LOWER BOUNDS FOR THE NORMALIZED HEIGHT AND NON-DENSE SUBSETS OF VARIETIES IN AN ABELIAN VARIETY

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ABSTRACT. This work is the third part of a series of papers. In the first two we considered curves and varieties in a power of an elliptic curve. Here we deal with subvarieties of an abelian variety in general.

Let V be an irreducible variety of dimension d embedded in an abelian variety A, both defined over the algebraic numbers. We say that V is weak-transverse if V is not contained in any proper algebraic subgroup of A, and transverse if it is not contained in any translate of such a subgroup.

Assume a conjectural lower bound for the normalized height of V. Then, for V transverse, we prove that the algebraic points of bounded height of V which lie in the union of all algebraic subgroups of A of codimension at least $d+1$ translated by the points close to a subgroup Γ of finite rank, are non Zariski-dense in V. If Γ has rank zero, it is sufficient to assume that V is weak-transverse. The notion of closeness is defined using a height function.

1. INTRODUCTION

All varieties we consider in this article are defined over $\overline{\mathbb{Q}}$ and we consider only algebraic points. Denote by A an abelian variety of dimension g. Consider an irreducible algebraic subvariety V of A of dimension d. We say that

- V is *transverse*, if V is not contained in any translate of a proper algebraic subgroup of A.
- V is *weak-transverse*, if V is not contained in any proper algebraic subgroup of A.

Given a subset V^e of V, an integer k with $1 \le k \le g$ and a subset F of A, we define the set

$$S_k(V^e, F) = V^e \cap \bigcup_{\mathrm{cod} B \ge k} (B + F), \tag{1}$$

where B varies over all abelian subvarieties of A of codimension at least k and

$$B + F = \{b + f \;\;:\;\; b \in B,\;\; f \in F\}.$$

We denote the set $S_k(V^e, \mathrm{Tor}_A)$ simply by $S_k(V^e)$, where Tor_A is the torsion of A.

Nowadays a vast number of theorems and conjectures claim the non-density of sets of the type (1). Among others, we recall the Manin-Mumford, Mordell, Mordell-Lang, Bogomolov and Zilber Conjectures. For more literature one can, for instance, refere to [7] and [19].

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Let $\mathcal{L}$ be a symmetric ample line bundel on A. Consider on A the semi-norm $||\cdot||$ induced by the $\mathcal{L}$-Néron-Tate height. For $\varepsilon \geq 0$, we define

$$\mathcal{O}_\varepsilon = \{\xi \in A : ||\xi|| \leq \varepsilon\}$$

Note that in the literature, often, the notation $\mathcal{O}_\varepsilon$ corresponds to the set $\{\xi \in A : ||\xi||^2 \leq \varepsilon\}$. Let Γ be a subgroup of finite rank in A. We denote $\Gamma_\varepsilon = \Gamma + \mathcal{O}_\varepsilon$.

Following Bombieri, Masser and Zannier [3], [4], [5], one can state the following:

Conjecture 1.1. *There exists $\varepsilon > 0$ such that:*

i. *If V is weak-transverse, then $S_{d+1}(V, \mathcal{O}_\varepsilon)$ is non Zariski-dense in V.*
ii. *If V is transverse, then $S_{d+1}(V, \Gamma_\varepsilon)$ is non Zariski-dense in V.*

For $\varepsilon = 0$, this conjecture part ii. is a special case of a conjecture by Zilber and by Pink. In view of several works, at present, it is clear that such a conjecture can be split in two parts: one for the height and the other for the non-density property.

Conjecture 1.2 (Bounded Height Conjecture)**.** *There exists $\varepsilon > 0$ and a non empty Zariski-open $V^u \subset V$ such that:*

i. *If V is weak-transverse, then $S_{d+1}(V^u, \mathcal{O}_\varepsilon)$ has bounded height.*
ii. *If V is transverse, then $S_{d+1}(V^u, \Gamma_\varepsilon)$ has bounded height.*

For $\theta \geq 0$, we denote

$$V(\theta) = V \cap \mathcal{O}_\theta.$$

Conjecture 1.3 (Non-density Conjecture)**.** *For all reals θ, there exists an effective $\varepsilon > 0$ such that:*

i. *If V is weak-transverse, then $S_{d+1}(V(\theta), \mathcal{O}_\varepsilon)$ is non Zariski-dense in V.*
ii. *If V is transverse, then $S_{d+1}(V(\theta), \Gamma_\varepsilon)$ is non Zariski-dense in V.*

These conjectures are optimal with respect to the codimension $d + 1$ of the algebraic subgroups.

In the present work, we focus our attention on the Non-density Conjecture. In section 5.3, we prove:

Theorem 1.4. *Conjecture 1.3 i. and ii. are equivalent.*

That i. implies ii. is quite elementary. The other implication is delicate. It is worth to note that, on the contrary, Conjecture 1.2 i. and ii. are not equivalent. It is true that i. implies ii., but the reverse does not hold in general.

In their work, Bombieri, Masser and Zannier present a method to tackle the non-density question based on the use of the Siegel Lemma and of the Generalized Lehmer Conjecture. In our previous works [17] and [18] we present a different method for varieties in a power of an elliptic curve. Our method avoids Siegel's Lemma and the Generalized Lehmer Conjecture. We use instead Dirichlet's Theorem and an effective version of the Bogomolov Conjecture. Here we extend our method to subvarieties of abelian varieties in general.

The essential minimum of a variety is defined as

$$\mu(V) = \inf\{\varepsilon > 0, \quad \overline{V(\varepsilon)} = V\},$$

where $\overline{V(\varepsilon)}$ is the Zariski closure of $V(\varepsilon)$. The Bogomolov conjecture, proven by Ulmo [16] and Zhang [20], claims that if V is not a union of translates of abelian subvarieties by torsion points, then $\mu(V) > 0$.

For $\theta \geq \varepsilon$, $S_{d+1}(V(\theta), \mathcal{O}_\varepsilon) \supset S_g(V(\theta), \mathcal{O}_\varepsilon) = V(\varepsilon)$. Then, for weak-transverse varieties, Conjecture 1.3 i. implies an effective lower bound for the essential minimum of V. Here, we are going to prove a strong reverse implication: an effective lower bound for the essential minimum of transverse varieties implies Conjecture 1.3. For V transverse, we need a lower bound for $\mu(V)$, which is the abelian analogue of [2] theorem 1.4.

Conjecture 1.5 (Effective Bogomolov Conjecture)**.** *Let $(A, \mathcal{L})$ be a polarized abelian variety. For all transverse subvarieties V of A of dimension d and for all $\eta > 0$*

$$\mu_{\mathcal{L}}(V) \geq c_0(A, \mathcal{L}, \eta)(\deg_{\mathcal{L}} V)^{-\frac{1}{\dim A - d} - \eta},$$

where $c_0(A, \mathcal{L}, \eta)$ is a positive constant depending on A, $\mathcal{L}$ and η.

Our main result is:

Theorem 1.6. *Conjecture 1.5 implies Conjecture 1.3.*

In a preprint Galateau [9] shows that Conjecture 1.5 holds under certain hypothesis on $(A, \mathcal{L})$, verified for instance for a product of elliptic curves with the natural line bundle. Then, in these cases we unconditionally prove Conjecture 1.3.

Even if our theorem is often conjectural, a nice aspect is that the codimension of the algebraic subgroups is the optimal $d + 1$. No other known methods, even conjectural (for example assuming the generalized Lehmer's Conjecture) give such an optimal result, at least for $\varepsilon > 0$.

To prove our main theorem, we first approximate an algebraic subgroup with a subgroup of degree bounded by a constant. This part is an extension of the method introduced in [17] for the ring $\mathbb{Z}$, to the ring of endomorphisms of an abelian variety.

The second step is to show that each intersection is non-dense. The proof relies on Cojecture 1.5 and on some properties of the stabilizer. This approach differs from the one adopted in [17] and [18].

The structure of the article is as follows. We first fix the notation and definitions. In chapter 3 we approximate the morphisms. In chapter 4 we prove the non-density of each intersection, under the assumption of Conjectur 1.5. In chapter 5 we prove a sequence of simplifications which lead to the proof of our main theorem.

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2. PRELIMINARIES

2.1. **The ambient variety.** In the first instance we analyse the ambient variety. Statements on the boundness of heights and on the non-density of sets are invariant under an isogeny of the ambient variety. Namely, given an isogeny $J : A \to A'$ between abelian varieties over $\overline{\mathbb{Q}}$, Conjecture 1.3 (as well as 1.1 and 1.2) holds for $V \subset A$ if and only if it holds for $J(V) \subset A'$. We want to fix a convenient isogeny which simplifies the setting.

Powers of simple abelian varieties behave quite similar to powers of elliptic curves, up to some extra technicality. A general abelian variety shall then be regarded as a product of such powers. In view of the decomposition theorem, an abelian variety A is isogenous to a product $A_1^{g_1} \times \cdots \times A_n^{g_n}$ where the A_i are non isogenous simple abelian varieties of dimension d_i. Thus we can assume

$$A = A_1^{g_1} \times \cdots \times A_n^{g_n}.$$

Note that the dimension of A is $\sum_i d_i g_i$.

In order to take advantage from the results on powers of elliptic curves, we often need to decompose our objects according to the decomposition of A in power of simple factors.

Given a multi-index $\underline{r} \in \mathbb{N}^n$ we denote by

$$A^{\underline{r}} = A_1^{r_1} \times \cdots \times A_n^{r_n},$$

where we simply forget the i factor if $r_i = 0$. Then $A = A^{\underline{g}}$ for $\underline{g} = (g_1, dots, g_n)$.

2.2. **Morphisms and their norm.** The ring of endomorphisms of $A^{\underline{g}}$ is far more complicated than the one of an elliptic curve. However, it is a free $\mathbb{Z}$-module of finite rank. Let $\mathcal{E}_i$ be the ring of endomorphism of A_i. This is a free $\mathbb{Z}$-module of rank t_i. We denote by $\tau_1^i, \ldots, \tau_{t_i}^i$ a set of integral generators of $\mathcal{E}_i$. Then, a morphism $\phi_i : A_i^{g_i} \to A_i^{r_i}$ is identified with a $r_i \times g_i$ matrix with entries in $\mathcal{E}_i$.

Since the simple factors of $A^{\underline{g}}$ are not isogenous, for $\underline{r} \in \mathbb{N}^n$,

$$\mathrm{Hom}(A^{\underline{g}}, A^{\underline{r}}) \cong \mathrm{Mat}_{r_1 \times g_1}(\mathcal{E}_1) \times \cdots \times \mathrm{Mat}_{r_n \times g_n}(\mathcal{E}_n).$$

More precisely, a morphism $\phi : A^{\underline{g}} \to A^{\underline{r}}$ is identified with a block matrix

$$\phi = [\phi_1, \ldots, \phi_n] = \begin{pmatrix} \phi_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & \phi_n \end{pmatrix}$$

with $\phi_i : A_i^{g_i} \to A_i^{r_i}$.

The Rosati involution defines a norm $|\cdot|$ on $\mathcal{E}_i$. The $\mathbb{Z}$-module $(\mathcal{E}_i, |\cdot|)$ is a lattice.

Note that we can identify $\mathcal{E}_i$ either with an order in a number field or with a quaternion ring. In an order, the Rosati-norm is identified with the standard Euclidean norm in $\mathbb{C}$. On the other hand, a quaternion ring can be identified with a ring of matrices with entries in an order. Then, the Rosati-norm of a is the trace of $a\bar{a}$.

For $\phi_i : A_i^{g_i} \to A_i^{r_i}$, we define $|\phi_i|$ as the maximum of the (Rosati-) norm of its entries. Note that $|\phi| = \max_i |\phi_i|$.

We finally remark that there are only finitely many morphism of norm smaller than a given constant.

2.3. **Algebraic Subgroups.** By the decomposition theorem for abelian varieties, we know that an abelian subvariety of $A^{\underline{g}}$ is isogenous to a product $A^{\underline{r}}$ for some multi-index $\underline{r} = (r_1, \ldots, r_n)$ with $r_i \leq g_i$. Masser and Wüstholz [13] Lemma 1.2, prove that the algebraic subgroups of $A^{\underline{g}}$ split as a product of algebraic subgroups of $A_i^{g_i}$. In fact non-split algebraic subgroups would define an isogeny between the non isogenous simple factors. Then,

Lemma 2.1. *An algebraic subgroup B of $A^{\underline{g}}$ is of the form $B_1 \times \cdots \times B_n$ for B_i an algebraic subgroup of $A_i^{g_i}$. Furthermore, the codimension of B_i is $d_i r_i$ for integers $0 \leq r_i \leq g_i$.* (Recall that d_i is the dimension of A_i).

Definition 2.2. *Let $B = B_1 \times \cdots \times B_n$ be an algebraic subgroup of $A^{\underline{g}}$. Let k_i be the codimension of B_i in $A_i^{g_i}$. The rank of B_i is $r_i = k_i/d_i$ and the rank of B is $\underline{r} = (r_1, \ldots, r_n)$.*

Let $\phi : A^{\underline{g}} \to A^{\underline{r}}$ be a surjective morphism. The codimension of ϕ is $\sum d_i r_i$, in other words it is the codimension of $\ker \phi$.

Lemma 2.1 implies that, as in the case of elliptic curves, an algebraic subgroup B of $A^{\underline{g}}$ of rank $\underline{r}$ is contained in the kernel of a surjective morphism $\phi_B : A^{\underline{g}} \to A^{\underline{r}}$ and the kernel B_ϕ of a surjective morphism $\phi_B : A^{\underline{g}} \to A^{\underline{r}}$ is an algebraic subgroup of rank $\underline{r}$. Furthermore, the codimension of B_ϕ is given by

$$\text{cod } B_\phi = \sum_i d_i r_i.$$

Also note that $\sum_i r_i$ is the rank of ϕ as matrix, and r_i is the rank of ϕ_i, for $\phi = [\phi_1, \ldots, \phi_n]$.

Clearly, in a product of elliptic curves, the rank and the codimension of an algebraic subgroup coincide.

2.4. **Subgroups.** Let R be a ring and M an R-module of rank s. By a set of free generators of M we mean a set of s elements of M which are R-linearly independent. If M is a free R module of rank s we call integral generators of M a set of s generators of M.

Let $\mathcal{E} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ be the ring of endomorphisms of $A_1 \times \cdots \times A_n$. Note that any subgroup of finite rank of $A^{\underline{g}}$ is contained in a $\mathcal{E}$-module of finite rank. In turn a $\mathcal{E}$-module of finite rank in $A^{\underline{g}}$ is a subgroup of finite rank.

Let Γ be a subgroup of $A^{\underline{g}}$ of finite rank.

Definition 2.3. *The i-th saturated module Γ_i of Γ is the submodule of A_i of rank s_i defined by*

$$\Gamma_i = \{\phi(y) \;\; : \;\; \phi \in \text{Hom}(A^{\underline{g}}, A_i) \text{ and } Ny \in \Gamma \text{ for } N \in \mathbb{N}^*\}.$$

The saturated module of Γ is $\overline{\Gamma} = \Gamma_1^{g_1} \times \cdots \times \Gamma_n^{g_n}$.

Note that, $\overline{\Gamma}$ is invariant with respect to the image and preimage of isogenies of $A^{\underline{g}}$. Furthermore it contains Γ and it is of finite rank. This shows that to prove finiteness statements for Γ, it is enough to prove them for $\overline{\Gamma}$. We also remark that $\overline{\Gamma} \supset \text{Tor}_{A^{\underline{g}}}$, where $\text{Tor}_{A^{\underline{g}}}$ is the torsion group of $A^{\underline{g}}$.

In order to pass form a transverse variety and a non trivial Γ to a weak-transverse variety, we need to associate to Γ a point γ. For the reverse operation, we need to associate to a point p a subgroup Γ_p.

Definition 2.4. *Let $\gamma_1^i, \ldots, \gamma_{s_i}^i$ be a set of free generators of Γ_i and let $\underline{s} = (s_1, \ldots, s_n)$. We define*

$$\gamma^i = (\gamma_1^i, \ldots, \gamma_{s_i}^i) \in A_i^{s_i},$$
$$\gamma = (\gamma^1, \ldots, \gamma^n) \in A^{\underline{s}}.$$

Since the coordinates of γ^i generate Γ_i, one can easily associate to the coordinates of γ a set of free generators of $\Gamma_1 \times \cdots \times \Gamma_n$.

Definition 2.5. *Let $p^i = (p^i_1, \dots, p^i_{s_i}) \in A_i^{s_i}$ and $p = (p^1, \dots, p^n) \in A^{\underline{s}}$. We define the submodule of $A^{\underline{g}}$ associated to the point p by*

$$\Gamma_p = \langle p^1_1, \dots, p^1_{s_1} \rangle^{g_1} \times \cdots \times \langle p^n_1, \dots, p^n_{s_n} \rangle^{g_n}.$$

We say that p has rank $\underline{s} = (s_1, \dots, s_n)$ if $\langle p^i_1, \dots, p^i_{s_i} \rangle$ has rank s_i as $\mathcal{E}_i$-module.

2.5. **Relations between weak-transverse and transverse varieties.** We discuss here, how we can associate to the couple V transverse and Γ, a weak-transverse variety V', and vice versa.

Let V be transverse in $A^{\underline{g}}$. Let Γ be a subgroup of finite rank of $A^{\underline{g}}$. Consider a point $\gamma \in A^{\underline{s}}$ as in Definition 2.4. We define

$$V' = V \times \gamma.$$

Note that V' is not contained in any proper algebraic subgroup, because the coordinates of γ are linearly independent and V is transverse. So V' is weak-transverse in $A^{\underline{g}+\underline{s}}$.

Let V' be weak-transverse in $A^{\underline{n}}$. Let H_0 be the abelian subvariety of smallest dimension such that $V' \subset H_0 + p$, for $p \in H_0^\perp$ and $H_0^\perp$ an orthogonal complement of H_0. Then H_0 is isogneous to $A^{\underline{g}}$ for a multi-index $\underline{g}$ and $H_0^\perp$ is isogenous to $A^{\underline{s}}$, for $\underline{s} = \underline{n} - \underline{g}$. We fix an isogeny

$$J : A^{\underline{n}} \to H_0 \times H_0^\perp \to A^{\underline{g}} \times A^{\underline{s}},$$

which sends H_0 to $A^{\underline{g}}$ and $H_0^\perp$ to $A^{\underline{s}}$. Then $J(p) \in 0 \times A^{\underline{s}}$. Since V' is weak transverse the projection of $J(p)$ on $A^{\underline{s}}$ has rank $\underline{s}$.

We consider the natural projection

$$\begin{aligned} \pi : A^{\underline{g}+\underline{s}} &\to A^{\underline{g}} \\ J(V') &\to \pi J(V'). \end{aligned}$$

We define

$$V = \pi J(V'),$$

and

$$\Gamma = \Gamma_{J(p)}$$

Since H_0 has minimal dimension, the variety V is transverse in $A^{\underline{g}}$.
Note that

$$V' = (V \times 0) + J(p).$$

Statements on the boundness of height and on the Zariski non-density of sets are invariant under an isogeny. Then, without loss of generality, we can assume that a weak-transverse variety in $A^{\underline{n}}$ is of the form

$$V \times p$$

with

- V a transverse subvariety of $A^{\underline{g}}$,
- p a point in $A^{\underline{s}}$ of rank $\underline{s}$,
- $\underline{n} = \underline{g} + \underline{s}$.

2.6. **Polarization and height.** In the previous sections we fixed isogenies of the ambient variety A such that A is a product of powers of non isogenous simple abelian varieties and a weak-transvers variety has the shape $V \times p$. We now fix a polarization of A. According to this polarization degrees and heights are computed.

On each A_i, we fix a symmetric ample line bundle $\mathcal{L}_i$. By $\mathcal{L}$ we denote the polarization on the ambient variety $A^{\underline{g}}$ given as the tensor product of the pull-backs of $\mathcal{L}_i$ via the natural projections on the factors. Let $x^i = (x^i_1, \dots, x^i_{g_i}) \in A_i^{g_i}$. On $A^{\underline{g}}$, we consider the height of the maximum defined as

$$h(x^1, \dots, x^n) = \max_{ij}(h(x^i_j)),$$

where $h(\cdot)$ on A_i is the canonical Néron-Tate height induced by $\mathcal{L}_i$. The height h is the square of a norm $||\cdot||$ on $A^{\underline{g}} \otimes \mathbb{R}$. For a point $x \in A^{\underline{g}}$, we write $||x||$ for $||x \otimes 1||$.

By Kronecker's Theorem, the only points of height zero are torsion points. Then, for $\varepsilon \geq 0$, $\mathcal{O}_\varepsilon \supset \mathrm{Tor}_{A^{\underline{g}}}$ and $\Gamma_\varepsilon = \Gamma + \mathcal{O}_\varepsilon \supset \mathrm{Tor}_{A^{\underline{g}}}$. Note that, for any $x \in A^{\underline{g}}$ and any morphism $\phi : A^{\underline{g}} \to A^{\underline{r}}$,

$$||\phi(x)|| \leq (\max_i g_i)|\phi| \cdot ||x||.$$

For any multi-index $\underline{r} \in \mathbb{N}^n$ and product variety $A^{\underline{r}}$, we extend the above definitions. By abuse of notation we still denote $\mathcal{L}$ the polarization on $A^{\underline{r}}$ given as the tensor product of the pull-backs of $\mathcal{L}_i$ via the natural projections on the factors.

2.7. **The fixed data of the problem.** For the convenience of the reader, we give here a summary of notation for the objects that are fixed in the problem. This objects will be used all along the article with no further clarification.

The ambient variety

- For $i = 1, \dots, n$, let A_i be non isogenous simple abelian varieties of dimension d_i.
- Let $A = A^{\underline{g}} = A_1^{g_1} \times \cdots \times A_n^{g_n}$ be the ambient variety of dimension $\sum_i d_i g_i$.
- Let $\mathcal{E}_i$ be the ring of endomorfisms of A_i and let t_i be its rank over $\mathbb{Z}$.
- Let $\tau^i_1, \dots, \tau^i_{t_i}$ be a set integral generators of $\mathcal{E}_i$ as $\mathbb{Z}$-module.
- Let $\mathcal{L}$ be a polarization on A given as tensor product of the pull-back of polarizations $\mathcal{L}_i$ on the factors A_i.

The subgroup

- Let Γ be a submodule of $A^{\underline{g}}$ of finite rank.
- Let s_i be the rank of the i-th saturated module Γ_i of Γ and $\underline{s} = (s_1, \dots, s_n)$.
- Let $\overline{\Gamma} = \Gamma_1^{g_1} \times \cdots \times \Gamma_n^{g_n}$ be the saturated module of Γ.

The subvariety

- Let V be a transverse subvariety of $A^{\underline{g}}$ of dimension d and codimension $\operatorname{cod} V$.
- Let $\theta > 0$ be a (large) real and $V(\theta) = V \cap \mathcal{O}_\theta$.
- Let p be a point in $A^{\underline{s}}$ of rank $\underline{s}$.
- Let $V \times p$ be the weak-transverse subvariety of $A^{\underline{g}+\underline{s}}$.

2.8. **Dependence of the constants.** We denote by $\ll$ an inequality up to a positive multiplicative effective constant which depends on the invariants of the problem. Most often such a constant will depend on the choice of a set of free generators of the rings of endomorphisms $\mathcal{E}_i$ of A_i, for $i = 1, \dots, n$. Some constants will also depend on the polarization $\mathcal{L}$, more precisely on the height and degree of the ambient variety A. Finally, some constants will depend on $\deg V$, $||p||$ and θ, as well. The dependence of the constants on other parameters will be specified in the brackets.

2.9. **Weighted and Special morphisms.** As in the elliptic case, there are matrices which have certain advantages. We generalize the definitions given in [17] for a power of an elliptic curve. The following definitions are less restrictive, in the sense that we allow common factors of the entries and we work up to an absolute positive constant depending on the endomorphisms ring of A_i. Let $\underline{f}$ and $\underline{r} \in \mathbb{N}^n$ with $r_i \leq f_i$. Up to reordering of columns which does not mix the blocks, a weighted matrix has the form

$$\phi = \begin{pmatrix} a & \dots & 0 & L_1^1 & 0 & \dots & & \dots & 0 \\ & \ddots & & & & & & & \\ 0 & \dots & a & L_{r_1}^1 & 0 & \dots & & \dots & 0 \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ 0 & & \dots & & 0 & a & \dots & 0 & L_1^n \\ & & & & & & \ddots & & \\ 0 & & \dots & & 0 & 0 & \dots & a & L_{r_n}^n \end{pmatrix}$$

where, $L_j^i : A_i^{f_i - r_i} \to A_i$ and $|\phi| \ll |a|$. If $r_i = f_i$, we simply forget L_j^i. A nice property of such a morphism is that its restriction to the first $A^{\underline{r}}$ factors is simply the multiplication $[a]$.

Definition 2.6 (Weighted Morphisms). *Let $\underline{f}$ and $\underline{r} \in \mathbb{N}^n$. We say that a surjective morphism $\phi = [\phi_i, \dots, \phi_n] : A^{\underline{f}} \to A^{\underline{r}}$ is weighted if:*

i. *There exists $a \in \mathbb{N}^*$ such that aI_r is a submatrix of ϕ, for $r = \sum_i r_i$.*
ii. $|\phi| \ll a$.

We associate to a weighted morphism ϕ an embedding $i_{\underline{r}} : A^{\underline{r}} \to A^{\underline{f}}$ such that $\phi \cdot i_{\underline{r}} = [a]$.

Definition 2.7 (Special Morphisms). *Let $\underline{r} \in \mathbb{N}^n$. We say that $\tilde{\phi} = (\phi|\phi') : A^{\underline{g}} \times A^{\underline{s}} \to A^{\underline{r}}$ is special if:*

i. *ϕ is weighted, and*
ii. $|\tilde{\phi}| \ll |\phi|$.

Note that a special morphism is weighted. Moreover the embedding $i_{\underline{r}} : A^{\underline{r}} \to A^{\underline{g}+\underline{s}}$ maps $A^{\underline{r}}$ to some of the factors of $A^{\underline{g}}$.

3. The approximation of the morphisms

As for curves, we want to approximate a morphism with a morphism of norm bounded by a constant. We reduce the problem of approximating a morphism of abelian varieties, to the approximation of a morphism with entries in $\mathbb{Z}$. This is done by considering the ring of endomorphis of A_i as a free $\mathbb{Z}$-module.

Dirichlet's Theorem on the rational approximation of reals claims:

Theorem 3.1 (Dirichlet 1842, see [15] Theorem 1 page 24). *Suppose that $\alpha_1, \dots, \alpha_m$ are n real numbers and that $Q \geq 2$ is an integer. Then there exist integers $b, \beta_1, \dots, \beta_m$ with*

$$1 \leq b < Q^m \quad \text{and} \quad |\alpha_i b - \beta_i| \leq \frac{1}{Q}$$

for $1 \leq i \leq m$.

The ring of endomorphisms of $A_1 \times \cdots \times A_n$ is the $\mathbb{Z}$-module of rank t

$$\mathcal{E} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_n.$$

For $\underline{l} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ we define

$$\mathcal{E}^{\underline{l}} = \mathcal{E}_1^{l_1} \times \cdots \times \mathcal{E}_n^{l_n}.$$

Lemma 3.2. *Let $\underline{l} \in \mathbb{N}^n$ and $l = \max_i l_i$. There exists Q_0 such that for all integers $Q \geq Q_0$ and all $\overline{a} \in \mathcal{E}^{\underline{l}}_{\setminus 0}$ there exists $b \in \mathbb{N}$ and $\overline{b} \in \mathcal{E}^{\underline{l}}_{\setminus 0}$ satisfying:*

i. $1 \leq b < Q^{ntl}$
ii. $|\overline{b}| \ll b \ll |\overline{b}|$,
iii. $\left|\frac{\overline{a}}{|\overline{a}|} - \frac{\overline{b}}{b}\right| \ll \frac{1}{Qb}$.

Proof. We first reduce the lemma to the case $\overline{a} \in \mathcal{E}^l$. To see this, it is sufficient to consider the natural immersion $\mathcal{E}^{\underline{l}} \to \mathcal{E}^l$ which identifies $\mathcal{E}_i^{l_i}$ to the first l_i factors of $\mathcal{E}_i^l$.

We now prove the lemma for $\overline{a} \in \mathcal{E}^l$. Let $\tau_1, \ldots, \tau_t$ be a set of integral generators of the ring $\mathcal{E}$. We define $\lambda_{\mathcal{E}} = \min_{a \in \mathcal{E}_{\setminus 0}} |a|$ and

$$Q_0 = 2 \max\left(1, \frac{\sum_i |\tau_i|}{\lambda_{\mathcal{E}}}\right).$$

The ring $\mathcal{E}$ and $\mathbb{Z}^t$ are isomorphic. Fix the isomorphism that associate to $\overline{a} = \alpha^1\tau_1 + \cdots + \alpha^t\tau_t$ with $\alpha^i \in \mathbb{Z}^l$ the point $\alpha = (\alpha^1, \ldots, \alpha^t) \in \mathbb{Z}^{lt}$.

Applying Dirichlet's Theorem 3.1 with $m = lt$ and $(\alpha_1, \ldots, \alpha_m) = \frac{1}{|\overline{a}|}\alpha$, we deduce that there exist an integer b and integer vectors $\beta^1, \ldots, \beta^l \in \mathbb{Z}^l$ such that

$$1 \leq b < Q^m \tag{2}$$

and

$$\left|\frac{\alpha^i}{|\overline{a}|} - \frac{\beta^i}{b}\right| \leq \frac{1}{Qb}. \tag{3}$$

The relation (2) proves part i.

Define $\overline{b} = \sum_i \beta^i\tau_i$ and $\beta = (\beta^1, \ldots, \beta^t)$. By relation (3) and the triangle inequality,

$$\left|\frac{\overline{a}}{|\overline{a}|} - \frac{\overline{b}}{b}\right| = \left|\frac{\sum_i \alpha^i\tau_i}{|\overline{a}|} - \frac{\sum_i \beta^i\tau_i}{b}\right| \leq \left|\frac{\alpha^i}{|\overline{a}|} - \frac{\beta^i}{b}\right| \sum_j |\tau_j| \leq \frac{\sum_j |\tau_j|}{Qb} \ll \frac{1}{Qb}. \tag{4}$$

This proves part iii.

From relations (3) we deduce

$$\frac{|\beta^i|}{b} \leq \frac{1}{Qb} + \frac{|\alpha^i|}{|\overline{a}|}.$$

The Rosati norm and the Euclidean norm induced by $\mathbb{Z}^t$ on $\mathcal{E}$ are equivalent, because the rank is finite. Then $|\alpha^i| \ll |\overline{a}|$. In addition $Qb \geq 1$. Therefore $\frac{1}{Qb} + \frac{|\alpha^i|}{|\overline{a}|} \ll 1$. So

$$|\beta^i| \ll b \text{ and } |\beta| \ll b.$$

Whence

$$|\overline{b}| \leq |\beta| \sum_i |\tau_i| \ll b.$$

This shows the first inequality in part ii.

Let k be an index such that $|\overline{a}| = |a_k|$. By relation (4) we have

$$\left|\frac{a_k}{|\overline{a}|} - \frac{b_k}{b}\right| \leq \frac{\sum_i |\tau_i|}{Qb}.$$

Whence

$$b = b\frac{|a_k|}{|\overline{a}|} \leq \frac{\sum_i |\tau_i|}{Q} + |b_k|.$$

Since $Q > \sum_i |\tau_i|/\lambda_{\mathcal{E}}$,

$$b \ll |b_k|.$$

This shows the second inequality of part ii.

□

Lemma 3.3. *Let $\underline{f}$ and $\underline{r} \in \mathbb{N}^n$ with $r_i \leq f_i$. Define $m = nt\max(r_i f_i - r_i^2 + 1)$. There exists $Q_0 > 0$ such that for all $Q \geq Q_0$ and for all weighted morphism $\phi : A^{\underline{f}} \to A^{\underline{r}}$ there exists a surjective morphism $\psi : A^{\underline{f}} \to A^{\underline{r}}$ satisfying*

i. $1 \leq b < Q^m$,
ii. $|\psi| \ll b$,
iii. $\left|\frac{\psi}{b} - \frac{\phi}{|\phi|}\right| \ll \frac{1}{Qb}$,
iv. $\psi \cdot i_{\underline{r}} = [b]$.

As a consequence of ii. and iv. ψ is weighted.

Proof. Let

$$\phi = \begin{pmatrix} a & \dots & 0 & L_1^1 & 0 & \dots & & \dots & 0 \\ & \ddots & & & & & & & \\ 0 & \dots & a & L_{r_1}^1 & 0 & \dots & & \dots & 0 \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ 0 & & \dots & & 0 & a & \dots & 0 & L_1^n \\ & & & & & & \ddots & & \\ 0 & & \dots & & 0 & 0 & \dots & a & L_{r_n}^n \end{pmatrix}$$

where $L_j^i : A_i^{f_i - r_i} \to A_i$ and

$$|\phi| \ll |a| \ll |\phi|. \tag{5}$$

If $|\phi| \leq Q^m$, no approximation is needed, as ϕ itself satisfies the claim of the lemma.

Suppose now that $|\phi| \geq Q^m$. Define $\underline{l} = (r_1 f_1 - r_1^2 + 1, r_2 f_2 - r_2^2, \dots, r_n f_n - r_n^2)$. We associate to ϕ a vector

$$\overline{a} = (a, L_1^1, \dots, L_{r_1}^1, \dots, L_1^n, \dots, L_{r_n}^n) \in \mathcal{E}^{\underline{l}}.$$

Note that $|\overline{a}| = |\phi|$. Apply Lemma 3.2 to the vector $\overline{a}$. Then, there exists an integer b and a vector $\overline{b}$ such that

1) $1 \leq b < Q^m$,
2) $|\overline{b}| \ll b \ll |\overline{b}|$
3) $\left|\frac{\overline{a}}{|\overline{a}|} - \frac{\overline{b}}{b}\right| \ll \frac{1}{Qb}$

We reconstruct a matrix ψ from $\overline{b}$ respecting exactly the same positional rule we used for constructing $\overline{a}$ from ϕ. Namely, let $\overline{b} = (b, L'^1_1, \dots, L'^1_{r_1}, \dots, L'^n_1, \dots, L'^n_{r_n})$, we define

$$\psi = \begin{pmatrix} b & \dots & 0 & L'^1_1 & 0 & \dots & & \dots & 0 \\ & \ddots & & & & & & & \\ 0 & \dots & b & L'^1_{r_1} & 0 & \dots & & \dots & 0 \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ 0 & & \dots & & 0 & b & \dots & 0 & L'^n_1 \\ & & & & & & \ddots & & \\ 0 & & \dots & & 0 & 0 & \dots & b & L'^n_{r_n} \end{pmatrix}$$

Then,
1) is exactly part i.
2) implies part ii, because $|\bar{b}| = |\psi|$.
3) gives part iii.
Part iv. is evident. □

Theorem 3.4. *Let $\underline{r} \in \mathbb{N}^n$. Given $\varepsilon > 0$, there exists a positive real M, depending on ε, such that to each special morphism $\tilde{\phi} : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ one can associate a special morphism $\tilde{\psi} : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ satisfying:*

i. $|\tilde{\psi}| \ll M$,
ii. $\big((V(\theta) \times p) \cap (B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/M})\big) \subset \big((V(\theta) \times p) \cap (B_{\tilde{\psi}} + \mathcal{O}_{\varepsilon'/|\tilde{\psi}|})\big)$, *with $\varepsilon' \ll \varepsilon$.*

Proof. Define

$$Q = \left(Q_0, \frac{1}{\varepsilon}\right) \quad \text{where } Q_0 \text{ is as in Lemma 3.2}$$
$$m = nt \max_i (r_i(g_i + s_i) - r_i^2 + 1)$$
$$M = Q^m.$$

If $|\tilde{\phi}| \le M$, we simply define $\tilde{\psi} = \tilde{\phi}$. Then $\varepsilon/M \le \varepsilon/|\tilde{\phi}|$ and

$$(V(\theta) \times p) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/M}\right)$$

is contained in the right hand side.

Now, suppose that $|\tilde{\phi}| \ge M$. Apply Lemma 3.3 with $\underline{f} = \underline{g} + \underline{s}$, $\phi = \tilde{\phi}$, and $i_{\underline{r}}$ is the immersion of $A^{\underline{r}}$ to some of the factors of $A^{\underline{g}}$. Then, there exists an integer b and a matrix $\tilde{\psi}$ such that

1) $1 \le b < Q^m = M$.
2) $|\tilde{\psi}| \ll b \ll |\tilde{\psi}|$,
3) $\left|\frac{\tilde{\phi}}{|\tilde{\phi}|} - \frac{\tilde{\psi}}{b}\right| \ll \frac{1}{Qb}$.
4) $\tilde{\psi} \cdot i_{\underline{r}} = [b]$.

Since $\tilde{\phi}$ is special, then 2) and 4) imply that $\tilde{\psi}$ is special, as well.

Let $(x, p) \in V(\theta) \times p$. We want to show that, if

$$\tilde{\phi}((x, p) + \xi) = 0$$

for $\xi \in \mathcal{O}_{\varepsilon/M}$, then

$$\tilde{\psi}((x, p) + \xi') = 0$$

for some $\xi' \in \mathcal{O}_{\varepsilon'/|\tilde{\psi}|}$ and $\varepsilon' \ll \varepsilon$.

Let ξ'' be a point in $A^{\underline{r}}$ such that

$$[b]\xi'' = -\tilde{\psi}(x, p).$$

Define $\xi' = i_{\underline{r}}(\xi'')$. Then

$$\tilde{\psi}(\xi') = [b]\xi'' = -\tilde{\psi}(x,p)$$

and

$$\tilde{\psi}((x,p)+\xi') = 0.$$

It follows

$$(x,p) \in (V(\theta) \times p) \cap (B_{\tilde{\psi}} + \mathcal{O}_{||\xi'||}),$$

where $\tilde{\psi}$ is special and $|\tilde{\psi}| \ll M$.

It remains to prove that

$$||\xi'|| \ll \frac{\varepsilon}{|\tilde{\psi}|}.$$

Obviously

$$|\tilde{\phi}|\tilde{\psi}(x,p) = b\left(\tilde{\phi}(x,p) - \tilde{\phi}(x,p)\right) + |\tilde{\phi}|\tilde{\psi}(x,p).$$

It holds

$$\begin{aligned} ||\xi'|| = ||\xi''|| = \frac{||\tilde{\psi}(x,p)||}{b} &= \frac{1}{|\tilde{\phi}|b}\left|\left|b\left(\tilde{\phi}(x,p) - \tilde{\phi}(x,p)\right) + |\tilde{\phi}|\tilde{\psi}(x,p)\right|\right| \\ &\le \frac{1}{|\tilde{\phi}|}\left|\left|\tilde{\phi}(x,p)\right|\right| + \frac{1}{|\tilde{\phi}|b}\left|\left||\tilde{\phi}|\tilde{\psi}(x,p) - b\tilde{\phi}(x,p)\right|\right|. \end{aligned}$$

We estimate the two norms on the right.

On one hand

$$\frac{||\tilde{\phi}(x,p)||}{|\tilde{\phi}|} = \frac{||\tilde{\phi}(\xi)||}{|\tilde{\phi}|} \ll ||\xi|| \le \frac{\varepsilon}{M} \le \frac{\varepsilon}{b},$$

where in the last inequality we use that $b \le M$.

On the other hand, we assumed

$$||(x,p)|| \le \theta + ||p||.$$

Using relation 3) and that $Q \ge \frac{1}{\varepsilon}$, we estimate

$$\frac{1}{|\tilde{\phi}|b}\left|\left||\tilde{\phi}|\tilde{\psi}(x,p) - b\tilde{\phi}(x,p)\right|\right| \le \left|\frac{\tilde{\phi}}{|\tilde{\phi}|} - \frac{\tilde{\psi}}{b}\right| ||(x,p)|| \ll \frac{\varepsilon}{b}.$$

By 2), we conclude

$$||\xi'|| \ll \frac{\varepsilon}{b} + \frac{\varepsilon}{b} \ll \frac{\varepsilon}{|\tilde{\psi}|}.$$

□

4. The non-density of each intersection

In this section, we use and compare several polarizations. If not otherwise specified, it is well understood that the polarization is $\mathcal{L}$. The main results in this section are conditioned to the validity of Conjecture 1.5.

4.1. **Helping-variety and isogenies.** Let $\phi : A^{\underline{g}} \to A^{\underline{r}}$ be a weighted morphism. We associate to ϕ some isogenies of $A^{\underline{g}}$.

Definition 4.1. *Let $\underline{r} \in \mathbb{N}^n$. To a weighted morphism $\phi = [\phi_1, \ldots, \phi_n] : A^{\underline{g}} \to A^{\underline{r}}$ with $\phi_i = (aI_{r_i}|L^i)$ we associate the following isogenies of $A^{\underline{g}}$:*

$$\Phi = [\Phi_1, \ldots, \Phi_n],$$
$$\hat{\Phi} = [\hat{\Phi}_1, \ldots, \hat{\Phi}_n],$$
$$L = [L_1, \ldots, L_n],$$

defined by

$$\Phi_i = \begin{pmatrix} & \phi_i & \\ 0 & | & I_{g_i - r_i} \end{pmatrix} = \begin{pmatrix} aI_{r_i} & L^i \\ 0 & I_{g_i - r_i} \end{pmatrix}$$
$$\hat{\Phi}_i = \begin{pmatrix} I_{r_i} & 0 \\ 0 & aI_{g_i - r_i} \end{pmatrix}$$
$$L_i = \begin{pmatrix} I_{r_i} & L^i \\ 0 & I_{g_i - r_i} \end{pmatrix}.$$

We now associate to a transverse variety $V \subset A^{\underline{g}}$ of dimension d, a Helping-variety $W \subset A^{\underline{g}}$ of dimension d. We define

$$W = L\hat{\Phi}^{-1}(V)$$

Then

$$[a]W = \Phi(V).$$

4.2. **Functorial behavior of the essential minimum.** We deduce from Conjecture 1.5 a lower bound for the essential minimum of $\Phi(V)$ which is functorial with respect to Φ (see Theorem 4.4).

In the first instance we need some properties of the stabilizer.

Lemma 4.2. *Let $\psi : A \to A$ be an isogeny. Let V be an irreducible algebraic subvariety of A of dimension d. Then*

$$\deg_{\mathcal{L}} \psi_*(V) = |\mathrm{Stab}\, V \cap \ker \psi| \deg_{\mathcal{L}} \psi(V).$$

Proof. We denote by f_x the fiber of $\psi_{|V}$ at a point x. Since an isogeny is generically proper, there exists an open O of $\psi(V)$ such that, for $x \in O$, the fiber f_x has constant order. Then $\deg \psi_*(V) = |f_x| \deg \psi(V)$. We shall show that, for $x \in O$,

$$|\mathrm{Stab}\, V \cap \ker \psi| = |f_x|.$$

Let $x \in O$ and $y_x \in f_x$. If $t \in \mathrm{Stab}\, V \cap \ker \psi$ then $y_x + t \in f_x$. This shows

$$|\mathrm{Stab}\, V \cap \ker \psi| \leq |f_x|.$$

Suppose by contradiction that, for $x \in O$, $|f_x| > |\mathrm{Stab}\, V \cap \ker \psi|$. If $y_x, y'_x \in f_x$ then $y_x - y'_x \in \ker \psi$. Therefore

$$f_x \subset y_x + \ker \psi.$$

Since $\ker \psi$ is finite, there exists a dense subset D of O and $t \in \ker \psi_{\setminus \mathrm{Stab}\, V \cap \ker \psi}$ such that for all for $x \in D$

$$y_x + t \in f_x.$$

Then

$$\psi((V + t) \cap V) \supset D.$$

Since $(V + t) \cap V$ is closed and isogenies are closed morphisms,

$$\psi(V + t) \cap V) \supset \overline{D},$$

where $\overline{D}$ is the Zariski closure of D. However $\overline{D} \supset O$ and so

$$\psi((V+t)\cap V) \supset \overline{O} = \psi(V).$$

Isogenies preserve dimensions, thus $\dim((V+t)\cap V) = \dim V$. Whence $V+t=V$. Therefore

$$t \in \operatorname{Stab} V \cap \ker \psi.$$

□

Lemma 4.3. *In the above notations,*

$$|\text{Stab } W \cap \ker[a]| = |\ker \hat{\Phi}|\,|\text{Stab } v \ker \Phi|\,.$$

Proof. First remark that, by the definitions,

- L is an isomorphism,
- Stab $W = L\hat{\phi}^{-1}$Stab V,
- $\hat{\phi}^{-1}\ker\Phi = \ker[a]$,
- $\ker L\Phi = \ker \Phi$.

Then,

$$\begin{aligned}
|\text{Stab } W \cap \ker[a]| &= |\text{Stab } W \cap \ker[a]| \\
&= \left| L\hat{\Phi}\text{Stab } V \cap \ker \hat{\Phi}\Phi \right| \\
&= \left| L\hat{\Phi}\text{Stab } V \cap \hat{\Phi}^{-1} \ker \Phi \right| \\
&= \left| \hat{\Phi}\text{Stab } V \cap L^{-1}\hat{\Phi}^{-1} \ker \Phi \right| \\
&= \left| \hat{\Phi}\text{Stab } V \cap \hat{\Phi}^{-1} \ker L\Phi \right| \\
&= |\ker \hat{\Phi}|\,|\text{Stab } V \cap \ker L\Phi| \\
&= |\ker \hat{\Phi}|\,|\text{Stab } V \cap \ker \Phi|\,.
\end{aligned}$$

□

Theorem 4.4 (Isogeny-Functorial Bound)**.** *Assume Conjecture 1.5. Let V be a transverse subvariety of A of dimension d. Then, there exists a positive constant $c(A,\mathcal{L},\eta)$ such that, for any weighted ϕ and isogeny Φ as above,*

$$\mu_{\Phi^*\mathcal{L}}(V) \geq c(A,\mathcal{L},\eta)|a|^{-2(\dim A-d)\eta}\left(\frac{\deg_{\Phi^*\mathcal{L}} A}{\deg_{\Phi^*\mathcal{L}} V}\right)^{\frac{1}{2(\dim A-d)}+\eta}.$$

Proof. Note that isogenies preserve dimensions, so $\dim W = \dim V = d$. Furthermore

$$[a]W = \Phi(V).$$

Then

$$\mu_{\Phi^*\mathcal{L}}(V) = \mu_{\mathcal{L}}(\Phi(V)) = \mu_{\mathcal{L}}([a]W) = |a|\mu_{\mathcal{L}}(W). \tag{6}$$

We denote by $\operatorname{cod} V$ the codimension of V in A. We now estimete $\mu_{\mathcal{L}}(W)$ using Conjecture 1.5. This gives

$$\mu_{\mathcal{L}}(W) \geq c_0(A,\mathcal{L},\eta)\,(\deg_{\mathcal{L}}(W))^{-\frac{1}{2\operatorname{cod} V}+\eta}. \tag{7}$$

By [10] Lemma 6 we obtain,

$$\deg_{\mathcal{L}} \Phi(V) = \deg_{\mathcal{L}}[a]W = \frac{|a|^{2d}}{|\text{Stab } W \cap \ker[a]|}\deg_{\mathcal{L}} W$$

or equivalently

$$\deg_{\mathcal{L}} W = \frac{|\text{Stab } W \cap \ker[a]|}{|a|^{2d}}\deg_{\mathcal{L}} \Phi(V).$$

Define

$$c(A,\mathcal{L},\eta)=c_0(A,\mathcal{L},\eta)\,(\deg_{\mathcal{L}} A)^{-\frac{1}{2\mathrm{cod}\,V}-\eta}\,.$$

Substituting in (7), and using Lemmas 4.2 and 4.3, we obtain

$$\begin{aligned}\mu_{\mathcal{L}}(W)&\geq c(A,\mathcal{L},\eta)\left(\frac{|a|^{2d}\deg_{\mathcal{L}} A}{|\mathrm{Stab}\;W\cap\ker[a]|\deg_{\mathcal{L}}\Phi(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\\&=c(A,\mathcal{L},\eta)\left(\frac{|a|^{2d}\deg_{\mathcal{L}} A}{|\ker\hat{\Phi}||\mathrm{Stab}\;V\cap\ker\Phi|\deg_{\mathcal{L}}\Phi(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\\&=c(A,\mathcal{L},\eta)\left(\frac{|a|^{2d}\deg_{\mathcal{L}} A}{|\ker\hat{\Phi}|\deg_{\mathcal{L}}\Phi_*(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}.\end{aligned}$$

We can substitute this last estimate in (6), so

$$\begin{aligned}\mu_{\Phi^*\mathcal{L}}(V)=|a|\mu_{\mathcal{L}}(W)&\geq|a|c(A,\mathcal{L},\eta)\left(\frac{|a|^{2d}\deg_{\mathcal{L}} A}{|\ker\hat{\Phi}|\deg_{\mathcal{L}}\Phi_*(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\\&=c(A,\mathcal{L},\eta)|a|^{-2\mathrm{cod}\,V\eta}\left(\frac{|a|^{2g-2d}|a|^{2d}|\ker\Phi|\deg_{\mathcal{L}} A}{|\ker\hat{\Phi}||\ker\Phi|\deg_{\mathcal{L}}\Phi_*(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\\&=c(A,\mathcal{L},\eta)|a|^{-2\mathrm{cod}\,V\eta}\left(\frac{|a|^{2g}\deg_{\Phi^*\mathcal{L}}(A)}{|a|^{2g}\deg_{\mathcal{L}}\Phi_*(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\\&=c(A,\mathcal{L},\eta)|a|^{-2\mathrm{cod}\,V\eta}\left(\frac{\deg_{\Phi^*\mathcal{L}}(A)}{\deg_{\Phi^*\mathcal{L}}(V)}\right)^{\frac{1}{2\mathrm{cod}\,V}+\eta}\end{aligned}$$

□

4.3. **A Lower bound for the essential minimum.** Using a lemma by Masser and Wüstholz, we now estimate degrees.

Lemma 4.5. *Let $\underline{r}\in\mathbb{N}^n$ and let $\phi:A^{\underline{g}}\to A^{\underline{r}}$ be a weighted morphism of codimension at least $d+1$. Let Φ as in definition 4.1. Then,*

i.
$$\deg_{\mathcal{L}}\Phi(V)\ll|\phi|^{2d}.$$

ii.
$$\deg_{\mathcal{L}}\phi(V)\ll|\phi|^{2d}.$$

Proof. Part i. is a non explicit version of [14] Lemma 2.3. Part ii. is deduced by part i. simply observing that $\phi(V)=\pi\Phi(V)$, for π a projection on some of the coordinates. In addition, in the chosen polarization, forgetting coordinates makes degrees decrease. □

Proposition 4.6. *Assume that Conjecture 1.5 holds. Let $\underline{r}\in\mathbb{N}^n$ and let $\phi:A^{\underline{g}}\to A^{\underline{r}}$ be a weighted morphism of codimension at least $d+1$. Then, for any $\eta>0$, there exist positive effective constants $\epsilon_1(\eta)$ and $\epsilon_2(\eta)$ such that, for all points $y\in A^{\underline{g}}$,*

i. *For Φ as in definition 4.1,*
$$\mu(\phi(V+y))>\epsilon_1(\eta)\frac{1}{|\phi|^{d+\eta}},$$

ii.
$$\mu\left(\Phi(V+y)\right)>\epsilon_2(\eta)|\phi|^{\frac{1}{\mathrm{cod}\,V}-\eta}.$$

Proof. i. Since V is irreducible, transverse and defined over $\overline{\mathbb{Q}}$, $\phi(V+y)$ is as well.

Recall that the codimension of ϕ is $\geq d+1$. Then $\phi(V+y) \subsetneq A^{\underline{r}}$ has codimension and dimension at least 1. Apply Theorem 4.4 to $\phi(V+y)$. Then

$$\mu_{\mathcal{L}}(\phi(V+y)) > c(A^{\underline{r}}, \mathcal{L}, V, \eta) \left(\frac{\deg_{\mathcal{L}} A^{\underline{r}}}{\deg_{\mathcal{L}} \phi(V+y)} \right)^{\frac{1}{2}+\eta}.$$

Degrees are preserved by translations, hence Proposition 4.5 ii. implies

$$\deg_{\mathcal{L}}(\phi(V+y)) = \deg_{\mathcal{L}} \phi(V) \ll |\phi|^{2d}.$$

It follows

$$\mu_{\mathcal{L}}(\phi(V+y)) > c'(A^{\underline{r}}, \mathcal{L}, \eta) \frac{1}{|\phi|^{d+2\eta}}.$$

For $\underline{r}$ ranging over all multi-indeces such that $\sum d_i r_i \geq d+1$ and $r_i \leq g_i$, define

$$\epsilon_1(\eta) = \min_{\underline{r}} c'(A^{\underline{r}}, \mathcal{L}, V, \frac{\eta}{2}).$$

Then

$$\mu_{\mathcal{L}}(\phi(V+y)) > \frac{\epsilon_1(\eta)}{|\phi|^{d+\eta}}.$$

ii. Recall that, for any variety X,

$$\mu_{\Phi^*\mathcal{L}} X = \mu_{\mathcal{L}}(\Phi(X)),$$
$$\deg_{\Phi^*\mathcal{L}} X = \deg_{\mathcal{L}} \Phi_* X.$$

Apply Theorem 4.4 to $V+y$. We obtain

$$\mu_{\mathcal{L}}(\Phi(V+y)) > c(A^{\underline{g}}, \mathcal{L}, \eta) a^{-2\operatorname{cod} V \eta} \left(\frac{\deg_{\Phi^*\mathcal{L}} A^{\underline{g}}}{\deg_{\Phi^*\mathcal{L}}(V+y)} \right)^{\frac{1}{2\operatorname{cod} V}+\eta}$$
$$= c(A^{\underline{g}}, \mathcal{L}, \eta) a^{-2\operatorname{cod} V \eta} \left(\frac{\deg_{\Phi^*\mathcal{L}} A^{\underline{g}}}{\deg_{\Phi^*\mathcal{L}}(V)} \right)^{\frac{1}{2\operatorname{cod} V}+\eta}.$$

Recall that (see, for instance, [11] (6.6) Corollary page 68)

$$\deg_{\Phi^*\mathcal{L}} A^{\underline{g}} = |\ker \Phi| \deg_{\mathcal{L}} A^{\underline{g}} = a^{2(\sum_i d_i r_i)} \deg_{\mathcal{L}} A^{\underline{g}}.$$

By assumption $\sum_i d_i r_i \geq d+1$ and $|\phi| \ll a$. So

$$\deg_{\Phi^*\mathcal{L}} A^{\underline{g}} \geq a^{2(d+1)} \deg_{\mathcal{L}} A^{\underline{g}} \gg |\phi|^{2(d+1)}.$$

By Lemma 4.5 i.,

$$\deg_{\Phi^*\mathcal{L}}(V) = \deg_{\mathcal{L}}(\Phi_*(V)) \ll |\phi|^{2d}.$$

Thus

$$\mu_{\mathcal{L}}(\Phi(V+y)) > c'(A^{\underline{g}}, \mathcal{L}, V, \eta) |\phi|^{-2\operatorname{cod} V \eta} |\phi|^{\frac{1}{\operatorname{cod} V}+2\eta}.$$

Define

$$\epsilon_2(\eta) = c' \left(A^{\underline{g}}, \mathcal{L}, V, \frac{\eta}{2(\operatorname{cod} V - 1)} \right).$$

□

4.4. **The non-density property.** We come to the main proposition of this section; each set in the union is non Zariski-dense.

Theorem 4.7. *Assume Conjecture 1.5. Let* $\underline{r} \in \mathbb{N}^n$*. Then, there exists an effective* $\varepsilon_1 > 0$ *such that for* $\varepsilon \leq \varepsilon_1$*, for all weighted morphisms* $\phi : A^{\underline{g}} \to A^{\underline{r}}$ *of codimension* $\geq d+1$ *and for all* $y \in i_{\underline{r}}(A^{\underline{r}}) \subset A^{\underline{g}}$*, the set*

$$(V(\theta) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/|\phi|}\right)$$

is non Zariski-dense in $V + y$*.*

Proof. Let

$$\eta = 1/2$$
$$\epsilon_1 = \epsilon_1(1/2)$$
$$\epsilon_2 = \epsilon_2(1/2)$$

where $\epsilon_1(\eta)$ and $\epsilon_2(\eta)$ are as in Proposition 4.6. Define

$$m = \left(\frac{\theta}{\epsilon_2}\right)^{\frac{\operatorname{cod} V}{1-(\operatorname{cod} V)/2}},$$
$$\varepsilon_1 = \frac{1}{|\underline{g}|} \min\left(\theta, \frac{\epsilon_1}{m^{d+1}}\right),$$

where $|\underline{g}| = \max_i g_i$. Choose

$$\varepsilon \leq \varepsilon_1.$$

We distinguish two cases: either $|\phi| \leq m$ or $|\phi| \geq m$.

Case (1) $|\phi| \leq m$.
Let $x + y \in (V(\theta) + y) \cap (B_\phi + \mathcal{O}_{\varepsilon/|\phi|})$, where $y \in i_{\underline{r}}(A^{\underline{r}})$. Then

$$\phi(x + y) = \phi(\xi)$$

for $||\xi|| \leq \varepsilon/|\phi|$. Since $\varepsilon \leq \frac{\epsilon_1}{|\underline{g}| m^{d+1}}$ and $|\phi| \leq m$,

$$||\phi(x + y)|| = ||\phi(\xi)|| \leq |\underline{g}|\varepsilon \leq \frac{\epsilon_1}{m^{d+1}} \leq \frac{\epsilon_1}{|\phi|^{d+1}}.$$

In Proposition 4.6 i. with $\eta = 1/2$, we have proven

$$\frac{\epsilon_1}{|\phi|^{d+1}} < \mu(\phi(V + y)).$$

We deduce that $\phi(x + y)$ belongs to the non Zariski-dense set

$$Z_1 = \phi(V + y) \cap \mathcal{O}_{\epsilon_1/m^{d+1}}.$$

Since V is transverse, the dimension of $\phi(V + y)$ is at least 1. Consider the restriction morphism $\phi_{|V+y} : V + y \to \phi(V + y)$. Then $x + y$ belongs to the non Zariski-dense set $\phi^{-1}_{|V+y}(Z_1)$.

Case (2) $|\phi| \geq m$.
Let $x + y \in (V(\theta) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/|\phi|}\right)$, where $y \in i_{\underline{r}}(A^{\underline{r}})$. Then

$$\phi(x + y) = \phi(\xi)$$

for $||\xi|| \leq \varepsilon/|\phi|$ and

$$\Phi(x + y) = (\phi^1(x + y), \overline{x}^1, \ldots, \phi^n(x + y), \overline{x}^n),$$

where $\overline{x}^i$ are some of the coordinates of x. So

$$||\Phi(x + y)|| \leq \max\left(||\phi(\xi)||, ||x||\right).$$

Since $||\xi|| \leq \frac{\varepsilon}{|\phi|}$ and $\varepsilon \leq \frac{\theta}{|\underline{g}|}$, then

$$||\phi(\xi)|| \leq |\underline{g}|\varepsilon \leq \theta.$$

Also $||x|| \leq \theta$, because $x \in V(\theta)$. Thus

$$||\Phi(x+y)|| \leq \theta.$$

Since $|\phi| \geq m = \left(\frac{\theta}{\epsilon_2}\right)^{\frac{\operatorname{cod} V}{1-(\operatorname{cod} V)/2}}$,

$$\theta \leq \epsilon_2 |\phi|^{\frac{1}{\operatorname{cod} V} - \frac{1}{2}}.$$

In Proposition 4.6 with $\eta = 1/2$, we have proven

$$\epsilon_2 |\phi|^{\frac{1}{\operatorname{cod} V} - \frac{1}{2}} < \mu(\Phi(V+y)).$$

So

$$||\Phi(x+y)|| \leq \theta < \mu(\Phi(V+y)).$$

We deduce that $\Phi(x+y)$ belongs to the non Zariski-dense set

$$Z_2 = \Phi(V+y) \cap \mathcal{O}_\theta.$$

The restriction morphism $\Phi_{|V+y} : V+y \to \Phi(V+y)$ is finite, because Φ is an isogeny. Then $x+y$ belongs to the non Zariski-dense set $\Phi^{-1}_{|V+y}(Z_2)$.

□

Proposition 4.8. *Let $\underline{r} \in \mathbb{N}^n$ and let $\tilde{\phi} = (\phi|\phi') : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ be a special morphism. Then, there exists $y \in i_{\underline{r}}(A^{\underline{r}}) \subset A^{\underline{g}}$ such that, for any $\varepsilon > 0$, the map $(x,p) \to x+y$ defines an injection*

$$\left((V(\theta) \times p) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/|\phi|}\right)\right) \hookrightarrow \left((V(\theta)+y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon'/|\phi|}\right)\right),$$

where $\varepsilon' \ll \varepsilon$.

Proof. By definition of special, for $r = \sum_i r_i$ the matrix aI_r is a submatrix of ϕ and $|\phi| \ll a$. Recall that $i_{\underline{r}} : A^{\underline{r}} \to A^{\underline{g}}$ is such that $\phi \cdot i_{\underline{r}} = [a]$.

Let $y' \in A^{\underline{r}}$ be a point such that

$$[a]y' = \phi'(p).$$

Define

$$y = i_{\underline{r}}(y').$$

Then

$$\phi(y) = [a]y' = \phi'(p) \tag{8}$$

Let

$$(x,p) \in (V(\theta) \times p) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/|\phi|}\right).$$

Then, there exists $\xi \in \mathcal{O}_{\varepsilon/|\phi|}$ such that

$$\tilde{\phi}((x,p)+\xi) = 0.$$

Equivalently

$$\phi(x) + \phi'(p) + \tilde{\phi}(\xi) = 0.$$

By relation (8) we deduce

$$\phi(x+y) + \tilde{\phi}(\xi) = 0.$$

Let $\xi'' \in A^{\underline{r}}$ be a point such that

$$[a]\xi'' = \tilde{\phi}(\xi).$$

Define $\xi' = i_{\underline{r}}(\xi'')$, then

$$\phi(\xi') = [a]\xi'' = \tilde{\phi}(\xi),$$

and

$$\phi(x + y + \xi') = 0.$$

Since $\tilde{\phi}$ is special $|\tilde{\phi}| \ll |\phi|$. Furthermore $||\xi|| \leq \frac{\varepsilon}{|\phi|}$. We deduce

$$||\xi'|| = ||\xi''|| = \frac{||\tilde{\phi}(\xi)||}{|\phi|} \ll \frac{\varepsilon}{|\phi|}.$$

In conclusion

$$(x + y) \in (V(\theta) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon'/|\phi|}\right).$$

□

Corollary 4.9. *Assume Conjecture 1.5. Let* $\underline{r} \in \mathbb{N}^n$. *Then, there exists an effective* $\varepsilon_2 > 0$ *such that for* $\varepsilon \leq \varepsilon_2$ *and for all special morphisms* $\tilde{\phi} = (\phi|\phi') : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ *of codimension at least* $d + 1$ *the set*

$$(V(\theta) \times p) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/|\phi|}\right)$$

is non Zariski-dense in $V \times p$.

Proof. This is an immediate consequence of Theorem 4.7 and Proposition 4.8.

□

5. The proof of Theorem 1.6: Reductions

5.1. **Reducing to weighted morphisms.** Using the Gauss algorithm we show:

Lemma 5.1. *Let* $\Delta_i \in M_{r_i \times r_i}(\mathcal{E}_i)$ *be a matrix of rank* r_i. *Then, there exists an integer* a *and a matrix* $\Delta'_i \in M_{r_i \times r_i}(\mathcal{E}_i)$ *of rank* r_i *such that*

$$\Delta'_i \Delta_i = a I_{r_i}.$$

Proof. Note that $\mathcal{E}_i$ is not necessarily commutative, it can be a quaternion, however given non zero elements $x, y \in \mathcal{E}_i$ there exist $a, b \in \mathcal{E}_i$ such that $ax = by$. This shows that one can operate a Gauss reduction using only operations on the left and without commuting elements in $\mathcal{E}_i$. In other words there exists a matrix Δ of rank r_i such that $\Delta\Delta_i$ is a diagonal matrix. Using the norm, we can find a matrix Δ' of maximal rank r_i such that $\Delta'\Delta\Delta_i = [a_1, \ldots, a_{r_i}]$ with $a_i \in \mathbb{Z}^*$. Let m be the minimum common multiple of $a_1, \ldots, a_{r_i}$. We define $\Delta'_i = [\frac{m}{|a_1|}, \ldots, \frac{m}{|a_{r_i}|}]\Delta'\Delta$. □

This has some immediate consequences.

Lemma 5.2. *Let* $\underline{r} \in \mathbb{N}^n$. *Let* $\psi : A^{\underline{g}} \to A^{\underline{r}}$ *be a surjective morphism. Then, there exists an isogeny* Δ *of* $A^{\underline{r}}$ *such that* $\phi = \Delta\psi$ *is a weighted morphism. As a consequence,*

i. $B_\psi \subset B_\phi + \mathrm{Tor}_{A^{\underline{g}}}$.
ii. *For all* $\varepsilon \geq 0$,

$$\bigcup_{\mathrm{rk}(\psi)=\underline{r}} (B_\psi + \overline{\Gamma}_\varepsilon) \subset \bigcup_{\substack{\phi \ \text{weighted} \\ \mathrm{rk}(\phi)=\underline{r}}} (B_\phi + \overline{\Gamma}_\varepsilon).$$

Proof. Let $\psi = [\psi_1, \ldots, \psi_n]$. Let Δ_i be a submatrix of ψ_i of rank r_i with maximal pivots. By Lemma 5.1 applied to each Δ_i, there exists Δ'_i such that $\Delta'_i\Delta_i = a_i I_{r_i}$ with $a_i \in \mathbb{N}^*$. Let m be the mimum common multiple of the a_i. Define

$$\Delta = \left[\frac{m}{a_1} I_{r_1}, \ldots, \frac{m}{a_n} I_{r_n}\right] [\Delta'_1, \ldots, \Delta'_n].$$

□

5.2. **Reducing to special morphisms.** We want to fix a convenient set of generators of Γ. Recall that Γ_i is the i-th saturated module of rank s_i and $\underline{s} = (s_1, \dots, s_n)$, (see Definition 2.3).

Lemma 5.3. *Let $\theta_0 \geq 0$. There exist points γ^i of $A_i^{s_i}$ such that the coordinates of γ^i are free generators of Γ_i. Moreover, for $\gamma = (\gamma^1, \dots, \gamma^n) \in A^{\underline{s}}$ and for all $\phi : A^{\underline{s}} \to A_1 \times \cdots \times A_n$,*

$$\theta_0 |\phi| \ll ||\phi(\gamma)||.$$

Proof. Let $\phi = [\phi_1, \dots, \phi_n]$. Then $||\phi(\gamma)|| = \max_i ||\phi_i(\gamma^i)||$. It is then sufficient to prove the lemma for each i.

Fix the isomorphism from $\mathcal{E}_i$ to $\mathbb{Z}^{t_i}$ that maps τ^i_j to the j-th elements of the standard basis of $\mathbb{Z}^{t_i}$. Then Γ_i is also a $\mathbb{Z}$-module. Let $v_1, \dots, v_{s_i}$ be a set of $\mathcal{E}_i$-free generators of Γ_i. Apply [17] Lemma 3.4 with $\Gamma = \langle v_1, \dots, v_{s_i} \rangle_{\mathbb{Z}}$, $K = \theta_0$ and $b_i \in \mathbb{Z}$. Then, there exists a set of $\mathbb{Z}$ free generators $\gamma^i_1, \dots, \gamma^i_{s_i}$ of $\Gamma_0 = \langle v_1, \dots, v_{s_i} \rangle_{\mathbb{Q}}$ such that

$$\theta_0 \leq ||\gamma^i_j||$$

and

$$\frac{1}{3} \sum_j |b_j| ||\gamma^i_j|| \leq || \sum_j b_j \gamma^i_j ||$$

where $b_i \in \mathbb{Z}$. Note that $\Gamma_i = \Gamma_0 \otimes_{\mathbb{Z}} \mathcal{E}_i$. The Rosati norm and the Euclidean norm induced by $\mathbb{Z}^{t_i}$ on $\mathcal{E}_i$ are equivalent, because the rank is finite. Then, the above inequalities imply

$$\theta_0 |\phi_i| \ll \sum_k |\phi_{i,k}| \, ||\gamma^i_k|| \ll ||\phi_i(\gamma^i)||.$$

□

We prove here an important inclusion.

Proposition 5.4. *Let $\gamma \in A^{\underline{s}}$ be as in Lemma 5.3 for $\theta_0 = 1$. Let $\underline{r} \in \mathbb{N}^n$. To each weighted morphism $\phi : A^{\underline{g}} \to A^{\underline{r}}$ we can associate a special morphism $\tilde{\phi} = (\phi|\phi') : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ such that, for all $0 \leq \varepsilon \leq \theta$, the map $x \to (x, \gamma)$ defines an injection*

$$\big(V(\theta) \cap (B_\phi + \overline{\Gamma}_\varepsilon)\big) \hookrightarrow \Big((V(\theta) \times \gamma) \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon) \Big).$$

Proof. Let $x \in V(\theta) \cap (B_\phi + \overline{\Gamma}_\varepsilon)$. Then, there exist points $y \in \overline{\Gamma}$ and $\xi \in \mathcal{O}_\varepsilon \subset A^{\underline{g}}$ such that

$$\phi(x + y + \xi) = 0.$$

As γ is a set of free generators, there exist a positive integer N and a morphism $G : A^{\underline{g}} \to A^{\underline{r}}$ such that

$$Ny = G\gamma.$$

We define

$$\tilde{\phi} = (N\phi|\phi G).$$

Then

$$\tilde{\phi}((x, \gamma) + (\xi, 0)) = 0. \tag{9}$$

We already know that ϕ is weighted and therefore $N\phi$ is weighted too. Then, to prove that $\tilde{\phi}$ is special, it remains to prove that $|\tilde{\phi}| \ll N|\phi|$. Equivalently, we shall show that $|\phi'| \ll N|\phi|$. Let l and j be indices such that $|\phi'| = |\phi'_{lj}|$. Consider the l row of the equation (9). For φ_l and φ'_l the l-th rows of ϕ and ϕ' respectively, we have

$$||N\varphi_l(x + \xi)|| = ||\varphi'_l(\gamma)||.$$

So

$$||\varphi'_l(\gamma)|| = ||N\varphi_l(x + \xi)|| \ll |N\phi|(||x|| + ||\xi||).$$

By assumption $||x|| \leq \theta$ and $||\xi|| \leq \varepsilon \leq \theta$. So

$$||\varphi_l'(\gamma)|| \ll N|\phi|.$$

In view of Lemma 5.3, we deduce

$$|\varphi_l'| \ll N|\phi|.$$

Thus

$$|\varphi_l'| = |\phi'| \ll N|\phi|.$$

□

5.3. **Reducing to Conjecture 1.3 ii.** In this subsection we are going to prove Theorem 1.4. In the first instance, we study some properties of a morphism vanishing on a point of large rank. For this we need a lemma of the geometry of numbers.

Lemma 5.5. *Let $1 \leq i \leq n$. Let $q^i = (q_1^i, \dots, q_{s_i}^i)$ be a point of $A_i^{s_i}$ of rank s_i. There exist positive effective constants $c_0(q^i)$ and $\varepsilon_0(q^i)$, depending on q^i, such that*

$$c_0(q^i) \sum_j |b_j|^2 ||q_j^i||^2 \ll || \sum_j b_j(q_j^i - \xi_j)||^2$$

for all $b_1, \dots, b_{s_i} \in \mathcal{E}_i$ and for all $\xi_1, \dots, \xi_{s_i} \in \mathcal{O}_{\varepsilon_0(q^i)} \subset A_i^{s_i}$.

Proof. The Rosati involution defines a norm on $\mathcal{E}_i$ which is compatible with the height norm on A_i. Namely $||b_j q_j^i|| = |b_j| ||q_j^i||$. Thus $(\mathcal{E}_i, |\cdot|)$ is a hermitian free $\mathbb{Z}$-module of rank t_i and $(A_i, ||\cdot||)$ is a hermitian $\mathcal{E}_i$-module.

The proof is then the analogue of the proof of [17] Proposition 3.3, where one shall read A_i instead of E and consider $b = 0$. □

Corollary 5.6. *Let $q \in A^{\underline{s}}$ be a point of rank $\underline{s}$ and let $\psi : A^{\underline{s}} \to A_1 \times \cdots \times A_n$ be a morphism. Then, there exist positive constants $c_0(q)$ and $\varepsilon_0(q)$ such that*

$$c_0(q)|\psi| \ll ||\psi(q - \xi)||,$$

for all $\xi \in \mathcal{O}_{\varepsilon_0(q)} \subset A^{\underline{s}}$.

Proof. We simply apply the previous proposition to each block. Let $\psi = [\psi_1, \dots, \psi_n]$ with $\psi_i : A_i^{s_i} \to A_i$. Let $q = (q^1, \dots, q^n)$ with $q^i \in A_i^{s_i}$ and $\xi = (\xi^1, \dots, \xi^n)$ with $\xi^i \in A_i^{s_i}$. Note that

$$\max_i ||\psi_i(q^i - \xi^i)|| = ||\psi(q - \xi)||.$$

Apply Lemma 5.5 with $(b_1, \dots, b_{s_i}) = \psi_i$, $(\xi_1, \dots, \xi_{s_i}) = \xi^i$. Choose $c_0(q)$ to be the minimum of $c_0(q^i)$ and $\varepsilon_0(q)$ to be the minimum of $\varepsilon_0(q^i)$, for $i = 1, \dots, n$. □

Lemma 5.7. *Let $q \in A^{\underline{s}}$ be a point of rank $\underline{s}$. Let $\tilde{\phi} = (\phi|\phi') : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ be a surjective morphism. Let $\varepsilon \leq \varepsilon_0(q)$ where $\varepsilon_0(q)$ is as in Corollary 5.6.*

If there exists a point $(x, q) \in B_{\tilde{\phi}} + \mathcal{O}_\varepsilon$, then

i. *ϕ has rank $\underline{r}$,*
ii. *There exists $\tilde{\psi} = (\psi|\psi') : A^{\underline{g}+\underline{s}} \to A^{\underline{r}}$ with ψ weighted, such that*

$$B_{\tilde{\phi}} \subset B_{\tilde{\psi}} + \mathrm{Tor}_{A^{\underline{g}}}.$$

Proof. i- Suppose that the rank of ϕ is less than $\underline{r}$. Then, there exists $\lambda = [\lambda_1, \dots, \lambda_n]$ with $\lambda_i \in \mathcal{E}_i^{r_i}$ such that

$$\lambda\phi = 0.$$

Let $(x, q) \in B_{\tilde{\phi}} + \mathcal{O}_\varepsilon$. Then, there exists $(\xi, \xi') \in \mathcal{O}_\varepsilon$ such that

$$\tilde{\phi}((x, q) + (\xi, \xi')) = 0.$$

So

$$\lambda\phi'(q+\xi') = -\lambda\phi(x+\xi) = 0.$$

Corollary 5.6, applied with $\xi = -\xi'$ and any non-trivial ψ, implies that $q+\xi$ has rank $\underline{s}$, whence $\lambda\phi' = 0$. So $\lambda\tilde{\phi} = 0$. This contradicts that $\tilde{\phi}$ has full rank $\underline{r}$.

ii- By part i we can assume that rank ϕ is $\underline{r}$. By Lemma 5.2 applied to ϕ, there exists an invertible Δ such that $\Delta\phi$ is weighted. Then $\tilde{\psi} = \Delta\tilde{\phi}$ satisfies ii. □

We can now prove a statement slightly more precise than Theorem 1.4.

Theorem 5.8 (Reformulation of Theorem 1.4)**.** *Let $\varepsilon \geq 0$ and $0 \leq k \leq \dim A$,*

i. *The map $x \to (x,\gamma)$ defines an injection*

$$S_k(V,\Gamma_\varepsilon) \hookrightarrow S_k(V\times\gamma, \mathcal{O}_\varepsilon).$$

ii. *If $\varepsilon \leq \varepsilon_0(p)$, where $\varepsilon_0(p)$ is as in Corollary 5.6. Then, the map $(x,p) \to x$ defines an injection*

$$S_k(V(\theta)\times p, \mathcal{O}_\varepsilon) \hookrightarrow S_k\left(V(\theta), (\overline{\Gamma_p})_{\varepsilon'}\right),$$

where $\varepsilon' \ll \varepsilon$ and $\overline{\Gamma_p}$ is the saturated module of Γ_p.

Proof. Part i. is an immediate consequence of Proposition 5.4, if $\varepsilon \leq \theta$. In general, relation (9) gives that if $x \in S_k(V,\Gamma_\varepsilon)$, then $(x,\gamma) \in V \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon) \subset S_k(V\times\gamma, \mathcal{O}_\varepsilon)$.

ii. Let $(x,p) \in S_k(V(\theta)\times p, \mathcal{O}_\varepsilon)$. Then, there exists a block matrix $\tilde{\phi} = [\tilde{\phi}_1, \dots, \tilde{\phi}_n]$ of rank $\underline{r}$ with $k \leq \sum_i d_i r_i$, and $(\zeta,\zeta') \in \mathcal{O}_\varepsilon$ such that

$$\tilde{\phi}((x,p) + (\zeta,\zeta')) = 0. \tag{10}$$

In view of Lemma 5.7 ii. for $q = p$, we can assume that $\tilde{\phi} = (\phi|\phi')$ with ϕ weighted. Let $aI_{\underline{r}}$ be a submatrix of ϕ with $|\phi| \ll a$ and $i_{\underline{r}} : A^{\underline{r}} \to A^{\underline{g}}$ such that $\phi \cdot i_{\underline{r}} = [a]$.

We want to show that $|\phi'| \ll |\phi|$. Let l and j be indices such that $|\phi'| = |\phi'_{lj}|$. Consider the l-th row of the equation (10). For φ_l and φ'_l the l-th rows of ϕ and ϕ' respectively, we have

$$||\varphi'_l(p+\xi')|| = ||\varphi_l(x+\xi)|| \ll |\phi|\left(||x|| + ||\xi||\right).$$

By assumption $||x|| \leq \theta$ and $||\xi|| \leq \varepsilon \leq \varepsilon_0(p)$. So

$$||\varphi'_l(p+\xi')|| \ll |\phi|.$$

By Corollary 5.6 applied with $q = p$, $\psi = \varphi'_l$ and $\xi = -\xi'$, we deduce

$$|\phi'| = |\varphi'_l| \ll |\phi| \ll a. \tag{11}$$

Define

$$[a](y') = \phi'(p) \quad \text{and} \quad y = i_{\underline{r}}(y') \in \overline{\Gamma_p},$$
$$[a](\zeta') = \tilde{\phi}(\xi,\xi') \quad \text{and} \quad \zeta = i_{\underline{r}}(\zeta') \in A^{\underline{g}}.$$

Then, for $y \in \overline{\Gamma_p}$,

$$\phi(x+y+\zeta) = 0.$$

We shall still show that $||\xi|| \leq \varepsilon'$. By relation (11), $\frac{|\tilde{\phi}|}{a} \ll 1$. In addition $||(\xi,\xi')|| \leq \varepsilon$. We then obtain

$$||\zeta|| = \frac{||\zeta'||}{a} \ll \frac{|\tilde{\phi}|}{a}||(\xi,\xi')|| \ll \varepsilon.$$

We conclude that

$$x \in V(\theta) \cap (B_\phi + \overline{\Gamma_p} + \mathcal{O}_{\varepsilon'}),$$

with $||\zeta|| \leq \varepsilon' \ll \varepsilon$.

□

5.4. **Conclusion.**

Proof of Theorem 1.6. Thanks to Theorem 1.4, it is sufficient to prove that Conjecture 1.5 implies Conjecture 1.3 ii.

Note that $\Gamma \subset \overline{\Gamma}$. By Lemma 5.2 ii., for all $\varepsilon \geq 0$,

$$S_{d+1}(V(\theta), \Gamma_\varepsilon) \subset \Big(V(\theta) \cap \bigcup_{\substack{\phi \text{ weighted} \\ \mathrm{cod}\phi \geq d+1}} \big(B_\phi + \overline{\Gamma}_\varepsilon \big) \Big).$$

Let γ be as in Lemma 5.3 for $\theta_0 = 1$. By Proposition 5.4, for $\varepsilon \leq \theta$,

$$\Big(V(\theta) \cap \bigcup_{\substack{\phi \text{ weighted} \\ \mathrm{cod}\, \phi \geq d+1}} \big(B_\phi + \overline{\Gamma}_\varepsilon \big) \Big) \hookrightarrow \Big((V(\theta) \times \gamma) \cap \bigcup_{\substack{\tilde{\phi} = (\phi|\phi') \text{ special} \\ \mathrm{cod}\, \tilde{\phi} \geq d+1}} \Big(B_{\tilde{\phi}} + \mathcal{O}_\varepsilon \Big) \Big).$$

By Theorem 3.4, for $\varepsilon > 0$, there exist a positive real M depending on ε and $\varepsilon' \ll \varepsilon$ such that

$$\bigcup_{\substack{\tilde{\phi} = (\phi|\phi') \text{ special} \\ \mathrm{cod}\, \phi \geq d+1}} \Big((V(\theta) \times \gamma) \cap \Big(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/M} \Big) \Big)$$

$$\subset \bigcup_{\substack{\tilde{\phi} = (\phi|\phi') \text{ special} \\ \mathrm{cod}\, \tilde{\phi} \geq d+1 \\ |\tilde{\phi}| \ll M}} \Big((V(\theta) \times \gamma) \cap \Big(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon'/|\tilde{\phi}|} \Big) \Big).$$

Note that on the right hand side the union is taken over finitely many $\tilde{\phi}$, because $|\tilde{\phi}| \ll M$.

Let ε_2 be as in Corollary 4.9. Choose $\varepsilon' \leq \varepsilon_2$ (and consequently choose ε). Note that $|\phi| \leq |\tilde{\phi}|$. By Corollary 4.9,

$$(V(\theta) \times \gamma) \cap \Big(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon'/|\tilde{\phi}|} \Big)$$

is non Zariski-dense in $V \times \gamma$.

We conclude that $S_{d+1}(V(\theta), \Gamma_{\varepsilon/M})$ is embedded in a finite union of non Zariski-dense sets.

□

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