Electromagnetic pair production with capture

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Calculations of the electron-positron pair production by a single photon in the Coulomb field of a nucleus with simultaneous capture of the electron into the K shell are presented. Numerical results are given for some nuclear charges up to 92 and energies from threshold to 15 MeV. Using the equivalent-photon method of Weizsäcker and Williams, reliable estimates of the cross sections for the production of e^+e^- (K orbit) pairs by very-high-energy fully stripped heavy ions can be made for arbitrary nuclear charges.

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I. INTRODUCTION

There is now some experimental interest in the process of single-photon pair production. It is hoped that reliable predictions of the cross section for K-shell capture can be made in the case of relativistic heavy-ion collisions, using the equivalent photon method of Weizsäcker and Williams [1,9]. In the relativistic heavy-ion colliders like the relativistic heavy-ion collider (RHIC) or the large-hadron collider (LHC) there is a serious loss process for high nuclear charges Z [2]. In addition to the fragmentation due to the Weizsäcker-Williams process there is the e^+e^- pair production with e^- capture, mainly into the most tightly bound K shell (all higher shells will contribute about 20% of the K capture). As this is a pure QED process, it can be calculated reliably. Recently, the same mechanism has also been proposed to produce antihydrogen atoms, where the antiproton captures the positron of the produced pair [3].

II. PAIR PRODUCTION WITH CAPTURE BY A SINGLE PHOTON

The cross section for single-quantum pair production is given by

$$\sigma^K_{e^+e^-} = \frac{\alpha}{4\pi}\frac{E_p k_p}{\omega}\int d\Omega \sum_{\varsigma,\nu,\hat{e}} \mid T_{e^+e^-} \mid^2, \tag{1}$$

where the matrix element $T_{e^+e^-}$ is given by

$$\begin{aligned} T_{e^+e^-} &= -i\langle \Psi^e_\nu \mid F \mid \Psi^p_\varsigma\rangle \\ &= -i\int d^3\mathbf{r}\,\Psi^{e\dagger}_\nu(\mathbf{r})\hat{\alpha}\hat{e}\Psi^p_\varsigma(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{2}$$

Throughout the paper, we will use relativistic units $\hbar = c = m_e = 1$. $\hat{\alpha}$ are standard Dirac matrices. E_p and k_p denote energy and momentum of the positron and $\hat{e}$ the photon polarization vector. The electron binding energy is given by $E_e = \gamma_e = \sqrt{1-\alpha^2 Z^2}$. Ψ^e_ν and Ψ^p_ς are the Coulomb field Dirac wave functions of a K-shell electron with magnetic quantum number ν and a free positron with spin ς, respectively.

The K-shell wave function for a Coulomb field

$$V(r) = -\frac{\xi}{r}, \quad \xi = \alpha Z \tag{3}$$

is given by [4]

$$\Psi^\nu_{-1} = \begin{pmatrix} g_{-1}(r)\chi^\nu_1(\hat{r}) \\ if_{-1}(r)\chi^\nu_{-1}(\hat{r}) \end{pmatrix}, \quad \nu = \pm\frac{1}{2}. \tag{4}$$

The radial functions g_{-1} and f_{-1} are

$$g_{-1} = -\frac{(2\xi)^{\gamma_e+\frac{1}{2}}}{[2\Gamma(2\gamma_e+1)]^{1/2}}(1+\gamma_e)^{1/2}r^{\gamma_e-1}e^{-\xi r}, \tag{5}$$

$$f_{-1} = \frac{(2\xi)^{\gamma_e+\frac{1}{2}}}{[2\Gamma(2\gamma_e+1)]^{1/2}}(1-\gamma_e)^{1/2}r^{\gamma_e-1}e^{-\xi r}. \tag{6}$$

The angular dependence in Eq. (4) is expressed by spherical spinors

$$\chi^\mu_\kappa = \sum_{\tau=\pm 1/2}(-1)^{l+\mu-\frac{1}{2}}\hat{j}\begin{pmatrix} l & \frac{1}{2} & j \\ \mu-\tau & \tau & -\mu \end{pmatrix} Y^{\mu-\tau}_l(\hat{\mathbf{r}})\chi_\tau, \tag{7}$$

in which χ_τ are Pauli spinors and

$$j = |\kappa| - \frac{1}{2}, \quad l = j + \frac{1}{2}\mathrm{sgn}(\kappa), \quad \hat{j} = \sqrt{2j+1}. \tag{8}$$

The positron Coulomb wave function, which is chosen to represent asymptotically a distorted plane wave with an outgoing spherical wave, is [5]

$$\begin{aligned} \Psi^p_\varsigma = 4\pi\sqrt{\frac{E_p+1}{2E_p}}\sum_{\kappa,\mu} i^{l'}\hat{j}(-1)^{\mu-\frac{1}{2}}\begin{pmatrix} l' & \frac{1}{2} & j \\ \mu+\varsigma & -\varsigma & -\mu \end{pmatrix} \\ \times Y^{\mu+\varsigma *}_{l'}(\hat{\mathbf{k}}_\mathbf{p})\begin{pmatrix} f_\kappa\chi^\mu_{-\kappa} \\ ig_\kappa\chi^\mu_\kappa \end{pmatrix}, \quad l' = j - \frac{1}{2}\mathrm{sgn}(\kappa), \end{aligned} \tag{9}$$

where

$$\begin{pmatrix} f_\kappa \\ g_\kappa \end{pmatrix} = \begin{pmatrix} -i\sqrt{\frac{E_p-1}{E_p+1}} \\ 1 \end{pmatrix} (k_p r)^{\gamma_p-1} 2^{\gamma_p-1} e^{-i\gamma_p\pi/2-\eta_p\pi/2} \times \frac{\Gamma(\gamma_p+i\eta_p)}{\Gamma(2\gamma_p+1)} e^{-ik_pr} \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \quad \eta_p = \xi\frac{E_p}{k_p}, \tag{10}$$

and

$$F_\pm = (\gamma_p + i\eta_p)F(\gamma_p - i\eta_p; 2\gamma_p+1; 2ik_pr) \pm(\kappa + i\xi/k_p)F(\gamma_p+1-i\eta_p; 2\gamma_p+1; 2ik_pr). \tag{11}$$

The negative energy solution for the positron is related by charge conjugation to a positive energy solution in a repulsive Coulomb potential. The evaluation of the matrix element for the total cross section leads after some algebra to a remarkably simple expression for the cross section [6]:

$$\begin{aligned} \sigma^K_{e^+e^-} = 16\pi\alpha\frac{E_pk_p}{\omega}\sum_{n=1}^{\infty} n\Big(&\mid J_{n,n}\mid^2 + \mid J_{-n,n-1}\mid^2 \\ &+\frac{1}{2n-1}\{[nI_{n,n} + (n-1)I_{k,k-2}]J^*_{n,n} + \text{c.c.}\} + \frac{1}{2n+1}\{[nI_{-n,n-1} + (n+1)I_{-n,n+1}]J^*_{-n,n-1} + \text{c.c.}\} \\ &+\frac{1}{(2n-1)^2}[(2n^2-1)\mid I_{n,n}\mid^2 + 2n(n-1)\mid I_{n,n-2}\mid^2 - (n-1)(I_{n,n}I^*_{n,n-2} + \text{c.c.})] \\ &+\frac{1}{(2n+1)^2}[2n(n+1)\mid I_{-n,n+1}\mid^2 + (2n^2-1)\mid I_{-n,n-1}\mid^2 + (n+1)(I_{-n,n-1}I^*_{-n,n+1} + \text{c.c.})]\Big), \end{aligned} \tag{12}$$

where the occurring form factors are given by

$$I_{\kappa l} = i^l \int_0^\infty r^2 j_l(kr) f_{-1}(r) f_\kappa(r) dr, \tag{13}$$

$$J_{\kappa l} = i^l \int_0^\infty r^2 j_l(kr) g_{-1}(r) g_\kappa(r) dr. \tag{14}$$

The summation index n in Eq. (12) is related to the positron angular momentum by $n = j + \frac{1}{2}$. The radial functions g_κ and f_κ fulfill the following coupled differential equations:

$$\frac{d}{dr}\begin{pmatrix} g_\kappa \\ f_\kappa \end{pmatrix} = \begin{pmatrix} -\frac{\kappa+1}{r} & E_p+1+V \\ -(E_p-1+V) & \frac{\kappa-1}{r} \end{pmatrix} \begin{pmatrix} g_\kappa \\ f_\kappa \end{pmatrix}. \tag{15}$$

From these differential equations, it is possible to obtain an expansion of the radial functions

$$\begin{pmatrix} g_\kappa \\ f_\kappa \end{pmatrix} = (k_pr)^{\gamma_p-1}\sum_{n=0}^{\infty}\begin{pmatrix} a_{\kappa;n} \\ b_{\kappa;n} \end{pmatrix}(k_pr)^n, \tag{16}$$

where the coefficients $a_{\kappa;n}$ and $b_{\kappa;n}$ are given by the coupled recursion relations

$$(n+1)(2\gamma_p+n+1)k_pa_{\kappa;n+1} - \xi(E_p-1)a_{\kappa;n} -(\gamma_p+n+1-\kappa)(E_p+1)b_{\kappa;n} = 0, \tag{17}$$

$$(n+1)(2\gamma_p+n+1)k_pb_{\kappa;n+1} - \xi(E_p+1)b_{\kappa;n} +(\gamma_p+n+1+\kappa)(E_p-1)a_{\kappa;n} = 0, \tag{18}$$

and the starting values follow from Eq. (9). Using the above decomposition of the radial wave functions, the radial form factors can be evaluated very fast by a method described by Trautmann, Baur, and Roesel [7]. This method is numerically stable and converges for arbitrary energies.

III. CAPTURE IN RELATIVISTIC HEAVY-ION COLLISIONS

Pair production with capture is of special interest for the development of relativistic heavy-ion colliders. The simplest way to get an estimate of the cross section in RHI collisions is provided by the equivalent photon method, which is originally due to Fermi [8], and later on developed by Weizsäcker and Williams [1,9]. A quite complete description of the method can be found in the textbook of Jackson on classical electrodynamics; we mention only the main idea involved in the method. The target nucleus is considered as fixed. The projectile is assumed to move in a straight line with relativistic Lorentz factor $\gamma_p = 1/\sqrt{1-v^2} \gg 1$ and impact parameter b, accompanied by the contracted electromagnetic field. This contracted field corresponds to a spectrum of equivalent photons, given by

$$N(\omega, b) = \frac{Z_p^2\alpha}{\pi^2}\left(\frac{\omega^2}{\gamma_p^2v^4}\right)\left[K_1^2(x) + \frac{1}{\gamma_p^2}K_0^2(x)\right], \tag{19}$$

where $K_0(K_1)$ are the modified Bessel functions of order zero (one) and $x = \omega b/\gamma v$. The cross section for an electromagnetic process in a highly relativistic collision is obtained now by integrating the single-quantum cross section over the frequency spectrum and from a minimum impact parameter $b_{\min} = R$ (which was, in our case, chosen to be the Compton wavelength) to infinity:

$$n(\omega) = \int_R^\infty 2\pi b N(\omega, b) db, \tag{20}$$

$$\sigma = \int n(\omega)\sigma_\gamma(\omega)\frac{d\omega}{\omega}, \tag{21}$$

where σ is now the total cross section of the electromagnetic process.

The integration of $N(\omega, b)$ over b can be carried out to give [10]

$$n(\omega) = 2\pi \int_R^\infty bN(\omega, b)db \qquad (22)$$

$$= \frac{2}{\pi}\frac{Z_p^2\alpha}{v^2}\left[\zeta K_0(\zeta)K_1(\zeta) - \frac{v^2\zeta^2}{2}[K_1^2(\zeta) - K_0^2(\zeta)]\right], \qquad (23)$$

where $\zeta = \omega R/\gamma_p v$ is an adiabatic cutoff parameter. For $\gamma \gg 1$ (except for extreme low-energy frequencies, satisfying the relationship $\omega R \ll 1$) one can use the approximation:

$$n(\omega) = \frac{1}{\pi}Z_p^2\alpha \ln\left[\left(\frac{\delta}{\zeta}\right)^2 + 1\right] \cong \frac{2}{\pi}Z_p^2\alpha\ln\left(\frac{\delta}{\zeta}\right), \qquad (24)$$

where $\delta = 0.681...$ is a number related to the Euler's constant.

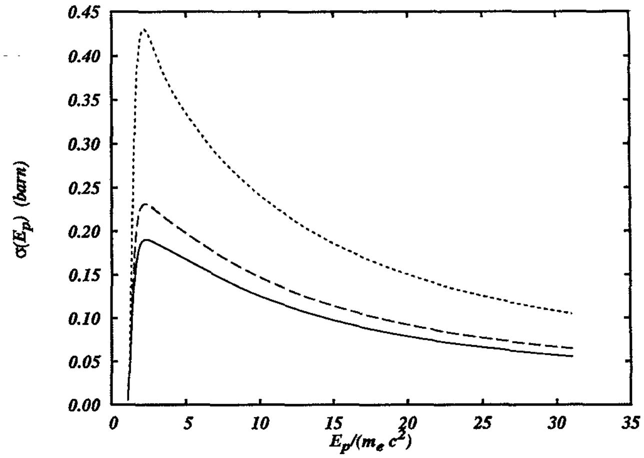

FIG. 1. Differential cross section $\sigma(E_p)$ for the pair production by a single photon is given as a function of the positron energy E_p for different ions. The dotted curve is the result for U, the dashed curve for Pb and the solid curve for Au.

IV. RESULTS AND CONCLUSIONS

Equation (12) was evaluated for arbitrary values of charge, and as a function of energy from threshold up to 30 MeV. For these high energies the summation over n, that is over the angular momentum, has to go to several hundreds. The accuracy of the computation was maintained at better than 0.1% throughout the range of charge and energies below 10 MeV. Results of the calculation for energies up to about 15 MeV are shown in Fig. 1. We were able to reproduce the results of Johnson, Buss, and Carroll [6] by restricting the summation in Eq. (12) to $n \leq 10$. Also a Fortran program to compute the cross section for arbitrary Z and E_p is available now.

Theoretical calculations using semirelativistic Sommerfeld-Maue wave functions show that for $\alpha Z \ll 1$ and $E_p \gg 1$ the cross section is approximately given by

$$\sigma^K_{e^+e^-} = 4\pi\alpha^6 Z^5 \frac{1}{E_p}. \qquad (25)$$

The numerical calculations confirm this simple formula. For nuclei with great Z the formula can be modified by a purely heuristic factor:

$$\sigma^K_{e^+e^-} = 4\pi\alpha^6 Z^5 \left(\frac{1}{2} + \frac{\alpha Z}{4}\right)^{2\pi\alpha Z} \frac{1}{E_p}. \qquad (26)$$

This modification agrees pretty well with the numerical values for physical Z and positron energies above 15 MeV. From the considerations made above we expect that the cross section for capture in relativistic collisions behaves like

$$\sigma^K_{\mathrm{RHI}} = 8Z_p^2 Z^5\alpha^7 \left(\frac{1}{2} + \frac{\alpha Z}{4}\right)^{2\pi\alpha Z} (c_1 \ln\gamma_p - c_2). \qquad (27)$$

Adapting the coefficients to our numerical results gives c_1 (c_2) = 0.250 (0.619) for uranium as target, 0.238 (0.603) for lead and 0.235 (0.598) for gold. For the calculation of these coefficients, we have used Eq. (26) for energies higher than 30 MeV. In $U^{92} + U^{92}$ collisions the numerical values for incident energies of E_p = 100 GeV amu^{-1} and 20 TeV amu^{-1} are 73b and 250b, respectively. The values are in good agreement with those of Becker, Grün, and Scheid [11], which predicted cross sections of 68b and 400b. For small Z, the

TABLE I. Parameters A and B that are to be used in Eq. (28) are given for different ions. The choice of the ions follows those of [2]. Also given are predictions of the total cross section σ^K_{RHI} for LHC ($\gamma = 3400$).

Z	A (b)	B (b)	σ^K_{RHI} (b)
^{92}U	33.4	-82.7	484
^{82}Pb	15.4	-39.0	222
^{79}Au	12.1	-30.7	173
^{41}Nb	1.87 $\times10^{-1}$	-5.02 $\times10^{-1}$	2.67
^{20}Ca	1.95 $\times10^{-3}$	-5.19 $\times10^{-3}$	2.78 $\times10^{-2}$
^{8}O	4.50 $\times10^{-6}$	-1.20 $\times10^{-5}$	6.43 $\times10^{-5}$
^{2}He	3.35 $\times10^{-10}$	-8.85 $\times10^{-10}$	4.79 $\times10^{-9}$

results confirm those of Bertulani and Baur [10] within a range of a few percent. Indeed, Baltz, Rhoades-Brown, and Weneser [12] established in a recent paper that the cross section for very high energies is of the form

$$\sigma^{K}_{\rm RHI} = A \ln\gamma_p + B, \tag{28}$$

and the A and B are independent of γ_p. In a later paper [13], they give for the specific case of ^{79}Au-^{79}Au the perturbational results $A = 11.2$b and $B = -28.3$b, a result which agrees well with our values $A = 12.1$b and $B = -30.7$b. They also predict an approximate Z^x scaling, with $x \sim 6.6$ for U to Au, and ~ 6.4 for the much larger jump Au to I. Our corresponding values are $x \sim 6.7$ and $x \sim 6.4$, respectively. Values for A and B are shown in Table I for the experimentally most important charges.

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