\mu$ is surjective and all its fibres have the same dimension $km + pn(m-k)$. Indeed, for $(A_1, \ldots, A_p, B)$ to lie in the fibre over $(C_1, \ldots, C_p)$ we may choose $B \in (M_{m,k})_{\mathrm{reg}}$ arbitrarily, and then each $A_i$ is determined on the $k$-dimensional image of $B$, but can still be freely prescribed on an $(n-k)$-dimensional complement. As $X_l$ has dimension $nl + pkl - l^2$ [9], $Z$ has dimension $km + pn(m-k) + nl + pkl - l^2$. Now $\mathrm{GL}_k$ acts freely on the fibres of $\pi$ by $g((A_i)_i, B) := ((A_i)_i, Bg^{-1})$, so that

$$\dim Q = \dim \pi(Z) \leq \dim Z - k^2 = pnm - (pk(n-l) - k(m-k) - l(n-l)).$$

This implies the first statement of the proposition.

For the dimension of $Q$ we compute the dimension of a generic fibre $\pi^{-1}\pi(z)$ by computing the Zariski tangent space $T_z \pi^{-1}\pi(z)$, as follows. First, we show that $Z$ is irreducible and determine $T_z Z$ for generic $z \in Z$. Observe for this that the group $\mathrm{GL}_m$ acts on the fibres of $\mu$ by $g((A_i)_i, B) := ((A_i g^{-1})_i, gB)$. Now the map

$$\phi : \mathrm{GL}_m \times M_{n,pk} \times M_{n,m-k}^p \to M_{n,k}^p \times M_{m,k},$$

$$(g, (C_1| \ldots |C_p), (E_i)_i) \mapsto g((C_i|E_i)_i, \begin{pmatrix} I_k \\ \hline 0_{m-k,k} \end{pmatrix})$$

maps $\mathrm{GL}_m \times X_l \times M^p_{n,m-k}$ surjectively onto $Z$, so $Z$ is irreducible as claimed. Furthermore, the map

$$s : M^p_{n,m-k} \to M^p_{n,k} \times M_{m,k},\ x \mapsto \phi(1, x, (0)_i)$$

is a right inverse of $\mu$, so by the chain rule $d_z\mu$ maps $M^p_{n,m} \times M_{m,k}$ surjectively onto $T_{\mu(z)}X_l$ for all $z \in M^p_{n,m} \times M_{m,k}$. In particular, if $z$ lies in $Z$ and $\mu(z)$ has rank exactly $l$ so that it is a smooth point of $X_l$, then we have

$$T_z Z = (d_z\mu)^{-1} T_{\mu(z)} X_l. \tag{2}$$

Now recall that if $\mu(z)$ has rank $l$, then

$$T_{\mu(z)}X_l = \{N \in M_{n,pk} \mid N \ker \mu(z) \subseteq \operatorname{im} \mu(z)\}; \tag{3}$$

see [9, Example 14.16]. This will enable us to interpret the right-hand side in (2). On the other hand, because $\operatorname{char} K = 0$, we have

$$T_z \pi^{-1}\pi(z) = \ker(d_z\pi : T_z Z \to T_{\pi(z)}Q) \tag{4}$$

for generic $z \in Z$. Now let $z = ((A_i)_i, B) \in Z$ be generic. In particular, we require (2) and (4), and what further open conditions on $z$ are needed will become clear along the way. By the action of $\mathrm{GL}_m$ above we may assume that $B$ is of the form

$$B = \begin{bmatrix} I_k \\ 0_{m-k,k} \end{bmatrix},$$

and we split each $A_i = (A_{i,1}|A_{i,2})$, accordingly. By genericity of the $A_i$ the matrix $\mu(z) = (A_{1,1}|\dots|A_{p,1})$ has rank $l$, and by (2), (3), and (4) we find that $T_z\pi^{-1}(\pi(z))$ is isomorphic to the space of all $m \times k$-matrices

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

such that

$$(A_{1,1}D_1 + A_{1,2}D_2|\dots|A_{p,1}D_1 + A_{p,2}D_2) \ker \mu(z) \subseteq \operatorname{im} \mu(z).$$

This is clearly the case for $D_2 = 0$ (this reflects the $\mathrm{GL}_k$-action used earlier), hence to determine what other $D$ have this property we may assume that $D_1 = 0$. The kernel of $\mu(z)$ has dimension $pk-l$, so we can choose $p$ matrices $Y_1, \dots, Y_p \in M_{k,pk-l}$ such that the columns of the matrix

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}$$

form a basis of the kernel of $\mu(z)$. Again by genericity—the $A_{i,2}$ are "independent" of the $A_{i,1}$—the pre-image of $\operatorname{im} \mu(z)$ under $(A_{1,2}|\dots|A_{p,2})$ has codimension $c := \min\{p(m-k), n-l\}$ in $K^{p(m-k)}$, and we may choose matrices $X_1, \dots, X_p \in M_{c,m-k}$ such that the rows of $(X_1|\dots|X_p)$ give linear equations for that inverse image. We now have

$$\begin{aligned}
&\{D_2 \in M_{m-k,k} \mid (A_{1,2}D_2|\dots|A_{p,2}D_2) \ker(A_{1,1}|\dots|A_{p,1}) \subseteq \operatorname{im}(A_{1,1}|\dots|A_{p,1})\} \\
&= \{D_2 \in M_{m-k,k} \mid \sum_i X_i D_2 Y_i = 0\} \\
&= \ker(\mathrm{CP}_{(X_i,Y_i)_i} : M_{m-k,k} \to M_{c,pk-l}).
\end{aligned}$$

Finally, because the $X_i$ and $Y_i$ are generic along with the $A_i$, the dimension of this space is $(m-k)k - \mathrm{cp}^{(p)}(m-k,k,c,pk-l)$. The dimension of the fibre $\pi^{-1}(\pi(z))$ is therefore $k^2$ plus this number, and we find

$$\begin{aligned}\dim \pi(Z) =& \dim Z - \dim \pi^{-1}\pi(z)\\ =& km + pn(m-k) + nl + pkl - l^2\\ &-(k^2 + (m-k)k - \mathrm{cp}^{(p)}(m-k,k,\min\{p(m-k),n-l\},pk-l))\\ =& pmn - (pk-l)(n-l)\\ &+\mathrm{cp}^{(p)}(m-k,k,\min\{p(m-k),n-l\},pk-l),\end{aligned}$$

as claimed. □

*Remark* 3. The difference $\dim \pi^{-1}(\pi(z)) - k^2$, expressed above as the nullity of a certain cut-and-paste map, is the dimension of the variety of $k$-dimensional subspaces $U$ for which $\sum_i A_i U$ is at most $l$-dimensional.

**Example 1.** Proposition 2 is particularly useful to prove the existence of tuples of matrices not mapping any subspace of dimension $k$ into a subspace of dimension $l$. Consider the following two questions.

(1) Do all triples $(A_1, A_2, A_3)$ of $8 \times 5$-matrices map some 4-dimensional subspace into some 7-dimensional subspace? Set $(m,n,k,l,p) = (5,8,4,7,3)$ and compute
$$\frac{l}{k} + \frac{m-k}{n-l} = \frac{7}{4} + \frac{1}{1} < 3 = p,$$
hence by the proposition the answer is no: there exist triples $(A_1, A_2, A_3)$ such that for all $U$ of dimension 4 we have $\sum A_i U = K^8$. This may not come as a surprise; however, it is not entirely obvious how to construct such a "generic" triple. For instance, we cannot choose them such that each $A_i$ is monomial in the sense that it maps every standard basis vector of $K^5$ to some multiple of a standard basis vector of $K^8$: if this is the case, then the inequality $8 \cdot 2 > 5 \cdot 3$ implies that there is a basis vector $e_i$ of $K^8$ which is "hit only once" by some $A_p$ applied to some $e_k$. But then $U = \bigoplus_{l \neq k} Ke_l$ is mapped into $\bigoplus_{j \neq i} Ke_j$.

(2) Do all triples of $5 \times 5$-matrices map some 2-dimensional space into some 3-dimensional space? Set $(m,n,k,l,p) = (5,5,2,3,3)$ in the proposition. Now we find
$$\frac{l}{k} + \frac{m-k}{n-l} = \frac{3}{2} + \frac{3}{2} = 3 = p,$$
so we need a more detailed analysis. The cut-and-paste rank in the proposition is
$$\mathrm{cp}^{(3)}(3,2,2,3),$$
which is $3 \cdot 2 = 6$ as one can cut a $3 \times 2$-matrix into $p = 3$ rectangular pieces that can be put together without overlap to make up a $2 \times 3$-matrix. It follows that the dimension in the proposition is in fact $pmn$, i.e., that indeed, every triple of $5 \times 5$-matrices maps some 2-dimensional space into some 3-dimensional space. To prove this is a nice exercise for students in linear algebra. (It is also true in positive characteristic.)

To conclude the proof of Theorem 1 we need the following lemma.

**Lemma 1.** *Let $V, W, m = \dim V, n = \dim W$, and the $C_k^{(p)}$ for $k = 1, \ldots, m$ and $p \in \mathbb{N}$ be as in Theorem 1. Fix $k \in \{1, \ldots, m\}$ and let $l$ be the maximal integer with $l/k < n/m$. Then the following two statements are equivalent:*

1. *$C_k^{(p)}$ is* not *contained in $C_{k'}^{(p)}$ for any $k' \neq k$.*
2. *There exist a $p$-tuple $(A'_1, \ldots, A'_p) \in M_{l,k}^p$ such that*

$$\sum_i A'_i K^k = K^l \tag{*}$$

*and*

$$\dim(\sum_i A'_i U') \geq \frac{n}{m} \dim U' \tag{**}$$

*for all* proper *subspaces $U' \subsetneq K^k$; as well as a $p$-tuple $(A''_1, \ldots, A''_p) \in M_{n-l,m-k}$ such that*

$$(l + \dim(\sum_i A''_i U'')) \geq \frac{n}{m}(k + \dim U'') \tag{***}$$

*for all* non-zero *subspaces $0 \neq U'' \subseteq K^{m-k}$.*

*Proof.* First suppose that the second condition is not satisfied, let $(A_1, \ldots, A_p)$ be in $C_k^{(p)}$, and let $V', W'$ be subspaces of $V, W$ of dimensions $k, l$, respectively, such that $A_i V' \subseteq W'$ for all $i = 1, \ldots, p$.

Suppose that no $p$-tuple $(A'_i)$ as above exists. Then for some $k' < k$ the closed set consisting of all $(A'_i) \in M_{l,k}^p$ for which there is a $k'$-dimensional $U'$ satisfying $\dim(\sum_i A'_i U) < k'n/m$ fills the entire space $M_{l,k}^p$. Taking for the $A'_i$ the restrictions $A_i|_{V'} : V' \to W'$ we conclude that $C_k^{(p)} \subseteq C_{k'}^{(p)}$.

Similarly, if no $p$-tuple $(A''_i)$ as above exists, then some $k'' \in \{1, \ldots, m-k\}$ has the property that any $p$-tuple $(A''_i) \in M_{n-l,m-k}$ maps some $k''$-dimensional space into a space of dimension $< (k + k'')n/m - l$. In particular, for the $p$-tuple of induced linear maps $\overline{A_i} : V/V' \to W/W'$ there is a $k''$-dimensional space $U''$ for which $\dim \sum_i \overline{A_i} U' < (k + k'')n/m - l$. But then the preimage $U$ of $U''$ in $V$ is a space of dimension $k + k''$ that is mapped into a space of dimension $< (k + k'')n/m - l + l = (k + k'')n/m$, and we conclude that $C_k^{(p)} \subseteq C_{k+k''}^{(p)}$.

Conversely, suppose that $p$-tuples $(A'_i)$ and $(A''_i)$ as above *do* exist. For $i = 1, \ldots, p$ let $A_i \in M_{n,m}$ be the block matrix

$$A_i = \begin{bmatrix} A'_i & \\ & A''_i \end{bmatrix},$$

and let $U$ be a subspace of $K^m$ unequal to $K^k$. Let $U'$ be the intersection of $U$ with $K^k$ and let $U''$ be the projection of $U$ on $K^{m-k}$ along $K^k$. Then $\dim U = \dim U' + \dim U''$ and one readily sees that

$$\dim(\sum_i A_i U) \geq \dim(\sum_i A'_i U') + \dim(\sum_i A''_i U''). \tag{5}$$

Now there are two possibilities: either $U' \neq K^k$, or $U' = K^k$ but $U'' \neq 0$. In the first case one finds that the right-hand side is at least

$$\frac{n}{m} \dim U' + \frac{n}{m} \dim U'' = \frac{n}{m} \dim U,$$

where we have used (**) for the first term, and (***) with $k$ and $l$ replaced by 0 for the second term—note that under this replacement (***) remains valid for $U'' \neq 0$ by the choice of $l$, and becomes valid for $U'' = 0$, as well.

If, on the other hand, $U' = K^k$ but $U'' \neq 0$, then using (*) and (***) we find that the right-hand side in (5) is at least

$$l + \dim(\sum_i A_i'' U'') \geq \frac{n}{m}(k + \dim U'') = \frac{n}{m} \dim U.$$

In other words, the pair $(K^k, K^l)$ is the *only* witness for the nilpotency of $(A_1, \ldots, A_p)$, and *a fortiori* this $p$-tuple lies in a unique $C_k^{(p)}$.

□

*Proof of Theorem 1, part two.* It is clear that if $p < q := \lceil \frac{n}{m} \rceil$, then for *any* subspace $V'$ of $V$ we have $\dim(\sum_{i=1}^p A_i V') \leq p \dim V' < \frac{n}{m} \dim V'$, so that all $C_k^{(p)}$ are equal to $M^p = \mathrm{Hom}(V, W)^p$. In other words: there are no invariants on $M^p$ for $p < q$.

Next suppose that $p \geq q + 1$; then we have to show that there are no inclusions among the $C_k^{(p)}$. For every $k \in \{1, \ldots, m\}$ let $l_k := \lceil k \frac{n}{m} \rceil - 1$ denote the maximal $l \in \{0, \ldots, n-1\}$ with $\frac{l}{k} < \frac{n}{m}$. One readily verifies that

$$1 \leq l_{k+1} - l_k \leq q \text{ for all } k \in \{1, \ldots, m-1\} \tag{6}$$

(the first inequality follows from our standing assumption $n \geq m$). Fix $k \in \{1, \ldots, m\}$ and set $l := l_k$, so that every $p$-tuple in $C_k^{(p)}$ maps some $k$-space into an $l$-space. We will prove the existence of $p$-tuples $(A_i') \in M_{l,k}^p$ and $(A_i'') \in M_{n-l,m-k}^p$ as in Lemma 1, so that $C_k^{(p)}$ is not contained in any $C_{k'}^{(p)}$ with $k' \neq k$.

To find the $A_i'$ we show that for all $k' \in \{1, \ldots, k-1\}$ and $l' \in \{0, \ldots, l-1\}$ with $\frac{l'}{k'} < \frac{n}{m}$ the dimension of the set of $p$-tuples $(A_1', \ldots, A_p') \in M_{l,k}$ that map a $k'$-space into an $l'$-space is smaller than $plk$. To this end we want to apply the sufficient condition of Proposition 2 with $m, n, k, l$ replaced by $k, l, k', l'$, respectively. Compute therefore

$$\frac{l'}{k'} + \frac{k - k'}{l - l'} < \frac{n}{m} + 1 \leq q + 1 \leq p,$$

where for the second term we used $l' \leq l_{k'}$ and the strict increasingness of the $l_k$. This shows the existence of $A_1', \ldots, A_p'$ as required.

Similarly, to find the $A_i''$ we show that for all $k' \in \{k+1, \ldots, m\}$ and $l' \in \{l, \ldots, n-1\}$ with $\frac{l'}{k'} < \frac{n}{m}$ there exists a $p$-tuple $(A_1'', \ldots, A_p'') \in M_{m-k,n-l}$ that does not map any $(k'-k)$-dimensional space into an $l'-l$-dimensional space. Again, we apply the proposition, but now with $m, n, k, l$ replaced by $m-k, n-l, k'-k, l'-l$, respectively. Consider therefore the expression

$$\frac{l' - l}{k' - k} + \frac{m - k'}{n - l'}$$

As $l' \leq l_{k'}$ and $l = l_k$ the first term is at most $q$ by (6). On the other hand, as $l' < \frac{n}{m} k'$, the denominator of the second term satisfies

$$n - l' > n - \frac{n}{m} k' = \frac{n}{m}(m - k') \geq m - k',$$

hence the second term is smaller than 1. We conclude that

$$p \geq q+1 > \frac{l'-l}{k'-k} + \frac{m-k'}{n-l'},$$

hence by Proposition 2 there exists a $p$-tuple $(A_i'')$ as required, and by Lemma 1 we conclude that $C_k^{(p)}$ is not contained in any $C_{k'}^{(p)}$ with $k' \neq k$. This concludes the case where $p > q$.

Finally, we assume that $p = q$. First suppose that there exists a $k \in \{1, \ldots, m-1\}$ with $l_{k+1} - l_k = q$. Then any $q$-tuple $(A_1, \ldots, A_q) \in C_k^{(q)}$ maps a $k$-space into an $l_k$-space, and adding one arbitrary dimension to that $k$-space yields a $(k+1)$-space mapped by all $A_i$ into a space of dimension $l_k + q = l_{k+1}$. In other words, we have $C_k^{(q)} \subseteq C_{k+1}^{(q)}$, so that there are indeed inclusions among the $C_k^{(q)}$. Next suppose that no such $k$ exists. Then we have

$$n-1 = l_m \leq l_1 + (m-1)(q-1) = m(q-1) < m\frac{n}{m} = n,$$

so that $n = m(q-1)+1$, where $q \geq 2$. In this case $l_k = (q-1)k$ for all $k$, and for $q > 2$ the inequalities

$$\frac{l_{k'}}{k'} + \frac{k-k'}{l_k - l_{k'}} = (q-1) + \frac{1}{q-1} < q \text{ for } k' < k$$

and

$$\frac{l_{k'} - l_k}{k'-k} + \frac{m-k'}{n-l_{k'}} = (q-1) + \frac{m-k'}{(q-1)(m-k')+1} < q \text{ for } k' > k$$

readily imply that the construction of the $A_i$ above still works to show that $C_k^{(q)}$ is not contained in any other $C_{k'}^{(q)}$. The last case to be considered is $q = 2$ and $n = m+1$. Then $l_k = k$ for all $k$, and any pair of matrices mapping a $k$-space into a $k$-space also maps a $(k-1)$-space into a $(k-1)$-space, so that the null-cone on $q = 2$ copies is irreducible. □

We should point out that, although Theorem 1 does settle the question of when all irreducible components of the null-cone in $\mathrm{Hom}(V,W)^p$ become visible, it does *not* conclusively describe the irreducible components in the case where $p = q := \lceil n/m \rceil$. Frankly, we do not fully understand the null-cone in this representation: although an easy dimension count shows that $\mathrm{SL}(V) \times \mathrm{SL}(W)$ cannot have a dense orbit on $\mathrm{Hom}(V,W)^q$, so that the null-cone does not fill up the entire space, it seems hard to predict which inclusions there exist among the $C_k^{(q)}$. The only thing that we venture to say in general is that there seem to be *many* inclusions when $n$ is close or equal to $qm$ and *few* inclusions when $q \geq 3$ and $n$ is close to $(q-1)m$. In concrete cases, however, Lemma 1 and Proposition 2 allow one to determine explicitly which of the $C_k^{(q)}$ are maximal. We have thus reduced the problem of determining the irreducible components of the null-cone on $q$ copies to the computation of cut-and-paste ranks—as this is the only non-trivial thing one has to do to apply Lemma 1 and Proposition 2. We conclude the discussion of the null-cone on $\mathrm{Hom}(V,W)^q$ with a few examples.

**Example 2.** (1) If $n = qm$, then $C_k^{(q)}$ is the set of $q$-tuples mapping some $k$-dimensional space into a $(kq-1)$-dimensional space. Clearly, they form a chain $C_1^{(q)} \subseteq C_2^{(q)} \subseteq \ldots \subseteq C_m^{(q)}$, so that the null-cone is equal to the last term and irreducible.

(2) Let $m = 4, n = 6, p = q = 2$. Then $C_k := C_k^{(2)}$ is the set of pairs of linear maps $K^4 \to K^6$ mapping some $k$-dimensional space into an $l_k$-dimensional space, where $l_k = 1, 2, 4, 5$ for $k = 1, 2, 3, 4$, respectively. One has the inclusions $C_1, C_2, C_4 \subseteq C_3$, so that the null-cone is equal to $C_3$ and irreducible (we do not claim that these are *all* inclusions among the $C_i$). Indeed, the inclusion $C_2 \subseteq C_3$ is easy. To see that $C_4 \subseteq C_3$ we apply Proposition 2 with $(m, n, k, l, p)$ equal to $(4, 5, 3, 4, 2)$: the dimension of the variety $Q$ there equals

$$40 - 2 \cdot 1 + \mathrm{cp}^{(2)}(1, 3, 1, 2) = 40,$$

so that every pair of $5 \times 4$-matrices maps some 3-dimensional space into a 4-dimensional space (this can, of course, also be seen directly).

Similarly, to see that $C_1 \subseteq C_3$ we apply Proposition 2 with $(m, n, k, l, p)$ equal to $(3, 5, 2, 3, 2)$. The dimension of $Q$ is now

$$30 - 1 \cdot 2 + \mathrm{cp}^{(2)}(1, 2, 2, 1) = 30,$$

so that every pair of $5 \times 3$-matrices maps some 2-dimensional space into a 3-dimensional space. Applying, as in Lemma 1, this fact to the linear maps induced by a pair $(A_1, A_2) \in C_1$, which go from a 3-dimensional quotient space to a 5-dimensional quotient space, we find that $C_1 \subseteq C_3$.

(3) Let $m = 5, n = 12, p = 3$. Then $l_k = 2, 4, 7, 9, 11$ for $k = 1, 2, 3, 4, 5$, respectively; write $C_k := C_k^{(3)}$. We readily find $C_2 \subseteq C_3$. We claim that no $C_k$ with $k \neq 2$ is contained in any $C_{k'}$ with $k' \neq k$. Again, one can prove this using Lemma 1 and Proposition 2. Indeed, it turns out that for $k = 1, 3, 4, 5$ the sufficient criterion

$$3 = p > l'/k' + (m' - k')/(n' - l')$$

of Proposition 2 is verified for all values $m' := k, n' := l_k, k' < k, l' := l_{k'}$ as well as for all values $m' := m - k, n' := n - l_k, 1 < k' \leq m - k, l' := l_{k+k'} - l_k$. Using Lemma 1, this proves that the null-cone has 4 irreducible components, namely $C_1, C_3, C_4, C_5$.

As promised in the Introduction, we now investigate when the polarisations of invariants on one copy of $\mathrm{Hom}(V, W)$ define the null-cone on $p$ copies. This question is interesting only in the case where there *are* non-trivial invariants on one copy—hence if $\dim V = \dim W$, in which case we may as well assume $V = W$. The invariant ring of $\mathrm{SL}(V) \times \mathrm{SL}(V)$ on $\mathrm{End}(V)$ is generated by the determinant; this readily follows from the fact that every invertible matrix $A$ has the matrix $\mathrm{diag}(\det A, 1, \ldots, 1)$ in its orbit. Note that by Theorem 1 the $p$-tuples in the null-cone on $\mathrm{End}(V)^p$ are precisely those whose span in $\mathrm{End}(V)$ is a "compression space" in the sense that it maps some subspace of $V$ into a strictly smaller subspace; see [6] for this terminology. On the other hand, the $p$-tuples on which all polarisations of det vanish are those that span a "singular space", i.e., a vector space in which every linear map is singular. Hence, the polarisations of det define the null-cone on $\mathrm{End}(V)^p$ if and only if every singular space in $\mathrm{End}(V)$ spanned by $p$ matrices is is a compression space. See [4] for interesting *small* examples of singular non-compression spaces.

**Theorem 2.** *The null-cone in* $\mathrm{End}(V)^p$ *is defined by the polarisations of* det *if and only if* $\dim V \leq 2$ *or* $p \leq 2$.

*Proof of Theorem 2.* The result for $p = 2$ follows from the Kronecker-Weierstrass theory of matrix pencils, see [7]; for completeness we include a short proof in our terminology. By Theorem 1 we have to show that if $A, B \in \mathrm{End}(V)$ satisfy $\det(sA + tB) = 0$ for all $s, t \in K$, then there exists a witness $V', W' \subseteq V$ for the nilpotency of $(A, B)$. Indeed, regarding $s, t$ as variables, $sA + tB$ has a non-zero vector $u(s, t)$ in $K[s, t] \otimes_K V$ in its kernel. But then any non-zero homogeneous component of $u(s, t)$, say of degree $d$, is also annihilated by $sA + tB$; hence we find $u_0, \ldots, u_d \in V$ such that $(sA+tB)(s^d u_0 + s^{d-1}tu_1 + \ldots + t^d u_d) = 0$, where we may assume that $u_0 \neq 0$. Taking the of coefficients of $s^{d+1}, s^d t, \ldots, t^{d+1}$, we find

$$Au_0 = 0,\ Au_1 = -Bu_0,\ \ldots,\ Au_d = -Bu_{d-1},\ \text{and}\ Bu_d = 0.$$

But then every element of $KA + KB$ maps the space $V' := \sum_i Ku_i$ into the space $U' := \sum_i KAu_i$, which is strictly smaller because $Au_0 = 0$ while $u_0 \neq 0$.

The statement for $\dim V = 2$ is easy: in a linear space of matrices of rank $\leq 1$ either all matrices have the same image, or all matrices have the same kernel (otherwise the space contains an $A = \lambda \otimes u$ and a $B = \mu \otimes v$ such that both $\lambda, \mu \in V^*$ and $u, v \in V$ are linearly independent—but then $A + B$ has rank 2). Now suppose that $m, n \geq 3$. To show that the null-cone in $\mathrm{End}(V)^3$ is then *not* defined by the polarisations of det, it suffices to construct a 3-dimensional singular subspace of $\mathrm{End}(V)$ for which there do not exist $V', W'$ as above. The space

$$\left\{ \begin{bmatrix} 0 & a & b & & & & \\ -a & 0 & c & & & & \\ -b & -c & 0 & & & & \\ & & & a & & & \\ & & & & a & & \\ & & & & & \ddots & \\ & & & & & & a \end{bmatrix} \;\middle|\; a, b, c \in K \right\} \text{ (empty entries are always zero)},$$

is such a space, as one easily verifies. □

## 3. SL($V$) on symmetric bilinear forms

The group $\mathrm{SL}(V)$ acts on bilinear forms as follows: if $\alpha$ is a bilinear form and $g \in \mathrm{SL}(V)$, then $(g\alpha)(v, w) = \alpha(g^{-1}v, g^{-1}w)$. It will be convenient to associate to every bilinear a linear map as follows: we fix, once and for all, a non-degenerate, *symmetric* bilinear form $(.,.)$ on $V$, and denote the *transpose* of $A \in \mathrm{End}(V)$ relative to this form by $A^t$. If $\alpha$ is a bilinear form on $V$, then we associate to $\alpha$ a linear map $A$ by the requirement that $\alpha(x, y) = (x, Ay)$ for all $x, y \in V$. Then $g$ acts on $A$ by $g \cdot A := (g^{-1})^t A g^{-1}$. Note that the image of $\mathrm{SL}(V)$ in $\mathrm{GL}(\mathrm{End}(V))$ under this representation is contained in the image of $\mathrm{SL}(V) \times \mathrm{SL}(V)$ under the representation of Section 2.

If $\alpha$ is a symmetric or skew symmetric bilinear form on $V$, and if $U$ is a subspace of $V$, then we will call the space $\{v \in V \mid \alpha(v, U) = 0\}$ the *$\alpha$-perp* of $U$. If $A$ is the linear map associated to $\alpha$, then this also the $(.,.)$-perp of $AU$.

As in Section 2 the invariants of $\mathrm{SL}(V)$ on $S^2(V^*)$ are generated by the determinant of (the linear map associated to) the form, and the null-cone on one copy is therefore the irreducible variety of singular forms.

**Theorem 3.** *For $p \geq 2$ and $n := \dim V$, the null-cone of* $\mathrm{SL}(V)$ *on* $S^2(V^*)^p$ *has* $\lfloor \frac{n+1}{2} \rfloor$ *irreducible components given by*

$$C_k^{(p)} := \{(\alpha_1, \ldots, \alpha_p) \mid \exists U \subseteq W \subseteq V : \dim U = k, \dim W = n - k + 1, \text{ and } \alpha_i(U, W) = 0 \text{ for all } i = 1, \ldots, p\},\ k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor.$$

In contrast to our proof for tuples of matrices, we will give explicit pairs of symmetric forms representing the various components of the null-cone; for this the following lemma is useful.

**Lemma 2.** *Let $m, n, k$ be non-negative integers and let $\pi_1, \ldots, \pi_p$ be partially defined strictly increasing functions $\{1, \ldots, m\} \to \{1, \ldots, n\}$, that is, every $\pi_l$ is defined on a subset* $\mathrm{dom}(\pi_l)$ *of $\{1, \ldots, m\}$ and satisfies*

$$i < j \Rightarrow \pi_l(i) < \pi_l(j) \text{ whenever the right-hand side is defined.}$$

*For $l = 1, \ldots, p$ let $A_l : K^m \to K^n$ be a linear map mapping $e_i$ to a non-zero multiple of $e_{\pi_l(i)}$ if $\pi_l$ is defined at $i$, and to zero otherwise. Let $U$ be a subspace of $K^m$ and set*

$$\mathrm{gr}\, U := \{i \in \{1, \ldots, m\} \mid U \cap (e_i + \langle e_1, \ldots, e_{i-1} \rangle_K) \neq \emptyset\}.$$

*Then*

$$\dim \sum_l A_l U \geq \left| \bigcup_l \pi_l(\mathrm{gr}\, U \cap \mathrm{dom}\, \pi_l) \right|$$

We will call a $p$-tuple $(A_1, \ldots, A_p)$ of linear maps as in this lemma *standard*.

*Proof.* We have $|\mathrm{gr}(U)| = \dim U$, and defining $\mathrm{gr}\, W$ for subspaces $W$ of $K^n$ in a similar way the conditions on the $A_i$ guarantee that

$$\mathrm{gr}(\sum_l A_l U) \supseteq \bigcup_l \pi_l(\mathrm{gr}\, U \cap \mathrm{dom}\, \pi_l),$$

whence the lemma follows immediately. □

*Proof of Theorem 3.* Suppose that $(\alpha_1, \ldots, \alpha_p)$ lies in the null-cone, and let $A_i$ be the linear map associated to $\alpha_i$. Then $(A_1, \ldots, A_p)$ lies in the null-cone of $\mathrm{SL}(V)$ acting on $\mathrm{End}(V)$ as indicated above and, *a fortiori*, in the null-cone of $\mathrm{SL}(V) \times \mathrm{SL}(V)$ on $\mathrm{End}(V)$ discussed in Section 2. Hence by Theorem 1 there exist subspaces $U'$ and $W'$ of $V$ with $\dim W' = n - \dim U' + 1$ and such that every $A_i$ maps $U'$ into the $(.,.)$-perp of $W'$ relative to $(.,.)$ (So $W'$ here is the $(.,.)$-perp of the space $W'$ in Theorem 1.) But then $\alpha_i(w, u) = (w, A_i u) = 0$ for all $u \in U'$ and $w \in W'$. Now set $U := U' \cap W'$ and $W := U' + W'$. Then clearly $U \subseteq W$, $\dim U + \dim W = \dim U' + \dim W' = n + 1$, and $\alpha_i(U, W) = 0$ for all $i$.

The $C_k^{(p)}$ are closed and irreducible as usual (see the Introduction), and so it only remains to check that there are no inclusions among them for $p \geq 2$. To this end, let $k \in \{1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$; we will construct a pair $(\alpha, \beta) \in C_k^{(2)}$ that does not lie in any $C_{k'}^{(2)}$ with $k \neq k'$. Take $V = K^n$ and $(x, y) := \sum_{i=1}^n x_i y_{n+1-i}$, so that transposition relative to this form corresponds to reflection of the matrix in the "skew diagonal";

we will refer to this symmetric form as the *skew diagonal symmetric form.* Now take the standard pair $(A, B)$ for which

$$sA + tB = \left[\begin{array}{cccc|cccc|ccc} s & t & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & s & t & & & & & & & \\ \hline & & & & t & & & & & & \\ & & & & s & \ddots & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & s & t & & & \\ \hline & & & & & & & & t & & \\ & & & & & & & & s & \ddots & \\ & & & & & & & & & \ddots & t \\ & & & & & & & & & & s \end{array}\right],$$

where the diagonal block sizes are, from top left to bottom right, $(k-1) \times k$, $(n-2k+1)\times(n-2k+1)$, and $k\times(k-1)$. Let $\alpha$ and $\beta$ be the forms defined by $A$ and $B$, respectively. Now if $U$ and $W$ are subspaces of $K^n$ with $\dim U + \dim W = n+1$ and $\alpha(U, W) = \beta(U, W) = 0$, then one finds $\dim(AU + BU) < \dim U$. But by Lemma 2 the only pair of subspaces of $K^n$ having this property are $U = \langle e_1, \ldots, e_k\rangle_K$ and $W = \langle e_1, \ldots, e_k, \ldots, e_{n-k+1}\rangle_K$. This shows that $(U, W)$ is the unique witness for the nilpotency of $(\alpha, \beta)$, and hence $(\alpha, \beta)$ does not lie in any other component $C^{(2)}_{k'}$. □

We now proceed with our second fundamental problem: for which $p, n$ is the null-cone on $p$-tuples of symmetric bilinear forms on $V$ defined by the polarisations of det? Suppose that $(\alpha_1, \ldots, \alpha_p)$ lies in $C^{(p)}_k$, and that $U$ and $W$ are a witness of its nilpotency as in Theorem 3. A dimension argument shows that $U$ must intersect the radical of each $\alpha_i$ non-trivially; in particular, if $\alpha_i$ has rank $n-1$, then its radical is contained in $U$, and $W$ is precisely the $\alpha_i$-perp of $U$.

Suppose now that all $\alpha_i$ have rank $n-1$. Then a geometric interpretation of $U, W$ as in the theorem is the following: $\mathbb{P}U$ is a linear subspace of $\mathbb{P}V$ common to all quadrics $Q_i = \{x \in \mathbb{P}V \mid \alpha_i(v, v) = 0\}$ and containing their radicals, and for each $i$, $\mathbb{P}W$ is the space tangent to $Q_i$ at all of $\mathbb{P}U$. For example, if $n = 4$ and $p = 2$, then a pair $(\alpha_1, \alpha_2)$ of rank 3 forms lies in $C^{(2)}_1$ if and only if $\alpha_1$ and $\alpha_2$ have the same radical (a projective point); if $(\alpha_1, \alpha_2) \notin C^{(2)}_1$, then the pair lies in $C^{(2)}_2$ if and only if the quadrics $Q_1, Q_2$ are tangent along the (projective) line through their radicals. This interpretation yields a nice proof of the following theorem.

**Theorem 4.** *The null-cone on $S^2(V^*)^p$ is defined by the polarisations of* det *if and only if* $\dim(V) \leq 4$ *or* $p \leq 2$.

*Proof of Theorem 4.* On $p = 2$ copies the null-cone is defined by the polarisations of the determinant. This follows either from the Kronecker-Weierstrass theory of pencils of forms [7] or from a direct construction of $U$ and $W$ as in Theorem 3 for any two-dimensional space of singular forms.

Next we prove that for $n \leq 4$ the null-cone on *any* number $p$ of copies is defined by the polarisations of det, or, in other words, that any space $\mathcal{A}$ of singular symmetric bilinear forms is spanned by a tuple $(\alpha_1, \ldots, \alpha_p)$ lying in some $C^{(p)}_k$; slightly

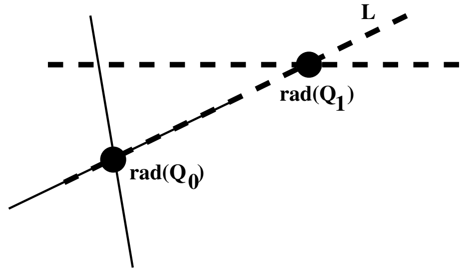


FIGURE 2. Proof of Theorem 4 for $n = 3$

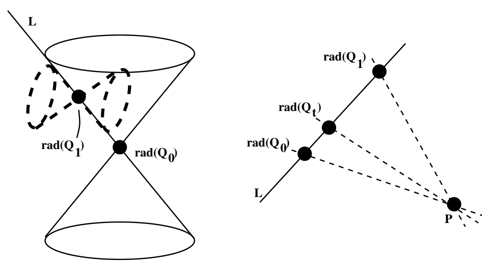


FIGURE 3. Proof of Theorem 4 for $n = 4$

inaccurately, we will then say that $\mathcal{A}$ *lies in* $C_k$. Note that we need only prove this for maximal spaces of singular forms; in particular, we may assume that $\mathcal{A}$ contains forms of rank $n-1$, because if it does not, we may add any rank 1 form to $\mathcal{A}$ without creating non-degenerate forms. In what follows we heavily use the fact that any 2-dimensional space of singular forms does already lie in some $C_k$.

For $n = 2$, the quadric of a rank 1 form is a point on the projective line $\mathbb{P}V$. As for any two non-zero forms in $\mathcal{A}$ this point coincides, it is the same for *all* forms in $\mathcal{A}$. Hence $\mathcal{A}$ lies in $C_1$.

For $n = 3$, the quadric of a rank 2 form $\alpha$ is the union of two lines in the projective plane $\mathbb{P}V$, whose intersection is the radical of $\alpha$. If the radicals of any two forms in $\mathcal{A}$ of rank 2 coincide, then $\mathcal{A}$ lies in $C_1$; suppose, therefore, that there exist forms $\alpha_0, \alpha_1$ in $\mathcal{A}$ of rank 2 whose radicals are distinct. We have $(\alpha_0, \alpha_1) \in C_2$, so that their quadrics $Q_0$ and $Q_1$ have a line $L$ in common (see Figure 2). Now a generic element $\beta \in \mathcal{A}$ has rank 2, does not have the same radical as $\alpha_0$ or $\alpha_1$, and its quadric $Q_\beta$ is not the union of the non-common lines of $Q_0$ and $Q_1$. But $Q_\beta$ must have lines in common with both $Q_0$ and $Q_1$, and therefore it contains $L$. But then $L$ is isotropic relative to all forms in $\mathcal{A}$, and $\mathcal{A}$ lies in $C_2$.

For $n = 4$, suppose that there exist forms $\alpha_0, \alpha_1 \in \mathcal{A}$ of rank 3 whose radicals do not coincide (otherwise $\mathcal{A}$ lies in $C_1$). The corresponding quadrics $Q_0, Q_1 \subseteq \mathbb{P}V$ are tangent along the line $L$ connecting their radicals (see Figure 3, left). For $t \in K$ set $\alpha_t := (1-t)\alpha_0 + t\alpha_1$ and

$$T := \{t \in K \mid \mathrm{rk}(\alpha_t) = 3\}.$$

For each $t \in T$, the quadric $Q_t$ of $\alpha_t$ is tangent to $Q_0$ along $L$, and its radical lies on $L$; the set of all radicals thus obtained forms a dense set of $L$.

If all rank 3 forms in $\mathcal{A}$ have their radicals on $L$, then their quadrics are all tangent to $Q_0$ along $L$ and $\mathcal{A}$ lies in $C_2$. Suppose, on the other hand, that there exists a rank 3 form $\beta \in \mathcal{A}$ whose radical does not lie on $L$. Then its quadric $Q_\beta$ is tangent to each $Q_t$ with $t \in T$ along the line connecting $P := \mathbb{P}\,\mathrm{rad}(\beta)$ and $\mathbb{P}\,\mathrm{rad}(\alpha_t)$; in particular, $Q_\beta$ contains all lines connecting $P$ with a dense subset of $L$ (see Figure 3, right). The closure of the union of these lines—the projective plane spanned by $L$ and $P$—is therefore contained in $Q_\beta$. Hence, the pre-image in $V$ of this plane is a 3-dimensional $\beta$-isotropic space—but this contradicts the assumption that $\mathrm{rk}(\beta) = 3$.

Finally, we need to show that if $n \geq 5$ and $p \geq 3$, then the null-cone is *not* defined by the polarisations of det. To this end, take for $(.,.)$ on $V = K^n$ the orthogonal sum of the skew diagonal symmetric form on $K^5$ and the skew diagonal symmetric form on $K^{n-5}$. Consider the triple $(\alpha_1, \alpha_2, \alpha_3)$ of bilinear forms on $K^n$ for which the linear map associated to $s\alpha + t\beta + u\gamma$ relative to $(.,.)$ equals

$$sA_1 + tA_2 + uA_3 = \left[\begin{array}{ccc|cc|c} s & t & 0 & 0 & 0 & \\ 0 & s & t & 0 & 0 & \\ \hline -u & 0 & 0 & t & 0 & \\ 0 & 2u & 0 & s & t & \\ 0 & 0 & -u & 0 & s & \\ \hline & & & & & sI_{n-5} \end{array}\right].$$

A direct computation shows that $\det(sA_1 + tA_2 + uA_3) = 0$. On the other hand, by Lemma 2 there exists no subspace $U$ of $K^n$ with $\dim(\sum_i A_i U) < \dim U$. We conclude that $(\alpha_1, \alpha_2, \alpha_3)$ is not nilpotent, and this concludes the proof of Theorem 4. □

*Remark* 4. The description of the null-cone in Theorem 3 already appears in [17, Theorem 0.1(ii)]. However, Wall claims in Corollary 1 of *loc. cit.* that the null-cone on *any* number of copies is defined by the polarisations of det—which, as we have just seen, is only the case for $n < 5$.

## 4. SL($V$) on skew-symmetric forms

Our results for skew-symmetric forms are similar to those for symmetric forms, except that the irreducible components of the null-cone become visible only from 3 or 4 copies onwards. Recall that if $n := \dim(V)$ is odd, then all skew bilinear forms are singular and there are no invariants on one copy of $\bigwedge^2(V^*)$, so that the null-cone is the whole space. If $n$ is even, then the invariant ring is generated by the Pfaffian and the null-cone is irreducible.

**Theorem 5.** *The null-cone* SL($V$) *on* $\bigwedge^2(V^*)^p$ *is equal to*

$$\{(\alpha_1, \ldots, \alpha_p) \mid \exists U \subseteq W \subseteq V \text{ with } \dim U + \dim W = n + 1 \text{ and } \alpha_i(U, W) = 0 \text{ for all } i = 1, \ldots, p\}.$$

*Let* $C_k^{(p)}$ *denote the subset of the null-cone where* $U$ *can be chosen of dimension* $k (= 1, \ldots, \lceil \frac{n}{2} \rceil =: q)$. *Then the irreducible components of the null-cone are as follows.*

(1) *If* $n = 2q \geq 2$ *is even, then the null-cone on* $p = 2$ *copies is* $C_q^{(2)}$ *(hence irreducible), while the null-cone on* $p \geq 3$ *copies has precisely* $q$ *components, namely* $C_k^{(p)}$ *for* $k = 1, \ldots, q$.

(2) *If $n = 2q-1 \geq 3$ is odd, then the null-cone on $p = 2$ copies is all of $\bigwedge^2(V^*)^p$; on $p = 3$ copies there are non-trivial invariants, and the components of the null-cone are precisely the $C_k^{(3)}$ with $k \in \{1, 2, \ldots, q-4, q\}$ (in particular, for $n \leq 7$ the null-cone is irreducible); on $p = 4$ copies the components of the null-cone are precisely the $C_k^{(4)}$ with $k \in \{1, 2, \ldots, q-3, q\}$ (in particular, for $n \leq 5$ the null-cone is irreducible); and on $p \geq 5$ copies the components of the null-cone are precisely the $C_k^{(p)}$ with $k \in \{1, 2, \ldots, q-2, q\}$ (in particular, for $n \leq 3$ the null-cone is irreducible).*

For the proof of this theorem we need a result from [11], which uses the following notation: $d(n, p)$ is the minimum, taken over all $p$-tuples $(\alpha_1, \ldots, \alpha_p)$ of skew bilinear forms on $K^n$, of the maximal dimension of a subspace that is isotropic with respect to all $\alpha_i$. In other words, $d(n, p)$ is the maximal dimension of a common isotropic subspace of a generic $p$-tuple of skew bilinear forms on $K^n$.

**Theorem 6** ([11, Main Theorem])**.** $d(n, p) = \lfloor \frac{2n+p}{p+2} \rfloor$.

**Corollary 1.** *For $n = 0, 2, 4, 6$ any triple of skew bilinear forms on $K^n$ has a common isotropic subspace of dimension $n/2$. On the other hand, for all odd $n \geq 3$ and for all even $n \geq 8$ there exist triples $(\alpha_1, \alpha_2, \alpha_3)$ of skew bilinear forms on $K^n$ for which there are* no *subspaces $0 \subsetneq U \subseteq W$ of $K^n$ with $\dim U + \dim W = n$ and $\alpha_i(U, W) = 0$ for all $i$.*

*Proof.* The first statement is immediate from Theorem 6. Now let $n = 2q \geq 8$ be even, fix $k \in \{1, \ldots, q\}$, and suppose that for any triple $\alpha_1, \alpha_2, \alpha_3$ of skew bilinear forms on $K^n$ there exist subspaces $0 \neq U \subseteq W$ of $K^n$ with $\dim U = k = n - \dim W$ and $\alpha_i(U, W) = 0$ for all $i = 1, 2, 3$. The induced forms $\bar{\alpha}_i$, $i = 1, 2, 3$, on the space $W/U$ of dimension $2(q-k)$ have a common isotropic subspace $U' \subseteq W/U$ of dimension $d(2(q-k), 3)$, by definition of the latter quantity. The pre-image of $U'$ in $W$ is then isotropic relative to all $\alpha_i$ and has dimension $d(2(q-k), 3) + k$. We thus find the inequality $d(2q, 3) \geq d(2(q-k), 3) + k$, which by Theorem 6 reads

$$\lfloor \frac{4q+3}{5} \rfloor \geq \lfloor \frac{4(q-k)+3}{5} \rfloor + k = \lfloor \frac{4q+3}{5} + \frac{k}{5} \rfloor. \tag{7}$$

For $n = 2q = 8$, however, this inequality does not hold for any $k \in \{1, 2, 3, 4\}$. For $n = 2q = 10$ the only $k \in \{1, \ldots, 5\}$ for which it holds is $k = 1$, but it is easy to construct a triple of bilinear forms on $K^{10}$ for which there are no $U, W$ as above of dimensions $1, 9$—indeed, one can use for this the construction that follows.

Suppose that $n = 2q \geq 12$, and note that inequality (7) can only hold for $k \leq 5$. On the other hand, let $\alpha_1, \alpha_2, \alpha_3$ be the skew bilinear forms on $K^n$ corresponding to the triple $(A_1, A_2, A_3)$ of matrices, standard in the sense of Lemma 2, satisfying

$$t_1A_1 + t_2A_2 + t_3A_3 = \begin{bmatrix} t_2 & t_3 & & & & & & \\ t_1 & t_2 & \ddots & & & & & \\ & \ddots & \ddots & t_3 & & & & \\ & & t_1 & t_2 & & & & \\ & & & & -t_2 & -t_3 & & \\ & & & & -t_1 & \ddots & \ddots & \\ & & & & & \ddots & -t_2 & -t_3 \\ & & & & & & -t_1 & -t_2 \end{bmatrix}.$$

Using Lemma 2 one verifies that any subspace $U$ of $K^n$ satisfying $\dim(A_1U + A_2U + A_3U) \leq \dim U$ has dimension $0, n/2$, or $n$. In particular, we should have $k \in \{0, q, n\}$—but we saw above that $1 \leq k \leq 5$, a contradiction.

We conclude that for $n = 2q \geq 8$ and fixed $k \in \{1, \dots, q\}$ there exist triples $(\alpha_1, \alpha_2, \alpha_3)$ of skew bilinear forms on $K^n$ for which there are no subspaces $U \subseteq W$ of $K^n$ with $\dim U = k = n - \dim W$ and $\alpha_i(U, W) = 0$ for all $i$. As the non-existence of such a pair $U, W$ with $\dim U = k$ is an open condition on the triple $(\alpha_1, \alpha_2, \alpha_3)$, there also exist triples for which there is no pair $(U, W)$ with $U$ of *any* dimension. This proves the corollary for even $n$.

For $n = 2q - 1 \geq 3$ odd we can construct $\alpha_1, \alpha_2, \alpha_3$ explicitly by a construction similar to that above: choose them corresponding to a standard triple $(A_1, A_2, A_3)$ of matrices satisfying

$$t_1A_1 + t_2A_2 + t_3A_3 = \begin{bmatrix} t_2 & t_3 & & & & & \\ t_1 & \ddots & \ddots & & & & \\ & \ddots & t_2 & t_3 & & & \\ & & t_1 & 0 & -t_3 & & \\ & & & -t_1 & -t_2 & \ddots & \\ & & & & \ddots & \ddots & -t_3 \\ & & & & & -t_1 & -t_2 \end{bmatrix}.$$

Using Lemma 2 one verifies that there are no subspaces $U \neq 0, K^n$ of $K^n$ with $\dim(\sum_i A_iU) \leq \dim U$. □

*Proof of Theorem 5.* The description of the null-cone is proved in exactly the same way as for symmetric bilinear forms; we do not repeat the argument here. We proceed to prove the inclusions $C_k^{(p)} \subseteq C_q^{(p)}$ for the following values of the parameters:

(1) $n$ arbitrary, $k$ arbitrary, and $p = 2$;
(2) $n = 2q - 1 \geq 3$, $k = q - 1$, and $p$ arbitrary;
(3) $n = 2q - 1 \geq 5$, $k = q - 2$, and $p \in \{3, 4\}$; or
(4) $n = 2q - 1 \geq 7$, $k = q - 3$, and $p = 3$.

These statements are proved as follows: let $(\alpha_1, \dots, \alpha_p) \in C_k^{(p)}$ and let $U \subseteq W$ be a pair with $\dim U = k, \dim W = n - k + 1$, and $\alpha_i(U, W) = 0$ for all $i$. Then the $\alpha_i$ induce bilinear forms $\bar{\alpha}_i$ on the space $W/U$ of dimension $n - 2k + 1$, and we find a subspace $U'$ of $W/U$ of dimension $d(n - 2k + 1, p)$ that is isotropic relative to all $\bar{\alpha}_i$. The pre-image of $U'$ in $W$ is then a space of dimension $d(n - 2k + 1, p) + k$ and isotropic relative to all $\alpha_i$. Using Theorem 6 one finds that for the above values of the parameters this value $d(n - 2k + 1, p) + k$ is at least $\lfloor \frac{n}{2} \rfloor + 1$, which shows that $(\alpha_1, \dots, \alpha_p) \in C_q^{(p)}$. This proves all inclusions above.

Now we prove that there are no other inclusions among the $C_k^{(p)}$ for other values of $n, k$, and $p$. Suppose first that $n = 2q$ is even, $p \geq 3$ and $k \in \{1, \dots, q\}$. Then we find a $p$-tuple in $C_k^{(p)}$ not lying in any other $C_{k'}^{(p)}$ by a construction similar to the constructions in the proof of Corollary 1: Let $\alpha_1, \alpha_2, \alpha_3$ be forms with matrices

$A_1, A_2, A_3$ for which $t_1A_1 + t_2A_2 + t_3A_3$ equals

$$\begin{bmatrix} t_2 & t_3 & & & & & & & & \\ t_1 & t_2 & t_3 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & t_1 & t_2 & t_3 & & & & & \\ & & & & & t_1A'_1 + t_2A'_2 + t_3A'_3 & & & & \\ & & & & & & -t_3 & & & \\ & & & & & & -t_2 & \ddots & & \\ & & & & & & -t_1 & \ddots & -t_3 & \\ & & & & & & & \ddots & -t_2 & -t_3 \\ & & & & & & & & -t_1 & -t_2 \end{bmatrix}, \tag{8}$$

where the diagonal blocks have sizes $(k-1) \times k$, $(n-2k+1) \times (n-2k+1)$, and $k \times (k-1)$ from top left to bottom right, and where the $A'_i$ are chosen such (skew relative to the skew diagonal) that they map no subspace $U \neq 0, K^{n-2k+1}$ of $K^{n-2k+1}$ into a subspace of dimension $< \dim U$; such $A'_i$ exist by Corollary 1. Write $V_1 := \langle e_1, \ldots, e_k \rangle_K$, $V_2 := \langle e_{k+1}, \ldots, e_{n-k} \rangle_K$, and $V_3 := \langle e_{n-k+1}, \ldots, e_n \rangle_K$. Now suppose that $U$ is a subspace of $K^n$ for which $\dim \sum A_iU < \dim U$. Let $U_1 := U \cap V_1$, let $U_2$ be the projection of $U \cap (V_1 \oplus V_2)$ to $V_2$ along $V_1$, and let $U_3$ be the projection of $U$ to $V_3$ along $V_1 \oplus V_2$. Then $\dim \sum_i A_iU_1 \geq \dim U_1$ unless $U_1 = V_1$, $\dim \sum_i A_iU_2 > \dim U_2$ unless $U_2 = 0$ or $V_2$, and $\dim \sum_i A_iU_3 > \dim U_3$ unless $U_3 = 0$. Summing up these dimensions, we find $\dim \sum_i A_iU < \dim U$ implies $U_1 = V_1$, $U_2 = 0$ or $U_2 = V_2$, and $U_3 = 0$. We conclude that $(V_1, V_1 \oplus V_2)$ is the only pair of subspaces $U \subseteq W$ with $\alpha_i(U, W) = 0$ and $\dim U + \dim W > n$. Hence $(\alpha_1, \alpha_2, \alpha_3)$ lies in $C_k^{(3)}$ but not in any other $C_{k'}^{(3)}$.

Next suppose that $n = 2q - 1 \geq 9$ is odd. Then we have to show that that $C_k^{(3)}$ for $k \notin \{q-1, q-2, q-3\}$ is *not* contained in any other $C_k^{(3)}$. This goes using a construction similar to that above for even $n$, choosing the $A'_i$—now square skew matrices of size $n - 2k + 1 = 2(q-k) \geq 8$—such that for all spaces $U$ with $0 \subsetneq U \subsetneq K^{2(q-k)}$ we have $\dim A'_1U + A'_2U + A'_3U > \dim U$; such matrices exist by Corollary 1.

Next, assuming $n = 2q - 1 \geq 7$, suppose that $p \geq 4$ and $k \in \{1, \ldots, q-3, q\}$. By writing down an appropriate standard quadruple of skew matrices $(A_1, \ldots, A_4)$ we show that $C_k^{(p)}$ is not contained in any other $C_{k'}^{(p)}$: take $A_1, A_2, A_3, A_4$ such that $\sum_i t_iA_i$ has the block shape of (8), where the outer two blocks are unchanged (i.e., $A_4$ has no non-zero entries there), but the inner block of size $2(q-k) \geq 6$ is as follows:

$$\begin{bmatrix} t_2 & t_3 & t_4 & & & & & \\ 0 & t_2 & \ddots & \ddots & & & & \\ t_1 & 0 & \ddots & t_3 & t_4 & & & \\ & \ddots & \ddots & t_2 & 0 & -t_4 & & \\ & & t_1 & 0 & -t_2 & -t_3 & \ddots & \\ & & & -t_1 & 0 & \ddots & \ddots & -t_4 \\ & & & & \ddots & \ddots & -t_2 & -t_3 \\ & & & & & -t_1 & 0 & -t_2 \end{bmatrix}$$

Again, applying Lemma 2, one readily verifies that this quadruple of skew matrices does not map any space $U$ into a space of dimension $\leq \dim U$.

A similar construction for $n = 2q - 1 \geq 5$ with the following $4 \times 4$-block in the middle:

$$\begin{bmatrix} t_3 & t_4 & t_5 & 0 \\ t_2 & t_3 & 0 & -t_5 \\ t_1 & 0 & -t_3 & -t_4 \\ 0 & -t_1 & -t_2 & -t_3 \end{bmatrix}$$

shows that on $p \geq 5$ copies the set $C^{(p)}_{q-2}$ is not contained in any other $C^{(p)}_k$, either. □

Finally, we settle the question, for $n$ even, of when the null-cone on $p$ copies of $\bigwedge^2(V^*)$ is defined by the polarisations of the Pfaffian.

**Theorem 7.** *The null-cone $\mathcal{N}(\bigwedge^2(V^*)^p)$ with $\dim V =: n$ even is defined by the polarisations of the Pfaffian if and only if either $p = 2$ or $n \in \{2, 4\}$.*

*Proof.* The proof for $p = 2$ goes exactly as for symmetric bilinear forms, and for $n = 2$ the statement is trivial.

Suppose therefore that $n = 4$. As the referee of this paper kindly pointed out, the theorem for this case can be proved using classical invariant theory: the image of $\mathrm{SL}_4$ in $\mathrm{GL}(\bigwedge^2 K^4)$ is precisely $\mathrm{SO}_6$, hence by the First Fundamental Theorem for $\mathrm{SO}_6$ [8] one knows precisely the invariants on $p$ copies of this representation, and from this knowledge one can deduce that the null-cone is defined by the polarisations of the invariants on one copy.

In keeping with the more geometrical arguments used for the case of symmetric bilinear forms, we include a short, self-contained proof for the case where $n = 4$ that does not rely on classical invariant theory: Let $\mathcal{A}$ be a vector space consisting of singular skew forms on $K^4$. We have to show that either the radicals of all forms in $\mathcal{A}$ intersect in a projective point, or there exist a line $U$ and a plane $W \supseteq U$ in $\mathbb{P}^3$ with $\alpha(U, W) = 0$ for all $\alpha \in A$. By the statement for $p = 2$ we know that any *pair* of elements in $\mathcal{A}$ is of one of these two types.

We prove that in fact every pair $\alpha, \beta \in \mathcal{A}$ is of the first type. Indeed, take $\alpha, \beta \in \mathcal{A}$ non-zero (and hence of rank 2), suppose that $\operatorname{rad} \alpha$ and $\operatorname{rad} \beta$ are disjoint lines in $\mathbb{P}^3$, and let $U \subseteq W$ be a line and a plane in $\mathbb{P}^3$ such that $\alpha(U, W) = \beta(U, W) = 0$. For dimension reasons, $U$ must intersect both $\operatorname{rad} \alpha$ and $\operatorname{rad} \beta$, and hence $U$ is distinct from both of these lines. But then the $\alpha$-perp of $U$ and the $\beta$-perp of $U$ are both planes containing $W$, and hence equal to $W$. On the other hand, the radicals of $\alpha$ and $\beta$ are contained in the $\alpha$-perp and the $\beta$-perp of $U$, respectively, hence in $W$. But this contradicts the assumption that the projective lines $\mathbb{P}\operatorname{rad}\alpha$ and $\mathbb{P}\operatorname{rad}\beta$ do not intersect.

We conclude that all radicals of elements in $\mathcal{A}$ intersect. But then they all lie in some plane $W$. Now if $U$ is any line in $W$, then $\alpha(U, W) = 0$ for all $\alpha \in \mathcal{A}$, so that $\mathcal{A}$ "lies in" $C_2$. This proves the theorem for $n = 4$.

Finally, for $n \geq 6$, we have to exhibit a triple of skew bilinear forms that is not nilpotent but whose span lies in the null-cone on $\bigwedge^2 V^*$. Choose for instance $\alpha_1, \alpha_2, \alpha_3$ with matrices $A_1, A_2, A_3$ such that

$$t_1A_1+t_2A_2+t_3A_3 = \begin{bmatrix} t_2 & & & & & & & \\ & \ddots & & & & & & \\ & & t_2 & & & & & \\ & & & 0 & S & & & \\ & & & S & 0 & & & \\ & & & & & -t_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -t_2 \end{bmatrix}, \text{ where } S = \begin{bmatrix} t_2 & t_3 & 0 \\ t_1 & 0 & -t_3 \\ 0 & -t_1 & -t_2 \end{bmatrix}.$$

Using arguments like those for Lemma 2 one verifies that no subspace of $K^n$ is mapped by all $A_i$ into a strictly smaller subspace. This concludes the proof of the theorem. □

## 5. SL($V$) ON ARBITRARY BILINEAR FORMS

The invariants of $\mathrm{SL}(V)$ on $(V^* \otimes V^*)$ are known [1], but in contrast to the situation for linear maps and symmetric bilinear forms, it is not clear from them that the null-cone on one copy of $V^* \otimes V^*$ is irreducible. The following theorem states that it is, and also describes the components in several copies.

**Theorem 8.** *For $p \geq 2$, the null-cone of $\mathrm{SL}(V)$ on $(V^* \otimes V^*)^p$ has $q := \lfloor \frac{n+1}{2} \rfloor$ irreducible components given by*

$$C_k^{(p)} := \{(\alpha_1, \dots, \alpha_p) \mid \exists U \subseteq W \subseteq V : \dim U = k, \dim W = n - k + 1, \text{ and } \alpha_i(U, W) = \alpha_i(W, U) = 0 \text{ for all } i = 1, \dots, p\},\ k = 1, \dots, q.$$

*On $p = 1$ copy the sets $C_k^{(1)}$ form a chain $C_1^{(1)} \subseteq \dots \subseteq C_q^{(1)}$, and hence the null-cone equals the irreducible set $C_q^{(1)}$.*

In the proof of this theorem we use the following lemma.

**Lemma 3.** *Let $\beta$ be a symmetric form and $\gamma$ a skew form on the vector space $V$ of dimension $\geq 2$. Then there exists a $\beta$-isotropic $v_0 \in V$ for which*

$$\dim\{v \in V \mid \beta(v_0, v) = \gamma(v_0, v) = 0\} \geq \dim V - 1$$

*Proof.* If the radical of $\gamma$ has dimension $\geq 2$, we may take for $v_0$ any $\beta$-isotropic vector in $\operatorname{rad}\gamma$. If $\operatorname{rad}\gamma$ has dimension 1 and is spanned by $v_1$, then there are two cases: either $v_1$ is $\beta$-isotropic and we may set $v_0 := v_1$, or $V = Kv_1 \oplus V'$, where $V' := v_1^{\perp_\beta}$. Then $\gamma$ is non-degenerate on $V'$ and if we find a $v_0$ in $V'$ satisfying the conclusion of the lemma for $V'$ instead of $V$, it also does the trick for $V$, as $\beta(v_1, v_0) = \gamma(v_1, v_0) = 0$.

Hence the case remains where $\gamma$ is non-degenerate. Let $B, C$ be the linear maps corresponding to $\beta, \gamma$ relative to $(.,.)$ and choose any eigenvector $v_0$ of $C^{-1}B$. Then we have $Bv_0 \in KCv_0$ so that $\gamma(v, v_0)(= (v, Cv_0)) = 0$ implies $\beta(v, v_0)(= (v, Bv_0)) = 0$. In particular, $v_0$ is $\beta$-isotropic, and the vector space on the left-hand side in the lemma is the $\gamma$-perp of $v_0$. □

*Proof of Theorem 8.* For the first statement, let $(\alpha_1, \ldots, \alpha_p)$ be a nilpotent $p$-tuple of bilinear forms and write $\alpha_i = \beta_i + \gamma_i$ for all $i$, with $\beta_i$ symmetric and $\gamma_i$ skew. Let $B_i, C_i$ be the linear maps associated $\beta_i, \gamma_i$, respectively. By assumption there exists a one-parameter subgroup $\lambda : K^* \to \mathrm{SL}(V)$ with $\lim_{t\to 0} \lambda(t)\alpha_i = 0$ for all $i$. But this implies that also $\lambda(t)\beta_i, \lambda(t)\gamma_i \to 0$ for $t \to 0$. *A fortiori*, the $2p$-tuple $(B_1, \ldots, B_p, C_1, \ldots, C_p)$ is nilpotent under the larger group $\mathrm{SL}(V) \times \mathrm{SL}(V)$, and by Theorem 1 there exist subspaces $U', U'' \subseteq V$ of dimensions $k$ and $k-1$ such that $B_iU', C_iU' \subseteq U''$ for all $i$. Let $W'$ be the perp of $U'$ relative to our fixed form $(.,.)$, set $U := U' \cap W'$ and $W := W' + U'$. Then $U \subseteq W$, $\dim U + \dim W = n + 1$, and $\beta_i(U, W) = \gamma_i(U, W) = 0$. But then also $\alpha_i(U, W) = \alpha_i(W, U) = 0$, as claimed.

Now we prove $C_k^{(1)} \subseteq C_{k+1}^{(1)}$ for $k < q$. To this end, let $U \subseteq W$ be subspaces of $V$ with $\dim U + \dim W = n + 1$. We want to prove that a form $\alpha \in V^* \otimes V^*$ lying in $C_k^{(1)}$ by virtue of $\alpha(U, W) = \alpha(W, U) = 0$ also lies in $C_{k+1}^{(1)}$. Indeed, write $\alpha = \beta + \gamma$, where $\beta$ is symmetric and $\gamma$ is skew. The forms $\beta, \gamma$ induce a symmetric form $\bar\beta$ and a skew-symmetric form $\bar\gamma$ on $W/U$, respectively, and by the preceding lemma there exists a $\bar w_0 \in W/U$ for which

$$\dim\{\bar w \in W/U \mid \bar\beta(\bar w, \bar w_0) = \bar\gamma(\bar w, \bar w_0) = 0\} \geq \dim W/U - 1.$$

Let $w_0$ be a pre-image of $\bar w$ in $W$, set $U' := U \oplus Kw_0$, and let $W' \subseteq W$ be a subspace of codimension 1 that contains $w_0$ and whose image in $W/U$ is contained in the space above. Then we still have $\alpha(U', W') = 0$ and $\dim U' + \dim W' = n + 1$, but now $\dim U' = k + 1$, as claimed.

Finally, we have to show that on $p \geq 2$ copies there are no inclusions among the sets $C^{(k)}$ with $k = 1, \ldots, q$ are distinct. But their intersections with the set of $p$-tuples of *symmetric* bilinear forms are already distinct, see Theorem 3. □

The last question to be answered here is whether the polarisations of the invariants on one copy of $V^* \otimes V^*$ define the null-cone on more copies. The answer can be deduced from the answers for symmetric forms and for skew forms.

**Theorem 9.** *The null-cone of* $\mathrm{SL}(V)$ *on* $(V^* \otimes V^*)^p$ *is defined by the polarisations to* $p \geq 2$ *copies of the invariants on* $V^* \otimes V^*$ *if and only if* $\dim V \leq 2$.

*Proof.* For $\dim V = 1$ the statement is trivial. Suppose that $\dim V = 2$ and let $\mathcal{A}$ be a space of nilpotent bilinear forms on $V$. If $\alpha \in \mathcal{A}$, then by theorem 8 both the symmetric component and the skew component of $\alpha$ are singular. As the skew component has even rank, it is then zero. Hence $\mathcal{A}$ consists of symmetric forms only, and therefore the existence of a common radical for forms in $\mathcal{A}$ follows from Theorem 4.

Suppose now that $n \geq 3$. Let $\beta_1, \beta_2, \gamma_1$ be the bilinear forms on $K^n$ whose matrices $B_1, B_2, C_1$ relative to the orthogonal sum $(.,.)$ of the skew diagonal forms on $K^3$ and $K^{n-3}$ satisfy

$$s_1B_1 + s_2B_2 + t_1C_1 = \left[\begin{array}{ccc|c} s_1 & s_2 & 0 & \\ t_1 & 0 & s_2 & \\ 0 & -t_1 & s_1 & \\ \hline & & & sI_{n-3} \end{array}\right].$$

A direct computation shows that $\det(s_1B_1 + s_2B_2 + t_1C_1)$ is identically zero. We claim that actually $\mathcal{A} := \langle \beta_1, \beta_2, \gamma_1 \rangle_K$ consists entirely of nilpotent bilinear forms; as the determinant is not the only invariant, the preceding computation does not prove this yet. But let $\alpha$ be in $\mathcal{A}$ with matrix $A$. Then $A^t$—where transposition, as always, is relative to the form $(.,.)$—defines the form $\alpha^t$, which by the definition of $\mathcal{A}$ also lies in $\mathcal{A}$ and the singular matrix pencil $\langle A, A^t \rangle_K$ has a subspace $U$ of $K^n$ for which $W' := A^tU + AU$ has dimension $< \dim U$. But then the perp $W$ of $W'$ relative to $(.,.)$ is a subspace of $K^n$ of dimension $> n - \dim U$ satisfying $\alpha(W,U) = \alpha^t(W,U)(= \alpha(U,W)) = 0$. Replacing $(U,W)$ by the pair $(U \cap W, U+W)$ as usual, we find a witness for the nilpotency of $\alpha$.

However, the pair $(\beta_1 + \gamma_1, \beta_2)$ of bilinear forms is not nilpotent. Indeed, if it were, then there would be $U \subseteq W$ with $\dim U + \dim W = n+1$ and $\beta_1(U,W) = \beta_2(U,W) = \gamma_1(U,W) = 0$, i.e., with $\dim B_1U + B_2U + C_1U < \dim U$. By Lemma 2 no $U$ with this property exists. □

## References


[1] Adamovich, O., Golovina, E.: Invariants of a pair of bilinear forms. Mosc. Univ. Math. Bull. **32**(2), 11–14 (1977)

[2] Bollobás, B.: Bla Modern graph theory, *Graduate Texts in Mathematics*, vol. 184. Springer-Verlag, New York (1998)

[3] Draisma, J.: Counting components of the null-cone on tuples. Preprint (2005). Presented at MEGA 2005, to appear in Transformation Groups.

[4] Draisma, J.: Small maximal spaces of non-invertible matrices. Preprint (2005). To appear in Bull. Lond. Math. Soc.

[5] Draisma, J., Kemper, G., Wehlau, D.: Polarization of separating invariants. preprint (2005). Submitted.

[6] Eisenbud, D., Harris, J.: Vector spaces of matrices of low rank. Adv. Math. **70**(2), 135–155 (1988)

[7] Gantmacher, F.: The theory of matrices. Vol. 2. AMS Chelsea Publishing, New York (1959)

[8] Goodman, R., Wallach, N.R.: Representations and Invariants of the Classical Groups. Cambridge University Press (1998)

[9] Harris, J.: Algebraic geometry. A first course. No. 133 in Graduate Texts in Mathematics. Springer-Verlag, Berlin etc. (1992)

[10] Hesselink, W.H.: Desingularizations of varieties of nullforms. Invent. Math. **55**, 141–163 (1979)

[11] Joe Buhler Renee Gupta, J.H.: Isotropic subspaces of skewforms and maximal abelian subgroups of p-groups. J. Algebra **108**, 269–279 (1987)

[12] Knop, F.: On noether's and weyl's bound in positive characteristic. In: Invariant theory in all characteristics, *CRM Proc. Lecture Notes*, vol. 35, pp. 175–188. Amer. Math. Soc., Providence, RI (2004)

[13] Kraft, H.: Geometrische Methoden in der Invariantentheorie. Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden (1984)

[14] Kraft, H., Wallach, N.R.: On the nullcone of representations of reductive groups. Preprint (2005). To appear in Pac. J. Math.

[15] Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*, vol. 34. Springer-Verlag, Berlin (1993)

[16] Popov, V., Vinberg, E.: Invariant Theory, *Encyclopaedia of Mathematical Sciences*, vol. 55, chap. II. Springer-Verlag, Berlin (1994)

[17] Wall, C.: Nets of quadrics, and theta-characteristics of singular curves. Philos. Trans. Roy. Soc. London Ser. A **289**(1357), 229–269 (1978)
[18] Weyl, H.: The classical groups, their invariants and representations. Princeton University Press, Princeton, New Jersey (1939)

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